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Abstract

Autocorrelation robust tests are notorious for suffering from size distortions and power problems. We investigate under which conditions the size of autocorrelation robust tests can be controlled by an appropriate choice of critical value.

1   Introduction

Autocorrelation robust tests have gained prominence in econometrics in the late 1980s and early 1990s, mainly through the work by Newey and West (1987, 1994), Andrews (1991), and Andrews and Monahan (1992). These tests are Wald-type tests that use a nonparametric variance estimator which tries to take the autocorrelation in the data into account. For more on the history of such tests, that can actually be traced back at least to Grenander and Rosenblatt (1957), see Section 1 of Preinerstorfer and Pötscher (2016). Critical values are usually obtained from the asymptotic null distribution, which is a chi-square distribution under the framework used in the before mentioned papers. Considerable size distortions of the resulting tests have been reported in several Monte Carlo studies. In an attempt to ameliorate this problem, Kiefer et al. (2000), Kiefer and Vogelsang (2002a,b, 2005) have suggested an alternative asymptotic framework ("fixed bandwidth asymptotics"), which resulted in another proposal for a critical value. While this proposal often leads to some reduction in the size distortion (typically at the expense of some loss in power), it does not eliminate the problem. A finite-sample theory explaining the aforementioned size distortions and power deficiencies of autocorrelation robust tests (in either

of their two variants) has been developed in Preinerstorfer and Pötscher (2016) in the framework of testing affine restrictions on the regression parameters of a linear regression model, the errors of which follow a Gaussian stationary process: Under the very mild assumption that the model for the autocorrelations encompasses the stationary autoregressive model of order one, Section 3 of Preinerstorfer and Pötscher (2016) shows, in particular, that the following cases typically arise for a given autocorrelation robust test statistic (which of the cases arises depending on observable quantities only):

1. The test has size one for every choice of the critical value C. This case arises, for example, when one wants to test a hypothesis concerning the intercept of a regression.

2. There exists an observable threshold c such that the size of the test is 1, if the critical value C is less than c. If the critical value C is larger than the threshold c, the infimal power of the test is zero.

We note that these results are to a large extent driven by a "concentration phenomenon" that arises for strong correlations in the error process. For a more detailed discussion and some intuition see Sections 3.1 and 5.2 as well as pp. 275-276 of Preinerstorfer and Pötscher (2016).

While the two cases just mentioned do not exhaust all possibilities, their union is shown to be *generic* in Preinerstorfer and Pötscher (2016), entailing that general autocorrelation robust tests are typically fraught with considerable size and/or power problems. Preinerstorfer and Pötscher (2016) also show that there is a case not covered by Cases 1 and 2 (and which hence is non-generic) for which the following holds, *provided* the model for the autocorrelation is precisely the stationary autoregressive model of order one (which, of course, is a quite restrictive assumption):

3. For every prescription of a significance value α, the size of the test can be made less than or equal to α by a proper choice of the critical value C. For this C, infimal power is positive (and more can be said about power properties, but we do not discuss this here).

While this case is nongeneric, Preinerstorfer and Pötscher (2016) also show how one can – under the restrictive assumption on the correlation structure just mentioned – often force the original testing problem into the situation of Case 3 by augmenting the model with auxiliary regressors and then using the autocorrelation robust test on the augmented model. We note that the above mentioned results in Preinerstorfer and Pötscher (2016) are given for commonly used autocorrelation robust test statistics where the bandwidth is nonrandom. Preinerstorfer (2017) extends these results to the case of random bandwidths and when prewithening is used. As shown in Section 3.3 of Preinerstorfer and Pötscher (2016), similar results also hold if a parametrically-based long-run variance estimator or feasible generalized least squares are used.

A fundamental question that is not satisfactorily answered by the results in Preinerstorfer and Pötscher (2016) is under which conditions one can control the size of the test (i.e., under which conditions one can find a critical value C in such a way that the resulting test has size less than or equal to the prescribed significance value α, $\alpha < 1$): While in Case 1 we have a

definite (negative) answer, the result in Case 2 as it stands tells us only that one can not choose C smaller than the threshold c if one wants to control the size by α (thus allowing one to rule out proposals for C if they fail this criterion), but it does not tell us anything about the behavior of the size if C is chosen larger than the threshold c. And the result in Case 3, which guarantees size control, is limited by a quite restrictive assumption on the correlation structure.

In the present paper we answer the question under which conditions the size of an auto-correlation robust test can be controlled. In particular, we provide sufficient conditions that guarantee that size control is possible even if we allow for *arbitrary* stationary autocorrelation in the data. We then show that these conditions are broadly satisfied, namely that they are satisfied *generically* (i.e., are – given the restriction to be tested – satisfied except for a Lebesgue null set in an appropriate universe of relevant design matrices). We also discuss how the critical value that leads to size control can be determined numerically and provide the R-package **acrt** (Preinerstorfer (2016)) for its computation. The usefulness of the proposed algorithms and their implementation in the R-package are illustrated on testing problems involving macroeconomic data taken from the FRED-MD database of McCracken and Ng (2016). In particular, we show that – even in situations where size control is possible – standard critical values in the literature based on asymptotic considerations are often substantially smaller than the size-controlling critical values devised in the present paper, and hence fail to deliver size control. Furthermore, in a subsequent paper Pötscher and Preinerstorfer (2017) we show that the sufficient conditions for size control we provide here are in a sense also necessary (for a large class of test statistics). While the main emphasis in the current paper is on stationary autocorrelation in the data, the general theory for size control developed in Section 5 applies to arbitrary models for the covariance structure of the data (including e.g., the case of heteroskedasticity or the case of spatial correlation). The present paper is only concerned with the size of autocorrelation robust tests and its control. Power properties of such tests and ways of improving power without loosing the size control property is the topic of a companion paper. While a trivial remark, we would like to note that the size control results given in this paper can obviously be translated into results stating that the minimal coverage probability of the associated confidence set obtained by "inverting" the test is not less than the nominal level.

The development in the present paper concentrates on controlling the size of autocorrelation robust tests by an appropriate choice of a 'fixed' (i.e., nonrandom) critical value. Extensions to size control by use of random critical values are the subject of ongoing research.

We also note that, on a broader scale, our results contribute to an important recent literature concerning size properties of tests and minimal coverage properties of confidence sets, see, e.g., Andrews and Guggenberger (2009), Guggenberger et al. (2012), Mikusheva (2007), Moreira (2009), or Romano and Shaikh (2012)).

The paper is organized as follows: Section 2 lays out the framework and introduces the basic concepts. Section 3 contains the size control results for commonly used autocorrelation robust tests: In Section 3.1 we present a theorem (Theorem 3.2) that lists easy to verify sufficient

conditions for size control under a very broad model for the autocorrelation structure in the data. Several examples illustrate this theorem. A similar result for a variant of the test statistic considered in Theorem 3.2 is provided in Theorem 3.8. In Section 3.2 we make precise in which sense the conditions for size control in Theorems 3.2 and 3.8 are generically satisfied. After a short section (Section 3.3) commenting on power properties, we discuss extensions of the results in Sections 3.1 and 3.2 to other classes of test statistics. Computational issues regarding the determination of the critical values effecting size control are discussed in Section 4. This section also contains a numerical illustration of the algorithms suggested. The results in Section 3 rest on a general theory for size control of tests under general forms of nonsphericity, which is developed in Section 5. In Section 6 we apply this general theory to the case where the nonsphericity is a consequence of stationary autocorrelation in the data and we obtain size control results more general (but also more complex) than the ones considered in Section 3. In Appendix E we show that the Gaussianity assumption underlying the paper can be substantially relaxed, whereas Appendices A-D contain the proofs and some auxiliary results for the main body of the paper. Appendices F-H contain material relevant for the numerical results in Section 4 including a detailed description of the algorithms used. Appendix I investigates a proposal for choosing critical values suggested by a referee, while Appendix J discusses what happens if, instead of stationary solutions, starting value solutions of autoregressive models are used as the error process in the linear model; in particular, in this appendix we show that under this alternative covariance model size control is always possible, but that standard critical values proposed in the literature fail to deliver size control. Appendix K provides some comments on the case of stochastic regressors. Finally, Appendix L contains some tables pertaining to Section 4.

2   Framework

Consider the linear regression model

$$\mathbf{Y} = X\beta + \mathbf{U}, \tag{1}$$

where X is a (real) nonstochastic regressor (design) matrix of dimension $n \times k$ and where $\beta \in \mathbb{R}^k$ denotes the unknown regression parameter vector. We always assume $\operatorname{rank}(X) = k$ and $1 \leq k < n$. We furthermore assume that the $n \times 1$ disturbance vector $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)'$ is normally distributed with mean zero and unknown covariance matrix $\sigma^2 \Sigma$, where Σ varies in a prescribed (nonempty) set $\mathfrak{C}$ of symmetric and positive definite $n \times n$ matrices and where $0 < \sigma^2 < \infty$ holds (σ always denoting the positive square root).[1] The set $\mathfrak{C}$ will be referred to as the covariance

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[1]Since we are concerned with finite-sample results only, the elements of $\mathbf{Y}$, X, and $\mathbf{U}$ (and even the probability space supporting $\mathbf{Y}$ and $\mathbf{U}$) may depend on sample size n, but this will not be expressed in the notation. Furthermore, the obvious dependence of $\mathfrak{C}$ on n will also not be shown in the notation.

model. We shall always assume that $\mathfrak{C}$ allows σ^2 and Σ to be uniquely determined from $\sigma^2\Sigma$.[2] [This entails virtually no loss of generality and can always be achieved, e.g., by imposing some normalization assumption on the elements of $\mathfrak{C}$ such as normalizing the first diagonal element of Σ or the norm of Σ to one, etc.] The leading case discussed in Section 3 will concern the situation where $\mathfrak{C}$ results from the assumption that the elements $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of the $n \times 1$ disturbance vector $\mathbf{U}$ are distributed like consecutive elements of a zero mean weakly stationary Gaussian process with an unknown spectral density.

The linear model described in (1) together with the Gaussianity assumption on $\mathbf{U}$ induces a collection of distributions on the Borel-sets of $\mathbb{R}^n$, the sample space of $\mathbf{Y}$. Denoting a Gaussian probability measure with mean $\mu \in \mathbb{R}^n$ and (possibly singular) covariance matrix A by $P_{\mu,A}$, the induced collection of distributions is then given by

$$\left\{ P_{\mu,\sigma^2\Sigma} : \mu \in \text{span}(X), 0 < \sigma^2 < \infty, \Sigma \in \mathfrak{C} \right\}. \tag{2}$$

Since every $\Sigma \in \mathfrak{C}$ is positive definite by assumption, each element of the set in the previous display is absolutely continuous with respect to (w.r.t.) Lebesgue measure on $\mathbb{R}^n$.

We shall consider the problem of testing a linear (better: affine) hypothesis on the parameter vector $\beta \in \mathbb{R}^k$, i.e., the problem of testing the null $R\beta = r$ against the alternative $R\beta \neq r$, where R is a $q \times k$ matrix always of rank $q \geq 1$ and $r \in \mathbb{R}^q$. Set $\mathfrak{M} = \text{span}(X)$. Define the affine space

$$\mathfrak{M}_0 = \{\mu \in \mathfrak{M} : \mu = X\beta \text{ and } R\beta = r\}$$

and let

$$\mathfrak{M}_1 = \{\mu \in \mathfrak{M} : \mu = X\beta \text{ and } R\beta \neq r\}.$$

Adopting these definitions, the above testing problem can then be written more precisely as

$$H_0 : \mu \in \mathfrak{M}_0, \ 0 < \sigma^2 < \infty, \ \Sigma \in \mathfrak{C} \quad \text{vs.} \quad H_1 : \mu \in \mathfrak{M}_1, \ 0 < \sigma^2 < \infty, \ \Sigma \in \mathfrak{C}. \tag{3}$$

The previously introduced concepts and notation will be used throughout the paper.

The assumption of Gaussianity is made mainly in order not to obscure the structure of the problem by technicalities. Substantial generalizations away from Gaussianity are possible and will be discussed in Appendix E. The assumption of nonstochastic regressors can be relaxed somewhat: If X is random and, e.g., independent of $\mathbf{U}$, the results of the paper apply after one conditions on X. For arguments supporting conditional inference see, e.g., Robinson (1979). In case X is random with X and $\mathbf{U}$ being dependent, our *theory* (at least in its current state) is not applicable. However, we show in Appendix K by means of numerical examples that in such a case the critical values generated by the *algorithm* proposed in Section 4 can lead to tests that perform at least as well as (and often better than) tests that are based on standard critical

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[2]That is, $\mathfrak{C}$ has the property that $\Sigma \in \mathfrak{C}$ implies $\delta\Sigma \notin \mathfrak{C}$ for every $\delta \neq 1$.

values suggested in the literature.[3] And this is so despite the fact that the asymptotic theory used in the literature to justify the latter critical values *is* applicable to cases where X and $\mathbf{U}$ are dependent (in particular to the examples considered in Appendix K).

We next collect some further terminology and notation used throughout the paper. A (non-randomized) *test* is the indicator function of a Borel-set W in $\mathbb{R}^n$, with W called the corresponding *rejection region*. The *size* of such a test (rejection region) is the supremum over all rejection probabilities under the null hypothesis H_0, i.e., $\sup_{\mu \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu, \sigma^2 \Sigma}(W)$. Throughout the paper we let $\hat{\beta}(y) = (X'X)^{-1} X'y$, where X is the design matrix appearing in (1) and $y \in \mathbb{R}^n$. The corresponding ordinary least squares (OLS) residual vector is denoted by $\hat{u}(y) = y - X\hat{\beta}(y)$. We use Pr as a generic symbol for a probability measure. Lebesgue measure on the Borel-sets of $\mathbb{R}^n$ will be denoted by $\lambda_{\mathbb{R}^n}$, whereas Lebesgue measure on an affine subspace $\mathcal{A}$ of $\mathbb{R}^n$ (but viewed as a measure on the Borel-sets of $\mathbb{R}^n$) will be denoted by $\lambda_{\mathcal{A}}$, with zero-dimensional Lebesgue measure being interpreted as point mass. The set of real matrices of dimension $l \times m$ is denoted by $\mathbb{R}^{l \times m}$ (all matrices in the paper will be real matrices) and Lebesgue measure on this set, equipped with its Borel σ-field, is denoted by $\lambda_{\mathbb{R}^{l \times m}}$. The Euclidean norm of a vector is denoted by $\|\cdot\|$, but the same symbol is also used to denote a norm of a matrix. Let B' denote the transpose of a matrix $B \in \mathbb{R}^{l \times m}$ and let $\mathrm{span}(B)$ denote the subspace in $\mathbb{R}^l$ spanned by its columns. For a symmetric and nonnegative definite matrix B we denote the unique symmetric and nonnegative definite square root by $B^{1/2}$. For a linear subspace $\mathcal{L}$ of $\mathbb{R}^n$ we let $\mathcal{L}^\perp$ denote its orthogonal complement and we let $\Pi_{\mathcal{L}}$ denote the orthogonal projection onto $\mathcal{L}$. We use the convention that the adjoint of a 1×1 dimensional matrix D, i.e., $\mathrm{adj}(D)$, equals one. Given an m-dimensional vector v we write $\mathrm{diag}(v)$ for the $m \times m$ diagonal matrix with main diagonal given by v. The j-th standard basis vector in $\mathbb{R}^n$ is written as $e_j(n)$. With e_+ we denote the $n \times 1$ vector of ones, i.e., $e_+ = (1, \ldots, 1)'$ and we define the $n \times 1$ vector $e_- = (-1, 1, \ldots, (-1)^n)'$. Furthermore, we let $\mathbb{N}$ denote the set of all positive integers. A sum (product, respectively) over an empty index set is to be interpreted as 0 (1, respectively). Finally, for a subset A of a topological space we denote by $\mathrm{cl}(A)$, $\mathrm{int}(A)$, and $\mathrm{bd}(A)$ the closure, interior, and boundary of A (w.r.t. the ambient space), respectively.

3 Size control of tests of affine restrictions in regression models with stationary autocorrelated errors: First results

In this section we are concerned with size control of autocorrelation robust tests that have been designed for use in case of stationary disturbances. We thus assume throughout this section that the elements $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of the $n \times 1$ disturbance vector $\mathbf{U}$ are distributed like consecutive elements of a zero mean weakly stationary Gaussian process that has an unknown spectral density, which

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[3]That is, the algorithm is used acting as if X and $\mathbf{U}$ were independent.

is not almost everywhere equal to zero. Consequently, the covariance matrix of the disturbance vector is positive definite and can be written as $\sigma^2 \Sigma(f)$ where

$$\Sigma(f) = \left[\int_{-\pi}^{\pi} e^{-\iota(j-l)\omega} f(\omega) d\omega \right]_{j,l=1}^n,$$

with f varying in $\mathfrak{F}$, a prescribed (nonempty) family of *normalized* (i.e., $\int_{-\pi}^{\pi} f(\omega) d\omega = 1$) spectral densities, and where $0 < \sigma^2 < \infty$ holds. Here ι denotes the imaginary unit. The set $\mathfrak{F}$ may, for example, be $\mathfrak{F}_{\mathrm{all}}$, the set of *all* normalized spectral densities, or a subset thereof (e.g., the set of normalized spectral densities corresponding to stationary autoregressive or autoregressive moving average models of a certain order, or to fractional autoregressive moving average models, etc.). We define the associated covariance model $\mathfrak{C}(\mathfrak{F}) = \{\Sigma(f) : f \in \mathfrak{F}\}$ and note that the test problem (3) now becomes

$$H_0 : \mu \in \mathfrak{M}_0, \ 0 < \sigma^2 < \infty, \ f \in \mathfrak{F} \quad \text{vs.} \quad H_1 : \mu \in \mathfrak{M}_1, \ 0 < \sigma^2 < \infty, \ f \in \mathfrak{F}. \tag{4}$$

Remark 3.1. (i) As is well-known, the covariance model $\mathfrak{C}(\mathfrak{F}_{\mathrm{all}})$ is precisely the set of *all* $n \times n$ symmetric and positive definite Toeplitz matrices with ones on the main diagonal, cf. Lemma C.8 and Remark C.9 in Appendix C. It is thus maximal in the sense that it coincides with the set of *all* positive definite $n \times n$ correlation matrices that can be generated from sections of length n of stationary processes (possessing a spectral density or not).

(ii) Furthermore, as is well-known, $\mathfrak{C}(\mathfrak{F}_{\mathrm{all}})$ coincides with $\mathfrak{C}(\mathfrak{F}_{\mathrm{AR}(p)})$ if $p \geq n-1$, where $\mathfrak{F}_{\mathrm{AR}(p)}$ is the set of all normalized spectral densities corresponding to stationary autoregressive processes of order not larger than p, cf. Remark C.9 in Appendix C. As the testing problem depends on $\mathfrak{F}$ only via $\mathfrak{C}(\mathfrak{F})$, the testing problems with $\mathfrak{F} = \mathfrak{F}_{\mathrm{all}}$ and $\mathfrak{F} = \mathfrak{F}_{\mathrm{AR}(p)}$ (with $p \geq n-1$), respectively, coincide. Thus we can use that "parameterization" of the covariance model (represented by $\mathfrak{F}_{\mathrm{all}}$ and $\mathfrak{F}_{\mathrm{AR}(p)}$, respectively), which is more convenient for our purpose. More generally, if $\mathfrak{C}(\mathfrak{F}_1) = \mathfrak{C}(\mathfrak{F}_2)$ holds for two sets of normalized spectral densities $\mathfrak{F}_1$ and $\mathfrak{F}_2$, the same argument can be made.

Commonly used autocorrelation robust tests for the null hypothesis H_0 given by (4) are based on test statistics $T_w : \mathbb{R}^n \to \mathbb{R}$ of the form

$$T_w(y) = \begin{cases} (R\hat{\beta}(y) - r)' \hat{\Omega}_w^{-1}(y)(R\hat{\beta}(y) - r) & \text{if } \det \hat{\Omega}_w(y) \neq 0 \\ 0 & \text{if } \det \hat{\Omega}_w(y) = 0 \end{cases} \tag{5}$$

for $y \in \mathbb{R}^n$, where R and r are as in Section 2, and where

$$\hat{\Omega}_w(y) = nR(X'X)^{-1} \hat{\Psi}_w(y)(X'X)^{-1} R', \tag{6}$$

$$\hat{\Psi}_w(y) = \sum_{j=-(n-1)}^{n-1} w(j,n) \hat{\Gamma}_j(y). \tag{7}$$

Here $\hat{\Gamma}_j(y) = n^{-1} \sum_{t=j+1}^n \hat{v}_t(y) \hat{v}_{t-j}(y)'$ if $j \geq 0$ and $\hat{\Gamma}_j(y) = \hat{\Gamma}_{-j}(y)'$ else, where $\hat{v}_t(y) = \hat{u}_t(y)x_{t\cdot}'$, with $\hat{u}_t(y)$ denoting the t-th coordinate of the least squares residual vector $\hat{u}(y) = y - X\hat{\beta}(y)$ and with $x_{t\cdot}$ denoting the t-th row vector of X. Rejection is for large values of T_w. We make the following standard assumption on the weights.

Assumption 1. The weights $w(j,n)$ for $j = -(n-1), \ldots, n-1$ are data-independent and satisfy $w(0,n) = 1$ as well as $w(-j,n) = w(j,n)$. Furthermore, the symmetric $n \times n$ Toeplitz matrix $\mathcal{W}_n$ with elements $w(i-j,n)$ is positive definite.

This assumption implies that $\hat{\Psi}_w(y)$, and hence $\hat{\Omega}_w(y)$, is always nonnegative definite, see Lemma 3.1 in Preinerstorfer and Pötscher (2016). In many applications the weights take the form $w(j,n) = w_0(|j|/M_n)$, where the lag-window w_0 is an even function with $w_0(0) = 1$ and where $M_n > 0$ is a truncation lag (bandwidth) parameter. In this case the first part of the above assumption means that we are considering deterministic bandwidths only. For extensions of the results in this section to more general classes of tests statistics, including the case of data-dependent bandwidth choices and prewhitening, see Subsection 3.4. Assumption 1 is known to be satisfied, e.g., for the (modified) Bartlett, Parzen, or the Quadratic Spectral lag-window, but is not satisfied, e.g., for the rectangular lag-window (with $M_n > 1$). It is also satisfied for many exponentiated lag-windows as used in Phillips et al. (2006, 2007) and Sun et al. (2011). For more discussion of Assumption 1 see Preinerstorfer and Pötscher (2016).

Assigning the test statistic T_w the value zero on the set where $\hat{\Omega}_w(y)$ is singular is, of course, arbitrary. Given Assumption 1, the set where $\det \hat{\Omega}_w(y) = 0$ holds can be shown to be

$$\mathsf{B} = \{y \in \mathbb{R}^n : \operatorname{rank}(B(y)) < q\},$$

where

$$
\begin{aligned}
B(y) &= R(X'X)^{-1} X' \operatorname{diag}(\hat{u}_1(y), \ldots, \hat{u}_n(y)) \\
&= R(X'X)^{-1} X' \operatorname{diag}(e_1'(n) \Pi_{\operatorname{span}(X)^\perp} y, \ldots, e_n'(n) \Pi_{\operatorname{span}(X)^\perp} y),
\end{aligned}
\tag{8}
$$

see Lemma 3.1 of Preinerstorfer and Pötscher (2016). Clearly, $\operatorname{span}(X) \subseteq \mathsf{B}$ and $\mathsf{B} + \operatorname{span}(X) = \mathsf{B}$ always hold. Furthermore, Lemma 3.1 in Preinerstorfer and Pötscher (2016) shows that the set B is a $\lambda_{\mathbb{R}^n}$-null set if and only if Assumption 2 given below holds, and is the entire space $\mathbb{R}^n$ otherwise. It thus transpires that under Assumptions 1 and 2 the chosen assignment is irrelevant for size (and power) properties of the test (since all relevant distributions $P_{\mu, \sigma^2 \Sigma(f)}$ are absolutely continuous w.r.t. $\lambda_{\mathbb{R}^n}$ due to the fact that every $\Sigma(f)$ is positive definite); the case where Assumption 2 is violated is hopeless for autocorrelation robust tests based on T_w in that these then break down in the sense that the quadratic form appearing in (5) is not defined for any $y \in \mathbb{R}^n$ (as then $\mathsf{B} = \mathbb{R}^n$). [If one insists on using T_w as defined by (5) in this case, T_w then reduces to the trivial test statistic that is identically zero, an uninteresting situation.[4]] We

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[4]Of course, size control is then trivially possible, but leads to a test that never rejects.

stress that the subsequently given Assumption 2 can be readily checked in any given application and is not very restrictive since – for given restriction matrix R – it holds generically in X.

Assumption 2. Let $1 \leq i_1 < \ldots < i_s \leq n$ denote all the indices for which $e_{i_j}(n) \in \text{span}(X)$ holds where $e_j(n)$ denotes the j-th standard basis vector in $\mathbb{R}^n$. If no such index exists, set $s = 0$. Let $X'(\neg(i_1, \ldots i_s))$ denote the matrix which is obtained from X' by deleting all columns with indices i_j, $1 \leq i_1 < \ldots < i_s \leq n$ (if $s = 0$ no column is deleted). Then $\text{rank}\left(R(X'X)^{-1}X'(\neg(i_1, \ldots i_s))\right) = q$ holds.

For later use we note that under Assumption 2 the set B not only is a $\lambda_{\mathbb{R}^n}$-null set, but is a finite union of proper linear subspaces of $\mathbb{R}^n$, see Lemma 5.18 in Section 5.3. Also note that if $\mathsf{B} = \text{span}(X)$ holds, then Assumption 2 must hold (since $\text{span}(X)$ is a $\lambda_{\mathbb{R}^n}$-null set due to $k < n$).

The test statistic T_w defined in (5) is based on a long-run covariance estimator for $\mathbf{u}_t x'_{t\cdot}$.[5] For nonstochastic regressors (as considered here) an alternative is to use the weighted Eicker-test statistic $T_{E,\mathsf{W}}$ that is based on a long-run covariance estimator for $\mathbf{u}_t$. The test statistic $T_{E,\mathsf{W}}$ is of the same form as given in (5), but with the estimator $\hat{\Psi}_{E,\mathsf{W}}(y) = n^{-1}X'(\hat{K}(y) \bullet \mathsf{W})X$ being used instead of $\hat{\Psi}_w$, where $\hat{K}(y)$ is the symmetric $n \times n$ Toeplitz matrix with (i,j)-th element given by $\hat{K}_{ij}(y) = n^{-1}\sum_{l=|i-j|+1}^{n}\hat{u}_l(y)\hat{u}_{l-|i-j|}(y)$, where W is an $n \times n$ symmetric and nonnegative definite Toeplitz matrix of weights with ones on the main diagonal, and where $\bullet$ denotes the Hadamard product. The (unweighted) Eicker-test statistic T_E discussed on pp. 283-284 of Preinerstorfer and Pötscher (2016) corresponds to the case where W is the matrix of all ones. As discussed in that reference, the corresponding matrix $\hat{\Omega}_E(y)$ is always nonnegative definite and the set where $\hat{\Omega}_E(y)$ is singular is given by $\text{span}(X)$, which is a $\lambda_{\mathbb{R}^n}$-null set by our maintained assumption that $k < n$ holds. By a result of Schur (see Theorem 3.1 in Horn (1990)) it is easy to see that all this is also true in the weighted case, i.e., for $\hat{\Omega}_{E,\mathsf{W}}(y) = nR(X'X)^{-1}\hat{\Psi}_{E,\mathsf{W}}(y)(X'X)^{-1}R'$.

3.1 Results on size control

To state the main result of this section we need to introduce some further notation: Let $\omega \in [0, \pi]$ and let $s \geq 0$ be an integer. Define $E_{n,s}(\omega)$ as the $n \times 2$-dimensional matrix with j-th row equal to $(j^s \cos(j\omega), j^s \sin(j\omega))$.

Definition 3.1. Given a linear subspace $\mathcal{L}$ of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$, define for every $\omega \in [0, \pi]$

$$\rho(\omega, \mathcal{L}) = \min\{s \in \mathbb{N} \cup \{0\} : \text{span}(E_{n,s}(\omega)) \nsubseteq \mathcal{L}\}. \tag{9}$$

We note that the set on the right-hand side of (9) is nonempty for every $\omega \in [0, \pi]$, and hence ρ is well-defined and takes values in $\mathbb{N} \cup \{0\}$. Furthermore, $\rho(\omega, \mathcal{L}) > 0$ holds at most for finitely many $\omega \in [0, \pi]$. See Appendix A for a proof of these claims. We denote by $\mathfrak{M}_0^{lin}$ the linear space

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[5]This test statistic makes sense for stochastic as well as nonstochastic regressors under appropriate assumptions. Note that the asymptotic theory for T_w in a bandwidth-to zero as well as in a fixed-bandwidth scenario relies only on limit theorems for partial sums of $\mathbf{u}_t x'_{t\cdot}$ and of $x'_{t\cdot}x_t$ which are satisfied for various classes of stochastic as well as nonstochastic regressors.

parallel to the affine space $\mathfrak{M}_0$, i.e., $\mathfrak{M}_0^{lin} = \mathfrak{M}_0 - \mu_0$ for $\mu_0 \in \mathfrak{M}_0$ (clearly, this does not depend on the choice of $\mu_0 \in \mathfrak{M}_0$).

3.1.1 Result for T_w

Our first result concerning size control is given next and is an immediate consequence of Theorem 6.6 in Section 6.2. This result is given for the test statistic T_w. A similar result for the weighted Eicker-test statistic $T_{E,W}$ is given in Subsection 3.1.2.

Theorem 3.2. [6] *Suppose Assumptions 1 and 2 are satisfied and T_w is defined by (5). Then for every $0 < \alpha < 1$ there exists a real number $C(\alpha)$ such that*

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{f \in \mathfrak{F}_{\mathrm{all}}} P_{\mu_0, \sigma^2 \Sigma(f)}(T_w \geq C(\alpha)) \leq \alpha \tag{10}$$

holds, provided that

$$\mathrm{span}\left(E_{n, \rho(\gamma, \mathfrak{M}_0^{lin})}(\gamma)\right) \not\subseteq \mathsf{B} \quad \text{for every} \ \ \gamma \in [0, \pi]. \tag{11}$$

In case the set B coincides with $\mathrm{span}(X)$, condition (11) can equivalently be expressed as

$$\mathrm{rank}\left(X, E_{n, \rho(\gamma, \mathfrak{M}_0^{lin})}(\gamma)\right) > k \quad \text{for every} \ \ \gamma \in [0, \pi]. \tag{12}$$

Furthermore, under the same condition (11) even equality can be achieved in (10) by a proper choice of $C(\alpha)$, provided $\alpha \in (0, \alpha^*] \cap (0, 1)$, where α^* is defined in (28) (with T replaced by T_w) further below.

It turns out, see Subsection 3.2 below, that for many combinations of design matrices X and restriction matrices R the set B coincides with $\mathrm{span}(X)$, and hence (11) reduces to the simpler rank condition (12). Furthermore, although a trivial observation, it should be kept in mind that $C(\alpha)$ depends not only on α but also on the testing problem at hand (i.e., on X, R, r, and the covariance model (here $\mathfrak{C}(\mathfrak{F}_{\mathrm{all}})$) as well as on the choice of test statistic (here on the choice of weights $w(j, n)$); but see Remark 3.3 further below. We furthermore note that, under the conditions of Theorem 3.2, a smallest critical value $C_\Diamond(\alpha)$ exists that satisfies (10) and that this critical value achieves equality in (10) provided $\alpha \in (0, \alpha^*] \cap (0, 1)$. This follows from Remark 6.7 in Section 6.2.

The preceding theorem provides simple sufficient conditions under which size control of commonly used autocorrelation robust tests is possible over the class $\mathfrak{F}_{\mathrm{all}}$, and hence a fortiori over *any* $\mathfrak{F} \subseteq \mathfrak{F}_{\mathrm{all}}$.[7] Of course, critical values achieving size control over a subset $\mathfrak{F}$ (i.e., critical values

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[6] Condition (11) clearly implies that the set B is a proper subset of $\mathbb{R}^n$ and thus implies Assumption 2. Hence, we could have dropped this assumption from the formulation of the theorem. For clarity of presentation we have, however, chosen to explicitly mention Assumption 2.

[7] In particular, $\mathfrak{F} = \mathfrak{F}_{\mathrm{AR}(p)}$ or $\mathfrak{F} = \mathfrak{F}_{\mathrm{ARMA}(p,q)}$ is covered for arbitrary p ((p, q), respectively), where $\mathfrak{F}_{\mathrm{ARMA}(p,q)}$ denotes the set of *all* normalized spectral densities corresponding to stationary autoregressive moving average

satisfying (10) with $\mathfrak{F}_{\text{all}}$ replaced by $\mathfrak{F}$) may be smaller than critical values achieving size control over $\mathfrak{F}_{\text{all}}$.

As mentioned before, the theorem is a special case of Theorem 6.6 in Section 6.2, which provides more refined sufficient conditions for the possibility of size control at the expense of more complicated conditions and notation. Theorem 6.6 is especially of importance if one is interested in sufficient conditions for the possibility of size control over classes $\mathfrak{F}$ that are much smaller than $\mathfrak{F}_{\text{all}}$, since then the conditions in Theorem 3.2 may be unnecessarily restrictive; cf. Remarks 3.4 and 3.7 given below, but see also Remark 3.6. We further note that Theorem 6.6 is in turn a corollary to Theorem 6.2 in Section 6.2, which applies to a much larger class of test statistics than the one considered in the present section. Finally, we note that the sufficient conditions in Theorem 3.2, as well as the sufficient conditions provided in the theorems in Section 6.2, are also necessary for size control in a sense made precise in Pötscher and Preinerstorfer (2017), provided that $\mathfrak{F}$ is rich enough to encompass $\mathfrak{F}_{\text{AR}(2)}$.

Remark 3.3. *(Independence of value of r)* (i) Since $\mathfrak{M}_0^{lin}$ does not depend on the value of r, the sufficient conditions in Theorem 3.2 – while depending on X and R – do not depend on the value of r.

(ii) For a large class of test statistics (including T_w considered here) the size of the corresponding tests, and hence the size-controlling critical values $C(\alpha)$, do not depend on the value of r, see Lemma 5.15 in Section 5.2. This observation is of some importance, as it allows one easily to obtain confidence sets for $R\beta$ by "inverting" the test without the need of recomputing the critical value for every value of r. [Of course, it is here understood that the weights w are not related to the value of r.]

In the subsequent examples we apply Theorem 3.2 and discuss simple sufficient conditions under which size control over $\mathfrak{F}_{\text{all}}$ (and hence a fortiori over any $\mathfrak{F} \subseteq \mathfrak{F}_{\text{all}}$) is possible. We also discuss that these sufficient conditions are generically satisfied in the appropriate universe of design matrices. For these examples we always assume that T_w is given by (5) with Assumption 1 being satisfied; also recall that the $n \times k$-dimensional design matrix X always has rank k, that $1 \leq k < n$ holds, and that the restriction matrix R is always $q \times k$-dimensional and has rank equal to $q \geq 1$.

Example 3.1. Assume that the design matrix X satisfies $\text{rank}(X, E_{n,0}(\gamma)) > k$ for every $\gamma \in [0, \pi]$ (which is tantamount to $\text{span}(E_{n,0}(\gamma)) \not\subseteq \text{span}(X)$ for every $\gamma \in [0, \pi]$ since $\text{rank}(X) = k$ by our assumptions). It is then easy to see that $\rho(\gamma, \mathfrak{M}_0^{lin}) = 0$ for every $\gamma \in [0, \pi]$. Hence, the conditions enabling size control over $\mathfrak{F}_{\text{all}}$ in Theorem 3.2 are then all satisfied if X and the restriction matrix R are such that B coincides with $\text{span}(X)$. In particular, it follows easily from Theorem 3.9 further below that the just mentioned rank condition as well as the condition on B is – for given restriction matrix R – generically satisfied in the set of all $n \times k$ matrices. [For the

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processes with autoregressive (moving average) order not larger than p (q, respectively). Note that the definition of $\mathfrak{F}_{\text{AR}(p)}$ and $\mathfrak{F}_{\text{ARMA}(p,q)}$ does *not* require the zeros of the autoregressive or moving average polynomial to be bounded away from the unit circle.

last claim use Theorem 3.9 with F the empty matrix, and observe that the conditions of that theorem are clearly satisfied. Also observe that $\rho_F(\gamma) = 0$ holds for every $\gamma \in [0, \pi]$.]

The next example treats the case where an intercept is required to be in the model.

Example 3.2. Assume that the design matrix X contains an intercept, in the sense that X has e_+ as its first column (e_+ is defined in Section 2). We also assume $k \geq 2$ and write $X = (e_+, \tilde{X})$. We are interested in testing restrictions that do not involve the intercept, i.e., $R = (0, \tilde{R})$, with $\tilde{R}$ of dimension $q \times (k-1)$. [Recall that autocorrelation robust testing of restrictions that involve the intercept by means of T_w is futile whenever $\mathfrak{F} \supseteq \mathfrak{F}_{\mathrm{AR}(1)}$, see Preinerstorfer and Pötscher (2016), Example 3.1.] It is now obvious that the rank condition on $(X, E_{n,0}(\gamma))$ in the preceding example is violated for $\gamma = 0$, and hence the conclusions of the preceding example do not apply in the situation considered here. However, assume now instead that $\mathrm{rank}(X, E_{n,0}(\gamma)) > k$ for every $\gamma \in (0, \pi]$ and that $\mathrm{rank}(X, E_{n,1}(\gamma)) > k$ for $\gamma = 0$ hold. It then follows that $\rho(\gamma, \mathfrak{M}_0^{lin}) = 0$ for every $\gamma \in (0, \pi]$ and that $\rho(\gamma, \mathfrak{M}_0^{lin}) = 1$ for $\gamma = 0$. Then again the conditions for size control over $\mathfrak{F}_{\mathrm{all}}$ in Theorem 3.2 are seen to be satisfied if X and the restriction matrix R are such that B coincides with span(X). Similarly as in the preceding example, it follows from Theorem 3.9 further below that the new rank conditions as well as the condition on B are – for given restriction matrix $R = (0, \tilde{R})$ – generically satisfied in the set of all $n \times k$ matrices of the form $X = (e_+, \tilde{X})$; for a proof of this claim see Appendix A.

A completely analogous discussion as in the preceding example can be given for the case where $X = (e_-, \tilde{X})$ and is omitted.

Example 3.3. Assume that the design matrix X contains e_+ as its first and e_- (defined in Section 2) as its second column, i.e., $X = (e_+, e_-, \tilde{X})$, and assume $k \geq 3$. Further assume that $R = (0, \tilde{R})$, where now $\tilde{R}$ is of dimension $q \times (k-2)$. [Recall that testing restrictions that involve the intercept by means of T_w is futile whenever $\mathfrak{F} \supseteq \mathfrak{F}_{\mathrm{AR}(1)}$, see Preinerstorfer and Pötscher (2016), Example 3.1, and a similar remark applies to restrictions involving the coefficient of e_-.] Similar as before, the rank condition on $(X, E_{n,0}(\gamma))$ in the preceding Example 3.1 is now violated for $\gamma \in \{0, \pi\}$, and so is the rank condition in Example 3.2 for $\gamma = \pi$. However, if we require instead that (i) $\mathrm{rank}(X, E_{n,0}(\gamma)) > k$ for every $\gamma \in (0, \pi)$, (ii) $\mathrm{rank}(X, E_{n,1}(\gamma)) > k$ for every $\gamma \in \{0, \pi\}$, and that (iii) X and R are such that B coincides with span(X), then it is not difficult to see that $\rho(\gamma, \mathfrak{M}_0^{lin}) = 0$ for $\gamma \in (0, \pi)$, $\rho(\gamma, \mathfrak{M}_0^{lin}) = 1$ for $\gamma \in \{0, \pi\}$, and that the sufficient conditions in Theorem 3.2 are satisfied, implying that size control over $\mathfrak{F}_{\mathrm{all}}$ is possible. Similarly as in the preceding examples, it follows from Theorem 3.9 further below that conditions (i)-(iii) are – for given restriction matrix $R = (0, \tilde{R})$ – generically satisfied in the set of all $n \times k$ matrices of the form $X = (e_+, e_-, \tilde{X})$, provided only that the mild condition $q \leq (n/2) - 1$ is satisfied; for a proof of this claim see Appendix A.

In the next remark we exemplarily discuss how the sufficient conditions in the preceding examples obtained from Theorem 3.2 can be weakened, if one is concerned with size control only over a set of spectral densities much smaller than $\mathfrak{F}_{\mathrm{all}}$. But see also Remark 3.6 further below.

Remark 3.4. Suppose we are interested in size control of tests of the form (5) but now only over the much smaller set $\mathfrak{F}_{\mathrm{AR}(1)}$.

(i) Assume that the design matrix X satisfies $\mathrm{rank}(X, E_{n,0}(\gamma)) > k$ for every $\gamma \in \{0, \pi\}$ (which is tantamount to $e_+ \notin \mathrm{span}(X)$ and $e_- \notin \mathrm{span}(X)$ as we always assume $\mathrm{rank}(X) = k$). Assume also that X and the restriction matrix R are such that B coincides with $\mathrm{span}(X)$. Then Theorem 6.6 in Section 6.2 implies that size control over $\mathfrak{F}_{\mathrm{AR}(1)}$ is possible. [This is so since the set $\mathbb{S}(\mathfrak{F}_{\mathrm{AR}(1)}, \mathfrak{M}_0^{lin})$ appearing in that theorem is $\{\{0\}, \{\pi\}\}$ as shown in Example 6.3. Furthermore, it is easy to see that here $\rho(\gamma, \mathfrak{M}_0^{lin}) = 0$ for every $\gamma \in \{0, \pi\}$ holds.]

(ii) Assume that $X = (e_+, \tilde{X})$ with $k \geq 2$, that $R = (0, \tilde{R})$, with $\tilde{R}$ of dimension $q \times (k-1)$, and that B coincides with $\mathrm{span}(X)$. If $\mathrm{rank}(X, E_{n,0}(\pi)) > k$ (i.e., if $e_- \notin \mathrm{span}(X)$), then size control over $\mathfrak{F}_{\mathrm{AR}(1)}$ is possible. [This follows from Theorem 6.6 since $\mathbb{S}(\mathfrak{F}_{\mathrm{AR}(1)}, \mathfrak{M}_0^{lin})$ now equals $\{\{\pi\}\}$, see Example 6.3, and since $\rho(\pi, \mathfrak{M}_0^{lin}) = 0$ holds.] The case where $X = (e_-, X)$ can be treated analogously, and we do not provide the details.

(iii) Assume $X = (e_+, e_-, \tilde{X})$ with $k \geq 3$, that $R = (0, \tilde{R})$, with $\tilde{R}$ of dimension $q \times (k-2)$. Then size control over $\mathfrak{F}_{\mathrm{AR}(1)}$ is possible without any further conditions. [Again this follows from Theorem 6.6 upon observing that now $\mathbb{S}(\mathfrak{F}_{\mathrm{AR}(1)}, \mathfrak{M}_0^{lin})$ is empty in view of Example 6.3.] We note that this result is also in line with Theorem 3.7 of Preinerstorfer and Pötscher (2016).

We proceed to providing two more examples illustrating Theorem 3.2. The first one concerns the case where a linear trend is present in the model.

Example 3.4. Assume $k \geq 3$ and that the design matrix X contains e_+ as its first column and the vector $v = (1, 2, \ldots, n)'$ as its second column, i.e., $X = (e_+, v, \tilde{X})$. That is, the linear model contains a linear trend. Suppose $R = (0, \tilde{R})$, where $\tilde{R}$ is of dimension $q \times (k-2)$, i.e., the restriction to be tested does not involve the coefficients appearing in the trend. This is, of course, a special case of the model considered in Example 3.2. However, it is plain that the condition $\mathrm{rank}(X, E_{n,1}(\gamma)) > k$ for $\gamma = 0$, used in that example, is not satisfied in the present context (as $E_{n,1}(0) = (v : 0)$). A simple set of sufficient conditions for size control in the present example is now as follows: (i) $\mathrm{rank}(X, E_{n,0}(\gamma)) > k$ for every $\gamma \in (0, \pi]$, (ii) $\mathrm{rank}(X, E_{n,2}(\gamma)) > k$ for $\gamma = 0$, and that (iii) X and R are such that B coincides with $\mathrm{span}(X)$. It is then not difficult to see that $\rho(\gamma, \mathfrak{M}_0^{lin}) = 0$ for $\gamma \in (0, \pi]$, $\rho(\gamma, \mathfrak{M}_0^{lin}) = 2$ for $\gamma = 0$, and that the sufficient conditions in Theorem 3.2 are satisfied, implying that size control over $\mathfrak{F}_{\mathrm{all}}$ is possible. Again, it follows from Theorem 3.9 further below that conditions (i)-(iii) are – for given restriction matrix $R = (0, \tilde{R})$ – generically satisfied in the set of all $n \times k$ matrices of the form $X = (e_+, v, \tilde{X})$; for a proof of this claim see Appendix A.

The preceding example can easily be generalized to arbitrary polynomial trends, but we abstain from providing the details. The last example considers a model with a cyclical component.

Example 3.5. Assume that the design matrix X has the form $(e_+, E_{n,0}(\gamma_0), \tilde{X})$ for some $\gamma_0 \in (0, \pi)$ and that $k \geq 4$. That is, the model contains a cyclical component. Suppose $R = (0, \tilde{R})$, where $\tilde{R}$ is of dimension $q \times (k-3)$, i.e., the restriction to be tested does neither involve the

intercept nor the coefficients appearing in the cyclical component. While this is a special case of the model considered in Example 3.2, it is also plain that the conditions provided in that example do not work here (since $\text{rank}(X, E_{n,0}(\gamma)) > k$ is violated for $\gamma = \gamma_0$). A simple set of sufficient conditions enabling size control in the present example is now as follows: (i) $\text{rank}(X, E_{n,0}(\gamma)) > k$ for every $\gamma \in (0, \pi] \setminus \{\gamma_0\}$, (ii) $\text{rank}(X, E_{n,1}(\gamma)) > k$ for $\gamma = 0$ as well as $\gamma = \gamma_0$, and that (iii) X and R are such that B coincides with $\text{span}(X)$. It is then not difficult to see that $\rho(\gamma, \mathfrak{M}_0^{lin}) = 0$ for $\gamma \in (0, \pi] \setminus \{\gamma_0\}$, $\rho(\gamma, \mathfrak{M}_0^{lin}) = 1$ for $\gamma = 0$ as well as $\gamma = \gamma_0$, and that the sufficient conditions in Theorem 3.2 are satisfied, implying that size control over $\mathfrak{F}_{\text{all}}$ is possible. Again, it follows from Theorem 3.9 further below that conditions (i)-(iii) are – for given restriction matrix $R = (0, \tilde{R})$ – generically satisfied in the set of all $n \times k$ matrices of the form $X = (e_+, E_{n,0}(\gamma_0), \tilde{X})$, provided only that the mild condition $q \leq (n/3) - 1$ is satisfied; for a proof of this claim see Appendix A.

Remark 3.5. *(No size control)* The size control result given above, as well as the more refined Theorem 6.6 in Section 6.2, do – for example – not apply to the following testing problems: (i) testing the intercept if $\mathfrak{F} \supseteq \mathfrak{F}_{\text{AR}(1)}$, (ii) testing the coefficient of the regressor e_- if $\mathfrak{F} \supseteq \mathfrak{F}_{\text{AR}(1)}$, (iii) testing a hypothesis regarding the coefficients of a linear trend appearing in the model provided $\mathfrak{F} \supseteq \mathfrak{F}_{\text{AR}(2)}$, and (iv) testing a hypothesis regarding the coefficients of a cyclical component appearing in the model provided $\mathfrak{F} \supseteq \mathfrak{F}_{\text{AR}(2)}$. In fact, it is known that the size of autocorrelation robust tests based on T_w is one in any of these testing problems regardless of the choice of critical value. For testing problems (i) and (ii) this follows from Theorem 3.3 of Preinerstorfer and Pötscher (2016) (see also Example 3.1 in that reference). For testing problem (iv) this follows from Theorem 3.12 of Preinerstorfer and Pötscher (2016). For testing problem (iii), see Section 5 in Pötscher and Preinerstorfer (2017). [These results are closely related to the fact that the sufficient conditions for size control given in Theorem 3.2 and in Section 6.2 are in fact necessary in a certain sense; see Pötscher and Preinerstorfer (2017).]

Remark 3.6. Suppose that in the context of Theorem 3.2 we are interested in size control over a set $\mathfrak{F}$ with $\mathfrak{F} \supseteq \mathfrak{F}_{\text{AR}(2)}$. It then follows from Remark 6.9 (with $\mathcal{L} = \mathfrak{M}_0^{lin}$) and Remark 6.10 that the sufficient condition (11) given in Theorem 3.2 is in fact equivalent to the more refined sufficient conditions given in Part 1 of Theorem 6.6.

Remark 3.7. (i) Suppose $\mathfrak{F} \subseteq \mathfrak{F}_{\text{ARMA}(p,q)}$ consists only of normalized spectral densities corresponding to stationary autoregressive moving average processes with the property that the zeros of all the autoregressive polynomials are bounded away from the unit circle in the complex plane by a fixed amount $\delta > 0$, say. Then it is easy to see that $\mathfrak{F} \subseteq \mathfrak{F}_{\text{all}}^B$ for some $B < \infty$ holds, where $\mathfrak{F}_{\text{all}}^B$ is defined in Example 6.2 in Section 6.1. Theorem 6.6 together with Example 6.2 now shows that size control over $\mathfrak{F}$ is possible even if the condition (11) in Theorem 3.2 is not satisfied. [In fact, this conclusion is true for any of the sets $\mathfrak{F}_{\text{all}}^B$ themselves.] Note, however, that choosing δ small may nevertheless result in large critical values, especially, if X and R are such that condition (11) is violated (and thus we are not in general guaranteed that size control is possible over $\mathfrak{F}_{\text{ARMA}(p,q)}$).

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(ii) More generally, size control in the setting of Theorem 3.2 is always possible (i.e., even when (11) is violated) if, e.g., the covariance model $\mathfrak{C}(\mathfrak{F})$ employed does not have any singular limit points; cf. also Remarks 5.7 and 6.11 further below.

3.1.2 Result for $T_{E,\mathsf{W}}$

Here we give a result similar to Theorem 3.2 but for the weighted Eicker-test statistic $T_{E,\mathsf{W}}$. The theorem follows immediately from Remark 6.8(ii).

Theorem 3.8. *Let $T_{E,\mathsf{W}}$ be the weighted Eicker-test statistic where W is an $n \times n$ symmetric and nonnegative definite Toeplitz matrix of weights with ones on the main diagonal. Then for every $0 < \alpha < 1$ there exists a real number $C(\alpha)$ such that*

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{f \in \mathfrak{F}_{\mathrm{all}}} P_{\mu_0, \sigma^2 \Sigma(f)}(T_{E,\mathsf{W}} \geq C(\alpha)) \leq \alpha \tag{13}$$

holds, provided that

$$\mathrm{span}\left(E_{n, \rho(\gamma, \mathfrak{M}_0^{lin})}(\gamma) \right) \not\subseteq \mathrm{span}(X) \quad \text{for every} \ \ \gamma \in [0, \pi]. \tag{14}$$

This can equivalently be expressed as

$$\mathrm{rank}\left(X, E_{n, \rho(\gamma, \mathfrak{M}_0^{lin})}(\gamma) \right) > k \quad \text{for every} \ \ \gamma \in [0, \pi]. \tag{15}$$

Furthermore, under the same condition (14) even equality can be achieved in (13) by a proper choice of $C(\alpha)$, provided $\alpha \in (0, \alpha^*] \cap (0, 1)$, where α^* is defined in (28) (with T replaced by $T_{E,\mathsf{W}}$) further below.

Mutatis mutandis, the entire discussion in Subsection 3.1.1 following Theorem 3.2 also applies to Theorem 3.8.

3.2 Generic size control

In Theorem 3.9 below we now show that – for given restriction matrix R – the set of design matrices, for which the conditions in Theorem 3.2 (Theorem 3.8, respectively) are satisfied and hence size control is possible, is generic. We provide this genericity result for a variety of universes of design matrices. For example, if F in Theorem 3.9 is absent (more precisely, corresponds to the empty matrix), the genericity holds in the class of all $n \times k$ design matrices. If one is only interested in regression models containing an intercept, then one sets $F = e_+$, and the theorem delivers a genericity result in the subuniverse of all $n \times k$ design matrices that contain e_+ as its first column. In general, F stands for that subset of columns of the design matrix that are a priori held fixed in the genericity analysis. The subsequent theorem follows immediately by combining Theorem 3.2 (Theorem 3.8, respectively) with Lemmata A.2 and A.3 in Appendix A.

Recall that B as well as $\rho(\gamma, \mathfrak{M}_0^{lin})$ depend on X (and R), which, however, is not shown in the notation; and that the relation $\mathsf{B} = \mathrm{span}(X)$ implies that Assumption 2 holds. Also recall that $n > k$ is assumed throughout.

Theorem 3.9. *Let F be a given $n \times k_F$ matrix of rank k_F, where $0 \le k_F < k$ (with the convention that F is the empty matrix in case $k_F = 0$, that the rank of the empty matrix is zero, and that its span is $\{0\}$). Assume that the given $q \times k$ restriction matrix R of rank q has the form $(0, \tilde{R})$ with $\tilde{R}$ a $q \times (k - k_F)$ matrix. Suppose the columns of F and $e_{i_1}(n), \ldots, e_{i_q}(n)$ are linearly independent for every choice of $1 \le i_1 < \ldots < i_q \le n$. Furthermore, suppose that (i) $n > k + 2$ holds, or (ii) $\mathrm{rank}(F, E_{n,0}(\gamma^*)) = k_F + 2$ holds for some $\gamma^* \in (0, \pi)$. Then the following holds generically for design matrices X of the form $(F, \tilde{X})$ (i.e., holds on the complement of a $\lambda_{\mathbb{R}^{n \times (k - k_F)}}$-null set of matrices $\tilde{X}$):*

1. *$X = (F, \tilde{X})$ has rank k.*

2. *$\mathsf{B} = \mathrm{span}(X)$.*

3. *Assumption 2 is satisfied.*

4. *$\rho(\gamma, \mathfrak{M}_0^{lin}) = \rho_F(\gamma)$ holds for every $\gamma \in [0, \pi]$ where $\rho_F(\gamma) = \rho(\gamma, \mathrm{span}(F))$.*

5. *Conditions (11), (12), (14), and (15) hold.*

6. *Suppose T_w is defined by (5) with Assumption 1 being satisfied. Then for every $0 < \alpha < 1$ there exists a real number $C(\alpha)$ such that (10) holds; and if $\alpha \in (0, \alpha^*] \cap (0, 1)$ even equality can be achieved in (10), where α^* is as in Theorem 3.2.*

7. *Let $T_{E,\mathsf{W}}$ be the weighted Eicker-test statistic where W is an $n \times n$ symmetric and nonnegative definite Toeplitz matrix of weights with ones on the main diagonal. Then for every $0 < \alpha < 1$ there exists a real number $C(\alpha)$ such that (13) holds; and if $\alpha \in (0, \alpha^*] \cap (0, 1)$ even equality can be achieved in (13), where α^* is as in Theorem 3.8.*

Note that neither the assumptions nor the first five conclusions nor the $\lambda_{\mathbb{R}^{n \times (k - k_F)}}$-null set depend on the value of r at all (this is obvious upon noting that $\rho(\gamma, \mathfrak{M}_0^{lin})$ depends on $\mathfrak{M}_0$ only via $\mathfrak{M}_0^{lin}$, which is independent of the value of r). For the last two conclusions note that they hold whatever the value of r is.

Remark 3.10. (i) Theorem 3.9 assumes that the restrictions to be tested do not involve the coefficients of the regressors corresponding to the columns of F. This assumption can be traded-off with an assumption ensuring that no singular limit point of the covariance model $\mathfrak{C}(\mathfrak{F}_{\mathrm{all}})$ concentrates on the space spanned by the columns of F. We abstain from providing such results.

(ii) In case $k_F < \lfloor n/2 \rfloor$ the rank-condition in (ii) of the theorem is always satisfied: Suppose not, then $E_{n,0}(\gamma)v(\gamma) \in \mathrm{span}(F)$ for some nonzero vector $v(\gamma)$ and for every $\gamma \in (0, \pi)$. Choose $\gamma_i \in (0, \pi)$, $i = 1, \ldots, \lfloor n/2 \rfloor$, all γ_i being different. By Lemma C.1 in Appendix C, it follows

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that the corresponding collection of vectors $E_{n,0}(\gamma_i)v(\gamma_i)$ is linearly independent, implying that $k_F \geq \lfloor n/2 \rfloor$, a contradiction. [In case $k_F > n/2$ examples can be given where this condition is not satisfied.]

3.3 Comments on power properties

Classical autocorrelation robust tests can have, in fact not infrequently will have, infimal power equal to zero if the underlying set $\mathfrak{F}$ is sufficiently rich; cf. Theorem 3.3 and Corollary 5.17 in Preinerstorfer and Pötscher (2016) as well as Lemma 5.11 in Section 5. In the special case where $\mathfrak{F} = \mathfrak{F}_{AR(1)}$, it has been shown in Preinerstorfer and Pötscher (2016) and Preinerstorfer (2017) how adjusted tests can be constructed that have correct size and at the same time do not suffer from infimal power being zero. In a companion paper, which builds on the results of the present paper, we investigate power properties in more detail and provide adjustments to the test statistics T_w and $T_{E,\mathsf{W}}$ that typically lead to improvements in power properties, at least over certain important subsets of $\mathfrak{F}_{all}$, while retaining size control over $\mathfrak{F}_{all}$ as in Theorems 3.2 and 3.8.

We also note here that despite what has just been said, one can show for autocorrelation robust tests based on T_w or $T_{E,\mathsf{W}}$ (size corrected or not) that power goes to one as one moves away from the null hypothesis along sequences of the following form: Let (μ_l, σ_l^2, f_l) be such that μ_l moves further and further away from $\mathfrak{M}_0$ (the affine space of means described by the restrictions $R\beta = r$) in an orthogonal direction, where σ_l^2 converges to some finite and positive σ^2, and f_l is such that $\Sigma(f_l)$ converges to a positive definite matrix. Note, however, that this result rules out sequences f_l for which $\Sigma(f_l)$ degenerates as $l \to \infty$.

3.4 Extensions to other test statistics

A. *(Adjusted tests)* In Preinerstorfer and Pötscher (2016) we have discussed adjusted autocorrelation robust tests, which are nothing else than standard autocorrelation robust tests but computed from an augmented regression model that contains not only the regressors in X, but also strategically chosen auxiliary regressors. The above results can easily accommodate adjusted tests: Simply view the augmented model as the true model. Since the adjusted test then is just a standard autocorrelation robust test in the augmented model, the above results can be applied. Note that the null hypothesis in the augmented model encompasses the null hypothesis in the original model, hence size control over the null hypothesis in the augmented model certainly implies size control in the originally given model. For more discussion see Theorem 3.8, Proposition 5.23, and especially Remark 5.24(iii) in Preinerstorfer and Pötscher (2016).

B. *(Tests based on general quadratic covariance estimators)* The test statistics T_{GQ} in this class are of the form (5) but – instead of $\hat{\Psi}_w$ – use the estimator

$$\hat{\Psi}_{GQ}(y) = \sum_{t,s=1}^{n} w(t,s;n)\hat{v}_t(y)\hat{v}_s(y)' \tag{16}$$

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for $y \in \mathbb{R}^n$, where the $n \times n$ weighting matrix $\mathcal{W}_n^* = (w(t, s; n))_{t,s}$ is symmetric and data-independent. For some background on this more general class of estimators see Section 3.2.1 of Preinerstorfer and Pötscher (2016). Note that $\hat{\Psi}_{GQ}(y)$, and thus the corresponding $\hat{\Omega}_{GQ}(y) = nR(X'X)^{-1}\hat{\Psi}_{GQ}(y)(X'X)^{-1}R'$, is nonnegative definite for every $y \in \mathbb{R}^n$ provided $\mathcal{W}_n^*$ is nonnegative definite. In the important case where $\mathcal{W}_n^*$ is additionally positive definite, inspection of the proofs (together with Lemma 3.11 of Preinerstorfer and Pötscher (2016)) shows that all results given above for the test statistic T_w based on $\hat{\Psi}_w$ remain valid for T_{GQ} (provided Assumption 1 is replaced by the assumption on $\mathcal{W}_n^*$ made here including positive definiteness of $\mathcal{W}_n^*$); see also Remark 6.8(i). In the case where $\mathcal{W}_n^*$ is nonnegative definite, but not positive definite, conditions under which size control is possible can be derived from Theorem 6.5 in Section 6.2 (with the help of Lemma 3.11 of Preinerstorfer and Pötscher (2016)); we do not provide details. In fact, even cases where $\mathcal{W}_n^*$ is not nonnegative definite can be accommodated by this result under appropriate conditions.

C. A referee has suggested a test statistic T_{ref} which is of the form (5), but where $\hat{\Omega}_w(y)$ is replaced by $\hat{\omega}_w(y)R(X'X)^{-1}R'$. Here $\hat{\omega}_w(y) = \sum_{j=-(n-1)}^{n-1} w(j, n)\hat{K}_j(y)$ where $\hat{K}_j(y) = \hat{K}_{-j}(y) = n^{-1}\sum_{l=j+1}^{n} \hat{u}_l(y)\hat{u}_{l-j}(y)$ for $j \geq 0$. It is easy to see that a size control result can be established for T_{ref}: If Assumption 1 holds, the conclusion of Theorem 3.8 with $T_{E,\mathsf{W}}$ replaced by T_{ref} still holds. However, the form of the long-run covariance estimator $\hat{\omega}_w(y)R(X'X)^{-1}R'$ used by this test statistic takes its justification from a well-known result of Grenander (1954), which holds under certain conditions on the regressors only. A leading case where these conditions are satisfied is polynomial regression. Unfortunately, precisely for such regressors it turns out that the conditions for size control are violated and, in fact, it can be shown that in this case the test based on T_{ref} has size 1 for every choice of critical value; see Pötscher and Preinerstorfer (2017).

D. (*Random bandwidth, prewithening, flat-top kernels, GLS-based tests, general nonsphericity-corrected F-type tests*) Tests based on weighted autocovariance estimators $\hat{\Psi}_w$, but where the weights are allowed to depend on the data (e.g., lag-window estimators with data-driven bandwidth choice), or where prewithening is used, can be viewed as special cases of nonsphericity-corrected F-type tests (under appropriate conditions, see Preinerstorfer (2017)). The same is true for tests using long-run variance estimators based on flat-top kernels. Another example are tests based on parametric long-run variance estimators or tests based on feasible generalized least squares. A size control result for general nonsphericity-corrected F-type tests, i.e., Wald-type tests that may use estimators for β other than ordinary least squares or may use estimators for the long-run covariance matrix other than the ones mentioned so far, is given in Theorem 6.5 in Section 6.2. However, we abstain in this paper from making the conditions in that theorem more concrete for the subclasses of tests just mentioned. Theorem 6.2 in Section 6.2 furthermore even applies to classes of tests more general than the class of nonsphericity-corrected F-type tests.

4 Computational issues and numerical results

In the previous section we have obtained conditions under which the size of autocorrelation robust tests can be controlled by a proper choice of the critical value. In the present section we now turn to the question of how critical values guaranteeing size control can be determined numerically (provided they exist). Additionally, we also address the related question of how to numerically compute the size of an autocorrelation robust test for a given choice of the critical value (e.g., when choosing a critical value suggested by asymptotic theory). We illustrate our algorithms by computing size-controlling critical values for regression models using US macroeconomic time series from the FRED-MD database of McCracken and Ng (2016) as regressors and where the correlation structure of the errors in the regression is governed by a variety of families of spectral densities. In these models we also compute the actual size of standard autocorrelation robust tests that employ critical values suggested in the literature.

We emphasize that, although the algorithms we shall discuss below are designed for regression models with stationary autocorrelated errors and for the test statistics T_w defined in (5) or $T_{E,\mathsf{W}}$ defined in Section 3, the basic ideas extend to other test statistics (e.g., the ones discussed in Section 3.4) and also to other covariance models with mostly obvious modifications.

4.1 Computation of critical values and size

Consider testing the hypothesis given in (4) at the significance level $\alpha \in (0,1)$ by means of the test statistic T_w as defined in (5), and suppose that Assumptions 1 and 2 are satisfied. Furthermore, suppose that one knows, e.g., by having checked the sufficient conditions given in Theorem 3.2, that existence of a critical value $C(\alpha)$ satisfying

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{f \in \mathfrak{F}} P_{\mu_0, \sigma^2 \Sigma(f)} \left(T_w \geq C(\alpha) \right) \leq \alpha \tag{17}$$

is guaranteed, where $\mathfrak{F}$ is a user-specified subset of $\mathfrak{F}_{\mathrm{all}}$. Because such a critical value is certainly not unique and because of power considerations, it is reasonable to try to find the "smallest" critical value satisfying the inequality in the previous display. From the discussion following Theorem 3.2 and from Remark 6.7 we conclude that such a smallest critical value $C_\Diamond(\alpha)$ indeed exists; furthermore, if equality is achievable in the preceding display, $C_\Diamond(\alpha)$ then certainly also achieves it. We note the obvious facts that $C_\Diamond(\alpha)$ depends on $\mathfrak{F}$, and that any critical value smaller than $C_\Diamond(\alpha)$ will lead to a test that violates the size constraint (17).

Now, because of $G(\mathfrak{M}_0)$-invariance of T_w (see Lemma 5.16 and Remark 5.17 in Section 5.3) and because of Remark 5.5(iii) in Preinerstorfer and Pötscher (2016), the inequality (17) is equivalent to

$$\sup_{f \in \mathfrak{F}} P_{\mu_0, \Sigma(f)} \left(T_w \geq C(\alpha) \right) \leq \alpha$$

where we have chosen μ_0 as an arbitrary but fixed element of $\mathfrak{M}_0$ and have set $\sigma = 1$. Exploiting

the fact that $P_{\mu_0, \Sigma(f)}(T_w = C) = 0$ for every $C \in \mathbb{R}$ (since $\lambda_{\mathbb{R}^n}(T_w = C) = 0$ for every $C \in \mathbb{R}$, see Lemma 5.16 and Remark 5.17 in Section 5.3), the preceding display implies that

$$C_\Diamond(\alpha) = \sup_{f \in \mathfrak{F}} F_{\Sigma(f)}^{-1}(1 - \alpha), \tag{18}$$

where $F_{\Sigma(f)}$ denotes the cumulative distribution function (cdf) of $P_{\mu_0, \Sigma(f)} \circ T_w$ (since $\mu_0 \in \mathfrak{M}_0$ is fixed we do not need to show dependence on μ_0 in the notation). As usual, for a cdf F we denote by F^{-1} the corresponding quantile function $F^{-1}(x) = \inf\{z \in \mathbb{R} : F(z) \geq x\}$. In order to obtain $C_\Diamond(\alpha)$, one must hence solve the optimization problem in (18).

We shall now provide an heuristic optimization algorithm to solve (18) in the case where $\mathfrak{F} = \mathfrak{F}_{\mathrm{AR}(p)}$ with $1 \leq p \leq n - 1$. We write $C_\Diamond(\alpha, p)$ to emphasize the dependence of the critical value on the autoregressive order; apart from p, the critical value only depends on X, R, and the weights w, but not on the value of r (cf. Remark 3.3(ii)). [We do not show the dependence on X, R, and w in the notation.] Note that by Remark 3.1 the families $\mathfrak{F}_{\mathrm{AR}(n-1)}$ and $\mathfrak{F}_{\mathrm{all}}$ induce the same testing problem, and hence the critical value $C_\Diamond(\alpha, n - 1)$ achieves size control also over $\mathfrak{F}_{\mathrm{all}}$. Consequently, the subsequent discussion covers testing problems where $\mathfrak{F} = \mathfrak{F}_{\mathrm{all}}$ as a special case. We start by reparameterizing the optimization problem (18), exploiting the fact that $\mathfrak{F} = \mathfrak{F}_{\mathrm{AR}(p)}$ can be parameterized through the partial autocorrelation coefficients (reflection coefficients), cf. Barndorff-Nielsen and Schou (1973): To each p-vector of partial autocorrelation coefficients $\rho \in (-1, 1)^p$ there corresponds a unique normalized $\mathrm{AR}(p)$ spectral density f_ρ, say, and vice versa. Hence, writing F_ρ for $F_{\Sigma(f_\rho)}$, it follows that

$$C_\Diamond(\alpha, p) = \sup_{f \in \mathfrak{F}_{\mathrm{AR}(p)}} F_{\Sigma(f)}^{-1}(1 - \alpha) = \sup_{\rho \in (-1, 1)^p} F_\rho^{-1}(1 - \alpha). \tag{19}$$

That is, $C_\Diamond(\alpha, p)$ can be found by maximizing the objective function $\rho \mapsto F_\rho^{-1}(1 - \alpha)$ over $(-1, 1)^p$. Compared to other parameterizations of the set of all stationary $\mathrm{AR}(p)$ spectral densities, e.g., through the autoregression coefficients or the set of zeros of the AR polynomial, working with partial autocorrelation coefficients has the clear advantage that no cross-restrictions are present. One aspect that complicates the optimization problem, in addition to being potentially high-dimensional, is that the objective function $\rho \mapsto F_\rho^{-1}(1 - \alpha)$ needs to be approximated numerically, e.g., by a Monte Carlo algorithm, since an analytical expression for F_ρ^{-1} is unknown in general. Therefore, an optimization algorithm for determining the supremum in the previous display needs to determine a quantile via a Monte Carlo algorithm each time a function evaluation is required, which can be computationally intensive, but is amenable to parallelization.

The optimization algorithm we use for numerically determining $C_\Diamond(\alpha, p)$ is described in detail in Algorithm 1 which can be found in Appendix F. Roughly speaking, the algorithm starts with a preliminary step that selects candidate values ρ from $(-1, 1)^p$, which are then used as initial values in a (local) optimization step. This step returns improved candidate values of ρ, the best of which are in turn used as initial values in a second (local) optimization step that now uses a

larger number of replications in the Monte-Carlo evaluation of the objective function $F_\rho^{-1}(1-\alpha)$ than was used in the previous steps.

A related problem is to numerically determine the size of the test that rejects if T_w exceeds a certain *given* critical value C, i.e., one wants to obtain

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{f \in \mathfrak{F}_{\mathrm{AR}(p)}} P_{\mu_0, \sigma^2 \Sigma(f)} \left(T_w \geq C \right).$$

Similarly as above this can be reduced to determining

$$\sup_{f \in \mathfrak{F}_{\mathrm{AR}(p)}} P_{\mu_0, \Sigma(f)} \left(T_w \geq C \right) \tag{20}$$

for a fixed value of $\mu_0 \in \mathfrak{M}_0$. One can then use a variant of Algorithm 1, which is described in Algorithm 2 in Appendix F, to solve that problem.

Finally we emphasize that, as is typical for numerical optimization problems, there is no guarantee that the algorithms mentioned above do return the exact critical value $C_\Diamond(\alpha, p)$ or the exact size of a test given a critical value C. The algorithms are heuristics that numerically approximate the quantities of interest.

Remark 4.1. In case $p = 0$, i.e., when the errors are i.i.d., the algorithms simplify considerably in an obvious way as no optimization over ρ is then necessary.

Remark 4.2. *(Other test statistics)* The above development has been given for the test statistic T_w. It applies to any other test statistic T as long as (i) T is $G(\mathfrak{M}_0)$-invariant and (ii) satisfies $\lambda_{\mathbb{R}^n}(T = C) = 0$ for every $C \in \mathbb{R}$, and (iii) one can ensure that a size-controlling critical value exists. It thus, in particular, applies to the weighted Eicker-test statistic $T_{E,\mathsf{W}}$ as defined in Section 3 (cf. Lemma 5.16 and Remark 5.17 in Section 5.3). Note that for problem (20) the just given condition (ii) on T is actually not needed. [If T does not belong to the class of nonsphericity-corrected F-type test statistics (cf. Preinerstorfer and Pötscher (2016), Section 5.4), we can, however, no longer conclude that the corresponding critical values $C_\Diamond(\alpha, p)$ are independent of the value of r.] Furthermore, for covariance models $\mathfrak{C}$ not of the form $\mathfrak{C}(\mathfrak{F})$ the general principles underlying the reduction of (17) to (18) still apply, provided T satisfies (i)-(iii) given above. Algorithms that perform optimization of the so-obtained analogue of (18) can then be developed in a similar way by exploiting the structure of the given covariance model $\mathfrak{C}$.

4.2 An illustration for regression models based on US macroeconomic time series and autoregressive error processes

We now apply Algorithms 1 and 2 introduced above to regression models based on data from the FRED-MD database, which consists of 128 macroeconomic time series that have been subjected to stationarity-inducing transformations (see McCracken and Ng (2016) for detailed information

concerning the database). More specifically, we consider regression models of the form

$$\mathbf{y}_t = \beta_1 + \beta_2 t + \beta_3 x_t + \mathbf{u}_t \quad \text{for } t = 1, \dots, n,$$

where $\mathbf{u}_1, \dots, \mathbf{u}_n$ are distributed like consecutive observations from a mean zero stationary Gaussian process with spectral density $\sigma^2 f$, $f \in \mathfrak{F}_{\mathrm{AR}(p)}$. Here x_t is one of the 128 macroeconomic variables in the FRED-MD database, and where we use the most recent $n = 100$ observations from each time series.[8] For each of these 128 regressors and for every $p \in \{0, 1, 2, 5, 10, 25, 50, 99\}$ we consider the problem of testing a restriction on the coefficient β_3 at the 5% level, i.e., we consider testing problem (4) with $\mathfrak{F} = \mathfrak{F}_{\mathrm{AR}(p)}$, $R = (0, 0, 1)$, and with r arbitrary (the results presented below do not depend on r, cf. Remark 3.3(ii)). Recall that the case $p = 99$ realizes the testing problem for the case $\mathfrak{F} = \mathfrak{F}_{\mathrm{all}}$, and that the case $p = 0$ corresponds to i.i.d. disturbances. In each setting we consider the test statistic T_w as defined in (5) as well as $T_{E,\mathsf{W}}$ as defined in Section 3, with the design matrix X corresponding to the regression model in the previous display. Bartlett weights $w(j, n) = (1 - |j|/M_n)\mathbf{1}_{(-1,1)}(j/M_n)$ with $M_n = n/10$ (i.e., bandwidth equal to $1/10$) are used for the test statistic T_w, and the same weights are used for the matrix W appearing in $T_{E,\mathsf{W}}$. Since $q = 1$, rejecting for large values of T_w is equivalent to rejecting if the t-type test statistic corresponding to T_w, i.e., if

$$t_w(y) = \begin{cases} (\hat{\beta}_3(y) - r)/\hat{\Omega}_w^{1/2}(y) & \text{if } \hat{\Omega}_w(y) \neq 0 \\ 0 & \text{else }, \end{cases}$$

is large in absolute value. A similar observation applies to tests obtained from $T_{E,\mathsf{W}}$, the corresponding t-type test statistic being denoted by $t_{E,\mathsf{W}}$. For the test statistic t_w, critical values that are based on fixed-bandwidth asymptotics are provided in Kiefer and Vogelsang (2005), p. 1146, and this critical value (for the bandwidth and kernel chosen here) is given by 2.260568. For the sake of comparability, and because critical values for t-statistics are usually easier to interpret, we shall present critical values for the t-type version of the test statistics in what follows. Critical values for T_w and $T_{E,\mathsf{W}}$ can easily be obtained by taking the square. We also note that the critical values obtained below can be used for the construction of confidence intervals for β_3. We shall now apply Algorithm 1 to numerically compute the critical value that is needed to control size in each scenario. Additionally, we also apply Algorithm 2 to numerically compute the size of the test that rejects if $|t_w|$ exceeds the above mentioned critical value provided by Kiefer and Vogelsang (2005).[9] The particular settings used in Algorithms 1 and 2 for the computations in

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[8]The database was downloaded on October 25, 2016 from https://research.stlouisfed.org/econ/mccracken/fred-databases/

[9]A referee has questioned if using this critical value here is appropriate given that the regressors are treated as nonrandom and that a linear trend is included. However, note that the theory developed in Kiefer et al. (2000), Kiefer and Vogelsang (2002a,b, 2005) is based on high-level assumptions that are compatible with random as well as nonrandom regressors. Furthermore, linear trends (as long as their coefficients are not subject to tests as is the case here) can be accommodated in this framework in the random as well as in the nonrandom regressor case by a reasoning based on the Frisch-Waugh-Lovell theorem as in Kiefer et al. (2000) (under appropriate assumptions

this section are described in detail in Appendix H.

To ensure, for each of the 128 design matrices, existence of a critical value for T_w (and hence for t_w) that controls size for any (nonempty) family of (normalized) spectral densities, we now check the sufficient conditions of Theorem 3.2: That Assumption 1 is satisfied follows from the discussion after that assumption since we use the Bartlett kernel. Assumption 2 is satisfied for all 128 cases as none of these design matrices contains an element of the canonical basis in its span, which is easily verified numerically, and which then implies the assumption, since $\text{rank}(R(X'X)^{-1}X') = q$ always holds. It remains to check condition (11) for each of the 128 design matrices. This can successfully be done numerically and we describe the details of this computation in Appendix G. Since $\text{span}(X) \subseteq \mathsf{B}$ always holds, this then also implies validity of condition (14), and thus also implies existence of a corresponding critical value for $t_{E,\mathsf{W}}$ (cf. Theorem 3.8).

Figure 1(a) summarizes the numerical results for the size of the test that rejects if $|t_w|$ exceeds the Kiefer-Vogelsang critical value 2.260568: For each autoregressive order $p \in \{1, 2, 5, 10, 25, 50, 99\}$ as well as for the i.i.d. case (i.e., $p = 0$) and for each of the 128 models considered we obtained the size (i.e., the maximal rejection probability under the null) by means of Algorithm 2 and summarize them in Figure 1(a) in the form of boxplots (each boxplot representing the 128 sizes obtained for a given order p). For a complete list of results see Table 1 in Appendix L. As is apparent from Figure 1(a), the Kiefer-Vogelsang critical value does not control size (not even in the i.i.d. setting) at the desired 5% level for neither one of the 128 regression models. This observation a fortiori extends to critical values smaller than the Kiefer-Vogelsang critical value such as, e.g., the standard normal critical value 1.96, or the critical value obtained from third-order asymptotic expansions (for the location model) in Sun et al. (2008) which equals 2.242583 (for the bandwidth and kernel chosen here). Figure 1(a) furthermore shows a large increase in the size when passing from the i.i.d. to the AR(1) case, and another large increase in size when passing from the AR(1) to the AR(2) case.[10] In the AR(2) case severe size distortions are present for all of the 128 regression models. Figure 1(a) also suggests that the sizes for the cases with $p > 2$ are comparable to the sizes in the AR(2) case.

In Figure 1(b) we present the critical values which guarantee size control at the 5% level as computed by an application of Algorithm 1 for the test statistics t_w as well as $t_{E,\mathsf{W}}$. Again we present boxplots, and refer the reader to Tables 2 and 3 in Appendix L for a complete list of results. Figure 1(b) suggests that the critical values required to control size increase strongly when passing from the i.i.d. case to the AR(1) model, and again when passing from the AR(1) to the AR(2) model. For larger p, the critical values, while still increasing with p, seem to stabilize. Figure 1(b) also illustrates the dependence of the critical value on the design matrix:

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on the regressors and errors).

[10]The increase in size when passing form the i.i.d. to the AR(1) case is connected to the fact that there are no concentration spaces in the i.i.d. case, whereas in the AR(1) case two concentration spaces corresponding to angular frequencies $\gamma = 0$ and $\gamma = \pi$ exist. The further increase in size when passing form AR(1) to AR(2) is related to the fact that AR(2) models allow additional concentration spaces corresponding to angular frequencies $\gamma \in (0, \pi)$.

(a)

(b)

Figure 1: (a) Sizes of the test which rejects if $|t_w| \geq 2.260568$ (Kiefer-Vogelsang critical value). The horizontal dashed red line corresponds to 0.05. (b) Critical values guaranteeing a 5 % level for the t-type tests corresponding to T_w and $T_{E,\mathrm{W}}$. The horizontal dashed grey line corresponds to the Kiefer-Vogelsang critical value 2.260568.

For some of the 128 regressors in the FRED-MD database, the critical values needed to control size are very large, while for other regressors the critical values are about 2-3 times as large as the Kiefer-Vogelsang critical value (which, however, does *not* provide size control). Figure 1(b) further suggests that the critical values for t_w needed to control size at the 5% level tend to be larger than the corresponding critical values for $t_{E,W}$.

While it is plain that the size-controlling critical values can never fall when passing from an AR(p) model to an AR(p') model with $p < p'$, this is not always guaranteed for the *numerically* determined critical values due to numerical errors. We could have "monotonized" the results in Figure 1(b), but have decided not to. A similar remark applies to Figure 1(a) as well as to Figure 2 given further below.

A referee has suggested to examine also the critical value that is computed under the presumption that the errors would follow a random walk. We discuss this in Appendix I.

The final issue we shall investigate is how the size distortions of the test using the Kiefer-Vogelsang critical values, documented in Figure 1(a) for (unrestricted) AR(p) models, are influenced if we consider *restricted* AR(p) error processes where the restrictions amount to placing a bound on the partial autocorrelations. More precisely, we consider AR(p) models where now the partial autocorrelations are restricted to sets of the form $(-1 + \varepsilon, 1 - \varepsilon)^p \subseteq (-1, 1)^p$ for some choice of ε, $0 < \varepsilon < 1$. Figure 1(a) suggests that, in order to obtain some insight, we can focus on the AR(2) case. We apply a variant of Algorithm 2 to numerically compute the size of the test based on t_w together with the Kiefer-Vogelsang critical value, where $\mathfrak{F}$ is now the set of all normalized AR(2) spectral densities with maximal absolute partial autocorrelation coefficient not exceeding a certain threshold in absolute value. Furthermore, we apply a variant of Algorithm 1 to numerically determine the critical values needed to control the sizes of the tests based on t_w and $t_{E,W}$ at the 5% level over these sets of spectral densities. As discussed in Remark F.4 in Appendix F, the algorithms now have to be modified in such a way that the feasible set of the optimization problems in Stages 1 and 2 are restricted sets of partial autocorrelation coefficients of the form $(-1 + \varepsilon, 1 - \varepsilon)^2$, and so that the starting values in Stage 0 fall within this feasible set. The starting values are randomly generated as described above, but in order to force them into $(-1 + \varepsilon, 1 - \varepsilon)^2$, they are all multiplied by $1 - \varepsilon$.

The size of the test based on the Kiefer-Vogelsang critical value over the so restricted AR(2) models are summarized in Figure 2(a) for a range of values for $1 - \varepsilon$. From these results we see that even if one is willing to impose the (questionable) assumption that partial autocorrelation coefficients are known not to exceed 0.55 in absolute value, the size of the test rejecting whenever $|t_w|$ exceeds the Kiefer-Vogelsang critical value is considerable larger than 0.05 for most of the 128 regression models under consideration. Unsurprisingly, the degree of size distortion increases steadily as the bound for the maximal absolute partial autocorrelation increases, where we observe a steep increase from 0.95 to the unrestricted case. This shows that even if the practitioner has good reasons to believe that each partial autocorrelation is bounded away from one in modulus, critical values that are typically used in practice, such as the Kiefer-Vogelsang
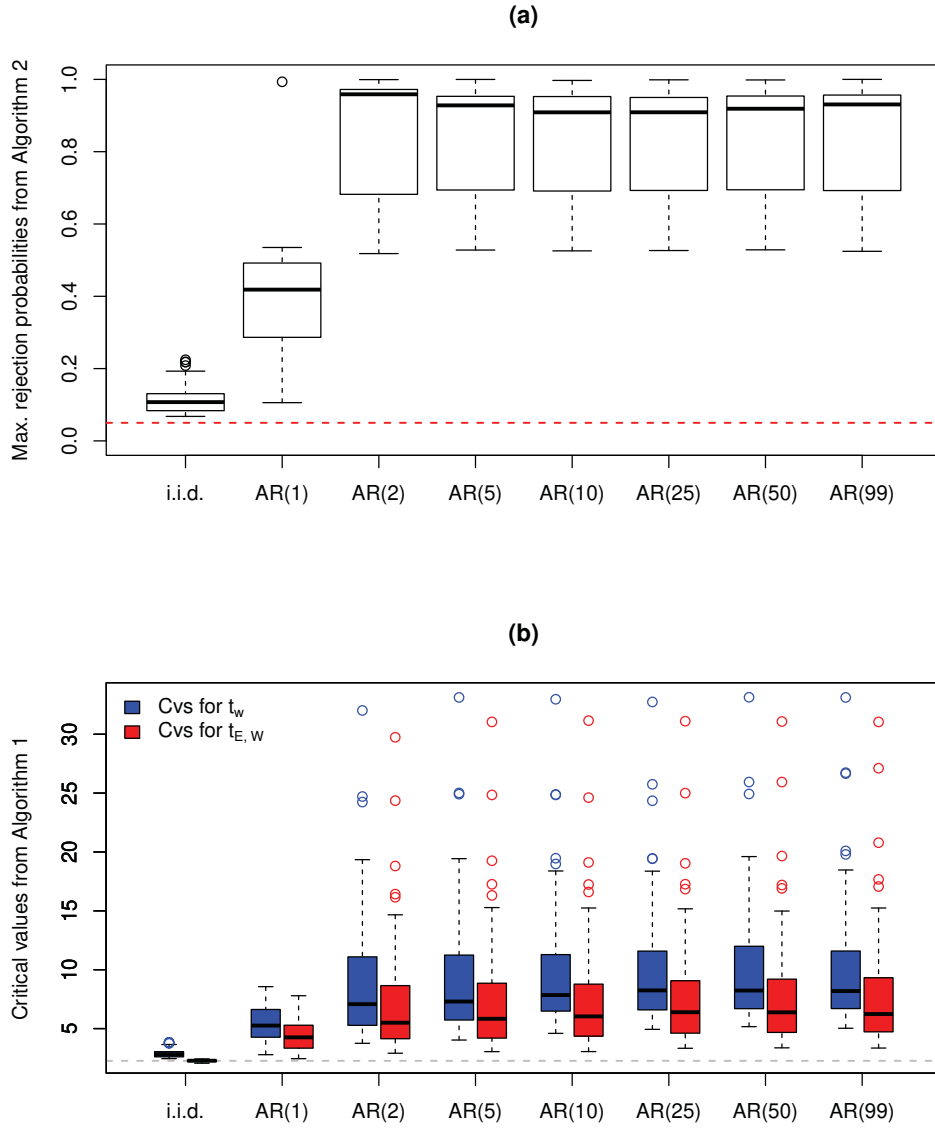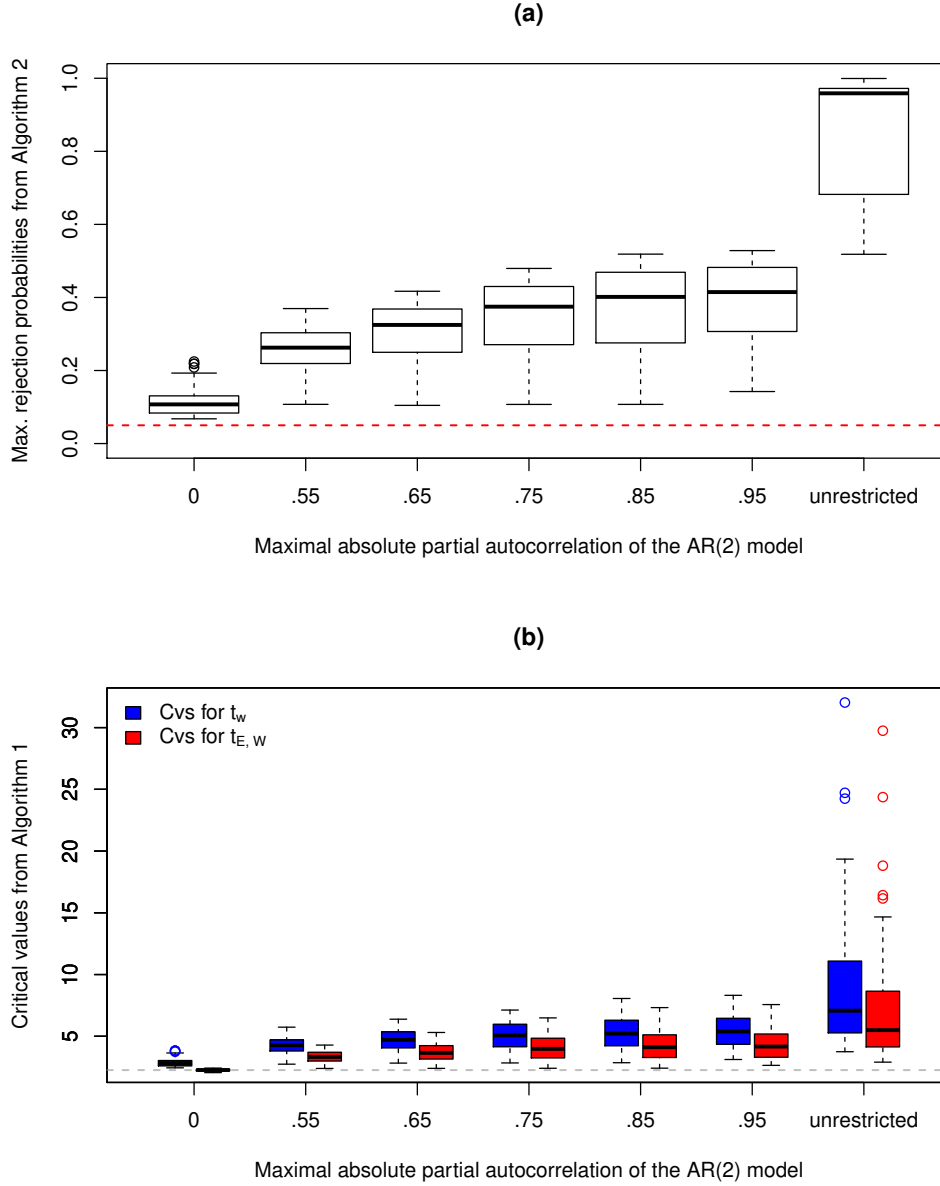
25

(a)

(b)

Figure 2: (a) Sizes of the test which rejects if $|t_w| \geq 2.260568$ (Kiefer-Vogelsang critical value). The horizontal dashed red line corresponds to 0.05. (b) Critical values guaranteeing a 5 % level for the t-type tests corresponding to T_w and $T_{E,\mathrm{W}}$. The horizontal dashed grey line corresponds to the Kiefer-Vogelsang critical value 2.260568.

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critical value, still fail to provide size control by a considerable margin. Numerical computations of critical values that do provide size control are given in Figure 2(b). Apart from the i.i.d. case, the corresponding boxplots do not "cover" the Kiefer-Vogelsang critical value. In line with Figure 2(a), the size-controlling critical values considerably exceed the Kiefer-Vogelsang critical value.

5 Size control of tests of affine restrictions in regression models with nonspherical disturbances: General theory

In this section we lay the foundation for all the size control results in the paper. Other than in the preceding Sections 3 and 4, we here do not require that the disturbance vector in the regression model (1) is induced by a stationary process, but we revert to the more general framework specified in Section 2. In Subsection 5.1 we obtain conditions under which the size of a rejection region W (satisfying certain invariance properties) is smaller than one when testing (3). This result is then further specialized to the important case when $W = \{T \geq C\}$ for a test statistic T satisfying weak regularity conditions. In Subsection 5.2 we consider a *family* of regions W_j for $j \in \mathbb{N}$ (satisfying certain invariance properties), and we obtain conditions under which for every $\alpha \in (0,1)$ there exists an element of this family $W_{j(\alpha)}$, say, the size of which does not exceed α. These results are then specialized to families of the form $W_j = \{T \geq C_j\}$ for $C_j \nearrow \infty$, and where the test statistic T satisfies certain weak regularity conditions. For such families we also obtain a lower bound for the critical values that possibly can lead to size control, and we study the problem under which conditions *exact* size control can be achieved, i.e., when for a given $\alpha \in (0,1)$ a critical value does exist so that the size of the corresponding test equals α. In Subsection 5.3 we then show how some of the conditions arising in the results in Subsections 5.1 and 5.2 can be implied from lower-level conditions; see also Section B.1 of Appendix B.

We start by defining a certain collection of linear subspaces of $\mathbb{R}^n$, where n is sample size, that plays a central rôle in the size control results. Loosely speaking, the linear spaces belonging to this collection are either (nontrivial) projections of concentration spaces of the covariance model $\mathfrak{C}$ (in the sense of Preinerstorfer and Pötscher (2016)) on $\mathcal{L}^\perp$, where $\mathcal{L}$ is an appropriately chosen subspace related to invariance properties of the tests under consideration, or are what one could call "higher-order" concentration spaces. For a more precise discussion see Section B.1 in Appendix B. Since the tests we are interested in are all at least $G(\mathfrak{M}_0)$-invariant, a typical choice for $\mathcal{L}$ is $\mathfrak{M}_0^{lin}$, the linear space parallel to $\mathfrak{M}_0$. However, it proves useful to allow for the more general case where $\mathcal{L}$ is an arbitrary linear space (typically containing $\mathfrak{M}_0^{lin}$). Recall from Section 5.1 of Preinerstorfer and Pötscher (2016) that $G(\mathfrak{M}_0)$ denotes the group of all maps of the form $y \mapsto \delta(y - \mu_0) + \mu_0'$, where $\delta \in \mathbb{R}$, $\delta \neq 0$, and where μ_0 as well as μ_0' belong to $\mathfrak{M}_0$.

Definition 5.1. Given a linear subspace $\mathcal{L}$ of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$ and a symmetric positive

definite $n \times n$-dimensional matrix Σ, we let

$$\mathcal{L}(\Sigma) = \frac{\Pi_{\mathcal{L}^\perp} \Sigma \Pi_{\mathcal{L}^\perp}}{\|\Pi_{\mathcal{L}^\perp} \Sigma \Pi_{\mathcal{L}^\perp}\|}. \tag{21}$$

Given a covariance model $\mathfrak{C}$, we let $\mathcal{L}(\mathfrak{C}) = \{\mathcal{L}(\Sigma) : \Sigma \in \mathfrak{C}\}$. Furthermore, we define

$$\mathbb{J}(\mathcal{L}, \mathfrak{C}) = \left\{ \mathrm{span}(\bar{\Sigma}) : \bar{\Sigma} \in \mathrm{cl}(\mathcal{L}(\mathfrak{C})), \ \mathrm{rank}(\bar{\Sigma}) < n - \dim(\mathcal{L}) \right\},$$

where the closure is here understood w.r.t. $\mathbb{R}^{n \times n}$.

Note that the denominator in (21) is always nonzero, that $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ can be empty, and that $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ neither contains $\mathcal{L}^\perp$ nor the trivial space $\{0\}$. Also note that $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ is independent of the particular choice of norm used in the above definition.

Remark 5.1. (i) Even in the special case where $\mathcal{L} = \{0\}$, the set $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ need not coincide with the set of all concentration spaces in the sense of Preinerstorfer and Pötscher (2016) (i.e., with the set, the union of which is $J(\mathfrak{C})$ defined in Section 5.3 of Preinerstorfer and Pötscher (2016)). Both sets coincide in case $\mathcal{L} = \{0\}$ if and only if $\mathfrak{C}$ is bounded away from zero, i.e., if there is no sequence $\Sigma_j \in \mathfrak{C}$ that converges to the zero matrix. Note that this latter condition on $\mathfrak{C}$ is satisfied for many covariance models, e.g., if $\mathfrak{C}$ consists of correlation matrices or under similar normalization assumptions.

(ii) In fact, as long as we are only concerned with $G(\mathfrak{M}_0)$-invariant tests and their null-rejection probabilities, we could without loss of generality always assume that the covariance model is bounded and is bounded away from zero: note that, e.g., passing from $\mathfrak{C}$ to the normalized covariance model $\{\Sigma / \|\Sigma\| : \Sigma \in \mathfrak{C}\}$ does not affect the null-rejection probabilities of $G(\mathfrak{M}_0)$-invariant tests, see Proposition 5.4 in Preinerstorfer and Pötscher (2016).[11] Furthermore, note that $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ does not change if $\mathfrak{C}$ is being rescaled.

5.1 Size less than one

The subsequent proposition gives simple sufficient conditions under which the size of a test is less than one.

Proposition 5.2. *Let $\mathfrak{C}$ be a covariance model, and let W be a $G(\mathfrak{M}_0)$-invariant rejection region that is also invariant w.r.t. addition of elements of a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$. Define $\mathcal{L} = \mathrm{span}(\mathfrak{M}_0^{lin} \cup \mathcal{V})$, i.e., $\mathcal{L}$ is the linear space generated by $\mathfrak{M}_0^{lin} \cup \mathcal{V}$, and assume that $\dim(\mathcal{L}) < n$. Suppose that*

(a) W^c is not a $\lambda_{\mathbb{R}^n}$-null set, and

(b) if $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ then $(\mathrm{cl}(W))^c \cap (\mu_0 + \mathcal{S}) \neq \emptyset$ for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$ in view of $G(\mathfrak{M}_0)$-invariance of W).

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[11]The effect of replacing Σ by $\Sigma / \|\Sigma\|$ on non-null-rejection probabilities is that σ^2 is replaced by $\sigma^2 \|\Sigma\|$. As a consequence, the nuisance-minimal power at any $\mu \in \mathfrak{M}_1$ remains unaffected.

Then the size of the test given by the rejection region W satisfies

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(W) < 1. \tag{22}$$

The leading case in this proposition and in the results in the subsequent section is the case where $\mathcal{V} = \{0\}$, and hence $\mathcal{L} = \mathfrak{M}_0^{lin}$.

Remark 5.3. (i) In case W satisfies the invariance assumptions appearing in Proposition 5.2 and $\dim(\mathcal{L}) = n$ holds, it follows that W is either empty or the entire space $\mathbb{R}^n$, both being trivial cases. Similar remarks apply to the other results in this and the next subsection and will not be stated.

(ii) If a rejection region W^* differs from a rejection region W that satisfies the assumptions of Proposition 5.2 only by a $\lambda_{\mathbb{R}^n}$-null set, then the conclusions of Proposition 5.2 also hold for W^* (even if it does not satisfy the assumptions of that proposition), since W and W^* have the same rejection probabilities. Similar remarks apply to the other results in this and the next subsection and will not be stated.

(iii) It is not difficult to see that $G(\mathfrak{M}_0)$-invariance together with invariance w.r.t. addition of the elements of $\mathcal{V}$ is equivalent to $G(\mu_0 + \mathcal{L})$-invariance for some $\mu_0 \in \mathfrak{M}_0$ (and hence every $\mu_0 \in \mathfrak{M}_0$).

Remark 5.4. (i) Proposition 5.2 applies, in particular, to rejection regions of the form $W = \{y \in \mathbb{R}^n : T(y) \geq C\}$ for some $-\infty < C < \infty$, where $T : \mathbb{R}^n \to \mathbb{R}$ is Borel-measurable, is $G(\mathfrak{M}_0)$-invariant and is also invariant w.r.t. addition of elements of a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$.

(ii) If, additionally to the conditions in (i), T is continuous on the complement of a closed set $N^\dagger$, then a sufficient condition for condition (b) in Proposition 5.2 is as follows: if $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$, then $W^c \cap (N^\dagger)^c \cap (\mu_0 + \mathcal{S}) \neq \emptyset$ for some $\mu_0 \in \mathfrak{M}_0$. This follows from Lemma B.1 in Appendix B. [An equivalent formulation of this sufficient condition is that whenever $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ then there exists an $s \in \mathcal{S}$ such that $T(\mu_0 + s) < C$ and $\mu_0 + s \notin N^\dagger$ for some $\mu_0 \in \mathfrak{M}_0$.]

(iii) Suppose the conditions on T in (ii) are satisfied. In case $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ is finite, i.e., $\mathbb{J}(\mathcal{L}, \mathfrak{C}) = \{\mathcal{S}_1, \ldots, \mathcal{S}_r\}$, and provided that for every $\mathcal{S}_i \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ one can find an element $s_i \in \mathcal{S}_i$ with $\mu_{0,i} + s_i \notin N^\dagger$ for some $\mu_{0,i} \in \mathfrak{M}_0$, any C satisfying $C > \max_{1 \leq i \leq r} T(\mu_{0,i} + s_i)$ gives rise to a critical region W that has size less than 1. To see this, observe that condition (b) in Proposition 5.2 is satisfied in view of (ii) and the construction of C. Furthermore, observe that $\mu_{0,i} + s_i \in W^c$. Since T is continuous at $\mu_{0,i} + s_i$ (as $\mu_{0,i} + s_i \notin N^\dagger$), a sufficiently small open ball with center at $\mu_{0,i} + s_i$ also belongs to W^c, showing that W^c is not a $\lambda_{\mathbb{R}^n}$-null set.

5.2 Size less than α

The next proposition is the basis for the size control results we want to obtain.

Proposition 5.5. *Let $\mathfrak{C}$ be a covariance model, and let W_j be a sequence of $G(\mathfrak{M}_0)$-invariant rejection regions that are also invariant w.r.t. addition of elements of a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$.*

Define $\mathcal{L} = \mathrm{span}(\mathfrak{M}_0^{lin} \cup \mathcal{V})$, and assume that $\dim(\mathcal{L}) < n$. Assume that the rejection regions satisfy $W_j \supseteq W_{j+1}$ for all $j \in \mathbb{N}$. Suppose that

(a) $\bigcap_{j \in \mathbb{N}} W_j$ is a $\lambda_{\mathbb{R}^n}$-null set, and

(b) if $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ then $\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$ is a $\lambda_{\mu_0 + \mathcal{S}}$-null set for some $\mu_0 \in \mathfrak{M}_0$ (and hence every $\mu_0 \in \mathfrak{M}_0$).

Then the sizes of the tests given by the rejection regions W_j satisfy

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(W_j) \to 0 \quad as \quad j \to \infty; \tag{23}$$

in particular, for every $\alpha \in (0, 1)$ there exists a $j(\alpha) \in \mathbb{N}$ so that

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(W_{j(\alpha)}) \leq \alpha.$$

In the important special case, where $W_j = \{y \in \mathbb{R}^n : T(y) \geq C_j\}$ for some real-valued test statistic T and where $C_j \nearrow \infty$ as $j \to \infty$, condition (a) in the preceding proposition is clearly always satisfied since the intersection is empty in this case. The subsequent corollary now provides sufficient conditions for condition (b) in case of such rejection regions W_j. The conditions imposed on T in this corollary are widely satisfied (at least in the leading case where $\mathcal{V} = \{0\}$), see Subsection 5.3.

Corollary 5.6. *Let $\mathfrak{C}$ be a covariance model, and assume that the test statistic $T : \mathbb{R}^n \to \mathbb{R}$ is Borel-measurable and is continuous on the complement of a closed set $N^\dagger$.[12] Assume that T and $N^\dagger$ are $G(\mathfrak{M}_0)$-invariant, and are also invariant w.r.t. addition of elements of a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$.[13] Define $\mathcal{L} = \mathrm{span}(\mathfrak{M}_0^{lin} \cup \mathcal{V})$, and assume that $\dim(\mathcal{L}) < n$. Let $W_j = \{y \in \mathbb{R}^n : T(y) \geq C_j\}$ for $-\infty < C_j < \infty$ with $C_j \nearrow \infty$ as $j \to \infty$. Then:*

1. *The conclusion of Proposition 5.5 holds, provided condition (b) in that proposition is satisfied.*

2. *A sufficient condition for condition (b) in Proposition 5.5 to hold is the following: if $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$, then the set $N^\dagger$ is a $\lambda_{\mu_0 + \mathcal{S}}$-null set for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$).*

3. *In case $N^\dagger$ is a finite or countable union of affine subspaces, the sufficient condition given in 2. is equivalent to: if $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$, then $\mu_0 + \mathcal{S} \not\subseteq N^\dagger$ for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$).*

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[12]While this condition is trivially satisfied for every Borel-measurable T upon choosing $N^\dagger$ equal to $\mathbb{R}^n$, satisfying the other conditions in the corollary will rule out this case, except if $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ is empty. Also note that for typical test statistics the set $N^\dagger$ will turn out to be a 'small' set, e.g., a $\lambda_{\mathbb{R}^n}$-null set.

[13]If T is Borel-measurable, is continuous on the complement of a closed set $N^\dagger$, and if T satisfies the invariance requirements in the corollary, then it is easy to see that one can always find a closed subset $N^{\dagger\dagger}$ of $N^\dagger$ such that T is continuous on the complement of $N^{\dagger\dagger}$ and such that $N^{\dagger\dagger}$ satisfies the invariance properties asked for in the corollary. Hence, requiring in the corollary that the set $N^\dagger$ satisfies the invariance conditions imposes no loss of generality.

Remark 5.7. The corollary implies, in particular, that size control is always possible in case $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ is empty.[14] This is, e.g., the case if $\mathfrak{C}$ has no singular limit points and is norm bounded, or more generally if $\mathfrak{C}$ is such that $\{\Sigma / \|\Sigma\| : \Sigma \in \mathfrak{C}\}$ has no singular limit points. Of course, this is in line with Theorems 5.10 and 5.12 in Preinerstorfer and Pötscher (2016).

Remark 5.8. If $\mathfrak{M}_0$ is a linear space (i.e., if $r = 0$), we can w.l.o.g. set $\mu_0 = 0$ in the sufficient conditions in Part 2 and 3 of Corollary 5.6, leading to a simplification of the conditions. Even if $\mathfrak{M}_0$ is not a linear space (i.e., if $r \neq 0$), the same simplification can be made provided that $N^\dagger$ is invariant under addition of elements of $\mathfrak{M}_0$. This latter invariance property is, in particular, satisfied whenever $N^\dagger$ is $G(\mathfrak{M})$-invariant, which is the case for a large class of test statistics including nonsphericity-corrected F-type test statistics that satisfy Assumption 5 in Preinerstorfer and Pötscher (2016) (cf. Lemma 5.16 further below).

Remark 5.9. (i) The argument that establishes Part 3 of the preceding corollary also shows that in condition (b) in Proposition 5.5 we can replace "$\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$ is a $\lambda_{\mu_0 + \mathcal{S}}$-null set" equivalently by "$\mu_0 + \mathcal{S} \nsubseteq \bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$" provided that $\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$ is a finite or countable union of affine spaces.

(ii) The condition that $\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$ ($N^\dagger$, respectively) is a $\lambda_{\mu_0 + \mathcal{S}}$-null set in Proposition 5.5 (Corollary 5.6, respectively) is – for 1-dimensional $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ – equivalent to $\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j) \cap (\mu_0 + \mathcal{S}) \subseteq \{\mu_0\}$ ($N^\dagger \cap (\mu_0 + \mathcal{S}) \subseteq \{\mu_0\}$, respectively); i.e., except possibly for μ_0, the entire set $\mu_0 + \mathcal{S}$ lies outside of $\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$ ($N^\dagger$, respectively). For a proof see Appendix B.

Corollary 5.6 implies that, under its conditions, for every $\alpha \in (0, 1)$ there exists a real number $C(\alpha)$ such that

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(T \geq C(\alpha)) \leq \alpha \tag{24}$$

holds. Under mild conditions a smallest such critical value exists as discussed in the next remark.

Remark 5.10. *(Existence of a smallest critical value guaranteeing size control)* (i) Let $\alpha \in (0, 1)$ and suppose T is a test statistic such that (24) holds. Let $\mathsf{CV}_{\leq}(\alpha)$ be the set of all real $C(\alpha)$ such that (24) holds. Then the set $\mathsf{CV}_{\leq}(\alpha)$ is clearly nonempty and is an interval either of the form $(C_\Diamond(\alpha), \infty)$ or of the form $[C_\Diamond(\alpha), \infty)$ for some real number $C_\Diamond(\alpha)$. If the map $C \mapsto \sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(T \geq C)$ is continuous from the right at $C = C_\Diamond(\alpha)$, then $\mathsf{CV}_{\leq}(\alpha) = [C_\Diamond(\alpha), \infty)$ must hold, i.e., a *smallest* critical value guaranteeing (24) does exist and is given by $C_\Diamond(\alpha)$. The just mentioned right-continuity property is easily seen to be satisfied, whenever the test statistic T has the property that $\lambda_{\mathbb{R}^n}(T = C_\Diamond(\alpha)) = 0$. This latter condition is satisfied by a large class of test statistics (cf. Lemma 5.16 further below).

(ii) Let $\alpha \in (0, 1)$ and suppose T is a test statistic such that equality in (24) holds for at least one real number $C(\alpha)$. Let $\mathsf{CV}_{=}(\alpha)$ denote the set of all such $C(\alpha)$, which then clearly is a nonempty subinterval of $\mathsf{CV}_{\leq}(\alpha)$ consisting of an initial (w.r.t. the order on the real line) segment

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[14]Observe that in this case setting $N^\dagger$ equal to $\mathbb{R}^n$ does not restrict the applicability of the corollary and hence voids the continuity requirement on T.

of $\mathsf{CV}_{\leq}(\alpha)$.[15] If $\mathsf{CV}_{\leq}(\alpha) = [C_{\Diamond}(\alpha), \infty)$, as is, e.g., the case under the condition discussed in (i), then $C_{\Diamond}(\alpha)$ is also the smallest element of $\mathsf{CV}_{=}(\alpha)$, i.e., a *smallest* critical value guaranteeing equality in (24) exists and is given by $C_{\Diamond}(\alpha)$.

The next lemma provides a lower bound C^* for the critical values C that possibly can lead to size control and is a building block for the next proposition.

Lemma 5.11. *Let $\mathfrak{C}$, T, $N^{\dagger}$, $\mathcal{V}$, and $\mathcal{L}$ be as in Corollary 5.6. Denote by $\mathbb{H}$ the set of all $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ such that T restricted to $\mu_0 + \mathcal{S}$ is equal to a constant $C(\mathcal{S})$, say, $\lambda_{\mu_0 + \mathcal{S}}$-almost everywhere for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$). Define $C_* = \inf_{\mathcal{S} \in \mathbb{H}} C(\mathcal{S})$ and $C^* = \sup_{\mathcal{S} \in \mathbb{H}} C(\mathcal{S})$, with the convention that $C_* = \infty$ and $C^* = -\infty$ if $\mathbb{H}$ is empty. Then the following hold:*

1. *Any one-dimensional $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ belongs to $\mathbb{H}$. [In particular, $\mathbb{H}$ is nonempty if a one-dimensional $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ exists.]*

2. *Suppose that for every $\mathcal{S} \in \mathbb{H}$ the set $N^{\dagger}$ is a $\lambda_{\mu_0 + \mathcal{S}}$-null set for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$). Then for $C \in (-\infty, C^*)$ the size of the test with critical region $\{T \geq C\}$ satisfies*

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(T \geq C) = 1. \tag{25}$$

   If, additionally, $N^{\dagger}$ is a $\lambda_{\mu_0 + \mathcal{S}}$-null set for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$) not only for $\mathcal{S} \in \mathbb{H}$ but for every $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$, then the l.h.s. in (25) converges to zero for $C \to \infty$ (implying that then $C^* < \infty$ holds).

3. *Suppose that for every $\mathcal{S} \in \mathbb{H}$ the set $N^{\dagger}$ is a $\lambda_{\mu_0 + \mathcal{S}}$-null set for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$). Then for $C \in (C_*, \infty)$*

$$\inf_{\mu_0 \in \mathfrak{M}_0} \inf_{0 < \sigma^2 < \infty} \inf_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(T \geq C) = 0. \tag{26}$$

Part 3 of the preceding lemma also provides some information on the infimal rejection probability under the null hypothesis (and thus on power properties "near" the null) of the test in case $\mathbb{H}$ is nonempty: In this case, we clearly have $C_* \leq C^*$. It follows that, under the assumptions of the lemma, a test based on T which has size less than or equal to $\alpha < 1$, *must* have a critical value C larger than or equal to $C^*(\geq C_*)$. Part 3 then implies severe biasedness of the test (except possibly if $C = C_* = C^*$ holds), which typically entails bad power properties in certain regions of the alternative hypothesis (in view of Part 3 of Theorem 5.7 and Remark 5.5(iii) in Preinerstorfer and Pötscher (2016)). However, note that in case $\mathbb{H}$ is empty, we have $C_* = \infty$ and consequently Part 3 of the lemma does not convey any information about rejection probabilities. Since we concentrate exclusively on size properties in this paper, we postpone a detailed discussion of power properties to a companion paper. Furthermore, in view of Remark

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[15]That is, if $c \in \mathsf{CV}_{=}(\alpha)$, then every $c' \in \mathsf{CV}_{\leq}(\alpha)$ with $c' \leq c$ also belongs to $\mathsf{CV}_{=}(\alpha)$.

5.5(iii) in Preinerstorfer and Pötscher (2016), relation (25) remains valid even after one removes the suprema over μ_0 and σ^2 in (25). A similar remark applies to (26).

The size control result in Corollary 5.6 can be sharpened to an exact size control result under some additional assumptions.

Proposition 5.12. *Let $\mathfrak{C}$, T, $N^{\dagger}$, $\mathcal{V}$, and $\mathcal{L}$ be as in Corollary 5.6. Suppose that for every $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ the set $N^{\dagger}$ is a $\lambda_{\mu_0 + \mathcal{S}}$-null set for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$). Let $\mathbb{H}$, $C(\mathcal{S})$, and C^* be as in Lemma 5.11 (note that $C^* < \infty$ holds in view of Part 2 of that lemma).*

A. *Suppose that for every $C \in (C^*, \infty)$*

(a) $\lambda_{\mathbb{R}^n}(T = C) = 0$, and

(b) if $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C}) \backslash \mathbb{H}$ then $\lambda_{\mu_0 + \mathcal{S}}(T = C) = 0$ for some $\mu_0 \in \mathfrak{M}_0$ (and hence for every $\mu_0 \in \mathfrak{M}_0$).

Then the following holds:

1. The function

$$C \mapsto \sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(T \geq C) \qquad (27)$$

is nonincreasing, equals one for $C \in (-\infty, C^*)$, is continuous on (C^*, ∞), and converges to 0 as $C \to \infty$.

2. Define

$$\alpha^* = \sup_{C \in (C^*, \infty)} \sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(T \geq C). \qquad (28)$$

Then, for every $\alpha \in (0, 1)$ there exists a $C(\alpha) \in (C^*, \infty)$ such that

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(T \geq C(\alpha)) \leq \min(\alpha, \alpha^*) \leq \alpha. \qquad (29)$$

[Note that the l.h.s. of (29) necessarily is less than or equal to α^* for every choice of $C(\alpha) \in (C^*, \infty)$.] Furthermore, for every $\alpha \in (0, \alpha^*)$ the constant $C(\alpha) \in (C^*, \infty)$ can be chosen such that ("exact size control")

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(T \geq C(\alpha)) = \alpha. \qquad (30)$$

B. *Suppose $C^{**} \geq C^*$ is a real number and suppose that the conditions (a) and (b) given above are only known to hold for every $C \in (C^{**}, \infty)$. Define α^{**} in the same way as α^*, but with C^{**} replacing C^*. Then the claims in Part A.1 continue to hold as they stand except for the fact that continuity is now only guaranteed on (C^{**}, ∞). As a consequence, all the conclusions of Part A.2 continue to hold provided C^* in that part is everywhere replaced by*

C^{**} and α^* by α^{**}. [Since $C^{**} \geq C^*$ and hence $\alpha^{**} \leq \alpha^*$, also the claim in (29) continues to hold as it stands.]

Again by Remark 5.5(iii) in Preinerstorfer and Pötscher (2016), the suprema w.r.t. μ_0 and σ^2 can be removed from (27)-(30), and the resulting expressions then do not depend on the particular choice for $\mu_0 \in \mathfrak{M}_0$ and $\sigma^2 \in (0, \infty)$.

Remark 5.13. *(Continuity properties and existence of smallest critical values)* Suppose the assumptions for Part A of Proposition 5.12 are satisfied.

(i) If $C^* = -\infty$ then the function given by (27) is continuous on $\mathbb{R}$, whereas if $C^* > -\infty$ it is only guaranteed to be continuous on $\mathbb{R} \backslash \{C^*\}$. If $C^* > -\infty$ and if, additionally, $\lambda_{\mathbb{R}^n}(T = C^*) = 0$ holds, it is easy to see that (27) is then at least continuous from the right at C^*.

(ii) If $C^* = -\infty$ then clearly $\alpha^* = 1$ and hence exact size control (30) is possible for every $\alpha \in (0, 1)$, whereas in case $C^* > -\infty$ we only can conclude that $\alpha^* \leq 1$ and that (30) holds for $\alpha \in (0, \alpha^*)$. If $C^* > -\infty$ and if, additionally, the map (27) is continuous from the right at C^*, it follows from (i) that (30) can also be achieved for $\alpha = \alpha^*$ (with a $C(\alpha)$ belonging to $[C^*, \infty)$).

(iii) An upper bound for α^* is obviously given by $\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(T \geq C^*)$. If $C^* = -\infty$ holds, or if $C^* > -\infty$ and the map (27) is continuous from the right at C^*, then this upper bound coincides with α^*.

(iv) If $C^* = -\infty$ holds, or if $C^* > -\infty$ and the map (27) is continuous from the right at C^*, then for every $\alpha \in (0, 1)$ a smallest critical value $C_\Diamond(\alpha) \in \mathbb{R}$ satisfying (29) exists (i.e., $\mathsf{CV}_{\leq}(\alpha) = [C_\Diamond(\alpha), \infty)$ holds) in view of (i) and Remark 5.10. [Note that for $\alpha \in (0, 1)$ here $C_\Diamond(\alpha) > C^*$ must hold if $\alpha < \alpha^*$, and that $C_\Diamond(\alpha) = C^*$ must hold if $\alpha = \alpha^*$.] Under the same conditions, for every $\alpha \in (0, \alpha^*] \cap (0, 1)$ there is a smallest critical value satisfying (30) which is again given by $C_\Diamond(\alpha)$ (in fact, $\mathsf{CV}_{=}(\alpha) = [C_\Diamond(\alpha), b]$ for some real $b \geq C_\Diamond(\alpha)$ holds). This follows from (i), (ii) above and Remark 5.10 (and the fact that (27) goes to zero for $C \to \infty$).

(v) For every $\alpha \in (0, \alpha^*)$ a smallest critical value $C_\Diamond(\alpha) \in \mathbb{R}$ satisfying (29) and (30) always exists, even without the right-continuity condition in case $C^* > -\infty$. Necessarily, $C_\Diamond(\alpha) > C^*$ has to hold.

(vi) The case $\alpha^* = 0$ can occur, e.g., if T is identically equal to a constant. However, for large classes of test statistics such as nonsphericity-corrected F-type test statistic as defined in (28) in Section 5.4 of Preinerstorfer and Pötscher (2016) and satisfying Assumptions 5 and 6 in that paper we always have $\alpha^* > 0$. This follows from Part 5 of Lemma 5.15 in that reference.

Remark 5.14. (i) Suppose $\dim(\mathcal{L}) = n - 1$. Then clearly $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ is empty. Furthermore, any T that satisfies the invariance properties mentioned in Corollary 5.6 is then constant $\lambda_{\mathbb{R}^n}$-almost everywhere, hence size control is trivially possible. Similarly, W in Proposition 5.2 then is an $\lambda_{\mathbb{R}^n}$-null set and the proposition holds trivially. Similarly, the sets W_j in Proposition 5.5 are then $\lambda_{\mathbb{R}^n}$-null sets from a certain j onwards.

(ii) In case $\mathcal{L} = \mathfrak{M}_0^{lin}$, the case $\dim(\mathcal{L}) = n - 1$ can not arise, since $k < n$ and $q \geq 1$ are always assumed.

The following observation applies to a large class of test statistics and is, e.g., useful when constructing confidence sets by "inverting" the corresponding test.

Lemma 5.15. *Suppose T is a nonsphericity-corrected F-type test statistic as defined in (28) in Section 5.4 of Preinerstorfer and Pötscher (2016) and that Assumption 5 in that paper is satisfied. Then the rejection probabilities under the null hypothesis described by (3) do not depend on the value of r. As a consequence, the size-controlling critical values $C(\alpha)$ (if they exist) do not depend on the value of r either. [It is understood here that the estimators $\check{\beta}$ and $\check{\Omega}$ used to define the test statistic T have been chosen independently of the value of r.]*

5.3 Some sufficient conditions

We collect here sufficient conditions for some of the assumptions on T in the preceding results. For sufficient conditions relating to $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ see Section B.1 in Appendix B.

Lemma 5.16. *Suppose T is a nonsphericity-corrected F-type test statistic as defined in (28) in Section 5.4 of Preinerstorfer and Pötscher (2016) and that Assumption 5 in that paper is satisfied. Then:*

1. *T is Borel-measurable and is continuous on the complement of a closed $\lambda_{\mathbb{R}^n}$-null set N^* (with N^* given in (27) of Preinerstorfer and Pötscher (2016)). Furthermore, T and N^* are $G(\mathfrak{M}_0)$-invariant (in fact, N^* is even $G(\mathfrak{M})$-invariant).*

2. *$\lambda_{\mathbb{R}^n}(T = C) = 0$ holds for $-\infty < C < \infty$.*

3. *The complement of the rejection region $\{T \geq C\}$ is not a $\lambda_{\mathbb{R}^n}$-null set for every $C > 0$.*

Remark 5.17. *(Special cases)* Lemma 5.16 applies, in particular, to the commonly used autocorrelation robust test statistic T_w given in (5) provided Assumptions 1 and 2 are satisfied, since such test statistics then are nonsphericity-corrected F-type test statistics and the above mentioned Assumption 5 is satisfied, cf. Lemma A.1 in Preinerstorfer and Pötscher (2016). The same is true, more generally, for the test statistics T_{GQ} defined in Section 3.4 whenever the weighting matrix $\mathcal{W}_n^*$ (also defined in that section) is positive definite and Assumption 2 holds (this is proved in the same way as Lemma A.1 in Preinerstorfer and Pötscher (2016) using Lemma 3.11 instead of Lemma 3.1 in that reference). Furthermore, the weighted Eicker-test statistic $T_{E,\mathsf{W}}$ defined in Section 3 (with W a symmetric and nonnegative definite $n \times n$ Toeplitz matrix with ones on the main diagonal) is a nonsphericity-corrected F-type test statistic with the above mentioned Assumption 5 being always satisfied; hence Lemma 5.16 also applies to the weighted Eicker-test statistic. [In fact, also Assumptions 6 and 7 in Preinerstorfer and Pötscher (2016) are satisfied for these three test statistics (under the respective assumptions mentioned above), since any one of $\hat{\Omega}_w(y)$, $\hat{\Omega}_{GQ}(y)$, and $\hat{\Omega}_{E,\mathsf{W}}(y)$ is then nonnegative definite for every $y \in \mathbb{R}^n$.]

For the test statistics mentioned in the preceding remark more can be said about the set N^*. For the weighted Eicker-test statistic (with W as in Remark 5.17) we always have $N^* = \mathrm{span}(X)$,

thus it is a proper linear subspace of $\mathbb{R}^n$. For autocorrelation robust test statistics of the form (5) and under Assumptions 1 and 2 the set N^* turns out to be the set B defined in Section 3, cf. Lemmata 3.1, A.1, and 5.15 of Preinerstorfer and Pötscher (2016); and this is more generally true for the test statistics T_{GQ} provided the weighting matrix $\mathcal{W}_n^*$ is positive definite and Assumption 2 holds, cf. Lemma 3.11 of Preinerstorfer and Pötscher (2016). The following is now true for the set B.

Lemma 5.18. *Suppose Assumption 2 holds. Then* B *is a finite union of proper linear subspaces of $\mathbb{R}^n$; in case $q = 1$,* B *is a proper linear subspace. [Consequently, the same applies to the set N^* associated with the autocorrelation robust test statistic T_w defined in (5) (or with T_{GQ}, respectively) if also Assumption 1 (or the condition $\mathcal{W}_n^*$ positive definite, respectively) applies.]*

Simple examples show, that in case $q > 1$, the set B in Lemma 5.18 need not be a linear space itself. However, generically $B = \text{span}(X)$ holds under typical assumptions as is shown in Lemma A.3 in Appendix A, cf. also Theorem 3.9. The next lemma verifies that the condition (b) in Proposition 5.12 is often satisfied.

Lemma 5.19. *(i) Let Assumptions 1 and 2 hold and let T_w be defined as in (5). Suppose $\mathcal{S}$ is a linear subspace of $\mathbb{R}^n$ and μ is an element of $\mathbb{R}^n$ such that T_w restricted to $\mu + \mathcal{S}$ is not equal to a constant $\lambda_{\mu+\mathcal{S}}$-almost everywhere. Then, for every real number C it holds that $\lambda_{\mu+\mathcal{S}}(T_w = C) = 0$.*

(ii) Let Assumption 2 hold and let T_{GQ} be the test statistic as defined in Section 3.4 with a positive definite weighting matrix $\mathcal{W}_n^*$. Then the same conclusion as in (i) holds with T_w replaced by T_{GQ}.

(iii) Let W *be an $n \times n$ symmetric and nonnegative definite Toeplitz matrix of weights with ones on the main diagonal. Then the same conclusion as in (i) holds with T_w replaced by $T_{E,W}$.*

Lemmata 5.16-5.19 can obviously be used to provide streamlined versions of Propositions 5.2, 5.5, 5.12, Lemma 5.11, as well as Corollary 5.6 in the case of nonsphericity-corrected F-type tests, and, in particular, in the case of autocorrelation robust tests based on T_w, T_{GQ}, or $T_{E,W}$. We abstain from presenting such results.

6 Size control of tests of affine restrictions in regression models with stationary autocorrelated errors: General results

It transpires from the results in Section 5 that characterizing the elements of the collection $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ is central to achieving explicit conditions for size control. In this section we undertake such a characterization for the important case where $\mathfrak{C} = \mathfrak{C}(\mathfrak{F})$, i.e., when the errors in the regression model come from a stationary process. In Subsection 6.1 we present the characterization result, which in turn forms the basis for the size control results in Subsection 6.2.

6.1 The structure of $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}))$

Before we can state the main results of this subsection we need to introduce some more notation.

Definition 6.1. For $\omega \in [0, \pi]$ and for $d \in \mathbb{N}$ we define $\kappa(\omega, d) = d$ if $\omega \in \{0, \pi\}$, and $\kappa(\omega, d) = 2d$ if $\omega \in (0, \pi)$. For a positive integer p, for $\underline{\omega} = (\omega_1, \ldots, \omega_p) \in [0, \pi]^p$, and for $\underline{d} = (d_1, \ldots, d_p) \in \mathbb{N}^p$ we define

$$\kappa(\underline{\omega}, \underline{d}) = \sum_{i=1}^{p} \kappa(\omega_i, d_i). \tag{31}$$

In case $p = 1$, we shall often simply write ω for $\underline{\omega}$ and d for $\underline{d}$.

It proves useful to introduce the convention that $\underline{\omega}$ and $\underline{d}$ are the 0-tupels for $p = 0$ and to set $\kappa(\underline{\omega}, \underline{d}) = 0$ in this case. The following notation will be helpful.

Definition 6.2. Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$. We write $\underline{\omega}(\mathcal{L})$ for the vector obtained by ordering the elements of

$$\{\omega \in [0, \pi] : \rho(\omega, \mathcal{L}) > 0\} \tag{32}$$

from smallest to largest, provided the set in (32) is nonempty, and we denote by $p(\mathcal{L})$ the dimension of this vector (clearly $p(\mathcal{L}) > 0$ then holds); furthermore, we set $d_i(\mathcal{L}) = \rho(\omega_i(\mathcal{L}), \mathcal{L})$ for every $i = 1, \ldots, p(\mathcal{L})$ (where $\omega_i(\mathcal{L})$ denotes the i-th coordinate of $\underline{\omega}(\mathcal{L})$), and we write $\underline{d}(\mathcal{L})$ for the vector with i-th coordinate equal to $d_i(\mathcal{L})$. If the set in (32) is empty, we take $\underline{\omega}(\mathcal{L})$ as well as $\underline{d}(\mathcal{L})$ as the 0-tupel (which we may identify with the empty set) and we set $p(\mathcal{L}) = 0$.

Recall that the set (32) is always a finite set as discussed subsequent to Definition 3.1 and hence $\underline{\omega}(\mathcal{L})$ is well-defined. Clearly, the set (32) coincides with the set $\{\omega \in [0, \pi] : \text{span}(E_{n,0}(\omega)) \subseteq \mathcal{L}\}$. Furthermore, $\kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) = 0$ if $p(\mathcal{L}) = 0$ in view of the above conventions.

Definition 6.3. For $\omega \in [0, \pi]$ define polynomials in the variable z as $\Delta_\omega(z) = 1 - \cos(\omega)z$ if $\omega \in \{0, \pi\}$ and as $\Delta_\omega(z) = 1 - 2\cos(\omega)z + z^2$ if $\omega \in (0, \pi)$. For p a positive integer, for $\underline{\omega} = (\omega_1, \ldots, \omega_p) \in [0, \pi]^p$, and for $\underline{d} = (d_1, \ldots, d_p) \in \mathbb{N}^p$ let the polynomial $\Delta_{\underline{\omega}, \underline{d}}(z)$ be defined as the product

$$\Delta_{\underline{\omega}, \underline{d}}(z) = \prod_{i=1}^{p} \Delta_{\omega_i}^{d_i}(z).$$

In case $p = 0$ (and thus $\underline{\omega}$ and $\underline{d}$ are the 0-tupels) we define $\Delta_{\underline{\omega}, \underline{d}}$ as the constant polynomial 1.

Note that the degree of $\Delta_{\omega_i}^{d_i}(z)$ is $\kappa(\omega_i, d_i)$ and that of $\Delta_{\underline{\omega}, \underline{d}}(z)$ is $\kappa(\underline{\omega}, \underline{d})$.

A finite Borel measure m on $[-\pi, \pi]$ is said to be symmetric, if $\mathsf{m}(A) = \mathsf{m}(-A)$ for every Borel subset A of $[-\pi, \pi]$. Recall that the finite and symmetric Borel measures on $[-\pi, \pi]$ are precisely the spectral measures of real weakly stationary processes. For a spectral density g, we denote by m_g the Borel measure on $[-\pi, \pi]$ with density g (w.r.t. Lebesgue measure $\lambda_{[-\pi, \pi]}$ on $[-\pi, \pi]$).

Definition 6.4. Let $\mathfrak{F} \subseteq \mathfrak{F}_{\text{all}}$ be nonempty, and let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$. We define $\mathbb{M}(\mathfrak{F}, \mathcal{L})$ to be the set of all finite and symmetric Borel measures m on $[-\pi, \pi]$ with finite support, such that (i) m is the weak limit of a sequence m_{g_j}, $j \in \mathbb{N}$, where

$$g_j(\nu) = \left|\Delta_{\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})}(e^{\iota\nu})\right|^2 f_j(\nu) / \int_{-\pi}^{\pi} \left|\Delta_{\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})}(e^{\iota\nu})\right|^2 f_j(\nu) d\nu \tag{33}$$

for some sequence $f_j \in \mathfrak{F}$, and (ii) $\sum_{\gamma \in \text{supp}(\mathsf{m}) \cap [0, \pi]} \kappa(\gamma, 1) < n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))$ holds. We furthermore define $\mathbb{S}(\mathfrak{F}, \mathcal{L}) = \{\text{supp}(\mathsf{m}) \cap [0, \pi] : \mathsf{m} \in \mathbb{M}(\mathfrak{F}, \mathcal{L})\}$.

We note that by construction the elements of $\mathbb{M}(\mathfrak{F}, \mathcal{L})$ all have total mass equal to one, and have a nonempty and finite support. Furthermore, $\mathbb{M}(\mathfrak{F}, \mathcal{L})$, and thus $\mathbb{S}(\mathfrak{F}, \mathcal{L})$, can be empty. Also recall that $\kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \leq \dim(\mathcal{L}) < n$ holds by Lemma D.1 in Appendix D.

To illustrate the concepts introduced in the preceding definition we shall now determine the set $\mathbb{S}(\mathfrak{F}, \mathcal{L})$ for a few choices of $\mathfrak{F}$. The proofs of the claims made in the next two examples can be found in Appendix D. Recall that $\mathfrak{F}_{\text{all}}$ denotes the set of all normalized spectral densities.

Example 6.1. Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$. Then

$$\mathbb{S}(\mathfrak{F}_{\text{all}}, \mathcal{L}) = \left\{ \Gamma \subseteq [0, \pi] : \text{card}(\Gamma) \in \mathbb{N}, \sum_{\gamma \in \Gamma} \kappa(\gamma, 1) < n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \right\}. \tag{34}$$

Note that this set is empty if $n = \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) + 1$, and is equal to $\{\{0\}, \{\pi\}\}$ if $n = \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) + 2$. In case $n > \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) + 2$, it is an infinite set with the property that $\bigcup \mathbb{S}(\mathfrak{F}_{\text{all}}, \mathcal{L})$ is equal to $[0, \pi]$; in fact, even $\{\gamma\} \in \mathbb{S}(\mathfrak{F}_{\text{all}}, \mathcal{L})$ holds for every $\gamma \in [0, \pi]$. [Note that any $\Gamma \in \mathbb{S}(\mathfrak{F}_{\text{all}}, \mathcal{L})$ in particular satisfies $2\,\text{card}(\Gamma) - 2 < n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))$.]

Example 6.2. Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$. For B such that $0 < B < \infty$ let $\mathfrak{F}_{\text{all}}^B$ denote the subset of elements of $\mathfrak{F}_{\text{all}}$ that are $\lambda_{[-\pi, \pi]}$-essentially bounded by B. Then $\mathbb{S}(\mathfrak{F}_{\text{all}}^B, \mathcal{L}) = \emptyset$ for every $0 < B < \infty$.

The result in the next example is easy to derive and we leave the proof to the reader.

Example 6.3. Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$ and consider $\mathfrak{F}_{\text{AR}(1)}$. Then we have the following: If $n = \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) + 1$ then $\mathbb{S}(\mathfrak{F}_{\text{AR}(1)}, \mathcal{L})$ is empty. Otherwise, we have four cases: (i) If neither e_+ nor e_- belong to $\mathcal{L}$, then $\mathbb{S}(\mathfrak{F}_{\text{AR}(1)}, \mathcal{L}) = \{\{0\}, \{\pi\}\}$. (ii) If e_+ belongs to $\mathcal{L}$, but e_- does not, then $\mathbb{S}(\mathfrak{F}_{\text{AR}(1)}, \mathcal{L}) = \{\{\pi\}\}$. (iii) If e_+ does not belong to $\mathcal{L}$, but e_- does, then $\mathbb{S}(\mathfrak{F}_{\text{AR}(1)}, \mathcal{L}) = \{\{0\}\}$. (iv) If both e_+ and e_- belong to $\mathcal{L}$, then $\mathbb{S}(\mathfrak{F}_{\text{AR}(1)}, \mathcal{L})$ is empty.

Proposition 6.1. *Let $\mathfrak{F} \subseteq \mathfrak{F}_{\text{all}}$ be nonempty. Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$.*

1. *For every $\bar{\Sigma} \in \text{cl}(\mathcal{L}(\mathfrak{C}(\mathfrak{F})))$ such that $\text{rank}(\bar{\Sigma}) < n - \dim(\mathcal{L})$ there exists a set $\Gamma \in \mathbb{S}(\mathfrak{F}, \mathcal{L})$ and positive real numbers $c(\gamma)$ for $\gamma \in \Gamma$, such that*

$$\bar{\Sigma} = \frac{\Pi_{\mathcal{L}^\perp} \sum_{\gamma \in \Gamma} c(\gamma) E_{n, \rho(\gamma, \mathcal{L})}(\gamma) E'_{n, \rho(\gamma, \mathcal{L})}(\gamma) \Pi_{\mathcal{L}^\perp}}{\|\Pi_{\mathcal{L}^\perp} \sum_{\gamma \in \Gamma} c(\gamma) E_{n, \rho(\gamma, \mathcal{L})}(\gamma) E'_{n, \rho(\gamma, \mathcal{L})}(\gamma) \Pi_{\mathcal{L}^\perp}\|} \tag{35}$$

holds. Furthermore, for every $\Gamma \in \mathbb{S}(\mathfrak{F}, \mathcal{L})$ there exists $\bar{\Sigma} \in \mathrm{cl}(\mathcal{L}(\mathfrak{C}(\mathfrak{F})))$ and positive real numbers $c(\gamma)$ for $\gamma \in \Gamma$, such that (35) holds (and clearly $\mathrm{rank}(\bar{\Sigma}) \leq n - \dim(\mathcal{L})$ holds).

2. The set $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}))$ coincides with the set of all linear subspaces of $\mathbb{R}^n$ that (i) have dimension smaller than $n - \dim(\mathcal{L})$, and (ii) that can be expressed as

$$\mathrm{span}\left(\Pi_{\mathcal{L}^{\perp}}\left(E_{n, \rho(\gamma_1, \mathcal{L})}(\gamma_1), \ldots, E_{n, \rho(\gamma_p, \mathcal{L})}(\gamma_p)\right)\right)$$

for some $\Gamma \in \mathbb{S}(\mathfrak{F}, \mathcal{L})$, where the γ_i's denote the elements of Γ and p denotes $\mathrm{card}(\Gamma)$.

3. Every element of $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}))$ contains a subspace of the form $\mathrm{span}(\Pi_{\mathcal{L}^{\perp}} E_{n, \rho(\gamma, \mathcal{L})}(\gamma))$ for some $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$.

We illustrate the proposition by two examples.

Example 6.4. *(Example 6.3 continued)* If $n = \dim(\mathcal{L}) + 1$ (which includes the case where $n = \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) + 1$), the set $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}_{\mathrm{AR}(1)}))$ is empty. Otherwise, we have the following cases: (i) If neither e_+ nor e_- belongs to $\mathcal{L}$, then $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}_{\mathrm{AR}(1)})) = \{\mathrm{span}(\Pi_{\mathcal{L}^{\perp}} e_+), \mathrm{span}(\Pi_{\mathcal{L}^{\perp}} e_-)\}$. (ii) If e_+ belongs to $\mathcal{L}$, but e_- does not, then $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}_{\mathrm{AR}(1)})) = \{\mathrm{span}(\Pi_{\mathcal{L}^{\perp}} e_-)\}$. (iii) If e_+ does not belong to $\mathcal{L}$, but e_- does, then $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}_{\mathrm{AR}(1)})) = \{\mathrm{span}(\Pi_{\mathcal{L}^{\perp}} e_+)\}$. (iv) If both e_+ and e_- belong to $\mathcal{L}$, then $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}_{\mathrm{AR}(1)}))$ is empty. [In this simple example these results can alternatively be obtained from the fact that the concentration spaces of $\mathfrak{C}(\mathfrak{F}_{\mathrm{AR}(1)})$ in the sense of Preinerstorfer and Pötscher (2016) are given by $\mathrm{span}(e_+)$ and $\mathrm{span}(e_-)$, combined with Lemma B.3 in Section B.1 of Appendix B as well as with Lemma G.1 of Preinerstorfer and Pötscher (2016).]

Example 6.5. Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$ and consider $\mathfrak{F} \subseteq \mathfrak{F}_{\mathrm{all}}$ such that $\mathfrak{F} \supseteq \mathfrak{F}_{\mathrm{AR}(2)}$ (which, in particular, covers the cases $\mathfrak{F} = \mathfrak{F}_{\mathrm{all}}$ as well as $\mathfrak{F} = \mathfrak{F}_{\mathrm{AR}(p)}$ for $p \geq 2$). The following is shown in Section 3 of Pötscher and Preinerstorfer (2017):

(i) Every $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$ satisfies $\{\gamma\} \in \mathbb{S}(\mathfrak{F}, \mathcal{L})$.

(ii) Suppose $\dim(\mathcal{L}) + 2 < n$. Then $\{\gamma\} \in \mathbb{S}(\mathfrak{F}, \mathcal{L})$ holds for every $\gamma \in [0, \pi]$. And it easily follows that $\mathrm{span}(\Pi_{\mathcal{L}^{\perp}} E_{n, \rho(\gamma, \mathcal{L})}(\gamma))$ belongs to $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}))$ for every $\gamma \in [0, \pi]$.

(iii) Suppose $\dim(\mathcal{L}) + 2 \geq n$. Then $\{\gamma\} \in \mathbb{S}(\mathfrak{F}, \mathcal{L})$ holds for $\gamma \in [0, \pi]$ precisely when $\kappa(\gamma, 1) < n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))$. Furthermore, $\mathrm{span}(\Pi_{\mathcal{L}^{\perp}} E_{n, \rho(\gamma, \mathcal{L})}(\gamma))$ belongs to $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}))$ for every $\gamma \in [0, \pi]$ that satisfies (a) $\kappa(\gamma, 1) < n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))$ and (b) $\dim \mathrm{span}(\Pi_{\mathcal{L}^{\perp}} E_{n, \rho(\gamma, \mathcal{L})}(\gamma)) < n - \dim(\mathcal{L})$.

6.2 Results on size control

Theorem 6.2. *Let $\mathfrak{F} \subseteq \mathfrak{F}_{\mathrm{all}}$ be nonempty, and assume that the test statistic $T : \mathbb{R}^n \to \mathbb{R}$ is Borel-measurable and is continuous on the complement of a closed set $N^{\dagger}$. Assume that T and $N^{\dagger}$ are $G(\mathfrak{M}_0)$-invariant, and are also invariant w.r.t. addition of elements of a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$. Define $\mathcal{L} = \mathrm{span}(\mathfrak{M}_0^{lin} \cup \mathcal{V})$, and assume that $\dim(\mathcal{L}) < n$.*

1. *Then for every $0 < \alpha < 1$ there exists a real number $C(\alpha)$ such that*

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{f \in \mathfrak{F}} P_{\mu_0, \sigma^2 \Sigma(f)}(T \geq C(\alpha)) \leq \alpha \tag{36}$$

   holds, provided every linear subspace $\mathcal{S}$, say, of $\mathbb{R}^n$ that (i) has dimension smaller than $n - \dim(\mathcal{L})$, and that (ii) can be written as

$$\mathcal{S} = \mathrm{span}\left(\Pi_{\mathcal{L}^\perp}\left(E_{n,\rho(\gamma_1, \mathcal{L})}(\gamma_1), \ldots, E_{n,\rho(\gamma_p, \mathcal{L})}(\gamma_p)\right)\right) \quad \text{for some} \quad \Gamma \in \mathbb{S}(\mathfrak{F}, \mathcal{L}), \tag{37}$$

   satisfies $\lambda_{\mu_0 + \mathcal{S}}(N^\dagger) = 0$ for some $\mu_0 \in \mathfrak{M}_0$ (and hence all $\mu_0 \in \mathfrak{M}_0$). Here the γ_i's denote the elements of Γ and $p = \mathrm{card}(\Gamma)$. [In case $N^\dagger$ is a finite or countable union of affine subspaces, we may replace $\lambda_{\mu_0 + \mathcal{S}}(N^\dagger) = 0$ by $\mu_0 + \mathcal{S} \nsubseteq N^\dagger$ in that condition.]

2. *Suppose $N^\dagger$ is a finite or countable union of affine subspaces. Then a sufficient condition for the condition given in Part 1 is that for all $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$ it holds that $\mu_0 + \mathrm{span}(\Pi_{\mathcal{L}^\perp} E_{n,\rho(\gamma, \mathcal{L})}(\gamma)) \nsubseteq N^\dagger$ for some $\mu_0 \in \mathfrak{M}_0$ (and hence all $\mu_0 \in \mathfrak{M}_0$). [A stricter sufficient condition is obtained if γ is required to range over $[0, \pi]$ rather than only $\bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$.]*

Remark 6.3. *(Special cases)* (i) The measurability and continuity assumption on T as well as the $G(\mathfrak{M}_0)$-invariance assumption on T and $N^\dagger$ are automatically satisfied for nonsphericity-corrected F-type test statistics as defined in (28) in Section 5.4 of Preinerstorfer and Pötscher (2016) provided that Assumption 5 in that paper is satisfied, see Lemma 5.16 in Section 5.3 above.

(ii) Specializing further (cf. Remark 5.17) to the case where T_{GQ} is the test statistic defined in Section 3.4 (with Assumption 2 being satisfied and with a positive definite weighting matrix $\mathcal{W}_n^*$), which, in particular, includes the case of the test statistic T_w defined in (5) (with Assumptions 1 and 2 being satisfied), or to the case of the weighted Eicker-test statistic $T_{E,\mathsf{W}}$ defined in Section 3 (with W a symmetric and nonnegative definite $n \times n$ Toeplitz matrix with ones on the main diagonal), we then have that the set $N^\dagger (= N^*)$ is additionally always guaranteed to be a proper linear subspace or a finite union of proper linear subspaces, see Lemma 5.18 in Section 5.3 and the discussion preceding this lemma; cf. also Theorem 6.6 below and the results in Section 3.

Remark 6.4. *(Some simplifications)* (i) The invariance assumptions on $N^\dagger$ in Theorem 6.2 imply, in particular, that $N^\dagger$ is invariant under addition of elements in $\mathcal{L}$. Hence, the sufficient condition in Part 1 of Theorem 6.2, namely that $\lambda_{\mu_0 + \mathcal{S}}(N^\dagger) = 0$ holds for every linear space $\mathcal{S}$ satisfying (i) and (ii) in that theorem, can equivalently be expressed as the condition that $\lambda_{\mu_0 + \mathcal{T}(\mathcal{S})}(N^\dagger) = 0$ holds for every linear space $\mathcal{S}$ satisfying (i) and (ii), where $\mathcal{T}(\mathcal{S})$ is shorthand for $\mathrm{span}((E_{n,\rho(\gamma_1, \mathcal{L})}(\gamma_1), \ldots, E_{n,\rho(\gamma_p, \mathcal{L})}(\gamma_p)))$. Similarly, the relation $\mu_0 + \mathcal{S} \nsubseteq N^\dagger$ given in the sentence in parenthesis in Part 1 of the theorem can equivalently be expressed as $\mu_0 + \mathcal{T}(\mathcal{S}) \nsubseteq N^\dagger$. Finally, the sufficient conditions given in Part 2 of the theorem can equivalently be expressed as $\mu_0 + \mathrm{span}(E_{n,\rho(\gamma, \mathcal{L})}(\gamma)) \nsubseteq N^\dagger$ for every $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$ (for every $\gamma \in [0, \pi]$, respectively).

(ii) If $r = 0$, or if $r \neq 0$ but $N^\dagger$ is invariant under addition of elements of $\mathfrak{M}_0$ (which is, e.g., the case if $N^\dagger$ is $G(\mathfrak{M})$-invariant), we may set $\mu_0 = 0$ in any of the sufficient conditions in Parts 1 and 2 of Theorem 6.2 that involve $N^\dagger$, cf. Remark 5.8 in Section 5.2; and the same applies to the equivalent formulations of these conditions discussed in (i) above. [Recall that $N^\dagger$ is $G(\mathfrak{M})$-invariant for a large class of test statistics including nonsphericity-corrected F-type test statistics that satisfy Assumption 5 in Preinerstorfer and Pötscher (2016); in particular, this applies to the test statistics mentioned in Remark 6.3(ii) above.]

(iii) Imposing that $\lambda_{\mu_0 + \mathcal{S}}(N^\dagger) = 0$ holds for every linear space $\mathcal{S}$ satisfying (ii) in Part 1 of Theorem 6.2 (but not necessarily (i)) leads to a potentially stricter sufficient condition. However, if $N^\dagger$ is an $\lambda_{\mathbb{R}^n}$-null set, this potentially stricter condition is in fact equivalent to the condition given in Part 1 of Theorem 6.2. For a proof see Appendix D.

(iv) In case $\dim(\mathcal{L}) = n - 1$, the test statistic T is $\lambda_{\mathbb{R}^n}$-almost everywhere constant and size control is hence trivially possible. Note also that the sufficient condition in Part 1 of Theorem 6.2 is trivially satisfied since $\mathbb{S}(\mathfrak{F}, \mathcal{L})$ is then empty and hence no $\mathcal{S}$ satisfies (37); cf. Remark 5.14.

Specializing to nonsphericity-corrected F-type test statistics as defined in Section 5.4 of Preinerstorfer and Pötscher (2016) and to the important case $\mathcal{L} = \mathfrak{M}_0^{lin}$ gives the following result.

Theorem 6.5. *Let $\mathfrak{F} \subseteq \mathfrak{F}_{\mathrm{all}}$ be nonempty, and suppose that T is a nonsphericity-corrected F-type test statistic as defined in (28) in Section 5.4 of Preinerstorfer and Pötscher (2016) and that Assumption 5 in that paper is satisfied.*

1. *Then for every $0 < \alpha < 1$ there exists a real number $C(\alpha)$ such that (36) holds, provided*

$$\lambda_{\mathrm{span}((E_{n,\rho(\gamma_1),\mathfrak{M}_0^{lin}}(\gamma_1),...,E_{n,\rho(\gamma_p),\mathfrak{M}_0^{lin}}(\gamma_p)))}(N^*) = 0 \tag{38}$$

   holds for every $\Gamma \in \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$. Here the γ_i's denote the elements of Γ and $p = \mathrm{card}(\Gamma)$.

2. *Suppose that N^* is a finite or countable union of affine subspaces. Then, for every $\Gamma \in \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$, we may rewrite (38) equivalently as*

$$\mathrm{span}\left((E_{n,\rho(\gamma_1),\mathfrak{M}_0^{lin}}(\gamma_1), \ldots, E_{n,\rho(\gamma_p),\mathfrak{M}_0^{lin}}(\gamma_p))\right) \not\subseteq N^*. \tag{39}$$

   Furthermore, a sufficient condition for this is given by $\mathrm{span}(E_{n,\rho(\gamma),\mathfrak{M}_0^{lin}}(\gamma)) \not\subseteq N^*$ for every $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$ (and an even stricter sufficient condition is obtained if γ is here required to range over $[0, \pi]$ instead of only over $\bigcup \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$).

Specializing Theorem 6.5 to commonly used autocorrelation robust tests based on T_w we obtain the following result. Since for these tests statistics also all assumptions of Proposition 5.12 can be shown to be satisfied (cf. Section 5.3), the size control result can furthermore be sharpened to an *exact* size control result.

Theorem 6.6. *Let $\mathfrak{F} \subseteq \mathfrak{F}_{\text{all}}$ be nonempty. Suppose Assumptions 1 and 2 are satisfied and T_w is defined by (5).*

1. *Then for every $0 < \alpha < 1$ there exists a real number $C(\alpha)$ such that (36) holds (with T replaced by T_w), provided*

$$\text{span}\left(\left(E_{n,\rho(\gamma_1,\mathfrak{M}_0^{lin})}(\gamma_1), \ldots, E_{n,\rho(\gamma_p,\mathfrak{M}_0^{lin})}(\gamma_p) \right) \right) \not\subseteq \mathsf{B} \tag{40}$$

holds for every $\Gamma \in \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$. Here the γ_i's denote the elements of Γ and $p = \text{card}(\Gamma)$. Furthermore, under the same condition even equality can be achieved in (36) (with T replaced by T_w) by a proper choice of $C(\alpha)$, provided $\alpha \in (0, \alpha^*] \cap (0, 1)$, where α^* is defined as in (28) (with T replaced by T_w).

2. *A sufficient condition for (40) to hold for every $\Gamma \in \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$ is given by*

$$\text{span}\left(E_{n,\rho(\gamma,\mathfrak{M}_0^{lin})}(\gamma) \right) \not\subseteq \mathsf{B} \tag{41}$$

for every $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$ (and an even stricter sufficient condition is obtained if we require (41) to hold for every $\gamma \in [0, \pi]$).

3. *In case the set B coincides with $\text{span}(X)$, condition (40) ((41), respectively) can equivalently be expressed as $\text{rank}(X, E_{n,\rho(\gamma_1,\mathfrak{M}_0^{lin})}(\gamma_1), \ldots, E_{n,\rho(\gamma_p,\mathfrak{M}_0^{lin})}(\gamma_p)) > k$ ($\text{rank}(X, E_{n,\rho(\gamma,\mathfrak{M}_0^{lin})}(\gamma)) > k$, respectively).*

For the following remark recall that $\mathsf{CV}_\leq(\alpha)$ and $\mathsf{CV}_=(\alpha)$ have been defined in Remark 5.10.

Remark 6.7. (i) Under the assumptions of Theorem 6.5 or Theorem 6.6 we have that $\mathsf{CV}_\leq(\alpha) = [C_\Diamond(\alpha), \infty)$ holds, i.e., a *smallest* critical value guaranteeing size control exists and is given by $C_\Diamond(\alpha)$. This follows from Remark 5.10, Lemma 5.16, and Remark 5.17.

(ii) Under the assumptions of Theorem 6.6 and if $\alpha \in (0, \alpha^*] \cap (0, 1)$, we have that $\mathsf{CV}_=(\alpha)$ has $C_\Diamond(\alpha)$ as its smallest element, i.e., a *smallest* critical value guaranteeing exact size control exists and is given by $C_\Diamond(\alpha)$.

(iii) Under the assumptions of Theorems 6.5 or 6.6 the size of the test, and hence the size-controlling critical values $C(\alpha)$, do not depend on the value of r; cf. Lemma 5.15. Also the sufficient conditions in both theorems do not depend on the value of r. [It is understood here that the estimator for β as well as the covariance matrix estimator used to define the test statistic have been chosen independently of the value of r.]

Remark 6.8. (i) Theorem 6.6 carries over to the test statistics T_{GQ} (see Section 3.4) if Assumption 1 is replaced by the assumption that the weighting matrix $\mathcal{W}_n^*$ is positive definite. This follows from an inspection of the proof of Theorem 6.6 and the fact that T_{GQ} is a special case of a nonsphericity-corrected F-type test statistic with $N^* = \mathsf{B}$, see Section 5.3. A statement analogous to Remark 6.7 also applies here.

(ii) A result for the weighted Eicker-test similar to Theorem 6.6 is obtained by replacing Assumptions 1 and 2 in that theorem by the assumption that W is a symmetric and nonnegative definite $n \times n$ Toeplitz matrix with ones on the main diagonal and by replacing the set B by $\operatorname{span}(X)$. This follows again from an inspection of the proof of Theorem 6.6 and the fact that $T_{E,\mathsf{W}}$ is a special case of a nonsphericity-corrected F-type test statistic with $N^* = \operatorname{span}(X)$, see Section 5.3. A statement analogous to Remark 6.7 also applies here.

Remark 6.9. Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$ and suppose $\mathfrak{F} \subseteq \mathfrak{F}_{\text{all}}$ has the property that $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$ implies $\{\gamma\} \in \mathbb{S}(\mathfrak{F}, \mathcal{L})$. [Note that this is always the case if $\mathfrak{F}$ contains $\mathfrak{F}_{\text{AR}(2)}$ as discussed in Example 6.5.]

(i) Suppose $\mathcal{L} = \mathfrak{M}_0^{lin}$. Then in the context of Theorem 6.6 it is obvious that the first sufficient condition in Part 2 of that theorem (i.e., the condition that (41) holds for every $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$) is actually equivalent to the sufficient condition in Part 1 (i.e., to the condition that (40) holds for every $\Gamma \in \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$). A similar remark applies to the versions of Theorem 6.6 for T_{GQ} and $T_{E,\mathsf{W}}$ outlined in Remark 6.8.

(ii) Suppose $\mathcal{L} = \mathfrak{M}_0^{lin}$. Then similarly in the context of Part 2 of Theorem 6.5 the condition that (39) holds for every $\Gamma \in \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$ is equivalent to $\operatorname{span}(E_{n,\rho(\gamma,\mathfrak{M}_0^{lin})}(\gamma)) \not\subseteq N^*$ for every $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$.

(iii) Under the assumptions of Part 2 of Theorem 6.2, the first sufficient condition given in Part 2 is in fact equivalent to the sufficient condition given in Part 1 of that theorem provided $N^\dagger \subsetneq \mathbb{R}^n$. This follows from Remark 6.4(iii), noting that $N^\dagger$, as a finite or countable union of affine subspaces, is then a $\lambda_{\mathbb{R}^n}$-null set. [In case $N^\dagger = \mathbb{R}^n$, the claim is clearly also true provided $\operatorname{rank}(\Pi_{\mathcal{L}^\perp} E_{n,\rho(\gamma,\mathcal{L})}(\gamma)) < n - \dim(\mathcal{L})$ holds for some $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$ or if $\bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$ is empty. A sufficient condition for the rank condition just mentioned is that $\kappa(\gamma, 1) < n - \dim(\mathcal{L})$ holds for such a γ; and this latter condition certainly holds if $\dim(\mathcal{L}) < n - 2$.][16]

Remark 6.10. Suppose $\mathfrak{F} \subseteq \mathfrak{F}_{\text{all}}$, but $\mathfrak{F} \supseteq \mathfrak{F}_{\text{AR}(2)}$. Under the assumptions of Part 2 of Theorem 6.2, the sufficient condition given in Part 2 as well as the stricter sufficient condition given in the final sentence in brackets are in fact equivalent provided $N^\dagger \subsetneq \mathbb{R}^n$.[17] Furthermore, in the context of Part 2 of Theorem 6.5, the sufficient condition $\operatorname{span}(E_{n,\rho(\gamma,\mathfrak{M}_0^{lin})}(\gamma)) \not\subseteq N^*$ for every $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$ is in fact equivalent to the stricter condition $\operatorname{span}(E_{n,\rho(\gamma,\mathfrak{M}_0^{lin})}(\gamma)) \not\subseteq N^*$ for every $\gamma \in [0, \pi]$. Similarly, in the context of Part 2 of Theorem 6.6, the sufficient condition that (41) holds for every $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathfrak{M}_0^{lin})$ is in fact equivalent to the stricter condition that (41) holds for every $\gamma \in [0, \pi]$. A similar remark applies to the versions of Theorem 6.6 for T_{GQ} and $T_{E,\mathsf{W}}$ outlined in Remark 6.8. A proof of these claims can be found in Appendix D.[18]

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[16]Suppose that $\operatorname{rank}(\Pi_{\mathcal{L}^\perp} E_{n,\rho(\gamma,\mathcal{L})}(\gamma)) < n - \dim(\mathcal{L})$ holds whenever $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$. Then actually a much simpler argument shows that the claimed equivalence is true, regardless of whether $N^\dagger \subsetneq \mathbb{R}^n$ or $N^\dagger = \mathbb{R}^n$. A sufficient condition for the just mentioned rank condition is that $\kappa(\gamma, 1) < n - \dim(\mathcal{L})$ holds whenever $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$, which is in turn implied by $\dim(\mathcal{L}) < n - 2$.

[17]If $\mathbb{S}(\mathfrak{F}, \mathcal{L})$ is nonempty, then the claimed equivalence is also true if $N^\dagger = \mathbb{R}^n$ as is easily seen. Note that $\mathbb{S}(\mathfrak{F}, \mathcal{L})$ is nonempty if and only if $\kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) < n - 1$ in view of Example 6.5. A sufficient condition for the latter inequality is that $\dim(\mathcal{L}) < n - 1$, cf. Lemma D.1.

[18]In case $\dim(\mathcal{L}) + 2 < n$, these claims as well as the claim in Footnote 17 are actually a simple consequence of

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Remark 6.11. If $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}))$ is empty (which is, e.g., the case if $\mathfrak{C}(\mathfrak{F})$ has no singular limit points), then size control in the contexts of the theorems in this subsection is always possible; cf. Remark 5.7.

A   Appendix: Proofs for Section 3

The quantities $V_n^{(0)}(\underline{\omega}, \underline{d})$ and $\kappa(\underline{\omega}, \underline{d})$ used in the subsequent proofs are defined in Appendix C and Section 6.1, respectively.

Proof of claims regarding Definition 3.1: Suppose $p \geq 1$ and suppose $\underline{\omega}$ is a $p \times 1$ vector with distinct coordinates $\omega_i \in [0, \pi]$, such that $\mathrm{span}(E_{n,j}(\omega_i)) \subseteq \mathcal{L}$ for $j = 0, \ldots, d_i - 1$, holds where $d_i \in \mathbb{N}$. Set $\underline{d} = (d_1, \ldots, d_p)$. Since $\mathcal{L}$ is a linear space, it follows that the span of the matrix $V_n^{(0)}(\underline{\omega}, \underline{d})$ is contained in $\mathcal{L}$, and thus $\mathrm{rank}(V_n^{(0)}(\underline{\omega}, \underline{d})) \leq \dim(\mathcal{L}) < n$ must hold. But as shown in Lemma C.1 in Appendix C, the rank of $V_n^{(0)}(\underline{\omega}, \underline{d})$ equals $\min(n, \kappa(\underline{\omega}, \underline{d}))$. Consequently, $\kappa(\underline{\omega}, \underline{d}) < n$ follows. Inspection of the definition of $\kappa(\underline{\omega}, \underline{d})$ now shows that this inequality implies an upper bound on p and the coordinates of $\underline{d}$. This obviously then implies that the set on the right-hand side of (9) contains 0 for every $\omega \in [0, \pi]$ except possibly for at most finitely many ω's; it also implies that the set on the right-hand side of (9) is nonempty for every $\omega \in [0, \pi]$. But this establishes the claims. ∎

Proof of claim in Example 3.2: To prove this claim we use Theorem 3.9 with $F = (e_+)$. First, observe that F is clearly linearly independent of every set of q standard basis vectors, since $q \leq k - 1 < n$. Second, condition (ii) in that theorem is satisfied since the rank of $(F, E_{n,0}(\gamma^*))$, $\gamma^* \in (0, \pi)$, coincides with the rank of the matrix $V_n^{(0)}(\underline{\omega}, \underline{d})$, where $\underline{\omega} = (0, \gamma^*)$ and $\underline{d} = (1, 1)$, and since $V_n^{(0)}(\underline{\omega}, \underline{d})$ has rank $\min(n, 3) = 3$ by Lemma C.1. Recall that $n \geq 3$ holds since $n > k \geq 2$. Also note that $\rho_F(\gamma) = 1$ for $\gamma = 0$ and $\rho_F(\gamma) = 0$ for $\gamma \in (0, \pi]$, again by Lemma C.1. ∎

Proof of claim in Example 3.3: To prove this claim we use Theorem 3.9 with $F = (e_+, e_-)$. First, observe that F is linearly independent of every set of q standard basis vectors provided $q \leq (n/2) - 1$ in view of of Part 2 of Lemma A.1 given below. Second, condition (ii) in Theorem 3.9 is satisfied since the rank of $(F, E_{n,0}(\gamma^*))$, $\gamma^* \in (0, \pi)$, coincides with the rank of the matrix $V_n^{(0)}(\underline{\omega}, \underline{d})$, where $\underline{\omega} = (0, \pi, \gamma^*)$ and $\underline{d} = (1, 1, 1)$, and since $V_n^{(0)}(\underline{\omega}, \underline{d})$ has rank $\min(n, 4) = 4$ by Lemma C.1. Recall that $n \geq 4$ holds since $n > k \geq 3$. Also note that $\rho_F(\gamma) = 1$ for $\gamma = 0, \pi$ and $\rho_F(\gamma) = 0$ for $\gamma \in (0, \pi)$, again by Lemma C.1. ∎

Proof of claim in Example 3.4: To prove this claim we use Theorem 3.9 with $F = (e_+, v)$. First, observe that F is linearly independent of every set of q standard basis vectors in view of Part 1 of Lemma A.1 below, since $q \leq k - 2 < n - 2$ and since every nonzero element of $\mathrm{span}(F)$ can have at most one zero coordinate. Second, condition (ii) in Theorem 3.9 is satisfied since the rank of $(F, E_{n,0}(\gamma^*))$, $\gamma^* \in (0, \pi)$, coincides with the rank of the matrix $V_n^{(0)}(\underline{\omega}, \underline{d})$, where $\underline{\omega} = (0, \gamma^*)$ and $\underline{d} = (2, 1)$, and since $V_n^{(0)}(\underline{\omega}, \underline{d})$ has rank $\min(n, 4) = 4$ by Lemma C.1. Recall

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Example 6.5(ii) as then $\bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$ coincides with $[0, \pi]$.

that $n \geq 4$ holds since $n > k \geq 3$. Also note that $\rho_F(\gamma) = 2$ for $\gamma = 0$ and $\rho_F(\gamma) = 0$ for $\gamma \in (0, \pi]$, again by Lemma C.1. ∎

Proof of claim in Example 3.5: To prove this claim we use Theorem 3.9 with $F = (e_+, E_{n,0}(\gamma_0))$. First, observe that F is linearly independent of every set of q standard basis vectors provided $q \leq (n/3) - 1$ in view of Part 2 of Lemma A.1 below. Second, condition (ii) in Theorem 3.9 is satisfied since the rank of $(F, E_{n,0}(\gamma^*))$, for $\gamma^* \in (0, \pi)$ with $\gamma^* \neq \gamma_0$, coincides with the rank of the matrix $V_n^{(0)}(\underline{\omega}, \underline{d})$ defined in Appendix C, where $\underline{\omega} = (0, \gamma_0, \gamma^*)$ and $\underline{d} = (1, 1, 1)$, and since $V_n^{(0)}(\underline{\omega}, \underline{d})$ has rank $\min(n, 5) = 5$ by Lemma C.1. Recall that $n \geq 5$ holds since $n > k \geq 4$. Also note that $\rho_F(\gamma) = 1$ for $\gamma = 0, \gamma_0$ and $\rho_F(\gamma) = 0$ for $\gamma \in (0, \pi] \backslash \{\gamma_0\}$, again by Lemma C.1. ∎

Lemma A.1. *Let F be an $n \times k_F$-dimensional matrix with $\operatorname{rank}(F) = k_F \geq 1$. Let $s \in \mathbb{N}$ and $s < n$.*

1. If the maximum of the number of zero-coordinates of nonzero elements of $\operatorname{span}(F)$ is smaller than $n - s$, then the columns of F and $e_{i_1}(n), \ldots, e_{i_s}(n)$ are linearly independent for every choice of $1 \leq i_1 < \ldots < i_s \leq n$.

2. Suppose F is such that $\operatorname{span}(F) = \operatorname{span}(V_n^{(0)}(\underline{\omega}, \underline{d}))$ holds for some $\underline{\omega} \in [0, \pi]^p$ with distinct coordinates, for some $\underline{d} \in \mathbb{N}^p$, and for some positive integer p. If $k_F \leq n/(s+1)$, then the columns of F and $e_{i_1}(n), \ldots, e_{i_s}(n)$ are linearly independent for every choice of $1 \leq i_1 < \ldots < i_s \leq n$.

Proof: The first claim is trivial. For the second claim we argue by contradiction: Suppose there exist indices $1 \leq i_1 < \ldots < i_s \leq n$ such that F and $e_{i_1}(n), \ldots, e_{i_s}(n)$ are linearly dependent. Then we can find an s-dimensional vector $b \neq 0$ so that $v := \sum_{j=1}^s b_j e_{i_j}(n) \in \operatorname{span}(V_n^{(0)}(\underline{\omega}, \underline{d}))$ holds. The finite sequence $v_1, \ldots, v_n$ of components of v must then contain a string of consecutive zeros of length at least $\lfloor n/(s+1) \rfloor$. Now, v is obviously nonzero and thus v must have a nonzero coordinate v_{i^*}, say, that is preceded or succeeded by at least $\lfloor n/(s+1) \rfloor$ consecutive zeros (note that $\lfloor n/(s+1) \rfloor \geq 1$ since $s < n$). Observe that $\kappa(\underline{\omega}, \underline{d}) = k_F$ must hold, since $k_F = \operatorname{rank}(F) = \operatorname{rank}(V_n^{(0)}(\underline{\omega}, \underline{d})) = \min(n, \kappa(\underline{\omega}, \underline{d}))$ (by Lemma C.1 in Appendix C) and since $k_F \leq n/(s+1) < n$. It then follows from Lemma C.3 in Appendix C that $D_n(\Delta_{\underline{\omega}, \underline{d}})v = 0$, where $D_n(\Delta_{\underline{\omega}, \underline{d}})$ is a $(n - k_F) \times n$ matrix as defined in Appendix C and where $\Delta_{\underline{\omega}, \underline{d}}$ is defined in Section 6.1. Note that the coefficient of the highest power occurring in $\Delta_{\underline{\omega}, \underline{d}}$ is always ± 1, that the constant term is 1, and that the degree of $\Delta_{\underline{\omega}, \underline{d}}$ is $\kappa(\underline{\omega}, \underline{d}) = k_F$. Inspection of the equation system $D_n(\Delta_{\underline{\omega}, \underline{d}})v = 0$ and noting that $k_F \leq n/(s+1)$ is equivalent to $k_F \leq \lfloor n/(s+1) \rfloor$, now reveals that one of the equations is of the form

$$\pm v_{i^*} + c_1 v_{i^*-1} + \ldots + c_{k_F-1} v_{i^*-(k_F-1)} + v_{i^*-k_F} = 0$$

where $v_{i^*-1} = \ldots = v_{i^*-k_F} = 0$, or of the form

$$\pm v_{i^*+k_F} + c_1 v_{i^*+k_F-1} + \ldots + c_{k_F-1} v_{i^*+1} + v_{i^*} = 0$$

where $v_{i^*+1} = \ldots = v_{i^*+k_F} = 0$. But this implies $v_{i^*} = 0$, a contradiction. ∎

Lemma A.2. *Let F be an $n \times k_F$ matrix with $\operatorname{rank}(F) = k_F$ where $0 \leq k_F < k$ (with the convention that F is the empty matrix in case $k_F = 0$, that the rank of the empty matrix is zero, and that its span is $\{0\}$). Define $\rho_F(\gamma) = \rho(\gamma, \operatorname{span}(F))$. Let*

$$\tilde{\mathfrak{X}}_1 = \left\{ \tilde{X} \in \mathbb{R}^{n \times (k-k_F)} : \operatorname{rank}(F, \tilde{X}) = k, \operatorname{rank}(F, \tilde{X}, E_{n,\rho_F(\gamma)}(\gamma)) > k \ \text{ for all } \gamma \in [0, \pi] \right\}. \quad (42)$$

1. *If the $q \times k$ restriction matrix R of rank q is of the form $(0, \tilde{R})$ where $\tilde{R}$ is $q \times (k - k_F)$, then for every $X = (F, \tilde{X})$ with $\tilde{X} \in \tilde{\mathfrak{X}}_1$ we have $\rho(\gamma, \mathfrak{M}_0^{lin}) = \rho_F(\gamma)$ for every $\gamma \in [0, \pi]$. [Note that $\rho(\gamma, \mathfrak{M}_0^{lin})$ depends on X and R, but this is not expressed in the notation.]*

2. *The set $\mathbb{R}^{n \times (k-k_F)} \backslash \tilde{\mathfrak{X}}_1$ is contained in a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set if $n > k + 2$ holds.*

3. *The set $\mathbb{R}^{n \times (k-k_F)} \backslash \tilde{\mathfrak{X}}_1$ is contained in a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set if $\operatorname{rank}(F, E_{n,0}(\gamma^*)) = k_F + 2$ holds for some $\gamma^* \in (0, \pi)$.*

Proof: 1. For every $X = (F, \tilde{X})$ with $\tilde{X} \in \mathbb{R}^{n \times (k-k_F)}$ and for R of the form assumed in the lemma we immediately see that the associated space $\mathfrak{M}_0^{lin}$ (depending on X and R, but the dependence not being shown in the notation) contains $\operatorname{span}(F)$. Consequently, $\rho(\gamma, \mathfrak{M}_0^{lin}) \geq \rho_F(\gamma)$ must hold for every $\gamma \in [0, \pi]$. To prove the converse, note that for $\tilde{X} \in \tilde{\mathfrak{X}}_1$ the second rank condition in (42) implies that $E_{n,\rho_F(\gamma)}(\gamma)$ is not contained in $\operatorname{span}(X) = \operatorname{span}((F, \tilde{X}))$, and hence a fortiori not in $\mathfrak{M}_0^{lin}$. This immediately implies $\rho(\gamma, \mathfrak{M}_0^{lin}) \leq \rho_F(\gamma)$ for every $\gamma \in [0, \pi]$.

2. Since the set of all $\tilde{X}$ such that $\operatorname{rank}(F, \tilde{X}) < k$ is obviously a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set, it suffices to show that the set $A = \bigcup_{\gamma \in [0,\pi]} A_\gamma$ where

$$A_\gamma = \left\{ \tilde{X} \in \mathbb{R}^{n \times (k-k_F)} : \operatorname{rank}(F, \tilde{X}) = k, \operatorname{rank}(F, \tilde{X}, E_{n,\rho_F(\gamma)}(\gamma)) = k \right\}$$

is contained in a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set. We first show that A_γ is a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set for each fixed γ: By definition of $\rho_F(\gamma)$ at least one of the columns of $E_{n,\rho_F(\gamma)}(\gamma)$ does not belong to $\operatorname{span}(F)$. Choose one such column and denote it by h. Then A_γ is contained in the set

$$B_\gamma = \left\{ \tilde{X} \in \mathbb{R}^{n \times (k-k_F)} : \operatorname{rank}(F, \tilde{X}, h) \leq k \right\}.$$

Since $(F, \tilde{X}, h)$ has dimension $n \times (k+1)$ the set B_γ is given by the zero-set of the polynomial $\det((F, \tilde{X}, h)'(F, \tilde{X}, h))$. We next construct a matrix $\tilde{X} \in \mathbb{R}^{n \times (k-k_F)}$ which does not belong to this set. Observe that F and h together span a linear space of dimension $k_F + 1$ and that $k_F + 1 \leq k < n$ holds since we have assumed $k_F < k$ and since we always maintain $k < n$. Hence we can find $k - k_F$ linearly independent vectors in $(\operatorname{span}((F, h)))^\perp$, which we use as the columns of $\tilde{X}$. Clearly, this $\tilde{X}$ does not belong to B_γ. Consequently, B_γ is a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set, and a fortiori the same is true for A_γ (Borel-measurability of A_γ being trivial). Let now U be the finite set $\{\gamma \in [0, \pi] : \rho_F(\gamma) > 0\} \cup \{0, \pi\}$ (cf. the discussion following Definition 3.1). Then $\bigcup_{\gamma \in U} A_\gamma$

is a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set. It remains to show that $\bigcup_{\gamma \in [0,\pi] \backslash U} A_\gamma$ is contained in a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set. Note that $[0,\pi] \backslash U \subseteq (0,\pi)$ is an open set and that $\rho_F(\gamma) = 0$ holds for every $\gamma \in [0,\pi] \backslash U$. Hence, $\text{span}(E_{n,0}(\gamma))$ is not contained in $\text{span}(F)$ for every $\gamma \in [0,\pi] \backslash U$. Thus, if $\tilde{X} \in A_\gamma$ with $\gamma \in [0,\pi] \backslash U$ we can then find an index $i(\gamma)$, $1 \le i(\gamma) \le k - k_F$ and a $(k+1) \times 1$ vector $v(\gamma)$, such that

$$\tilde{X}_{.i(\gamma)} = (F, \tilde{X}(\neg i(\gamma)), E_{n,0}(\gamma))v(\gamma)$$

holds, where $\tilde{X}_{.i(\gamma)}$ denotes the $i(\gamma)$-th column of $\tilde{X}$, and $\tilde{X}(\neg i(\gamma))$ denotes the matrix $\tilde{X}$ after the $i(\gamma)$-th column has been deleted. In other words, $\tilde{X}$ coincides with

$$((F, \tilde{X}(\neg i(\gamma)), E_{n,0}(\gamma))v(\gamma), \tilde{X}(\neg i(\gamma))),$$

up to a permutation of columns. [Note that A_γ may be empty, namely if $\text{rank}(F, E_{n,0}(\gamma)) = k_F + 2$ and $k - k_F = 1$.] Now, for every $(k - k_F) \times (k - k_F)$ permutation matrix Per define the map $\Xi_{Per} : \mathbb{R}^{n \times (k-k_F-1)} \times \mathbb{R}^{k+1} \times [0,\pi] \backslash U \to \mathbb{R}^{n \times (k-k_F)}$ via $(\bar{X}, \bar{v}, \gamma) \mapsto ((F, \bar{X}, E_{n,0}(\gamma))\bar{v}, \bar{X})Per$. [In case $k - k_F = 1$ the symbol $\mathbb{R}^{n \times (k-k_F-1)}$ is to be interpreted as $\{0\}$ and hence the map Ξ_{Per} is effectively defined on $\mathbb{R}^{k+1} \times [0,\pi] \backslash U$.] Because of what has been said before, we now see that $\bigcup_{\gamma \in [0,\pi] \backslash U} A_\gamma$ is contained in the union of the images of all the maps Ξ_{Per} when Per varies in the indicated set of permutation matrices. Note that this union is a finite union. It hence suffices to show that the image of each Ξ_{Per} is contained in a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set. Clearly, the domain of definition of each Ξ_{Per} is an open set in Euclidean space of dimension $n(k - k_F - 1) + k + 2$ and each Ξ_{Per} is a smooth map. Sard's Theorem (see, e.g., Milnor (1997)) now implies the desired conclusion provided $n(k - k_F - 1) + k + 2$ is smaller than the dimension $n(k - k_F)$ of the range space. But this is guaranteed by the assumption that $n > k + 2$.

3. We proceed as in 2. up to the point where the set U has been defined. Now define U^* as the union of U and the set $\{\gamma \in (0,\pi) : \text{rank}(F, E_{n,0}(\gamma)) < k_F + 2\}$. The latter set is clearly contained in the zero-set of the function $\gamma \mapsto \det((F, E_{n,0}(\gamma))'(F, E_{n,0}(\gamma)))$. Obviously this function can be expressed as a rational function in $\exp(\iota\gamma)$ and thus only has finitely many zeros, except if it is identically zero. But the latter cannot happen because of the assumption made for Part 3. We have now established that U^* is a finite set. It follows that $\bigcup_{\gamma \in U^*} A_\gamma$ is a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set. It remains to show that $\bigcup_{\gamma \in [0,\pi] \backslash U^*} A_\gamma$ is contained in a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set. Note that $[0,\pi] \backslash U^* \subseteq (0,\pi)$ is an open set and that $\rho_F(\gamma) = 0$ holds for every $\gamma \in [0,\pi] \backslash U^*$. In case $k - k_F = 1$, it follows that $\bigcup_{\gamma \in [0,\pi] \backslash U^*} A_\gamma$ is the empty set (since $\text{rank}(F, E_{n,0}(\gamma)) = k_F + 2$ for $\gamma \in [0,\pi] \backslash U^*$) and we are done. Hence assume $k - k_F \ge 2$. Then, if $\tilde{X} \in A_\gamma$ with $\gamma \in [0,\pi] \backslash U^*$ we can find indices $i_1(\gamma), i_2(\gamma)$ with $1 \le i_1(\gamma) < i_2(\gamma) \le k - k_F$, and two $k \times 1$ vectors $v_1(\gamma)$ and $v_2(\gamma)$ such that

$$(\tilde{X}_{.i_1(\gamma)}, \tilde{X}_{.i_2(\gamma)}) = (F, \tilde{X}(\neg i_1(\gamma), \neg i_1(\gamma)), E_{n,0}(\gamma))(v_1(\gamma), v_2(\gamma))$$

holds, where $\tilde{X}(\neg i_1(\gamma), \neg i_2(\gamma))$ denotes the matrix $\tilde{X}$ after the columns $i_1(\gamma)$ and $i_2(\gamma)$ have

been deleted. In other words, $\tilde{X}$ coincides with

$$((F, \tilde{X}(\neg i_1(\gamma), \neg i_1(\gamma)), E_{n,0}(\gamma))(v_1(\gamma), v_2(\gamma)), \tilde{X}(\neg i_1(\gamma), \neg i_1(\gamma))),$$

up to a permutation of columns. Now, for every $(k - k_F) \times (k - k_F)$ permutation matrix Per define the map $\Xi_{Per} : \mathbb{R}^{n \times (k-k_F-2)} \times \mathbb{R}^k \times \mathbb{R}^k \times [0, \pi] \backslash U^* \to \mathbb{R}^{n \times (k-k_F)}$ via $(\bar{X}, \bar{v}_1, \bar{v}_2, \gamma) \mapsto ((F, \bar{X}, E_{n,0}(\gamma))(\bar{v}_1, \bar{v}_2), \bar{X})Per$. [In case $k - k_F = 2$ the symbol $\mathbb{R}^{n \times (k-k_F-2)}$ is to be interpreted as $\{0\}$ and hence the map Ξ_{Per} is effectively defined on $\mathbb{R}^k \times \mathbb{R}^k \times [0, \pi] \backslash U$.] Because of what has been said before, we now see that $\bigcup_{\gamma \in [0,\pi] \backslash U^*} A_\gamma$ is contained in the union of the images of all the maps Ξ_{Per} when Per varies in the indicated set of permutation matrices. This union is again a finite union. It hence suffices to show that the image of each Ξ_{Per} is contained in a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set. Clearly, the domain of definition of each Ξ_{Per} is an open set in Euclidean space of dimension $n(k - k_F - 2) + 2k + 1$ and each Ξ_{Per} is a smooth map. Sard's Theorem again now implies the desired conclusion provided $n(k - k_F - 2) + 2k + 1$ is smaller than the dimension $n(k - k_F)$ of the range space. But this is guaranteed by our general assumption that $n > k$. ∎

Lemma A.3. *Let F be an $n \times k_F$ matrix with $\mathrm{rank}(F) = k_F$ where $0 \leq k_F < k$ (with the same convention as before if $k_F = 0$). Assume that the $q \times k$ restriction matrix R of rank q is of the form $(0, \tilde{R})$ where $\tilde{R}$ is $q \times (k - k_F)$. Furthermore, assume that the columns of F and $e_{i_1}(n), \ldots, e_{i_q}(n)$ are linearly independent for every choice of $1 \leq i_1 < \ldots < i_q \leq n$. Then the complement of the set*

$$\tilde{\mathfrak{X}}_2 = \left\{ \tilde{X} \in \mathbb{R}^{n \times (k-k_F)} : \mathrm{rank}(X) = k, \ \mathsf{B} = \mathrm{span}(X) \right\}$$

in $\mathbb{R}^{n \times (k-k_F)}$ is contained in a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set, where $X = (F, \tilde{X})$. [Recall that B depends on X, which, however, is not shown in the notation.]

Proof: Step 1: Let $1 \leq i_1 < \ldots < i_q \leq n$ be given and define the set

$$\tilde{\mathfrak{X}}(i_1, \ldots, i_q) = \left\{ \tilde{X} \in \mathbb{R}^{n \times (k-k_F)} : \mathrm{rank}(X) = k, \mathrm{rank}(R(X'X)^{-1}X'\left(e_{i_1}(n), \ldots, e_{i_q}(n)\right)) = q \right\},$$

where $X = (F, \tilde{X})$. First, we show that $\tilde{\mathfrak{X}}(i_1, \ldots, i_q)$ is nonempty: Since $\mathrm{span}((F, e_{i_1}(n), \ldots, e_{i_q}(n)))$ has dimension $k_F + q$ by the assumptions on F and since $k_F + q \leq k < n$ in view of the assumptions on R, we can find orthonormal $n \times 1$ vectors $a_1, \ldots, a_{k-(k_F+q)}$ in the orthogonal complement of $\mathrm{span}((F, e_{i_1}(n), \ldots, e_{i_q}(n)))$. Define $\tilde{X}^* = (a_1, \ldots, a_{k-(k_F+q)}, e_{i_1}(n), \ldots, e_{i_q}(n))$, with the convention that $\tilde{X}^* = (e_{i_1}(n), \ldots, e_{i_q}(n))$ in case $k_F + q = k$, and set $X^* = (F, \tilde{X}^*)$. Obviously X^* has rank equal to k. Let $\tilde{V}$ be a nonsingular $(k - k_F) \times (k - k_F)$ matrix such that $\tilde{R}\tilde{V}^{-1} = (0, I_q)$ and define V as the $k \times k$ block-diagonal matrix with first diagonal block I_{k_F} and second diagonal block $\tilde{V}$. Clearly, $RV^{-1} = (0, I_q)$ holds. Set $X = X^*V$ and note that $X = (F, \tilde{X})$ with $\tilde{X} = \tilde{X}^*\tilde{V}$ and that $\mathrm{rank}(X) = k$. Furthermore, we have

$$R(X'X)^{-1}X'\left(e_{i_1}(n), \ldots, e_{i_q}(n)\right) = (0, I_q)(X^{*'}X^*)^{-1}X^{*'}\left(e_{i_1}(n), \ldots, e_{i_q}(n)\right) = I_q,$$

showing that the so-constructed $\tilde{X}$ belongs to $\tilde{\mathfrak{X}}(i_1, \ldots, i_q)$. Second, observe that $\tilde{X} \notin \tilde{\mathfrak{X}}(i_1, \ldots, i_q)$ is equivalent to

$$\det(X'X) = 0 \text{ or } (\det(X'X) \neq 0 \text{ and } \det(R(X'X)^{-1}X'(e_{i_1}(n), \ldots, e_{i_q}(n))) = 0),$$

which in turn is equivalent to

$$\det(X'X) \det(R \operatorname{adj}(X'X)X'(e_{i_1}(n), \ldots, e_{i_q}(n))) = 0.$$

This is a polynomial in the entries of $\tilde{X}$ and does not vanish identically, because we have shown that $\tilde{\mathfrak{X}}(i_1, \ldots, i_q)$ is nonempty. Consequently, the complement of $\tilde{\mathfrak{X}}(i_1, \ldots, i_q)$ is a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set.

Step 2: It follows that the set $\tilde{\mathfrak{X}}_{20}$ defined as the intersection of all sets of the form $\tilde{\mathfrak{X}}(i_1, \ldots, i_q)$, where we vary over all possible combinations of indices satisfying $1 \leq i_1 < \ldots < i_q \leq n$, is the complement of a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set, since this is an intersection of finitely many sets.

Step 3: Let $1 \leq j_1 < \ldots < j_{n-q+1} \leq n$ be given and define the set

$$\tilde{\mathfrak{X}}^*(j_1, \ldots, j_{n-q+1}) = \left\{ \tilde{X} \in \mathbb{R}^{n \times (k-k_F)} : z_{j_1} = \ldots = z_{j_{n-q+1}} = 0 \text{ for some } z \in (\operatorname{span}(X))^{\perp} \backslash \{0\} \right\}.$$

We show that this set is a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set: If $q = 1$, then $\tilde{\mathfrak{X}}^*(j_1, \ldots, j_{n-q+1})$ is obviously empty. Hence consider the case $q > 1$. Observe that $\tilde{X} \in \tilde{\mathfrak{X}}^*(j_1, \ldots, j_{n-q+1})$ is equivalent to the equation system $A(\tilde{X})z = 0$ having a nonzero solution, where

$$A(\tilde{X}) = \left(e_{j_1}(n), \ldots, e_{j_{n-q+1}}(n), X \right)'.$$

Observe that $A(\tilde{X})$ is of dimension $(n - q + 1 + k) \times n$ and that the row-dimension is larger than n, since $q \leq k$. Consequently, $\tilde{X} \in \tilde{\mathfrak{X}}^*(j_1, \ldots, j_{n-q+1})$ is equivalent to

$$\det(A(\tilde{X})'A(\tilde{X})) = 0,$$

a polynomial equation in the elements of $\tilde{X}$. The solution set is thus a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set if we can exhibit an element $\tilde{X} \notin \tilde{\mathfrak{X}}^*(j_1, \ldots, j_{n-q+1})$. We now construct such an element as follows: Let $i_1, \ldots, i_{q-1}$ be the elements of $\{1, \ldots, n\}$ not appearing in the list $j_1, \ldots, j_{n-q+1}$. Choose linearly independent elements $a_1, \ldots, a_{k-(k_F+q-1)}$ of the orthogonal complement of $\operatorname{span}((F, e_{i_1}(n), \ldots, e_{i_{q-1}}(n)))$; note that such a choice is possible since the dimension of this span is $k_F + q - 1 < k < n$ in view of our assumptions. Now define

$$\tilde{X} = (e_{i_1}(n), \ldots, e_{i_{q-1}}(n), a_1, \ldots, a_{k-(k_F+q-1)})$$

and observe that clearly $\tilde{X} \notin \tilde{\mathfrak{X}}^*(j_1, \ldots, j_{n-q+1})$ is satisfied.

Step 4: Define $\tilde{\mathfrak{X}}_{21}$ as the complement (in $\mathbb{R}^{n \times (k-k_F)}$) of the set

$$\left(\bigcup \tilde{\mathfrak{X}}^*(j_1, \ldots, j_{n-q+1}) \right) \cup \left\{ \tilde{X} \in \mathbb{R}^{n \times (k-k_F)} : \operatorname{rank}(X) < k \right\},$$

where the union extends over all tuples $(j_1, \ldots, j_{n-q+1})$ satisfying $1 \leq j_1 < \ldots < j_{n-q+1} \leq n$. Then $\tilde{\mathfrak{X}}_{21}$ is the complement of a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set: This is obvious because of Step 3, the fact that the union is a union of finitely many sets, and because the set $\left\{ \tilde{X} \in \mathbb{R}^{n \times (k-k_F)} : \operatorname{rank}(X) < k \right\}$ is clearly a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set.

Step 5: It follows from the preceding steps that $\tilde{\mathfrak{X}}_{20} \cap \tilde{\mathfrak{X}}_{21}$ is the complement of a $\lambda_{\mathbb{R}^{n \times (k-k_F)}}$-null set. We now show that $\tilde{\mathfrak{X}}_{20} \cap \tilde{\mathfrak{X}}_{21}$ is contained in $\tilde{\mathfrak{X}}_2$: First, if $\tilde{X} \in \tilde{\mathfrak{X}}_{20} \cap \tilde{\mathfrak{X}}_{21}$, then $X = (F, \tilde{X})$ clearly has rank equal to k. Second, we need to show that $\tilde{X} \in \tilde{\mathfrak{X}}_{20} \cap \tilde{\mathfrak{X}}_{21}$ implies $\mathsf{B} \subseteq \operatorname{span}(X)$, the other inclusion being trivial: Now, suppose that $\tilde{X} \in \tilde{\mathfrak{X}}_{20} \cap \tilde{\mathfrak{X}}_{21}$, and that $y \in \mathsf{B}$ but $y \notin \operatorname{span}(X)$ would be possible. Then $z = \Pi_{(\operatorname{span}(X))^\perp} y \neq 0$ would follow. Hence, there would have to exist indices $1 \leq i_1 < \ldots < i_q \leq n$ such that

$$e'_{i_1}(n)z \neq 0, \ldots, e'_{i_q}(n)z \neq 0, \tag{43}$$

because of $\tilde{X} \in \tilde{\mathfrak{X}}_{21}$. But then we could conclude that

$$\begin{aligned}
q &\geq \operatorname{rank}(B(y)) \geq \operatorname{rank}\left(R(X'X)^{-1} X' \left(e_{i_1}(n) e'_{i_1}(n)z, \ldots, e_{i_q}(n) e'_{i_q}(n)z \right) \right) \\
&= \operatorname{rank}\left(R(X'X)^{-1} X' \left(e_{i_1}(n), \ldots, e_{i_q}(n) \right) \right) = q,
\end{aligned}$$

the last but one equality holding because of (43), and the last one holding in view of $\tilde{X} \in \tilde{\mathfrak{X}}_{20}$. This would entail $y \notin \mathsf{B}$, a contradiction. This now completes the proof. ∎

B   Appendix: Proofs for Section 5 and some sufficient conditions relating to $\mathbb{J}(\mathcal{L}, \mathfrak{C})$

Proof of Proposition 5.2: Fix an arbitrary $\mu_0 \in \mathfrak{M}_0$. Due to $G(\mathfrak{M}_0)$-invariance of W the l.h.s. of (22) coincides with $\sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \Sigma}(W)$, cf. Remark 5.5(iii) in Preinerstorfer and Pötscher (2016). Now, let Σ_j be a sequence in $\mathfrak{C}$ such that $P_{\mu_0, \Sigma_j}(W) \to \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \Sigma}(W)$ as $j \to \infty$. By relative compactness of $\mathcal{L}(\mathfrak{C})$, we may assume that $\mathcal{L}(\Sigma_j) \to \bar{\Sigma}$ (possibly after passing to a subsequence). For notational convenience, define the sequence $L_j = \mathcal{L}(\Sigma_j) + \Pi_{\mathcal{L}}$ and denote its limit by $L = \bar{\Sigma} + \Pi_{\mathcal{L}}$. We first claim that

$$P_{\mu_0, \Sigma_j}(W) = P_{\mu_0, \mathcal{L}(\Sigma_j)}(W) = P_{\mu_0, L_j}(W). \tag{44}$$

To see this, let $\mathbf{Z}$ be a standard Gaussian $n \times 1$ vector. Then

$$P_{\mu_0, \Sigma_j}(W) = \Pr\left(\mu_0 + \Sigma_j^{1/2} \mathbf{Z} \in W \right) = \Pr\left(\mu_0 + \Pi_{\mathcal{L}^\perp} \Sigma_j^{1/2} \mathbf{Z} \in W \right)$$

because W is easily seen to be invariant w.r.t. addition of elements of $\mathcal{L}$ and since $\Pi_{\mathcal{L}}\Sigma_m^{1/2}\mathbf{Z}$ clearly belongs to $\mathcal{L}$. The latter probability equals

$$\Pr\left(\mu_0 + \Pi_{\mathcal{L}^\perp}\Sigma_j^{1/2}\mathbf{Z}/\,\|\Pi_{\mathcal{L}^\perp}\Sigma_j\Pi_{\mathcal{L}^\perp}\|^{1/2} \in \mu_0 + (W - \mu_0)/\,\|\Pi_{\mathcal{L}^\perp}\Sigma_j\Pi_{\mathcal{L}^\perp}\|^{1/2}\right)$$

$$= \Pr\left(\mu_0 + \Pi_{\mathcal{L}^\perp}\Sigma_j^{1/2}\mathbf{Z}/\,\|\Pi_{\mathcal{L}^\perp}\Sigma_j\Pi_{\mathcal{L}^\perp}\|^{1/2} \in W\right) = P_{\mu_0,\mathcal{L}(\Sigma_j)}(W)$$

where the first equality follows from $G(\mathfrak{M}_0)$-invariance of W. Furthermore, using the invariance of W w.r.t. to addition of elements of $\mathcal{L}$ again we obtain

$$P_{\mu_0,\mathcal{L}(\Sigma_j)}(W) = \Pr\left(\mu_0 + \mathcal{L}(\Sigma_j)^{1/2}\mathbf{Z} \in W\right) = \Pr\left(\mu_0 + \left(\mathcal{L}(\Sigma_j)^{1/2} + \Pi_{\mathcal{L}}\right)\mathbf{Z} \in W\right) = P_{\mu_0,L_j}(W),$$

the last equality holding because $\mathcal{L}(\Sigma_j)^{1/2} + \Pi_{\mathcal{L}}$ is a square-root of L_j. This establishes (44).

We now distinguish two cases.

Case 1: Suppose L is positive definite. Then $P_{\mu_0,\Sigma_j}(W) \to P_{\mu_0,L}(W)$ follows from (44) in view of total variation convergence of P_{μ_0,L_j} to $P_{\mu_0,L}$. But note that $P_{\mu_0,L}(W) < 1$, since W^c is not a $\lambda_{\mathbb{R}^n}$-null set by assumption and since $\lambda_{\mathbb{R}^n}$ is equivalent to $P_{\mu_0,L}$.

Case 2: Suppose L is singular. We note that

$$\limsup_{j\to\infty} P_{\mu_0,\Sigma_j}(W) \leq \limsup_{j\to\infty} P_{\mu_0,\mathcal{L}(\Sigma_j)}(\mathrm{cl}(W)) \leq P_{\mu_0,\bar{\Sigma}}(\mathrm{cl}(W)),$$

where we have used Equation (44), the inclusion $W \subseteq \mathrm{cl}(W)$, weak convergence of $P_{\mu_0,\mathcal{L}(\Sigma_j)}$ to $P_{\mu_0,\bar{\Sigma}}$ (cf. Lemma E.1 in Preinerstorfer and Pötscher (2016)), and the Portmanteau theorem. Define $\mathcal{S} = \mathrm{span}(\bar{\Sigma})$ and observe that

$$P_{\mu_0,\bar{\Sigma}}(\mathrm{cl}(W)) = 1 - P_{\mu_0,\bar{\Sigma}}\left((\mathrm{cl}(W))^c \cap (\mu_0 + \mathcal{S})\right),$$

since $P_{\mu_0,\bar{\Sigma}}$ is concentrated on $\mu_0 + \mathcal{S}$. Furthermore, singularity of L implies $\mathrm{rank}(\bar{\Sigma}) < n - \dim(\mathcal{L})$, and thus $\mathcal{S} \in \mathbb{J}(\mathcal{L},\mathfrak{C})$. Hence, by assumption, $(\mathrm{cl}(W))^c \cap (\mu_0 + \mathcal{S}) \neq \emptyset$, and thus there exists a nonempty open subset $U \subseteq \mu_0 + \mathcal{S}$ (w.r.t. the topology induced from $\mathbb{R}^n$) such that $U \subseteq (\mathrm{cl}(W))^c \cap (\mu_0 + \mathcal{S})$. Equivalence of the measures $P_{\mu_0,\bar{\Sigma}}$ and $\lambda_{\mu_0+\mathcal{S}}$ then implies $P_{\mu_0,\bar{\Sigma}}(U) > 0$. Together with the preceding display this then gives $P_{\mu_0,\bar{\Sigma}}(\mathrm{cl}(W)) \leq 1 - P_{\mu_0,\bar{\Sigma}}(U) < 1$. $\blacksquare$

Proof of Proposition 5.5: Fix an arbitrary $\mu_0 \in \mathfrak{M}_0$. Due to $G(\mathfrak{M}_0)$-invariance of W_j the l.h.s. of (23) coincides with $\sup_{\Sigma \in \mathfrak{C}} P_{\mu_0,\Sigma}(W_j)$, cf. Remark 5.5(iii) in Preinerstorfer and Pötscher (2016). Let Σ_j be a sequence in $\mathfrak{C}$ such that

$$\left|P_{\mu_0,\Sigma_j}(W_j) - \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0,\Sigma}(W_j)\right| \to 0 \quad \text{as} \quad j \to \infty. \tag{45}$$

By relative compactness of $\mathcal{L}(\mathfrak{C})$, we may assume that $\mathcal{L}(\Sigma_j) \to \bar{\Sigma}$ (possibly after passing to a subsequence). Again we define the sequence $L_j = \mathcal{L}(\Sigma_j) + \Pi_{\mathcal{L}}$ and denote its limit by $L = \bar{\Sigma} + \Pi_{\mathcal{L}}$.

Using the assumed invariance properties of W_j we obtain as before that

$$P_{\mu_0, \Sigma_j}(W_j) = P_{\mu_0, \mathcal{L}(\Sigma_j)}(W_j) = P_{\mu_0, L_j}(W_j). \tag{46}$$

Case 1: Suppose L is positive definite. Let $\varepsilon > 0$ and note that $P_{\mu_0, L}$ is equivalent to $\lambda_{\mathbb{R}^n}$. Since $W_j \supseteq W_{j+1}$, and since $\bigcap_{j \in \mathbb{N}} W_j$ is a $\lambda_{\mathbb{R}^n}$-null set by assumption, we can find $j_0 \in \mathbb{N}$ such that $P_{\mu_0, L}(W_{j_0}) < \varepsilon$. Since P_{μ_0, L_j} converges to $P_{\mu_0, L}$ in total variation, we arrive at

$$\lim_{j \to \infty} P_{\mu_0, L_j}(W_{j_0}) < \varepsilon.$$

From $W_j \supseteq W_{j+1}$ we obtain

$$\limsup_{j \to \infty} P_{\mu_0, L_j}(W_j) \leq \limsup_{j \to \infty} P_{\mu_0, L_j}(W_{j_0}) < \varepsilon,$$

which proves the claim in view of (46) and (45).

Case 2: Suppose L is singular. Let $\varepsilon > 0$. Singularity of L implies $\mathrm{rank}(\bar{\Sigma}) < n - \dim(\mathcal{L})$, and therefore $\mathcal{S} = \mathrm{span}(\bar{\Sigma}) \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ must hold. By assumption, $\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$ is hence a $\lambda_{\mu_0 + \mathcal{S}}$-null set. Since furthermore $\mathrm{cl}(W_j) \supseteq \mathrm{cl}(W_{j+1})$ holds, and since $P_{\mu_0, \bar{\Sigma}}$ is equivalent to $\lambda_{\mu_0 + \mathcal{S}}$, we can hence find a $j_0 \in \mathbb{N}$ such that

$$P_{\mu_0, \bar{\Sigma}}(\mathrm{cl}(W_{j_0})) < \varepsilon.$$

But $\mathrm{cl}(W_j) \supseteq \mathrm{cl}(W_{j+1})$ implies

$$\limsup_{j \to \infty} P_{\mu_0, \mathcal{L}(\Sigma_j)}(W_j) \leq \limsup_{j \to \infty} P_{\mu_0, \mathcal{L}(\Sigma_j)}(\mathrm{cl}(W_j))$$
$$\leq \limsup_{j \to \infty} P_{\mu_0, \mathcal{L}(\Sigma_j)}(\mathrm{cl}(W_{j_0})).$$

Since $P_{\mu_0, \mathcal{L}(\Sigma_j)}$ converges to $P_{\mu_0, \bar{\Sigma}}$ weakly (cf. Lemma E.1 in Preinerstorfer and Pötscher (2016)), the Portmanteau theorem gives

$$\limsup_{j \to \infty} P_{\mu_0, \mathcal{L}(\Sigma_j)}(\mathrm{cl}(W_{j_0})) \leq P_{\mu_0, \bar{\Sigma}}(\mathrm{cl}(W_{j_0})) < \varepsilon,$$

which then proves the claim via (46) and (45). ∎

Lemma B.1. *Suppose $T : \mathbb{R}^n \to \mathbb{R}$ is continuous on the complement of a closed set $N^\dagger \subseteq \mathbb{R}^n$. Then for every $-\infty < C < \infty$ the set $\{y \in \mathbb{R}^n : T(y) \geq C\} \cup N^\dagger$ is closed, and hence contains $\mathrm{cl}(\{y \in \mathbb{R}^n : T(y) \geq C\})$. Furthermore, $\mathrm{bd}(\{y \in \mathbb{R}^n : T(y) \geq C\})$ is contained in the closed set $\{y \in \mathbb{R}^n : T(y) = C\} \cup N^\dagger$. [Analogous statements hold for the set $\{y \in \mathbb{R}^n : T(y) \leq C\}$.]*

Proof: Suppose y_0 is an accumulation point of $A := \{y \in \mathbb{R}^n : T(y) \geq C\} \cup N^\dagger$. If $y_0 \in N^\dagger \subseteq A$ we are done. Suppose next that $y_0 \notin N^\dagger$. Let $y_j \in A$ be such that $y_j \to y_0$. Then $y_j \notin N^\dagger$ eventually, since $N^\dagger$ is closed. By assumption T is continuous on $\mathbb{R}^n \backslash N^\dagger$. Hence

$T(y_j) \to T(y_0)$. Now, since $y_j \in A$ and since $y_j \notin N^\dagger$ eventually, we have $T(y_j) \geq C$ eventually. But then $T(y_0) \geq C$ follows, implying that $y_0 \in A$. We turn to the second claim. That $B := \{y \in \mathbb{R}^n : T(y) = C\} \cup N^\dagger$ is closed, is proved in a completely analogous way. Next we establish the claimed inclusion: If y_* is an element of $\mathrm{bd}(\{y \in \mathbb{R}^n : T(y) \geq C\})$ and y_* belongs to $N^\dagger$ we are done. Assume y_* does not belong to $N^\dagger$. Since y_* must also be an element of $\mathrm{cl}(\{y \in \mathbb{R}^n : T(y) \geq C\})$, and hence of A by what has already been shown, it follows that $T(y_*) \geq C$ must hold. If $T(y_*) > C$ would be true, then T would have to be larger than C on an open neighborhood of y_*, since $y_* \notin N^\dagger$. This would lead to the contradiction that y_* belongs to the interior of $\{y \in \mathbb{R}^n : T(y) \geq C\}$. The claim in parenthesis is proved completely analogously. ∎

Proof of Corollary 5.6: For the first statement we check the conditions of Proposition 5.5. The invariance properties of W_j follow from invariance of T. That $W_j \supseteq W_{j+1}$ holds is obvious. Condition (a) of Proposition 5.5 is satisfied because $\bigcap_{j \in \mathbb{N}} W_j$ is empty as T is real-valued and $C_j \nearrow \infty$ as $j \to \infty$. Hence, Part 1 follows. Next observe that

$$\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j) \subseteq \bigcap_{j \in \mathbb{N}} (W_j \cup N^\dagger) = \bigcap_{j \in \mathbb{N}} W_j \cup N^\dagger = \emptyset \cup N^\dagger = N^\dagger,$$

where we have used Lemma B.1 to obtain the first inclusion. Part 2 then follows immediately. For Part 3 note that $\lambda_{\mu_0 + \mathcal{S}}(N^\dagger) = 0$ obviously implies $\mu_0 + \mathcal{S} \not\subseteq N^\dagger$, since $\lambda_{\mu_0 + \mathcal{S}}$ is supported by $\mu_0 + \mathcal{S}$. The converse is seen as follows: Clearly, $(\mu_0 + \mathcal{S}) \cap N^\dagger$ is a finite or countable union of sets of the form $(\mu_0 + \mathcal{S}) \cap \mathcal{A}_j$, where the $\mathcal{A}_j$'s are affine spaces, the union of which is $N^\dagger$. Since $\mu_0 + \mathcal{S} \not\subseteq N^\dagger$, the sets $(\mu_0 + \mathcal{S}) \cap \mathcal{A}_j$ must be proper affine subspaces of $\mu_0 + \mathcal{S}$ or must be empty, entailing $\lambda_{\mu_0 + \mathcal{S}}((\mu_0 + \mathcal{S}) \cap \mathcal{A}_j) = 0$ for every j. Since $\lambda_{\mu_0 + \mathcal{S}}(N^\dagger) = \lambda_{\mu_0 + \mathcal{S}}((\mu_0 + \mathcal{S}) \cap N^\dagger)$, we conclude that $\lambda_{\mu_0 + \mathcal{S}}(N^\dagger) = 0$. ∎

Proof of claim in Remark 5.9(ii): First note that $\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$ and $N^\dagger$ are $G(\mathfrak{M}_0)$-invariant. The reverse implication is trivial since $\lambda_{\mu_0 + \mathcal{S}}$ is supported on $\mu_0 + \mathcal{S}$ and has a density there, and thus assigns zero mass to $\{\mu_0\}$ as $\dim \mathcal{S} > 0$. To prove the other implication suppose that $x \in \bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j) \cap (\mu_0 + \mathcal{S})$ but is different from μ_0. Then $\delta(x - \mu_0) + \mu_0$ also belongs to this set for every $\delta \neq 0$ in view of $G(\mathfrak{M}_0)$-invariance of $\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$ and because of the particular form of $\mu_0 + \mathcal{S}$. Note that $s = x - \mu_0$ is nonzero and belongs to $\mathcal{S}$, and hence spans $\mathcal{S}$. We conclude that $(\mu_0 + \mathcal{S}) \setminus \{\mu_0\}$ is a subset of $\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$. But this contradicts the assumption that $\bigcap_{j \in \mathbb{N}} \mathrm{cl}(W_j)$ is a $\lambda_{\mu_0 + \mathcal{S}}$-null set. The proof for $N^\dagger$ is completely analogous. ∎

Proof of Lemma 5.11: Since T is $G(\mathfrak{M}_0)$-invariant and since for a one-dimensional $\mathcal{S}$ every element of $\mu_0 + \mathcal{S}$ has the form $\mu_0 + \gamma s_0$ for a fixed $s_0 \in \mathcal{S}$, we conclude that

$$T(\mu_0 + \gamma s_0) = T\left(\gamma^{-1}((\mu_0 + \gamma s_0) - \mu_0) + \mu_0\right) = T(\mu_0 + s_0)$$

for every $\gamma \neq 0$. Hence, any one-dimensional $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ satisfies $\mathcal{S} \in \mathbb{H}$. This proves Part 1. In view of the assumed $G(\mathfrak{M}_0)$-invariance of T we may for the rest of the proof fix an arbitrary

$\mu_0 \in \mathfrak{M}_0$, set $\sigma^2 = 1$, and drop the suprema (infima) w.r.t. μ_0 and σ^2 from (25) and (26), cf. Remark 5.5(iii) in Preinerstorfer and Pötscher (2016). We turn to Part 2: Concerning (25), note that in case $\mathbb{H}$ is empty there is nothing to prove, hence assume that $\mathbb{H}$ is nonempty and choose a $C \in (-\infty, C^*)$. Since $C < C^*$, there exists an $\mathcal{S} \in \mathbb{H}$ such that the corresponding constant $C(\mathcal{S})$ satisfies $C < C(\mathcal{S}) \leq C^*$. By assumption, T is continuous on the complement of $N^\dagger$, a closed $\lambda_{\mu_0 + \mathcal{S}}$-null set. By definition of $C(\mathcal{S})$ we have that $\lambda_{\mu_0 + \mathcal{S}}(T \neq C(\mathcal{S})) = 0$ which together with $C(\mathcal{S}) > C$ and Lemma B.1 implies that

$$
\begin{aligned}
\lambda_{\mu_0 + \mathcal{S}}([\mathrm{int}(\{y \quad \in \quad \mathbb{R}^n : T(y) \geq C\})]^c) &= \lambda_{\mu_0 + \mathcal{S}}(\mathrm{cl}(\{y \in \mathbb{R}^n : T(y) < C\})) \\
&\leq \quad \lambda_{\mu_0 + \mathcal{S}}(\mathrm{cl}(\{y \in \mathbb{R}^n : T(y) \leq C\})) \leq \lambda_{\mu_0 + \mathcal{S}}(\{y \in \mathbb{R}^n : T(y) \leq C\}) + \lambda_{\mu_0 + \mathcal{S}}(N^\dagger) \\
&\leq \quad \lambda_{\mu_0 + \mathcal{S}}(T \neq C(\mathcal{S})) + \lambda_{\mu_0 + \mathcal{S}}(N^\dagger) = 0.
\end{aligned}
$$

Now, let Σ_j be a sequence in $\mathfrak{C}$ such that $\mathcal{L}(\Sigma_j) \to \bar{\Sigma}$ with $\mathrm{span}(\bar{\Sigma}) = \mathcal{S}$. Then by $G(\mathfrak{M}_0)$-invariance of T and its invariance w.r.t. addition of elements of $\mathcal{V}$ we have (cf. the proof of (44))

$$
\begin{aligned}
\sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \Sigma}(T \geq C) &\geq \liminf_{j \to \infty} P_{\mu_0, \Sigma_j}(T \geq C) = \liminf_{j \to \infty} P_{\mu_0, \mathcal{L}(\Sigma_j)}(T \geq C) \\
&\geq \liminf_{j \to \infty} P_{\mu_0, \mathcal{L}(\Sigma_j)}(\mathrm{int}(\{y \in \mathbb{R}^n : T(y) \geq C\})) \\
&\geq P_{\mu_0, \bar{\Sigma}}(\mathrm{int}(\{y \in \mathbb{R}^n : T(y) \geq C\})),
\end{aligned}
$$

where the last inequality follows from weak convergence of $P_{\mu_0, \mathcal{L}(\Sigma_j)}$ to $P_{\mu_0, \bar{\Sigma}}$ (cf. Lemma E.1 in Preinerstorfer and Pötscher (2016)) and the Portmanteau theorem. But absolute continuity of $P_{\mu_0, \bar{\Sigma}}$ w.r.t. $\lambda_{\mu_0 + \mathcal{S}}$ then implies

$$
P_{\mu_0, \bar{\Sigma}}(\mathrm{int}(\{y \in \mathbb{R}^n : T(y) \geq C\})) = 1 - P_{\mu_0, \bar{\Sigma}}([\mathrm{int}(\{y \in \mathbb{R}^n : T(y) \geq C\})]^c) = 1.
$$

This proves (25). The statement concerning the convergence behavior of the size as $C \to \infty$ is a consequence of Corollary 5.6.

Finally consider Part 3. Again, if $\mathbb{H}$ is empty there is nothing to prove. Hence assume $\mathbb{H} \neq \varnothing$. Let $C \in (C_*, \infty)$. Then we can find $\mathcal{S} \in \mathbb{H}$ with $C(\mathcal{S}) < C$. Furthermore, by definition we can find a sequence $\Sigma_j \in \mathfrak{C}$ such that $\bar{\Sigma}_j = \mathcal{L}(\Sigma_j)$ converges to $\bar{\Sigma}$ with $\mathrm{span}(\bar{\Sigma}) = \mathcal{S}$. It follows that $T = C(\mathcal{S}) < C$ holds $\lambda_{\mu_0 + \mathrm{span}(\bar{\Sigma})}$-a.e., since $\mathcal{S} \in \mathbb{H}$. Hence, $\lambda_{\mu_0 + \mathrm{span}(\bar{\Sigma})}(T \geq C) = 0$, which entails $P_{\mu_0, \bar{\Sigma}}(T \geq C) = 0$ by equivalence of the measures involved. Using (44), the Portmanteau

theorem, and Lemma B.1, we obtain

$$
\begin{aligned}
\inf_{\Sigma \in \mathfrak{C}} P_{\mu_0, \Sigma}(T \geq C) \quad &\leq \quad \limsup_{j \to \infty} P_{\mu_0, \Sigma_j}(T \geq C) = \limsup_{j \to \infty} P_{\mu_0, \bar{\Sigma}_j}(T \geq C) \\
&\leq \quad \limsup_{j \to \infty} P_{\mu_0, \bar{\Sigma}_j}\left(\mathrm{cl}\left(\{y \in \mathbb{R}^n : T(y) \geq C\}\right)\right) \\
&\leq \quad P_{\mu_0, \bar{\Sigma}}\left(\mathrm{cl}\left(\{y \in \mathbb{R}^n : T(y) \geq C\}\right)\right) \\
&\leq \quad P_{\mu_0, \bar{\Sigma}}\left(T \geq C\right) + P_{\mu_0, \bar{\Sigma}}(N^\dagger) = P_{\mu_0, \bar{\Sigma}}(N^\dagger) = 0,
\end{aligned}
$$

the last equality following from equivalence of $P_{\mu_0, \bar{\Sigma}}$ with $\lambda_{\mu_0 + \mathrm{span}(\bar{\Sigma})}$, from $\mathrm{span}(\bar{\Sigma}) \in \mathbb{H}$, and the assumptions made on $N^\dagger$. $\blacksquare$

Proof of Proposition 5.12: We start with the observation that the statements "and hence for all $\mu_0 \in \mathfrak{M}_0$" in parentheses follow from the corresponding statements involving "and hence for some $\mu_0 \in \mathfrak{M}_0$" in view of the assumed $G(\mathfrak{M}_0)$-invariance of T and $N^\dagger$. In view of the assumed $G(\mathfrak{M}_0)$-invariance of T we may for the rest of the proof fix an arbitrary $\mu_0 \in \mathfrak{M}_0$, set $\sigma^2 = 1$, and drop the suprema w.r.t. μ_0 and σ^2 from the displayed expressions shown in the lemma, cf. Remark 5.5(iii) in Preinerstorfer and Pötscher (2016).

We now establish Part A.1 of the lemma: The statement concerning nonincreasingness is obvious. The constancy property of the size as well as its convergence to zero for $C \to \infty$ has already been established in Lemma 5.11. Therefore, it remains to verify that the function $C \mapsto \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \Sigma}(T \geq C)$ is continuous on (C^*, ∞), where we note that $C^* < \infty$ holds. In order to achieve this we proceed in two steps:

Step 1: We show that the map $(C, \bar{\Sigma}) \mapsto P_{\mu_0, \bar{\Sigma}}(T \geq C)$ is continuous on $(C^*, \infty) \times \mathrm{cl}(\mathcal{L}(\mathfrak{C}))$ and that $\mathrm{cl}(\mathcal{L}(\mathfrak{C}))$ is compact. Compactness is obvious, since $\mathcal{L}(\mathfrak{C})$ is norm bounded by construction. In order to establish continuity, let $C_j \in (C^*, \infty)$ be a sequence such that $C_j \to C \in (C^*, \infty)$ and let $\bar{\Sigma}_j \in \mathrm{cl}(\mathcal{L}(\mathfrak{C}))$ converge to $\bar{\Sigma} \in \mathrm{cl}(\mathcal{L}(\mathfrak{C}))$. In view of the assumed invariance properties of T we have $P_{\mu_0, \bar{\Sigma}_j}(T \geq \cdot) = P_{\mu_0, \Omega_j}(T \geq \cdot)$ and $P_{\mu_0, \bar{\Sigma}}(T \geq \cdot) = P_{\mu_0, \Omega}(T \geq \cdot)$ where $\Omega_j = \bar{\Sigma}_j + \Pi_{\mathcal{L}}$ and $\Omega = \bar{\Sigma} + \Pi_{\mathcal{L}}$ (cf. the proof of (44)). Note that Ω_j converges to Ω, and thus P_{μ_0, Ω_j} converges to $P_{\mu_0, \Omega}$ weakly (cf. Lemma E.1 in Preinerstorfer and Pötscher (2016)), and in fact in total variation distance if Ω is nonsingular.

Case 1: Assume that Ω is nonsingular. Then convergence of P_{μ_0, Ω_j} to $P_{\mu_0, \Omega}$ in total variation distance implies

$$
P_{\mu_0, \bar{\Sigma}_j}(T \geq C_j) - P_{\mu_0, \bar{\Sigma}}(T \geq C_j) = P_{\mu_0, \Omega_j}(T \geq C_j) - P_{\mu_0, \Omega}(T \geq C_j) \to 0 \quad \text{for} \quad j \to \infty.
$$

Furthermore,

$$
P_{\mu_0, \bar{\Sigma}}(T \geq C_j) = P_{\mu_0, \Omega}(T \geq C_j) \to P_{\mu_0, \Omega}(T \geq C) = P_{\mu_0, \bar{\Sigma}}(T \geq C) \quad \text{for} \quad j \to \infty,
$$

since $P_{\mu_0, \Omega}(T = C) = 0$ holds in view of the assumption $\lambda_{\mathbb{R}^n}(T = C) = 0$ for $C > C^*$ and equivalence of $P_{\mu_0, \Omega}$ and $\lambda_{\mathbb{R}^n}$. Together this implies the desired convergence.

Case 2: Assume that Ω is singular. Then $\operatorname{span}(\bar{\Sigma}) \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ follows. We distinguish two subcases:

Case 2a: Assume that $\operatorname{span}(\bar{\Sigma}) \in \mathbb{H}$. Choose $\varepsilon > 0$ small enough such that $C - \varepsilon > C^*$ holds. Then $T \leq C^* < C - \varepsilon$ holds $\lambda_{\mu_0 + \operatorname{span}(\bar{\Sigma})}$-a.e., since $\operatorname{span}(\bar{\Sigma}) \in \mathbb{H}$. Consequently, $\lambda_{\mu_0 + \operatorname{span}(\bar{\Sigma})}(T \geq C - \varepsilon) = 0$, which entails $P_{\mu_0, \bar{\Sigma}}(T \geq C - \varepsilon) = 0$ by equivalence of the measures involved. Consequently, also $P_{\mu_0, \bar{\Sigma}}(T \geq C) = 0$ holds. Using the just established identities, the Portmanteau theorem, and Lemma B.1, we obtain

$$
\begin{aligned}
\limsup_{j \to \infty} P_{\mu_0, \bar{\Sigma}_j}(T \geq C_j) \quad & \leq \quad \limsup_{j \to \infty} P_{\mu_0, \bar{\Sigma}_j}\left(\operatorname{cl}\left(\{y \in \mathbb{R}^n : T(y) \geq C - \varepsilon\}\right)\right) \\
& \leq \quad P_{\mu_0, \bar{\Sigma}}\left(\operatorname{cl}\left(\{y \in \mathbb{R}^n : T(y) \geq C - \varepsilon\}\right)\right) \\
& \leq \quad P_{\mu_0, \bar{\Sigma}}\left(T \geq C - \varepsilon\right) + P_{\mu_0, \bar{\Sigma}}(N^\dagger) = P_{\mu_0, \bar{\Sigma}}(N^\dagger) = 0,
\end{aligned}
$$

the last equality following from equivalence of $P_{\mu_0, \bar{\Sigma}}$ with $\lambda_{\mu_0 + \operatorname{span}(\bar{\Sigma})}$, from $\operatorname{span}(\bar{\Sigma}) \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$, and the assumptions made on $N^\dagger$. This establishes the desired convergence.

Case 2b: Assume that $\operatorname{span}(\bar{\Sigma}) \in \mathbb{J}(\mathcal{L}, \mathfrak{C}) \backslash \mathbb{H}$. Choose $\varepsilon > 0$ as before. By Lemma B.1 and our assumptions on $N^\dagger$, and since $\lambda_{\mu_0 + \operatorname{span}(\bar{\Sigma})}(T = C - \varepsilon) = 0$ in view of $\operatorname{span}(\bar{\Sigma}) \in \mathbb{J}(\mathcal{L}, \mathfrak{C}) \backslash \mathbb{H}$ and of $C - \varepsilon > C^*$, it follows that

$$
\lambda_{\mu_0 + \operatorname{span}(\bar{\Sigma})}\left(\operatorname{bd}\left(\{y \in \mathbb{R}^n : T(y) \geq C - \varepsilon\}\right)\right) = 0.
$$

Using equivalence of $P_{\mu_0, \bar{\Sigma}}$ with $\lambda_{\mu_0 + \operatorname{span}(\bar{\Sigma})}$ we also have

$$
P_{\mu_0, \bar{\Sigma}}\left(\operatorname{bd}\left(\{y \in \mathbb{R}^n : T(y) \geq C - \varepsilon\}\right)\right) = 0,
$$

and the same is true if $C - \varepsilon$ is replaced by $C + \varepsilon$. From weak convergence we thus obtain

$$
P_{\mu_0, \bar{\Sigma}_j}(T \geq C \pm \varepsilon) \to P_{\mu_0, \bar{\Sigma}}(T \geq C \pm \varepsilon) \quad \text{for} \quad j \to \infty.
$$

This implies

$$
P_{\mu_0, \bar{\Sigma}}(T \geq C - \varepsilon) \geq \limsup_{j \to \infty} P_{\mu_0, \bar{\Sigma}_j}(T \geq C_j) \geq \liminf_{j \to \infty} P_{\mu_0, \bar{\Sigma}_j}(T \geq C_j) \geq P_{\mu_0, \bar{\Sigma}}(T \geq C + \varepsilon).
$$

Observe that $P_{\mu_0, \bar{\Sigma}}(T = C) = 0$ holds, since $\lambda_{\mu_0 + \operatorname{span}(\bar{\Sigma})}(T = C) = 0$ by our assumptions (note that $C > C^*$ and that $\operatorname{span}(\bar{\Sigma}) \in \mathbb{J}(\mathcal{L}, \mathfrak{C}) \backslash \mathbb{H}$). Letting ε go to zero in the above display then gives the desired convergence of $P_{\mu_0, \bar{\Sigma}_j}(T \geq C_j)$ to $P_{\mu_0, \bar{\Sigma}}(T \geq C)$.

Step 2: Note that by the assumed invariance properties and by the definition of $\mathcal{L}(\mathfrak{C})$ we have for every $C > C^*$ that

$$
\sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \Sigma}(T \geq C) = \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \mathcal{L}(\Sigma)}(T \geq C) = \sup_{\bar{\Sigma} \in \mathcal{L}(\mathfrak{C})} P_{\mu_0, \bar{\Sigma}}(T \geq C) = \sup_{\bar{\Sigma} \in \operatorname{cl}(\mathcal{L}(\mathfrak{C}))} P_{\mu_0, \bar{\Sigma}}(T \geq C),
$$

the last equality following from the continuity established in Step 1. But the right-most supremum is continuous on (C^*, ∞) as a consequence of the claim established in Step 1 and Lemma B.2 given below. [Note that (C^*, ∞) as well as $\mathrm{cl}(\mathcal{L}(\mathfrak{C}))$ are not empty, since $C^* < \infty$ has been established before and since $\mathfrak{C} \neq \varnothing$ by assumption.] This completes the proof of Part A.1.

The claims in Part A.2 are now immediate consequences of the already established Part A.1. Inspection of the proof of Part A.1 shows that under the assumptions of Part B continuity of the size on (C^{**}, ∞) follows. Everything else in Part B is then proved similarly as the corresponding claims in Part A. $\blacksquare$

The following lemma is a special case of Berge's maximum theorem, see Berge (1963), Chapter VI, Section 3.

Lemma B.2. *Let $f : A \times B \to \mathbb{R}$ be a continuous map, where A is a (nonempty) topological space and B is a (nonempty) compact topological space. Then $g(a) = \sup_{b \in B} f(a, b)$ for $a \in A$ defines a continuous map $g : A \to \mathbb{R}$.*

Proof of Lemma 5.15: Since T is $G(\mathfrak{M}_0)$-invariant (see Lemma 5.15 and Proposition 5.4 in Preinerstorfer and Pötscher (2016)) it follows that $P_{\mu_0, \sigma^2 \Sigma}(T \geq C)$ for $\mu_0 \in \mathfrak{M}_0$ does neither depend on the choice of $\mu_0 \in \mathfrak{M}_0$ nor on σ^2. We hence we may fix $\mu_0 \in \mathfrak{M}_0$ and set $\sigma^2 = 1$. Furthermore, $T(y) = T_0(y - \mu_0)$ with

$$T_0(y) = \begin{cases} (R\check{\beta}(y))'\check{\Omega}^{-1}(y)(R\check{\beta}(y)) & y \in \mathbb{R}^n \backslash N^* \\ 0 & y \in N^* \end{cases}$$

follows, because N^* is $G(\mathfrak{M})$-invariant by Lemma 5.15 in Preinerstorfer and Pötscher (2016) and because of the equivariance (invariance) requirements on $\check{\beta}$ ($\check{\Omega}$, respectively) made in Assumption 5 of Preinerstorfer and Pötscher (2016). Consequently,

$$P_{\mu_0, \Sigma}(T(y) \geq C) = P_{\mu_0, \Sigma}(T_0(y - \mu_0) \geq C) = P_{0, \Sigma}(T_0(y) \geq C)$$

holds, where the most right-hand expression does not depend on the value of r. $\blacksquare$

Proof of Lemma 5.16: Parts 1 and 3 have been established in Lemma 5.15 of Preinerstorfer and Pötscher (2016), Borel-measurability being trivial. Consider next Part 2: Lemma 5.15 of Preinerstorfer and Pötscher (2016) shows that $\lambda_{\mathbb{R}^n}(T = C) = 0$ holds for $C > 0$. It follows immediately, that this also holds for $C < 0$ (by passing from T to $-T$, absorbing the sign into $\check{\Omega}$, and by applying Lemma 5.15 in that reference to $-T$). That $\lambda_{\mathbb{R}^n}(T = C) = 0$ also holds for $C = 0$ is seen as follows: Write the set $O = \{y \in \mathbb{R}^n : T(y) = 0\}$ as $O_* \cup N^*$ where $O_* = \{y \in \mathbb{R}^n \backslash N^* : T(y) = 0\}$. Certainly $O_* \subseteq \mathbb{R}^n \backslash N^*$ by construction. It suffices to show that O_* is a $\lambda_{\mathbb{R}^n}$-null set: But this follows from repeating the arguments given in the proof of Part 4 of Lemma 5.15 in Preinerstorfer and Pötscher (2016) for the set O_* (instead of for O), with the only change that the argument that the set $O(y_2)$ constructed in the proof is empty if $y_2 \in N^* \cap \mathfrak{M}^\perp$ now has to be deduced from the observation that $y = y_1 + y_2 \in \mathbb{R}^n \backslash N^*$ is not possible if $y_2 \in N^* \cap \mathfrak{M}^\perp$, since $y_1 \in \mathfrak{M}$ and since N^* is $G(\mathfrak{M})$-invariant. $\blacksquare$

Proof of Lemma 5.18: By definition B is the set where

$$
\begin{aligned}
B(y) &= R(X'X)^{-1}X'\operatorname{diag}\left(e'_1(n)\Pi_{\operatorname{span}(X)^\perp}y,\ldots,e'_n(n)\Pi_{\operatorname{span}(X)^\perp}y\right)\\
&= R(X'X)^{-1}X'\left[e_1(n)e'_1(n)\Pi_{\operatorname{span}(X)^\perp}y,\ldots,e_n(n)e'_n(n)\Pi_{\operatorname{span}(X)^\perp}y\right]
\end{aligned}
$$

has rank less than q. Define the set

$$
D=\left\{(j_1,\ldots,j_s):1\le s\le n,1\le j_1<\ldots<j_s\le n,\operatorname{rank}\left(R(X'X)^{-1}X'\left[e_{j_1}(n),\ldots,e_{j_s}(n)\right]\right)<q\right\},
$$

which may be empty in case $q=1$. Consider first the case where D is nonempty: Since $R(X'X)^{-1}X'$ has rank q, it is then easy to see that we have $y\in\mathsf{B}$ if and only if there exists $(j_1,\ldots,j_s)\in D$ such that $e'_j(n)\Pi_{\operatorname{span}(X)^\perp}y=0$ for $j\ne j_i$ for $i=1,\ldots,s$. This shows, that B is a finite union of (not necessarily distinct) linear subspaces. In case D is empty, $\operatorname{rank}(R(X'X)^{-1}X')=q$ implies that $y\in\mathsf{B}$ if and only if $e'_j(n)\Pi_{\operatorname{span}(X)^\perp}y=0$ for all $1\le j\le n$, i.e., if and only if $y\in\operatorname{span}(X)$. That the linear subspaces making up B are proper, follows since otherwise B would be all of $\mathbb{R}^n$, which is impossible under Assumptions 2 as transpires from an inspection of Lemma 3.1 (and its proof) in Preinerstorfer and Pötscher (2016). To prove the second claim, observe that in case $q=1$ the condition that $\operatorname{rank}(B(y))$ is less than q is equivalent to $B(y)=0$. Since the expressions $R(X'X)^{-1}X'e_j(n)$ are now scalar, we may thus write the condition $B(y)=0$ equivalently as

$$
\left[R(X'X)^{-1}X'e_1(n)e_1(n),\ldots,R(X'X)^{-1}X'e_n(n)e_n(n)\right]\Pi_{\operatorname{span}(X)^\perp}y=0.
$$

But this shows that B is a linear space, namely the kernel of the matrix appearing on the l.h.s. of the preceding display. ∎

Proof of Lemma 5.19: We first prove (ii): Note that the set N^* on which $\hat{\Omega}_{GQ}(y)$ is singular coincides with B, and hence is a finite union of proper linear subspaces of $\mathbb{R}^n$ by Lemma 5.18. Since T_{GQ} is constant on N^* by definition, it follows that $\mu+\mathcal{S}\not\subseteq N^*$ must hold. An argument like the one discussed in Remark 5.9(i) then shows that N^* is a $\lambda_{\mu+\mathcal{S}}$ null set. Consequently, T_{GQ} restricted to $(\mu+\mathcal{S})\backslash N^*$ is not constant. Suppose now there exists a C so that $\lambda_{\mu+\mathcal{S}}(\{y\in\mathbb{R}^n:T(y)=C\})>0$. Then, since N^* is a $\lambda_{\mu+\mathcal{S}}$-null set, it follows that even $\lambda_{\mu+\mathcal{S}}(\{y\in\mathbb{R}^n\backslash N^*:T(y)=C\})>0$, which can be written as $\lambda_{\mu+\mathcal{S}}(\{y\in\mathbb{R}^n\backslash N^*:p(y)=0\})>0$, with $p(y)=(R\hat{\beta}(y)-r)'\operatorname{adj}(\hat{\Omega}_{GQ}(y))(R\hat{\beta}(y)-r)-\det(\hat{\Omega}_{GQ}(y))C$, a polynomial in y. This implies that p restricted to $\mu+\mathcal{S}$ vanishes on a set of positive $\lambda_{\mu+\mathcal{S}}$-measure. Since p can clearly be expressed as a polynomial in coordinates parameterizing the affine space $\mu+\mathcal{S}$, it follows that p vanishes identically on $\mu+\mathcal{S}$. But this implies that T_{GQ} restricted to $(\mu+\mathcal{S})\backslash N^*$ is constant equal to C, a contradiction. Part (i) follows as a special case of Part (ii). The proof of Part (iii) is completely analogous, noting that for the weighted Eicker-test statistic the set N^* is always $\operatorname{span}(X)$. ∎

B.1 Sufficient conditions relating to $\mathbb{J}(\mathcal{L}, \mathfrak{C})$

The subsequent lemma sheds light on the relation between the collection $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ and the set of concentration spaces of $\mathfrak{C}$ in an important case and leads to sufficient conditions discussed in the remark given below. Recall from Definition 2.1 in Preinerstorfer and Pötscher (2016) that a linear subspace $\mathcal{Z}$ of $\mathbb{R}^n$ is said to be a concentration space of $\mathfrak{C}$, if $\dim(\mathcal{Z}) < n$ and if there exists a sequence $\Sigma_m \in \mathfrak{C}$ such that $\Sigma_m \to \Sigma^*$ with $\mathrm{span}(\Sigma^*) = \mathcal{Z}$.

Lemma B.3. *Let $\mathfrak{C}$ be a covariance model and let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$. Then the following hold:*

1. *If $\mathcal{Z}$ is a concentration space of $\mathfrak{C}$, then either $\Pi_{\mathcal{L}^\perp}\mathcal{Z}$ is an element of $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ or $\Pi_{\mathcal{L}^\perp}\mathcal{Z} = \mathcal{L}^\perp$ or $\Pi_{\mathcal{L}^\perp}\mathcal{Z} = \{0\}$.*

2. *Suppose that $\mathfrak{C}$ is bounded. If $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ then either (i) there is a concentration space $\mathcal{Z}$ of $\mathfrak{C}$ with the property that $\mathcal{S} = \Pi_{\mathcal{L}^\perp}\mathcal{Z}$ or (ii) there exists a concentration space $\mathcal{Z}$ of $\mathfrak{C}$ satisfying $\Pi_{\mathcal{L}^\perp}\mathcal{Z} = \{0\}$ (i.e., $\mathcal{Z} \subseteq \mathcal{L}$) and a sequence $\Sigma_j \in \mathfrak{C}$ such that Σ_j converges to Σ^* satisfying $\mathcal{Z} = \mathrm{span}(\Sigma^*)$ and such that $\Pi_{\mathcal{L}^\perp}\Sigma_j\Pi_{\mathcal{L}^\perp} / \|\Pi_{\mathcal{L}^\perp}\Sigma_j\Pi_{\mathcal{L}^\perp}\|$ converges to $\bar{\Sigma}$ satisfying $\mathcal{S} = \mathrm{span}(\bar{\Sigma})$.*

Proof: 1. If $\mathcal{Z}$ is a concentration space, we can find a sequence $\Sigma_j \in \mathfrak{C}$ such that Σ_j converges to a singular matrix $\tilde{\Sigma}$ with $\mathcal{Z} = \mathrm{span}(\tilde{\Sigma})$. If $\Pi_{\mathcal{L}^\perp}\mathcal{Z} = \mathcal{L}^\perp$ or $\Pi_{\mathcal{L}^\perp}\mathcal{Z} = \{0\}$ holds, we are done, since neither $\mathcal{L}^\perp$ nor $\{0\}$ can belong to $\mathbb{J}(\mathcal{L}, \mathfrak{C})$.[19] Hence assume that $\{0\} \subsetneq \Pi_{\mathcal{L}^\perp}\mathcal{Z} \subsetneq \mathcal{L}^\perp$. Then it is easy to see that $\Pi_{\mathcal{L}^\perp}\tilde{\Sigma}\Pi_{\mathcal{L}^\perp} \neq 0$ must hold. But then $\Pi_{\mathcal{L}^\perp}\Sigma_j\Pi_{\mathcal{L}^\perp} / \|\Pi_{\mathcal{L}^\perp}\Sigma_j\Pi_{\mathcal{L}^\perp}\|$ converges to $\bar{\Sigma} := \Pi_{\mathcal{L}^\perp}\tilde{\Sigma}\Pi_{\mathcal{L}^\perp} / \left\|\Pi_{\mathcal{L}^\perp}\tilde{\Sigma}\Pi_{\mathcal{L}^\perp}\right\|$. Because of $\Pi_{\mathcal{L}^\perp}\mathcal{Z} \subsetneq \mathcal{L}^\perp$, it follows that $\mathrm{rank}(\bar{\Sigma}) < n - \dim(\mathcal{L})$ must hold, showing that $\mathrm{span}(\bar{\Sigma}) \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$. It remains to show that $\mathrm{span}(\bar{\Sigma}) = \Pi_{\mathcal{L}^\perp}\mathcal{Z}$, i.e., that $\mathrm{span}(\Pi_{\mathcal{L}^\perp}\tilde{\Sigma}\Pi_{\mathcal{L}^\perp}) = \Pi_{\mathcal{L}^\perp}\mathrm{span}(\tilde{\Sigma})$. But this follows since $\mathrm{span}(\Pi_{\mathcal{L}^\perp}\tilde{\Sigma}\Pi_{\mathcal{L}^\perp}) = \mathrm{span}(\Pi_{\mathcal{L}^\perp}\tilde{\Sigma}^{1/2}) = \Pi_{\mathcal{L}^\perp}\mathrm{span}(\tilde{\Sigma}^{1/2}) = \Pi_{\mathcal{L}^\perp}\mathrm{span}(\tilde{\Sigma})$.

2. Suppose $\mathcal{S}$ is as in Part 2 of the lemma. Since $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ there exists a sequence $\Sigma_j \in \mathfrak{C}$ such that $\Pi_{\mathcal{L}^\perp}\Sigma_j\Pi_{\mathcal{L}^\perp} / \|\Pi_{\mathcal{L}^\perp}\Sigma_j\Pi_{\mathcal{L}^\perp}\|$ converges to a singular matrix $\bar{\Sigma}$ with $\mathrm{rank}(\bar{\Sigma}) < n - \dim(\mathcal{L})$ and such that $\mathcal{S} = \mathrm{span}(\bar{\Sigma})$ holds. By the assumption on $\mathfrak{C}$, we can find a subsequence j_i along which Σ_{j_i} converges to a matrix Σ^*. Note that Σ^* must be singular, since otherwise $\Pi_{\mathcal{L}^\perp}\Sigma_{j_i}\Pi_{\mathcal{L}^\perp} / \|\Pi_{\mathcal{L}^\perp}\Sigma_{j_i}\Pi_{\mathcal{L}^\perp}\|$ would converge to the matrix $\Pi_{\mathcal{L}^\perp}\Sigma^*\Pi_{\mathcal{L}^\perp} / \|\Pi_{\mathcal{L}^\perp}\Sigma^*\Pi_{\mathcal{L}^\perp}\|$ which would have rank equal to $n - \dim(\mathcal{L})$, but at the same time would have to be equal to $\bar{\Sigma}$, which has smaller rank. Hence, $\mathcal{Z} := \mathrm{span}(\Sigma^*)$ is a concentration space of $\mathfrak{C}$. Consider first the case where $\|\Pi_{\mathcal{L}^\perp}\Sigma^*\Pi_{\mathcal{L}^\perp}\| \neq 0$. Then we can conclude that $\bar{\Sigma}$ and $\Pi_{\mathcal{L}^\perp}\Sigma^*\Pi_{\mathcal{L}^\perp}$ coincide up to a positive proportionality factor. By construction of $\bar{\Sigma}$ we have $\mathrm{span}(\bar{\Sigma}) = \mathcal{S}$. Hence, $\mathrm{span}(\Pi_{\mathcal{L}^\perp}\Sigma^*\Pi_{\mathcal{L}^\perp}) = \mathcal{S}$ holds. But the same argument as in the proof of Part 1 shows that $\mathrm{span}(\Pi_{\mathcal{L}^\perp}\Sigma^*\Pi_{\mathcal{L}^\perp}) = \Pi_{\mathcal{L}^\perp}\mathrm{span}(\Sigma^*)$, which leads to $\Pi_{\mathcal{L}^\perp}\mathcal{Z} = \Pi_{\mathcal{L}^\perp}\mathrm{span}(\Sigma^*) = \mathcal{S}$. Next consider the case where $\|\Pi_{\mathcal{L}^\perp}\Sigma^*\Pi_{\mathcal{L}^\perp}\| = 0$.[20] In this case we can clearly conclude that $\mathcal{Z} = \mathrm{span}(\Sigma^*)$ is

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[19]If $\mathcal{L} = \{0\}$, (i) the case $\Pi_{\mathcal{L}^\perp}\mathcal{Z} = \mathcal{L}^\perp$ is impossible, and (ii) the case $\Pi_{\mathcal{L}^\perp}\mathcal{Z} = \{0\}$ cannot arise if $\mathfrak{C}$ is bounded away from the zero matrix.

[20]Note that this case cannot arise in case $\mathcal{L} = \{0\}$ and $\mathfrak{C}$ is bounded away from the zero matrix.

a concentration space satisfying $\Pi_{\mathcal{L}^{\perp}}\mathcal{Z} = \{0\}$. The remaining claims follow for the sequence Σ_{j_i} just constructed. $\blacksquare$

The assumption that $\mathfrak{C}$ is bounded used in Part 2 of the lemma can be made without much loss of generality if the tests one is interested in are invariant under $G(\mathfrak{M}_0)$, see Remark 5.1(ii). The preceding lemma allows one in certain circumstances to reduce checking the conditions on $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ postulated in Proposition 5.2 or in Proposition 5.5 (or the corresponding sufficient conditions appearing in Remark 5.4(ii) or Corollary 5.6) to checking similar conditions that are expressed in terms of the concentration spaces $\mathcal{Z}$. This can be advantageous since the concentration spaces are sometimes easier to obtain than the spaces $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$. We illustrate this in the following remark. However, it is important to note that for many covariance models of interest this "reduction" trick does *not* work and $\mathbb{J}(\mathcal{L}, \mathfrak{C})$ has to be determined. For example, this is the case for $\mathfrak{C}(\mathfrak{F}_{\text{all}})$ and related covariance models, necessitating the developments in Section 6.1.

Remark B.4. Let $\mathfrak{C}$ be a covariance model that is bounded and is bounded away from the zero matrix. Assume that T, $N^{\dagger}$, and W are as in Corollary 5.6 and assume furthermore that $\mathcal{L} \subseteq \text{span}(X)$ (which is, in particular, the case if $\mathcal{V} = \{0\}$).

(i) Assume that $N^{\dagger}$ is a finite or countable union of affine subspaces. Suppose furthermore that we are in a scenario where we can show that no concentration space $\mathcal{Z}$ is entirely contained in $\text{span}(X)$, and thus no $\mathcal{Z}$ is entirely contained in $\mathcal{L}$. Then the sufficient condition given at the end of Corollary 5.6, namely that if $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ then $\mu_0 + \mathcal{S} \not\subseteq N^{\dagger}$ for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$) must hold, is satisfied whenever $\mu_0 + \mathcal{Z} \not\subseteq N^{\dagger}$ for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$) holds for every concentration space $\mathcal{Z}$ (this is so since $N^{\dagger}$ is invariant under addition of elements from $\mathcal{L}$ and since every $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ must be of the form $\Pi_{\mathcal{L}^{\perp}}\mathcal{Z}$ in view of the preceding lemma and our assumption on $\mathcal{Z}$). Hence, we can check the sufficient condition in Corollary 5.6 without explicitly computing the spaces $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$. Furthermore, in many cases of interest we have $N^{\dagger} = \text{span}(X)$ (cf. the discussion surrounding Lemma 5.18), in which case the condition $\mu_0 + \mathcal{Z} \not\subseteq N^{\dagger}$ is then an automatic consequence of the assumption that no concentration space $\mathcal{Z}$ is entirely contained in $\text{span}(X)$ (since $\text{span}(X)$ is a linear space containing $\mathfrak{M}_0$).

(ii) The example in (i) can be generalized a bit: Suppose now that for every concentration space $\mathcal{Z}$ we either have that (a) it is not contained in $\text{span}(X)$, or (b) that $\mathcal{Z}$ is contained in $\mathcal{L}$ but for every sequence Σ_j converging to some Σ^* satisfying $\mathcal{Z} = \text{span}(\Sigma^*)$ the limit points of $\Pi_{\mathcal{L}^{\perp}}\Sigma_j\Pi_{\mathcal{L}^{\perp}} / \|\Pi_{\mathcal{L}^{\perp}}\Sigma_j\Pi_{\mathcal{L}^{\perp}}\|$ are regular on $\mathcal{L}^{\perp}$. Then it suffices to check that $\mu_0 + \mathcal{Z} \not\subseteq N^{\dagger}$ for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$) for every concentration space $\mathcal{Z}$ that is not contained in $\text{span}(X)$ [this follows from the discussion in (i), since by property (b) and the preceding lemma any $\mathcal{S} \in \mathbb{J}(\mathcal{L}, \mathfrak{C})$ must be of the form $\Pi_{\mathcal{L}^{\perp}}\mathcal{Z}$]. Again if $N^{\dagger} = \text{span}(X)$, then this latter condition is automatically satisfied. Of course, a sufficient condition for the aforementioned limit points to be regular on $\mathcal{L}^{\perp}$ is the following condition: for each relevant $\mathcal{Z}$, Σ, and sequence Σ_j the limit points of the sequence $\Pi_{\mathcal{L}^{\perp}}\Sigma_j\Pi_{\mathcal{L}^{\perp}} / \|\Pi_{\mathcal{L}^{\perp}}\Sigma_j\Pi_{\mathcal{L}^{\perp}}\|$ are regular on $\text{span}(\Sigma^*)^{\perp}$.

(iii) Suppose now that we are in the situation of (ii) except that now $N^\dagger$ is not a finite or countable union of affine subspaces. Then the relevant sufficient condition in Corollary 5.6 can be shown to be implied by the condition that if $\mathcal{Z}$ is a concentration space of the covariance model $\mathfrak{C}$ that is not contained in span(X), then the set $N^\dagger$ is a $\lambda_{\mu_0+\mathcal{Z}}$-null set for some $\mu_0 \in \mathfrak{M}_0$ (and hence for all $\mu_0 \in \mathfrak{M}_0$). This follows since any $\mathcal{Z}$ with $\Pi_{\mathcal{L}^\perp}\mathcal{Z} = \mathcal{S}$ can be shown to be of the form of a direct sum $\mathcal{A} \oplus \mathcal{B}$, where $\mathcal{A}$ is a linear space that is linearly isomorphic to $\mathcal{S}$ and $\mathcal{B}$ is a linear subspace of $\mathcal{L}$, and since $N^\dagger - \mu_0$ is of the form $N^{\dagger\dagger} \oplus \mathcal{L}$ for an appropriate Borel-set $N^{\dagger\dagger} \subseteq \mathcal{L}^\perp$ and where the direct sum is in fact an orthogonal sum. We omit the details. Similar arguments can be applied if condition (b) in Proposition 5.5 is to be verified instead of the sufficient condition just considered.

(iv) Similar arguments apply to the sufficient condition given in Remark 5.4(ii) or to condition (b) of Proposition 5.2.

C   Appendix: Auxiliary results for Section 6

In this appendix we provide results that will be used in the proofs of the results of Section 6 that are provided in Appendix D.

Definition C.1. Let $\omega \in [0,\pi]$, $l \in \mathbb{Z}$, $m \in \mathbb{N}$, and let $s \geq 0$ be an integer. Define $E_{m,s}^{(l)}(\omega)$ as the $m \times 2$-dimensional matrix with j-th row equal to

$$((j+l)^s \cos((j+l)\omega), (j+l)^s \sin((j+l)\omega)),$$

where we shall often drop the superscript l in case $l = 0$. For a positive integer p, for $\underline{\omega} = (\omega_1, \ldots, \omega_p) \in [0,\pi]^p$, and for $\underline{d} = (d_1, \ldots, d_p) \in \mathbb{N}^p$ we define the $m \times 2\sum_{i=1}^p d_i$-dimensional matrix

$$V_m^{(l)}(\underline{\omega}, \underline{d}) = \left(E_{m,0}^{(l)}(\omega_1), \ldots, E_{m,d_1-1}^{(l)}(\omega_1), \ldots, E_{m,0}^{(l)}(\omega_p), \ldots, E_{m,d_p-1}^{(l)}(\omega_p) \right).$$

In case $p = 1$, we shall often simply write ω for $\underline{\omega}$ and d for $\underline{d}$.

For the following recall that $\kappa(\underline{\omega}, \underline{d})$ has been defined in Section 6.1, and that we use the convention that $\underline{\omega}$ and $\underline{d}$ are the 0-tupels for $p = 0$.

Lemma C.1. *Let p be a positive integer, let $\underline{\omega} \in [0,\pi]^p$ have distinct coordinates, and let $\underline{d} \in \mathbb{N}^p$. Then for every positive integer m and for every integer l it holds that*

$$\mathrm{rank}(V_m^{(l)}(\underline{\omega}, \underline{d})) = \min(m, \kappa(\underline{\omega}, \underline{d})).$$

Proof of Lemma C.1: Standard results concerning linear difference equations (e.g., Kelley and Peterson (2001), Chp. 3) can be used to verify that the collection made up of the functions $j \mapsto j^s \cos(j\omega_i)$ and $j \mapsto j^s \sin(j\omega_i)$ (defined for $j \in \mathbb{Z}$) for $\omega_i \in (0,\pi)$ and $s = 0, \ldots, d_i - 1$

as well as of $j \mapsto j^s \cos(j\omega_i)$ for $\omega_i \in \{0, \pi\}$ and $s = 0, \dots, d_i - 1$ forms a fundamental set of solutions of the difference equation $\Delta_{\underline{\omega},\underline{d}}(z)w_t = 0$ with $t \in \mathbb{Z}$, where we abuse the symbol z also to denote the backshift-operator. Hence any Casorati-matrix associated with this fundamental set is non-singular. Observe that striking the columns corresponding to $(j+l)^s \sin((j+l)\omega_i)$ for $\omega_i \in \{0, \pi\}$ from the matrix $V^{(l)}_{\kappa(\underline{\omega},\underline{d})}(\underline{\omega}, \underline{d})$, results in a $\kappa(\underline{\omega}, \underline{d}) \times \kappa(\underline{\omega}, \underline{d})$ matrix which is precisely a Casorati-matrix. This shows that $\text{rank}(V^{(l)}_{\kappa(\underline{\omega},\underline{d})}(\underline{\omega}, \underline{d})) = \kappa(\underline{\omega}, \underline{d})$. The claim is then an immediate consequence. ∎

Definition C.2. For a polynomial $\Theta(z) = 1 + \theta_1 z + \dots + \theta_a z^a$ of degree a we define for $m > a$ the $(m - a) \times m$ matrix $D_m(\Theta)$ via

$$\begin{pmatrix} \theta_a & \dots & \theta_1 & 1 & 0 & \dots & 0 \\ 0 & \theta_a & \dots & \theta_1 & 1 & \dots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \dots & 0 & \theta_a & \dots & \theta_1 & 1 \end{pmatrix}$$

with the convention that $D_m(\Theta) = I_m$ in case $a = 0$.

Remark C.2. (i) Obviously, $D_m(\Theta)$ has full row-rank, i.e., $\text{rank}(D_m(\Theta)) = m - a$.

(ii) Let $\Theta_1(z)$ and $\Theta_2(z)$ be polynomials of degree a_1 and a_2, respectively, satisfying $\Theta_1(0) = \Theta_2(0) = 1$. Then for $m > a_1 + a_2$ we have $D_m(\Theta_1\Theta_2) = D_{m-a_1}(\Theta_2)D_m(\Theta_1) = D_{m-a_2}(\Theta_1)D_m(\Theta_2)$ where $D_m(\Theta_1\Theta_2)$ denotes the matrix associated with the polynomial $\Theta_1(z)\Theta_2(z)$.

Lemma C.3. *Let p be a positive integer, let $\underline{\omega} \in [0, \pi]^p$ have distinct coordinates, and let $\underline{d} \in \mathbb{N}^p$. Then for every positive integer m satisfying $m > \kappa(\underline{\omega}, \underline{d})$, and for every integer l we have that the transposes of the row vectors of $D_m(\Delta_{\underline{\omega},\underline{d}})$ constitute a basis of $\text{span}(V^{(l)}_m(\underline{\omega}, \underline{d}))^\perp$. In particular, $\text{span}(V^{(l)}_m(\underline{\omega}, \underline{d}))$ does not depend on l.*

Proof of Lemma C.3: Since the columns of $V^{(l)}_m(\underline{\omega}, \underline{d})$ are either zero or segments of length m (with $m > \kappa(\underline{\omega}, \underline{d})$) from the fundamental set of solutions to the difference equation $\Delta_{\underline{\omega},\underline{d}}(z)w_t = 0$, we obviously have $D_m(\Delta_{\underline{\omega},\underline{d}})V^{(l)}_m(\underline{\omega}, \underline{d}) = 0$. This implies $\text{span}(D'_m(\Delta_{\underline{\omega},\underline{d}})) \subseteq \text{span}(V^{(l)}_m(\underline{\omega}, \underline{d}))^\perp$. Since the $m - \kappa(\underline{\omega}, \underline{d})$ rows of $D_m(\Delta_{\underline{\omega},\underline{d}})$ are linearly independent, cf. Remark C.2, and since $\text{rank}(V^{(l)}_m(\underline{\omega}, \underline{d})) = \kappa(\underline{\omega}, \underline{d})$ by Lemma C.1, the result follows. ∎

Remark C.4. In case $p = 1$, and hence $\underline{\omega} = \omega \in [0, \pi]$, and $\underline{d} = d \in \mathbb{N}$, the result in Lemma C.3 reduces to the fact that the transposes of the row vectors of $D_m(\Delta_\omega^d)$ constitute a basis of $\text{span}((E^{(l)}_{m,0}(\omega), \dots, E^{(l)}_{m,d-1}(\omega)))^\perp$. In particular, $D_m(\Delta_\omega^d)(E^{(l)}_{m,0}(\omega), \dots, E^{(l)}_{m,d-1}(\omega)) = 0$ holds.

Lemma C.5. *Let $\omega \in [0, \pi]$, $d \in \mathbb{N}$, and $l \in \mathbb{Z}$. If $m \in \mathbb{N}$ satisfies $m > \kappa(\omega, 1)$ then we have*

$$D_m(\Delta_\omega)E^{(l)}_{m,d}(\omega) = \begin{cases} 2\sum_{i=0}^{d-1} \binom{d}{i} \cos^{(d-i)}(\omega)E^{(l+1)}_{m-\kappa(\omega,1),i}(\omega)P^{i-d} & \text{for } \omega \in (0, \pi) \\ \cos(\omega)\sum_{i=0}^{d-1} \binom{d}{i} E^{(l)}_{m-\kappa(\omega,1),i}(\omega) & \text{for } \omega \in \{0, \pi\} \end{cases}, \quad (47)$$

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where $\cos^{(i)}(\omega)$ denotes the i-th order derivative of the cosine function and where P is the 2×2-dimensional orthogonal matrix with first row $(0,1)$ and second row $(-1,0)$. If $m \in \mathbb{N}$ satisfies $m > \kappa(\omega, d)$ then we have

$$D_m(\Delta_\omega^d)E_{m,d}^{(l)}(\omega) = \begin{cases} 2^d d! (-\sin(\omega))^d E_{m-\kappa(\omega,d),0}^{(l+d)}(\omega)P^{-d} & for \ \omega \in (0,\pi) \\ d! (\cos(\omega))^d E_{m-\kappa(\omega,d),0}^{(l)}(\omega) & for \ \omega \in \{0,\pi\} \end{cases}. \tag{48}$$

Proof of Lemma C.5: From Lemma C.3 and Remark C.4 we obtain that the identity $D_m(\Delta_\omega)E_{m,0}^{(l)}(\omega) = 0$ for every $\omega \in [0,\pi]$ holds. Consider first the case where $\omega \in (0,\pi)$: Since the left-hand side of this identity is a smooth function of ω in this range, we may differentiate the identity d times leading to

$$\sum_{i=0}^{d} \binom{d}{i} \left(\frac{d^{d-i}}{d\omega^{d-i}} D_m(\Delta_\omega) \right) \left(\frac{d^i}{d\omega^i} E_{m,0}^{(l)}(\omega) \right) = 0.$$

Rearranging terms and computing the derivatives gives

$$D_m(\Delta_\omega)E_{m,d}^{(l)}(\omega)P^d = 2 \sum_{i=0}^{d-1} \binom{d}{i} \cos^{(d-i)}(\omega)[0, I_{m-\kappa(\omega,1)}, 0]E_{m,i}^{(l)}(\omega)P^i.$$

But clearly $[0, I_{m-\kappa(\omega,1)}, 0]E_{m,i}^{(l)}(\omega) = E_{m-\kappa(\omega,1),i}^{(l+1)}(\omega)$ holds, from which we obtain (47) in case $\omega \in (0,\pi)$. Next, consider the case where $\omega \in \{0,\pi\}$ holds: Using the Binomial formula, together with $\omega \in \{0,\pi\}$, we have

$$(r+1)^d \cos((r+1)\omega) - \cos(\omega)r^d \cos(r\omega) = \cos(\omega) \sum_{i=0}^{d-1} \binom{d}{i} r^i \cos(r\omega).$$

Since the second column of $E_{m,i}^{(l)}(\omega)$ is 0 for any i and m, the claim (47) then also follows in case $\omega \in \{0,\pi\}$. We next prove (48) by induction over d. In case $d = 1$, the claim holds as it reduces to (47). Suppose that the induction hypothesis now holds for some $d \geq 1$ (and any $m > \kappa(\omega,d)$). Then for any $m \in \mathbb{N}$ satisfying $m > \kappa(\omega, d+1)$ we have, using Remark C.2(ii) and (47) with d replaced by $d+1$,

$$D_m(\Delta_\omega^{d+1})E_{m,d+1}^{(l)}(\omega) = D_{m-\kappa(\omega,1)}(\Delta_\omega^d)D_m(\Delta_\omega)E_{m,d+1}^{(l)}(\omega)$$
$$= \begin{cases} 2\sum_{i=0}^{d} \binom{d+1}{i} \cos^{(d+1-i)}(\omega)D_{m-\kappa(\omega,1)}(\Delta_\omega^d)E_{m-\kappa(\omega,1),i}^{(l+1)}(\omega)P^{i-d-1} & for \ \omega \in (0,\pi) \\ \cos(\omega)\sum_{i=0}^{d} \binom{d+1}{i}D_{m-\kappa(\omega,1)}(\Delta_\omega^d)E_{m-\kappa(\omega,1),i}^{(l)}(\omega) & for \ \omega \in \{0,\pi\} \end{cases}.$$

Observe that $m - \kappa(\omega,1) > \kappa(\omega,d)$ holds, since $m > \kappa(\omega,d+1)$ and $\kappa(\omega,d+1) = \kappa(\omega,1) + \kappa(\omega,d)$. Hence we may apply Lemma C.3 and Remark C.4 to obtain $D_{m-\kappa(\omega,1)}(\Delta_\omega^d)E_{m-\kappa(\omega,1),i}^{(l')}(\omega) = 0$

for $i < d$ and for $l' = l$ or $l' = l + 1$. We thus obtain from the preceding display that

$$D_m(\Delta_\omega^{d+1})E_{m,d+1}^{(l)}(\omega) = \begin{cases} 2(d+1)\cos^{(1)}(\omega)D_{m-\kappa(\omega,1)}(\Delta_\omega^d)E_{m-\kappa(\omega,1),d}^{(l+1)}(\omega)P^{-1} & \text{for } \omega \in (0,\pi) \\ (d+1)\cos(\omega)D_{m-\kappa(\omega,1)}(\Delta_\omega^d)E_{m-\kappa(\omega,1),d}^{(l)}(\omega) & \text{for } \omega \in \{0,\pi\} \end{cases}.$$

Together with the induction hypothesis (applied with m replaced by $m - \kappa(\omega,1)$) this establishes (48) for $d + 1$. ∎

Lemma C.6. *Let $(\omega_1, \omega_2) \in [0,\pi]^2$, $l \in \mathbb{Z}$, $m \in \mathbb{N}$, and $d \in \mathbb{N}$. Assume $m > \kappa(\omega_1, d)$ holds. Then*

$$D_m(\Delta_{\omega_1}^d)E_{m,0}^{(l)}(\omega_2) = E_{m-\kappa(\omega_1,d),0}^{(l)}(\omega_2)(A(\omega_1,\omega_2))^d,$$

where $A(\omega_1,\omega_2) = 2(\cos(\omega_2) - \cos(\omega_1))P(\omega_2)$ when $\omega_1 \in (0,\pi)$, and $A(\omega_1,\omega_2) = P(\omega_2) - \cos(\omega_1)I_2$ when $\omega_1 \in \{0,\pi\}$. The matrices $A(\omega_1,\omega_2)$ are multiples of orthogonal matrices; they are nonsingular if $\omega_1 \neq \omega_2$ and equal the zero matrix otherwise. [Here $P(\omega)$ denotes the 2×2-dimensional orthogonal matrix with first row $(\cos(\omega), \sin(\omega))$ and second row $(-\sin(\omega), \cos(\omega))$.]

Proof of Lemma C.6: Consider first the case where $d = 1$. We start with the following standard trigonometric identities for $j \in \mathbb{Z}$, which easily follow from the angle addition formulas,

$$\cos((j+2)\omega_2) - 2\cos(\omega_2)\cos((j+1)\omega_2) + \cos(j\omega_2) = 0 \tag{49}$$

$$\sin((j+2)\omega_2) - 2\cos(\omega_2)\sin((j+1)\omega_2) + \sin(j\omega_2) = 0. \tag{50}$$

Consider first the case where $\omega_1 \in (0,\pi)$ holds. From (49) and (50) it follows that

$$\cos((j+2)\omega_2) - 2\cos(\omega_1)\cos((j+1)\omega_2) + \cos(j\omega_2) = 2\cos((j+1)\omega_2)(\cos(\omega_2) - \cos(\omega_1)) \tag{51}$$

$$\sin((j+2)\omega_2) - 2\cos(\omega_1)\sin((j+1)\omega_2) + \sin(j\omega_2) = 2\sin((j+1)\omega_2)(\cos(\omega_2) - \cos(\omega_1)), \tag{52}$$

and thus (by the angle addition formulas)

$$D_m(\Delta_{\omega_1})E_{m,0}^{(l)}(\omega_2) = 2(\cos(\omega_2) - \cos(\omega_1))E_{m-\kappa(\omega_1,1),0}^{(l)}(\omega_2)P(\omega_2) = E_{m-\kappa(\omega_1,1),0}^{(l)}(\omega_2)A(\omega_1,\omega_2).$$

Next consider the case where $\omega_1 \in \{0,\pi\}$. It is then easy to see, using the angle addition formulas, that

$$D_m(\Delta_{\omega_1})E_{m,0}^{(l)}(\omega_2) = E_{m-\kappa(\omega_1,1),0}^{(l)}(\omega_2)(P(\omega_2) - \cos(\omega_1)I_2) = E_{m-\kappa(\omega_1,1),0}^{(l)}(\omega_2)A(\omega_1,\omega_2).$$

The case $d > 1$ now follows from (cf. Remark C.2)

$$D_m(\Delta_{\omega_1}^d) = D_{m-\kappa(\omega_1,d-1)}(\Delta_{\omega_1})\cdots D_{m-\kappa(\omega_1,2)}(\Delta_{\omega_1})D_{m-\kappa(\omega_1,1)}(\Delta_{\omega_1})D_m(\Delta_{\omega_1}),$$

together with a repeated application of the already established result for $d = 1$. ∎

Recall that the finite and symmetric Borel measures on $[-\pi, \pi]$ are precisely the spectral measures of real weakly stationary processes.

Definition C.3. For m a finite and symmetric Borel measure on $[-\pi, \pi]$ (symmetry here meaning $\mathsf{m}(A) = \mathsf{m}(-A)$ for every Borel subset A of $[-\pi, \pi]$) and for $m \in \mathbb{N}$ we define the $m \times m$ matrix

$$\Sigma(\mathsf{m}, m) = \left[\int_{-\pi}^{\pi} e^{-\iota\nu(j-j')} d\mathsf{m}(\nu) \right]_{j,j'=1}^{m}.$$

For a spectral density f, i.e., an even $\lambda_{[-\pi,\pi]}$-integrable function from $[-\pi, \pi]$ to $[0, \infty)$, we denote by m_f the (finite and symmetric) Borel measure on $[-\pi, \pi]$ with density f (w.r.t. Lebesgue measure $\lambda_{[-\pi,\pi]}$ on $[-\pi, \pi]$) and we abbreviate $\Sigma(\mathsf{m}_f, m)$ to $\Sigma(f, m)$. Finally, for m a finite and symmetric Borel measure on $[-\pi, \pi]$ and Θ a polynomial, we denote by $\Theta \odot \mathsf{m}$ the (finite and symmetric) Borel measure given by $(\Theta \odot \mathsf{m})(A) = \int_A |\Theta(e^{\iota\nu})|^2 d\mathsf{m}(\nu)$ for Borel sets $A \subseteq [-\pi, \pi]$.

Given a nonempty set of spectral densities $\mathfrak{F}$ and $m \in \mathbb{N}$, we shall write $\mathfrak{C}(\mathfrak{F}, m) = \{\Sigma(f, m) : f \in \mathfrak{F}\}$. In case $m = n$, where n is the sample size, we shall – in line with the notation introduced in Section 3 – often simply write $\Sigma(f)$ for $\Sigma(f, n)$ and $\mathfrak{C}(\mathfrak{F})$ for $\mathfrak{C}(\mathfrak{F}, n)$. The next lemma is an immediate consequence of a standard result concerning linear filters (e.g., Rozanov (1967), Chp I.8) applied to the linear filter $\Delta_{\underline{\omega},\underline{d}}(z)$ operating on a stationary process with spectral measure m.

Lemma C.7. *Let p be a positive integer, let $\underline{\omega} \in [0, \pi]^p$, let $\underline{d} \in \mathbb{N}^p$, and let $m \in \mathbb{N}$ satisfy $m > \kappa(\omega, d)$. Let m be a finite and symmetric Borel measure on $[-\pi, \pi]$. Then*

$$D_m(\Delta_{\underline{\omega},\underline{d}})\Sigma(\mathsf{m}, m)D'_m(\Delta_{\underline{\omega},\underline{d}}) = \Sigma(\Delta_{\underline{\omega},\underline{d}} \odot \mathsf{m}, m - \kappa(\underline{\omega}, \underline{d})).$$

The following result is well-known, but difficult to pinpoint in the literature in this form. The first claim of the lemma can, e.g., be found in Theorem 2.6 of Kreǐn and Nudel′man (1977). This is closely related to a theorem of Carathéodory (Section 4.1 of Grenander and Szegö (1958)), on which we have chosen to base the proof of the lemma.

Lemma C.8. *Let Φ be a real nonnegative definite symmetric Toeplitz matrix of dimension $m \times m$ with $m \in \mathbb{N}$. Then there exists a finite and symmetric Borel measure m on $[-\pi, \pi]$ such that $\Phi = \Sigma(\mathsf{m}, m)$. If Φ is singular, the measure m is unique. Furthermore, if Φ is singular and $\Phi \neq 0$, m is of the form $\sum_{i=1}^{p} c_i(\delta_{-\omega_i} + \delta_{\omega_i})$ for some $p \in \mathbb{N}$, for some positive c_i, $1 \leq i \leq p$, and for some $\underline{\omega} = (\omega_1, \ldots, \omega_p) \in [0, \pi]^p$ with $\omega_1 < \ldots < \omega_p$ such that $1 \leq \sum_{i=1}^{p} \kappa(\omega_i, 1) < m$; if $\Phi = 0$, the measure m is the zero measure.*

Proof of Lemma C.8: We start with a preparatory remark: Recall that for any finite and symmetric Borel measure m the matrix $\Sigma(\mathsf{m}, m)$ is nonnegative definite, since $x'\Sigma(\mathsf{m}, m)x = \int_{-\pi}^{\pi} |x(e^{\iota\nu})|^2 d\mathsf{m}(\nu) \geq 0$ for every $x \in \mathbb{R}^m$, where $x(e^{\iota\nu}) = \sum_{j=1}^{m} x_j e^{\iota j\nu}$. If $\Sigma(\mathsf{m}, m)$ is singular

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for some m, then $0 = x_*' \Sigma(\mathsf{m}, m) x_* = \int_{-\pi}^{\pi} |x_*(e^{\iota\nu})|^2 \, d\mathsf{m}(\nu)$ for some nonzero $x_* \in \mathbb{R}^m$ must hold. Consequently, the support of any such measure m must be contained in the zero-set of the trigonometric polynomial $|x_*(e^{\iota\nu})|^2$ (which has degree at most $m-1$). Hence, m must be the zero measure or must be of the form $\sum_{i=1}^{p} c_i (\delta_{-\omega_i} + \delta_{\omega_i})$ for some $p \in \mathbb{N}$, for positive c_i's, $1 \le i \le p$, and for some $\underline{\omega} = (\omega_1, \dots, \omega_p) \in [0, \pi]^p$ with $\omega_1 < \dots < \omega_p$; obviously, $\sum_{i=1}^{p} \kappa(\omega_i, 1) < m$ must also hold.

Now, if $\Phi = 0$, the zero measure satisfies $\Phi = \Sigma(\mathsf{m}, m)$ and obviously this is the only possible choice. Next consider the case where Φ is singular, but $\Phi \ne 0$. Then $m \ge 2$ must hold. In the following let $\Phi(|i - j|)$ denote the (i, j)-th element of Φ. Consider first the case $m = 2$: Then $|\Phi(1)| = \Phi(0)$ has to hold. Consequently, $\Phi = \Sigma(\mathsf{m}, m)$ holds for $\mathsf{m} = (\Phi(0)/2)(\delta_{-\omega_1} + \delta_{\omega_1})$ where $\omega_1 = 0$ or π, depending on whether $\Phi(1)$ is positive or negative (here $p = 1$ and $\kappa(\omega_1, 1) = 1 < m = 2$ is satisfied). If m' is another finite and symmetric Borel measure satisfying $\Phi = \Sigma(\mathsf{m}', m)$, then the preparatory remark shows that m' must be a discrete measure of the form as in the preparatory remark with $\sum_{i=1}^{p} \kappa(\omega_i, 1) < m = 2$. But this shows that $p = 1$ and that $\omega_1 = 0$ or π. Uniqueness then follows immediately. We next turn to the case $m > 2$. Observe that not all $\Phi(r)$ for $r = 1, \dots, m - 1$ can be zero, since $\Phi \ne 0$, Φ is singular, and is Toeplitz. It now follows from a theorem of Carathéodory (Section 4.1 of Grenander and Szegö (1958)) that

$$\Phi(r) = \int_{-\pi}^{\pi} e^{-\iota\nu r} d\mathsf{m}(\nu) \quad \text{for every } r = 1, \dots, m - 1 \tag{53}$$

holds for some measure m of the form $\sum_{i=1}^{p} c_i (\delta_{-\omega_i} + \delta_{\omega_i})$ with $p \ge 1$, $c_i > 0$, $\underline{\omega} = (\omega_1, \dots, \omega_p) \in [0, \pi]^p$ with $\omega_1 < \dots < \omega_p$, and with $\sum_{i=1}^{p} \kappa(\omega_i, 1) < m$; it furthermore follows from that theorem that p and the constants c_i, ω_i are uniquely determined. [The theorem in Section 4.1 of Grenander and Szegö (1958) is given for complex $\Phi(r)$ and shows that (53) holds for a Borel measure of the form $\sum_{j=1}^{q} a_j \delta_{\varpi_j}$, where $1 \le q < m$, $a_j > 0$, and where ϖ_j are distinct elements in the half-open interval $(-\pi, \pi]$. Furthermore, q and the constants a_j, ϖ_j are uniquely determined. Exploiting that $\Phi(r) = \overline{\Phi(r)}$ in our context, the just mentioned uniqueness immediately shows that those ϖ_j's, which are different from 0 or π, must appear in pairs symmetrically located around zero, and that $a_j = a_{j'}$ must hold if $\varpi_j = -\varpi_{j'}$. It is now not difficult to see that one can replace the Borel measure $\sum_{j=1}^{q} a_j \delta_{\varpi_j}$ by an appropriate m of the form as given above (note that $e^{-\iota\nu r} = e^{\iota\nu r}$ for $\nu = 0, \pi$) and that uniqueness of q, the a_j's and the ϖ_j's translates into uniqueness of p, the c_i's, and $\underline{\omega} \in [0, \pi]^p$. Also note that $q < m$ translates into $\sum_{i=1}^{p} \kappa(\omega_i, 1) < m$.] It thus suffices to show that $\Phi(0) = \int_{-\pi}^{\pi} d\mathsf{m}(\nu)$, uniqueness of m then already following from the preparatory remark, together with the uniqueness part of the cited theorem. Now, because of $\sum_{i=1}^{p} \kappa(\omega_i, 1) < m$, we can find a vector $x_0 \in \mathbb{R}^m$ such that $x_0(e^{\iota\nu})$ vanishes at $\nu = \omega_i$ and at $\nu = -\omega_i$ for every coordinate ω_i of $\underline{\omega}$. [Just set $x_0(e^{\iota\nu})$ equal to $\prod_{i:\omega_i \in (0, \pi)} (1 - e^{-\iota\omega_i} e^{\iota\nu})(1 - e^{\iota\omega_i} e^{\iota\nu}) \prod_{i:\omega_i \in \{0, \pi\}} (1 - e^{-\iota\omega_i} e^{\iota\nu})$ and observe that the coefficients are real.] This implies $x_0' \Sigma(\mathsf{m}, m) x_0 = 0$, and hence

that $\Sigma(\mathsf{m}, m)$ is singular. Obviously, $\Sigma(\mathsf{m}, m) = \Phi + (\int_{-\pi}^{\pi} d\mathsf{m}(\nu) - \Phi(0))I_m$ holds. We thus obtain

$$0 = \inf_{x'x=1} x'\Sigma(\mathsf{m}, m)x = \inf_{x'x=1} x'\Phi x + \int_{-\pi}^{\pi} d\mathsf{m}(\nu) - \Phi(0) = \int_{-\pi}^{\pi} d\mathsf{m}(\nu) - \Phi(0),$$

since $\inf_{x'x=1} x'\Phi x = 0$ in view of singularity of Φ. This completes the proof for singular Φ. Finally, if Φ is positive definite, write Φ as $\Phi_* + cI_m$ where $c > 0$ and Φ_* is nonnegative definite and singular. Obviously, Φ_* is symmetric and Toeplitz. Hence $\Phi_* = \Sigma(\mathsf{m}_*, m)$ for some measure m_* satisfying the conditions in the theorem. Setting $\mathsf{m} = \mathsf{m}_* + (c/2\pi)\lambda_{[-\pi,\pi]}$ completes the proof. ∎

Remark C.9. (i) While the measure m in Lemma C.8 is unique in case Φ is singular, it is never unique if Φ is positive definite. As shown in the proof, if Φ is positive definite, one such measure is given by the sum of a discrete measure and the spectral measure of white noise. As is well-known (see, e.g., Section 3.9.2 of Stoica and Moses (2005)), for positive definite Φ, another representation $\Phi = \Sigma(\mathsf{m}, m)$ can be found where m is the spectral measure of an appropriate stationary autoregressive process of order at most $m - 1$ (and thus is absolutely continuous w.r.t. Lebesgue measure).

(ii) An alternative proof of Lemma C.8 can be based on the just mentioned autoregressive representation result by applying a limiting argument to cover also the case of singular Φ.

(iii) As a converse to the last claim in Lemma C.8 we have: If $\mathsf{m} = 0$ or if m is of the form $\sum_{i=1}^{p} c_i(\delta_{-\omega_i} + \delta_{\omega_i})$ for some $p \in \mathbb{N}$, for some positive c_i, $1 \le i \le p$, and for some $\underline{\omega} = (\omega_1, \ldots, \omega_p) \in [0, \pi]^p$ with $\omega_1 < \ldots < \omega_p$ such that $1 \le \sum_{i=1}^{p} \kappa(\omega_i, 1) < m$ holds, then $\Sigma(\mathsf{m}, m)$ is singular. This follows since $x_0'\Sigma(\mathsf{m}, m)x_0 = 0$ for x_0 as defined in the proof.

(iv) The lemma is somewhat more general than what is needed later in the paper, but its extra generality is useful in other contexts.

D   Appendix: Proofs for Sections 6.1 and 6.2

Recall that by our conventions $\kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) = 0$ if $p(\mathcal{L}) = 0$.

Lemma D.1. *Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$. Then $\kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \le \dim(\mathcal{L})$ holds.*

Proof of Lemma D.1: If $p(\mathcal{L}) = 0$, there is nothing to prove in view of the above convention. Suppose now that $p(\mathcal{L}) > 0$. By construction, $\mathrm{span}(V_n^{(0)}(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))) \subseteq \mathcal{L}$. From Lemma C.1 we have $\mathrm{rank}(V_n^{(0)}(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))) = \min(n, \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})))$. Consequently, $\min(n, \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))) \le \dim(\mathcal{L}) < n$ must hold, which obviously proves the desired result. ∎

Definition D.1. For a positive integer p, $\underline{\omega} \in [0, \pi]^p$, $\underline{d} \in \mathbb{N}^p$, and for n satisfying $n > \kappa(\underline{\omega}, \underline{d})$

we define the $n \times (n - \kappa(\underline{\omega}, \underline{d}))$-dimensional matrix

$$H_n(\underline{\omega}, \underline{d}) := D'_n(\Delta_{\underline{\omega}, \underline{d}}) \left(D_n(\Delta_{\underline{\omega}, \underline{d}}) D'_n(\Delta_{\underline{\omega}, \underline{d}}) \right)^{-1}.$$

Observe that the inverse in the preceding display exists in view of Remark C.2.

Lemma D.2. *Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$. Furthermore, let m be a finite symmetric Borel measure on $[-\pi, \pi]$. Then*

$$\Pi_{\mathcal{L}^\perp} \Sigma(\mathsf{m}, n) \Pi_{\mathcal{L}^\perp} = \Pi_{\mathcal{L}^\perp} H_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Sigma(\Delta_{\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})} \odot \mathsf{m}, n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))) H'_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Pi_{\mathcal{L}^\perp}.$$
(54)

Proof of Lemma D.2: If $p(\mathcal{L}) = 0$, we have $\Delta_{\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})} = 1$ and $\kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) = 0$ by our conventions. Since then clearly $H_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) = I_n$ holds in view of the definition of $D_n(\Delta_{\underline{\omega}, \underline{d}})$, there is nothing to prove in this case. Assume that $p(\mathcal{L}) > 0$. By Lemma D.1 we have that $n > \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))$ holds. Lemma C.3 shows that the columns of $D'_n(\Delta_{\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})})$ constitute a basis of $\mathrm{span}(V_n^{(0)}(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})))^\perp$, which contains $\mathcal{L}^\perp$ in view of Definition 6.2. Hence, $\Pi_{\mathcal{L}^\perp} = \Pi_{\mathcal{L}^\perp} H_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) D_n(\Delta_{\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})})$ holds. Inserting this into the l.h.s. of (54) and applying Lemma C.7 completes the proof. ∎

Lemma D.3. *Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$ with $\dim(\mathcal{L}) < n$. Let $\omega \in [0, \pi]$. Then there exists a 2×2-dimensional regular matrix $B(\mathcal{L}, \omega)$ that is proportional to an orthogonal matrix such that*

$$\Pi_{\mathcal{L}^\perp} H_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) E_{n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})), 0}(\omega) = \begin{cases} \Pi_{\mathcal{L}^\perp} E_{n, d_i(\mathcal{L})}(\omega) B(\mathcal{L}, \omega) & \text{if } \omega = \omega_i(\mathcal{L}) \text{ for some } i \\ \Pi_{\mathcal{L}^\perp} E_{n, 0}(\omega) B(\mathcal{L}, \omega) & \text{else.} \end{cases}$$
(55)

Furthermore, $\Pi_{\mathcal{L}^\perp} E_{n, d_i(\mathcal{L})}(\omega_i(\mathcal{L})) B(\mathcal{L}, \omega_i(\mathcal{L})) \neq 0$ for every $i = 1, \ldots, p(\mathcal{L})$, and $\Pi_{\mathcal{L}^\perp} E_{n, 0}(\omega) B(\mathcal{L}, \omega) \neq 0$ for every ω not equal to some coordinate $\omega_i(\mathcal{L})$ of $\underline{\omega}(\mathcal{L})$.

Proof of Lemma D.3: If $p(\mathcal{L}) = 0$, the result is trivial with $B(\mathcal{L}, \omega)$ the identity matrix. Hence, suppose $p(\mathcal{L}) > 0$ holds. We start with the case where ω does not coincide with a coordinate of $\underline{\omega}(\mathcal{L})$. Because of Remark C.2 we may write

$$D_n(\Delta_{\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})}) = \prod_{i=1}^{p(\mathcal{L})} D_{n - g_i}(\Delta_{\omega_i(\mathcal{L})}^{d_i(\mathcal{L})})$$

where $g_i = \sum_{j=i+1}^{p(\mathcal{L})} \kappa_j$ and κ_j is shorthand for $\kappa(\omega_j(\mathcal{L}), d_j(\mathcal{L}))$. Now we may repeatedly apply Lemma C.6 (with $l = 0$) to obtain

$$D_n(\Delta_{\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})}) E_{n, 0}(\omega) = E_{n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})), 0}(\omega) A(\mathcal{L}, \omega),$$
(56)

where $A(\mathcal{L}, \omega)$ is proportional to an orthogonal matrix and is nonsingular. Next assume that ω

coincides with a coordinate of $\underline{\omega}(\mathcal{L})$, say $\omega_{i^*}(\mathcal{L})$. We may write

$$D_n(\Delta_{\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L})}) = \left(\prod_{i=1,i\neq i^*}^{p(\mathcal{L})} D_{n-g_i^*}(\Delta_{\omega_i(\mathcal{L})}^{d_i(\mathcal{L})}) \right) D_n(\Delta_{\omega_{i^*}(\mathcal{L})}^{d_{i^*}(\mathcal{L})})$$

where $g_i^* = \kappa_{i^*} + \sum_{j=i+1,j\neq i^*}^{p(\mathcal{L})} \kappa_j$. Applying first Lemma C.5 (with $l = 0$) and using the elementary fact that $E_{\cdot,0}^{(s)}(\omega) = E_{\cdot,0}(\omega)P(\omega)^s$, where $P(\omega)$ has been defined in Lemma C.6, and then repeatedly applying Lemma C.6 gives

$$D_n(\Delta_{\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L})})E_{n,d_{i^*}(\mathcal{L})}(\omega) = E_{n-\kappa(\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L})),0}(\omega)A(\mathcal{L},\omega), \qquad (57)$$

for some $A(\mathcal{L},\omega)$ that is proportional to an orthogonal matrix and is nonsingular. Multiplying (56) and (57) by $\Pi_{\mathcal{L}^\perp} H_n(\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L}))$ from the left as well as by the inverse of $A(\mathcal{L},\omega)$ from the right and using that $\Pi_{\mathcal{L}^\perp} = \Pi_{\mathcal{L}^\perp} H_n(\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L}))D_n(\Delta_{\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L})})$ holds, as noted in the proof of Lemma D.2, now completes the proof of (55). The final claim in the lemma follows immediately from the definition of $d_i(\mathcal{L})$ and $\omega_i(\mathcal{L})$, respectively, see Definition 6.2. ∎

Proof of (34) in Example 6.1: By its definition, $\mathbb{S}(\mathfrak{F}_{\text{all}}, \mathcal{L})$ is certainly contained in the set on the r.h.s. of (34). To prove the reverse inclusion, it suffices to show that any $\Gamma \subseteq [0,\pi]$ with $\text{card}(\Gamma) \in \mathbb{N}$ is the intersection of $[0,\pi]$ with the support of a finite and symmetric Borel measure m on $[-\pi,\pi]$ that arises as the weak limit of a sequence m_{g_j} with g_j as in (i) of Definition 6.4 (and with $\mathfrak{F} = \mathfrak{F}_{\text{all}}$). Let now $\Gamma \subseteq [0,\pi]$ with $\text{card}(\Gamma) \in \mathbb{N}$ be given. For every $\gamma \in \Gamma$ let $U_j(\gamma)$ be the intersection of the open interval $(\gamma - 1/j, \gamma + 1/j)$ with $[0,\pi]$. Define $V_j(\gamma)$ as $U_j(\gamma) \cup (-U_j(\gamma))$. For j large enough the elements of the collection $\{V_j(\gamma) : \gamma \in \Gamma\}$ are pairwise disjoint. For each $\gamma \in \Gamma$ one can then easily find functions $h_{\gamma,j}$ on $[-\pi,\pi]$ such that (i) $h_{\gamma,j}$ vanishes outside of $V_j(\gamma)$, (ii) $h_{\gamma,j}$ is positive on $V_j(\gamma)$, (iii) $h_{\gamma,j}$ is symmetric, and (iv) $h_{\gamma,j}$ is Borel-measurable and satisfies $\int_{-\pi}^{\pi} h_{\gamma,j}(\nu)d\nu = 1$. [E.g., let $h_{\gamma,j}$ be the indicator function of $V_j(\gamma)$, suitably normalized.] Define

$$h_{\gamma,j}^*(\nu) = \left|\Delta_{\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L})}(e^{\iota\nu})\right|^2 h_{\gamma,j}(\nu) \left/ \int_{-\pi}^{\pi} \left|\Delta_{\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L})}(e^{\iota\nu})\right|^2 h_{\gamma,j}(\nu)d\nu \right. ,$$

where we note that the integral in the denominator is certainly positive. It is now obvious that for each $\gamma \in \Gamma$ the measure with density $h_{\gamma,j}^*(\nu)$ converges weakly (as $j \to \infty$) to the convex combination of pointmass at γ and $-\gamma$, each with weight $1/2$, if $\gamma \neq 0$, and to unit pointmass at zero if $\gamma = 0$. Next define

$$h_j^{**} = \sum_{\gamma \in \Gamma} h_{\gamma,j}^*.$$

It follows that the measure on $[-\pi,\pi]$ with density h_j^{**} converges weakly to the discrete measure m^{**} that assigns mass $1/2$ to the points γ and $-\gamma$ with $\gamma \in \Gamma$, $\gamma \neq 0$, and assigns mass 1 to

$\gamma = 0$ in case $0 \in \Gamma$. Define

$$f_j = c_j^{-1} \sum_{\gamma \in \Gamma} \left[h_{\gamma,j}(\nu) \bigg/ \int_{-\pi}^{\pi} \left| \Delta_{\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L})}(e^{\iota\nu}) \right|^2 h_{\gamma,j}(\nu) d\nu \right],$$

where the normalization constant c_j is given by

$$c_j = \sum_{\gamma \in \Gamma} \left(\int_{-\pi}^{\pi} \left| \Delta_{\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L})}(e^{\iota\nu}) \right|^2 h_{\gamma,j}(\nu) d\nu \right)^{-1}.$$

Obviously, f_j belongs to $\mathfrak{F}_{\text{all}}$. Using this sequence f_j, construct the sequence g_j as in (33). Clearly, g_j coincides with $h_j^{**}/\operatorname{card}(\Gamma)$, and hence m_{g_j} converges to the finite and symmetric Borel measure $\mathsf{m} = \mathsf{m}^{**}/\operatorname{card}(\Gamma)$ which obviously is exactly supported on $\Gamma \cup (-\Gamma)$. This completes the proof. ■

Proof of the claims in Example 6.2: Suppose $\mathbb{S}(\mathfrak{F}_{\text{all}}^B, \mathcal{L})$ were nonempty. Then there has to exist a $\mathsf{m} \in \mathbb{M}(\mathfrak{F}, \mathcal{L})$. Let $\Gamma = \operatorname{supp}(\mathsf{m}) \cap [0, \pi]$. Then $\operatorname{card}(\Gamma) \in \mathbb{N}$ must hold. Let g_j and f_j be as in (33) with m_{g_j} converging weakly to m. Choose $\delta > 0$ such that $B\delta < 1$. Let U be the union of the finitely many intervals $(\eta - \varepsilon, \eta + \varepsilon) \cap [-\pi, \pi]$ where η runs through the union of $\Gamma \cup (-\Gamma)$ with the set of zeros of the trigonometric polynomial $\left| \Delta_{\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L})}(e^{\iota\nu}) \right|^2$. Here $\varepsilon > 0$ is chosen so small such that the intervals are disjoint and such that the Lebesgue measure of U is smaller than δ. Since the boundary of $V = [-\pi, \pi] \backslash U$ w.r.t. $[-\pi, \pi]$ has measure zero under m we have that $\int_V g_j(\nu) d\nu$ converges to $\mathsf{m}(V) = 0$. Furthermore, $\left| \Delta_{\underline{\omega}(\mathcal{L}),\underline{d}(\mathcal{L})}(e^{\iota\nu}) \right|^2$ is bounded from below on V by a positive constant, c say; and it is bounded from above by a finite constant, C say, on $[-\pi, \pi]$. We conclude that

$$\int_V g_j(\nu) d\nu \geq (2\pi CB)^{-1} c \int_V f_j(\nu) d\nu \geq (2\pi CB)^{-1} c(1 - B\delta) > 0,$$

a contradiction. ■

Proof of Proposition 6.1: 1. If $\bar{\Sigma} \in \operatorname{cl}(\mathcal{L}(\mathfrak{C}(\mathfrak{F})))$, then there must exist a sequence $f_j \in \mathfrak{F}$ so that $\mathcal{L}(\Sigma(f_j)) \to \bar{\Sigma}$ holds as $j \to \infty$. Lemma D.2 (together with homogeneity of $\Sigma(\cdot, \cdot)$ in its first argument) now shows that

$$\mathcal{L}(\Sigma(f_j)) = \frac{\Pi_{\mathcal{L}^\perp} H_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Sigma(g_j, n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))) H_n'(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Pi_{\mathcal{L}^\perp}}{\| \Pi_{\mathcal{L}^\perp} H_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Sigma(g_j, n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))) H_n'(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Pi_{\mathcal{L}^\perp} \|}, \quad (58)$$

where g_j is as in Definition 6.4, observing that the denominator in the preceding display, i.e., $\Pi_{\mathcal{L}^\perp} \Sigma(f_j) \Pi_{\mathcal{L}^\perp}$, is nonzero because of positive definiteness of $\Sigma(f_j)$ and the assumption $\dim(\mathcal{L}) < n$. Since the sequence m_{g_j} is tight (as each m_{g_j} is a probability measure on the compact set $[-\pi, \pi]$), it converges weakly, at least along a subsequence, to a finite and (necessarily) symmetric Borel probability measure m, say. We now show that $\Sigma(\mathsf{m}, n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})))$ must be singular if $\operatorname{rank}(\bar{\Sigma}) < n - \dim(\mathcal{L})$ holds: assume not, then $H_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Sigma(\mathsf{m}, n -$

$\kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})))H_n'(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))$ defines a bijection from $\text{span}(V_n^{(0)}(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})))^{\perp}$ onto itself (with the convention that this latter space is $\mathbb{R}^n$ if $p(\mathcal{L}) = 0$). Consequently,

$$A := \Pi_{\mathcal{L}^{\perp}} H_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Sigma(\mathsf{m}, n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))) H_n'(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Pi_{\mathcal{L}^{\perp}}$$

would have rank $n - \dim(\mathcal{L}) > 0$.[21] But passing to the limit in (58) along the above mentioned subsequence would then entail $\bar{\Sigma} = A/\|A\|$, contradicting $\text{rank}(\bar{\Sigma}) < n - \dim(\mathcal{L})$. Having established singularity of $\Sigma(\mathsf{m}, n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})))$ and noting that m can not be the zero measure (as it must have total mass 1), we can conclude from Lemma C.8 that $\mathsf{m} = \sum_{i=1}^{p} \bar{c}_i(\delta_{-\gamma_i} + \delta_{\gamma_i})$ for some $p \in \mathbb{N}$ and positive real numbers $\bar{c}_i$, $i = 1, \ldots, p$, and for $0 \le \gamma_1 < \gamma_2 < \ldots < \gamma_p \le \pi$, such that $\sum_{i=1}^{p} \kappa(\gamma_i, 1) < n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))$ holds. In particular, $\Gamma := \{\gamma_1, \ldots, \gamma_p\} \in \mathbb{S}(\mathfrak{F}, \mathcal{L})$ holds. We may now conclude that

$$\Sigma(\mathsf{m}, n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))) = \sum_{i=1}^{p} 2\bar{c}_i E_{n-\kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})), 0}(\gamma_i) E_{n-\kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})), 0}'(\gamma_i). \tag{59}$$

As a consequence of Lemma D.3 the matrix A is nonzero and coincides with the numerator in (35) for some positive $c(\gamma_i)$'s. Passing to the limit in (58) along the above mentioned subsequence then establishes (35). We turn to the second claim in Part 1 next: If $\Gamma \in \mathbb{S}(\mathfrak{F}, \mathcal{L})$, we can find a measure m and a sequence $f_j \in \mathfrak{F}$ satisfying all the requirements in Definition 6.4. In particular, (59) again holds for some positive constants c_i and with γ_i enumerating the elements of Γ. Consider the sequence $\Sigma(f_j)$. Again Lemma D.2 shows that (58) holds. It follows that the numerator of (58) converges to A defined above. Now, $A \ne 0$ follows again from Lemma D.3. But then we can conclude that $\mathcal{L}(\Sigma(f_j))$ converges to $\bar{\Sigma} := A/\|A\|$, implying $\bar{\Sigma} \in \text{cl}(\mathcal{L}(\mathfrak{C}(\mathfrak{F})))$. But Lemma D.3 now implies that (35) holds. The claim in parentheses is obvious since $\bar{\Sigma}$ vanishes on $\mathcal{L}$ in view of (35).

2.&3. Part 2 is a simple consequence of Part 1 since $\text{span}(\sum_{i=1}^{l} A_i A_i') = \text{span}((A_1, \ldots, A_l))$ holds for matrices A_i of the same row-dimension. Part 3 follows immediately from Part 2. ∎

Proof of Theorem 6.2: Follows from Corollary 5.6 (with $\mathfrak{C} = \mathfrak{C}(\mathfrak{F})$) together with Proposition 6.1. ∎

Proof of the claim in Remark 6.4(iii): Note that any $\mathcal{S}$ satisfying (ii), but not (i), coincides with $\mathcal{L}^{\perp}$. Since $N^{\dagger}$ is invariant under addition of elements from $\mathcal{L}$, we can write $N^{\dagger}$ as the direct sum $M \oplus \mathcal{L}$, where $M = N^{\dagger} \cap \mathcal{L}^{\perp}$. But then we have

$$0 = \lambda_{\mathbb{R}^n}(N^{\dagger}) = \lambda_{\mathcal{L}^{\perp} \oplus \mathcal{L}}(M \oplus \mathcal{L}),$$

---

[21] By construction of A, its rank can not exceed $n - \dim(\mathcal{L})$. If its rank were less than $n - \dim(\mathcal{L})$, we could find a nonzero $z \notin \mathcal{L}$ such that $Az = 0$ holds. Since A vanishes on $\mathcal{L}$ by its construction, we may assume w.l.o.g. that $z \in \mathcal{L}^{\perp} \backslash \{0\}$. But then $z'Az = 0$ and hence $z'H_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Sigma(\mathsf{m}, n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))) H_n'(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))z = 0$ follows as $\Pi_{\mathcal{L}^{\perp}} z = z$. By symmetry and nonnegative definiteness we arrive at $H_n(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) \Sigma(\mathsf{m}, n - \kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))) H_n'(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))z = 0$. Since $z \in \mathcal{L}^{\perp} \subseteq \text{span}(V_n^{(0)}(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})))^{\perp}$ in view of the definition of $V_n^{(0)}(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L}))$, this would contradict the bijectivity established before.

from which we can conclude that $\lambda_{\mathcal{L}^\perp}(M) = 0$, i.e., $\lambda_{\mathcal{S}}(N^\dagger) = 0$ holds (note that $\lambda_{\mathcal{S}}(N^\dagger) = \lambda_{\mathcal{S}}(N^\dagger \cap \mathcal{S})$). Let $\mu_0 \in \mathfrak{M}_0$ be arbitrary, and write $\mu_0 = l + l^\perp$, where $l \in \mathcal{L}$ and $l^\perp \in \mathcal{L}^\perp$. Since $l^\perp + \mathcal{S} = \mathcal{S}$ and since $N^\dagger$ is invariant under addition of elements from $\mathcal{L}$ we arrive at

$$\lambda_{\mu_0+\mathcal{S}}(N^\dagger) = \lambda_{l+l^\perp+\mathcal{S}}(N^\dagger) = \lambda_{l+\mathcal{S}}(N^\dagger) = \lambda_{\mathcal{S}}(N^\dagger - l) = \lambda_{\mathcal{S}}(N^\dagger) = 0.$$

But this shows that any $\mathcal{S}$ satisfying (ii), but not (i), automatically satisfies the condition $\lambda_{\mu_0+\mathcal{S}}(N^\dagger) = 0$ for every $\mu_0 \in \mathfrak{M}_0$. ∎

Proof of Theorem 6.5: Observe that here $\mathcal{L} = \mathfrak{M}_0^{lin}$ and that $\dim(\mathcal{L}) = k - q < n$ holds, since we always assume $k < n$ and $q \geq 1$. The claims in Part 1 then follow immediately from Part 1 of Theorem 6.2 and Remarks 6.3(i) and 6.4(i)-(iii) (recalling that $N^\dagger = N^*$ is an $G(\mathfrak{M})$-invariant $\lambda_{\mathbb{R}^n}$-null set, cf. Lemma 5.16). Part 2 follows from Part 1, noting that $\lambda_{\mathcal{T}}(N^*) = 0$ is equivalent to $\mathcal{T} \not\subseteq N^*$ if $\mathcal{T}$ is an affine space (cf. the proof of Corollary 5.6). ∎

Proof of Theorem 6.6: Recall from Section 5.3 that, under Assumptions 1 and 2, autocorrelation robust tests based on T_w are a special case of nonsphericity-corrected F-type tests and that Assumption 5 (as well as Assumptions 6 and 7) of Preinerstorfer and Pötscher (2016) are then satisfied. Furthermore, recall from Section 5.3 that the set N^* here is given by B, which is a finite union of proper linear subspaces as a result of Lemma 5.18. Then, except for the last claim, Part 1 follows immediately from Theorem 6.5. To prove the last claim in Part 1, note that the assumptions for Proposition 5.12 (with $\mathfrak{C} = \mathfrak{C}(\mathfrak{F})$ and $\mathcal{L} = \mathfrak{M}_0^{lin}$) are satisfied in view of Lemmata 5.16 and 5.19, keeping in mind the characterization of $\mathbb{J}(\mathcal{L}, \mathfrak{C}(\mathfrak{F}))$ provided in Proposition 6.1 and Remark 6.4. The result then follows from Proposition 5.12 in case $\alpha < \alpha^*$. In case $\alpha = \alpha^* < 1$ it follows from Remark 5.13(i),(ii) together with Lemma 5.16. Part 2 is now a trivial consequence of Part 1. Part 3 is obvious. ∎

Proof of the claims in Remark 6.10: Suppose the claimed equivalence does not hold. Then we can find a $\gamma \in [0, \pi]$ such that $\gamma \notin \bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$ and $\mu_0 + \mathrm{span}(\Pi_{\mathcal{L}^\perp} E_{n,\rho(\gamma,\mathcal{L})}(\gamma)) \subseteq N^\dagger$ holds for some $\mu_0 \in \mathfrak{M}_0$. Observe that $N^\dagger + \mathcal{L} \subseteq N^\dagger$ holds by the assumed invariance properties. Consequently,

$$\begin{aligned} \mathcal{A} \quad &: \quad = \mathrm{span}(E_{n,\rho(\gamma,\mathcal{L})}(\gamma)) + \mathcal{L} = \mathrm{span}(\Pi_{\mathcal{L}^\perp} E_{n,\rho(\gamma,\mathcal{L})}(\gamma) + \Pi_{\mathcal{L}} E_{n,\rho(\gamma,\mathcal{L})}(\gamma)) + \mathcal{L} \\ &\subseteq \quad \mathrm{span}(\Pi_{\mathcal{L}^\perp} E_{n,\rho(\gamma,\mathcal{L})}(\gamma)) + \mathcal{L} \subseteq N^\dagger - \mu_0 \neq \mathbb{R}^n \end{aligned}$$

since $N^\dagger \neq \mathbb{R}^n$. It follows that $\dim(\mathcal{A}) < n$. Trivially, $\mathrm{span}(E_{n,\rho(\gamma,\mathcal{L})}(\gamma)) \subseteq \mathcal{A}$ holds, and $\mathrm{span}(E_{n,\rho(\gamma,\mathcal{L})}(\gamma)) \not\subseteq \mathcal{L}$ in view of the definition of $\rho(\gamma, \mathcal{L})$. Consequently,

$$\kappa(\underline{\omega}(\mathcal{L}), \underline{d}(\mathcal{L})) + \kappa(\gamma, 1) \leq \kappa(\underline{\omega}(\mathcal{A}), \underline{d}(\mathcal{A})) \leq \dim(\mathcal{A}) < n,$$

where we have made use of Lemma D.1. But in view of Example 6.5(ii),(iii) this shows that $\{\gamma\} \in \mathbb{S}(\mathfrak{F}, \mathcal{L})$ and thus $\gamma \in \bigcup \mathbb{S}(\mathfrak{F}, \mathcal{L})$, a contradiction. This establishes the first claim. The remaining claims are proved analogously, observing that the relevant sets N^*, B, etc are $\lambda_{\mathbb{R}^n}$-null

sets and thus are proper subsets of $\mathbb{R}^n$. $\blacksquare$

E Appendix: Extensions to non-Gaussian models

As mentioned in the Introduction and in Section 2, Gaussianity is not essential to the results in the paper. Here we discuss various ways of substantially weakening the Gaussianity assumption.

E.1 Elliptically symmetric and related distributions

Consider again the linear model (1) with all the assumptions made in Section 2, except that the disturbance vector $\mathbf{U}$ now follows an elliptically symmetric distribution. More precisely, let Z_{spher} denote the set of spherically symmetric distributions ζ on $\mathbb{R}^n$ that have no atom at the origin. The vector $\mathbf{U}$ is assumed to be distributed as $\sigma\Sigma^{1/2}\mathbf{z}$, where $\mathbf{z}$ has a distribution $\zeta \in Z_{\mathrm{spher}}$ and $\Sigma \in \mathfrak{C}$. For background information on elliptically symmetric distributions see Cambanis et al. (1981). Three remarks are in order: First, we do not assume that ζ is absolutely continuous. Second, if ζ has a finite first moment (which we, however, do not assume), then $\mathbf{U}$ has mean zero; otherwise, we can only say that the origin is the (uniquely determined) center of elliptical symmetry of $\mathbf{U}$. Third, if ζ has finite second moments, then $\mathbf{U}$ has covariance matrix $c_\zeta \sigma^2 \Sigma$, where c_ζ is the variance of the first component of $\mathbf{z}$; otherwise, $\sigma^2\Sigma$ is a parameter describing the ellipticity of the distribution of $\mathbf{U}$ (which is unique if we consider ζ as fixed and which is only unique up to a scale factor if ζ can freely vary in Z_{spher}). Nevertheless, we shall – in abuse of terminology – continue to address the set $\mathfrak{C}$ as the covariance model. Let $Q_{\mu,\sigma^2\Sigma,\zeta}$ denote the distribution of $\mathbf{Y}$ resulting from model (1) under the preceding assumptions and where $\mu = X\beta$. Then, for *every* distribution $\zeta \in Z_{spher}$, for *any* $G(\mathfrak{M}_0)$-invariant rejection region W, and for *any* $\mu_0 \in \mathfrak{M}_0$ we have that

$$
\begin{aligned}
Q_{\mu_0,\sigma^2\Sigma,\zeta}(W) &= \Pr\left(\mu_0 + \sigma\Sigma^{1/2}\mathbf{z} \in W\right) = \Pr\left(\sigma\Sigma^{1/2}\mathbf{z}/\|\mathbf{z}\| \in (W-\mu_0)/\|\mathbf{z}\| + \mu_0 - \mu_0\right) \\
&= \Pr\left(\sigma\Sigma^{1/2}\mathbf{z}/\|\mathbf{z}\| \in W - \mu_0\right) = \Pr\left(\mu_0 + \sigma\Sigma^{1/2}\mathbf{z}/\|\mathbf{z}\| \in W\right),
\end{aligned}
$$

where we have used $G(\mathfrak{M}_0)$-invariance and the fact that $\|\mathbf{z}\| \neq 0$ almost surely in the last but one step. Note that the distribution of $\mathbf{z}/\|\mathbf{z}\|$ is the uniform distribution on the unit sphere in $\mathbb{R}^n$ and hence does not depend on ζ at all. Since the Gaussian case is a special case with ζ the n-dimensional standard normal distribution, we have

$$
P_{\mu_0,\sigma^2\Sigma}(W) = Q_{\mu_0,\sigma^2\Sigma,\zeta}(W) \tag{60}
$$

for every distribution $\zeta \in Z_{\mathrm{spher}}$ (and for every $\mu_0 \in \mathfrak{M}_0$, $0 < \sigma^2 < \infty$, $\Sigma \in \mathfrak{C}$). That is, the rejection probabilities of any $G(\mathfrak{M}_0)$-invariant rejection region W under the null hypothesis are the same whether we assume a Gaussian linear model or a linear model with elliptically

symmetric errors (satisfying the above made assumptions). In particular, for the size we have

$$\sup_{\zeta \in Z_{\mathrm{spher}}} \sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} Q_{\mu_0, \sigma^2 \Sigma, \zeta}(W) = \sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(W). \tag{61}$$

This shows that all results in the paper carry over to the case of elliptically distributed errors as they stand. In particular, note that the critical values $C(\alpha)$ computed under the assumption of Gaussianity automatically also deliver size control under the more general assumption of elliptical symmetry and thus are robust under this sort of deviations from Gaussianity. Note that the model discussed here allows for heavy-tailed disturbances.

The above discussion in fact shows that all results in the paper hold for a class of distributions even wider than the class of elliptically symmetric distributions: Let Z_{ua} denote the class of distributions ζ on $\mathbb{R}^n$ that (i) do not have any mass at the origin and (ii) have the property that the distribution of $\mathbf{z}/\|\mathbf{z}\|$ under ζ is the uniform distribution on the unit sphere in $\mathbb{R}^n$. [Clearly, Z_{ua} contains Z_{spher}, but also other distributions under which the radial component $\|\mathbf{z}\|$ may be dependent on the uniformly distributed angular component $\mathbf{z}/\|\mathbf{z}\|$.] Then it is plain that (60) and hence (61) continue to hold with Z_{spher} replaced by Z_{ua}. In particular, the critical values computed under Gaussianity are valid in this much wider context. Note, however, that now in general $X\beta$ and $\sigma^2 \Sigma$ no longer have the same interpretation as in the Gaussian or elliptically symmetric case.

Remark E.1. The above discussion obviously also applies in case that (i) ζ is restricted to a subset of Z_{spher} (or Z_{ua}, respectively) or (ii) there are cross-restrictions between $(\beta, \sigma^2, \Sigma)$ and ζ in the sense that depending on $(\beta, \sigma^2, \Sigma)$ the distribution ζ is restricted to a subset $Z(\beta, \sigma^2, \Sigma)$ of Z_{spher} (or Z_{ua}, respectively).

E.2   Other distributions

Again we consider the linear model (1) with all the assumptions as in Section 2, except that $\mathbf{U}$ is now assumed to be distributed as $\sigma \Sigma^{1/2} U \mathbf{z}$ where $\Sigma \in \mathfrak{C}$, where U is an orthogonal matrix and where $\mathbf{z}$ has a density h. It is assumed that the pair (h, U) belongs to a given subset $\mathcal{D}$ of $H \times \mathcal{U}$ where H is a given set of density functions and $\mathcal{U}$ denotes the set of orthogonal $n \times n$ matrices. [Important special cases are (i) the case where $U = I_n$ holds for every $(h, U) \in \mathcal{D}$, or (ii) when H is a singleton.] We assume that $\mathbf{z}$ has mean zero and unit covariance matrix under each $h \in H$.[22] Let $Q_{\mu, \sigma^2 \Sigma, h, U}$ denote the distribution of $\mathbf{Y}$ resulting from the preceding assumptions. Then we have the following result which will allow us to easily extend the size control results from the Gaussian case to the present setting. Observe that the condition on H in the subsequent lemma is trivially satisfied if $H = \{h\}$ and thus imposes no further condition on h in this case (and the same is true if H is a finite set).

---

[22] We make this assumption only in order for $X\beta$ and $\sigma^2 \Sigma$ to have the same interpretation as in the Gaussian case. Lemma E.2 also holds without this assumption. However, note that then the interpretation of $X\beta$ and $\sigma^2 \Sigma$ becomes somewhat obscure and there is no guarantee that the parameters are identified.

Lemma E.2. *Suppose the maintained assumptions of this subsection hold and that there is a $\lambda_{\mathbb{R}^n}$-integrable envelope h^* for H (i.e., $h(z) \leq h^*(z)$ holds $\lambda_{\mathbb{R}^n}$-a.e. for every $h \in H$ and $\int h^*(z)d\lambda_{\mathbb{R}^n}(z) < \infty$). Let W_m be a sequence of rejection regions such that*

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(W_m) \to 0 \quad for \quad m \to \infty. \tag{62}$$

Then

$$\sup_{(h,U) \in \mathcal{D}} \sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} Q_{\mu_0, \sigma^2 \Sigma, h, U}(W_m) \to 0 \quad for \quad m \to \infty. \tag{63}$$

Proof: Let $Q_{\mu_{0,m}, \sigma_m^2 \Sigma_m, h_m, U_m}$ with $\mu_{0,m} \in \mathfrak{M}_0$, $0 < \sigma_m^2 < \infty$, $\Sigma_m \in \mathfrak{C}$, and $(h_m, U_m) \in \mathcal{D}$ be a sequence such that $Q_{\mu_{0,m}, \sigma_m^2 \Sigma_m, h_m, U_m}(W_m)$ differs from the multiple supremum in (63) only by a null sequence. Then we have $P_{\mu_{0,m}, \sigma_m^2 \Sigma_m}(W_m) \to 0$ as $m \to \infty$ as a consequence of (62). Furthermore,

$$P_{\mu_{0,m}, \sigma_m^2 \Sigma_m}(W_m) = \Pr(\mu_{0,m} + \sigma_m \Sigma_m^{1/2} \mathbf{G} \in W_m) = \Pr(\mathbf{G} \in U_m B_m) = \Pr(\mathbf{G} \in B_m) = P_{0, I_n}(B_m)$$

and

$$Q_{\mu_{0,m}, \sigma_m^2 \Sigma_m, h_m, U_m}(W_m) = \Pr(\mu_{0,m} + \sigma_m \Sigma_m^{1/2} U_m \mathbf{z}_m \in W_m) = \Pr(\mathbf{z}_m \in B_m),$$

where $\mathbf{G}$ is a standard Gaussian $n \times 1$ random vector, $\mathbf{z}_m$ is a random vector with density h_m, and $B_m = U_m' \sigma_m^{-1} \Sigma_m^{-1/2}(W_m - \mu_{0,m})$. It suffices to show that $\Pr(\mathbf{z}_m \in B_m)$ converges to zero as $m \to \infty$. Let m' be an arbitrary subsequence of m. Since $P_{0, I_n}(B_{m'}) \to 0$, there exists a subsequence m'' of m' and a $\lambda_{\mathbb{R}^n}$-null set A_1 such that $\mathbf{1}_{B_{m''}}(z) \to 0$ for every $z \notin A_1$ (e.g., Theorem 3.12 in Rudin (1987)). Since the envelope h^* is $\lambda_{\mathbb{R}^n}$-integrable, it is finite everywhere, except possibly on a $\lambda_{\mathbb{R}^n}$-null set A_2. By assumption, $0 \leq h_{m''}(z) \leq h^*(z)$ holds for all m'' outside of a $\lambda_{\mathbb{R}^n}$-null set A_3. Set $A = A_1 \cup A_2 \cup A_3$. But then $\mathbf{1}_{B_{m''}}(z)h_{m''}(z) \to 0$ holds for every z outside the $\lambda_{\mathbb{R}^n}$-null set A. Furthermore, $\left|\mathbf{1}_{B_{m''}}(z)h_{m''}(z)\right| \leq h^*(z)$ holds for every m'' and for every $z \notin A$. But then the Dominated Convergence Theorem gives

$$\Pr(\mathbf{z}_{m''} \in B_{m''}) = \int_{\mathbb{R}^n} \mathbf{1}_{B_{m''}}(z)h_{m''}(z)d\lambda_{\mathbb{R}^n}(z) \to 0.$$

Since the subsequence m' was arbitrary, we conclude that $\Pr(\mathbf{z}_m \in B_m)$ converges to zero. $\blacksquare$

As a consequence of this lemma, versions of all the size control results in the paper (except for the exact size control results) can be given under the maintained assumptions of this subsection if H satisfies the assumption in Lemma E.2. We illustrate this exemplarily with the following version of Theorem 3.2, which is an immediate consequence of the just mentioned theorem combined with Lemma E.2. Observe that – other than with the extensions discussed in the preceding subsection – the critical values $C'(\alpha)$ in the subsequent theorem may now differ from the critical values one obtains in the Gaussian case.

Theorem E.3. *Suppose the maintained assumptions of this subsection are satisfied with H*

having a $\lambda_{\mathbb{R}^n}$-integrable envelope h^*. Suppose Assumptions 1 and 2 are satisfied and T_w is defined by (5). Then for every $0 < \alpha < 1$ there exists a real number $C'(\alpha)$ such that

$$\sup_{(h,U)\in\mathcal{D}} \sup_{\mu_0\in\mathfrak{M}_0} \sup_{0<\sigma^2<\infty} \sup_{f\in\mathfrak{F}_{all}} Q_{\mu_0,\sigma^2\Sigma(f),h}(T_w \geq C'(\alpha)) \leq \alpha \qquad (64)$$

holds, provided that

$$\mathrm{span}(E_{n,\rho(\gamma,\mathfrak{M}_0^{lin})}(\gamma)) \not\subseteq \mathsf{B} \quad \text{for every} \ \ \gamma \in [0,\pi]. \qquad (65)$$

In case the set B coincides with $\mathrm{span}(X)$, condition (65) can equivalently be expressed as

$$\mathrm{rank}(X, E_{n,\rho(\gamma,\mathfrak{M}_0^{lin})}(\gamma)) > k \quad \text{for every} \ \ \gamma \in [0,\pi].$$

Extensions of the other size control results in the paper to the present setting follow a similar pattern and will not be given.

Remark E.4. For the computation of critical values the cumbersome optimization over h in (64) can in principle be avoided by determining $C'(\alpha)$ from

$$\sup_{U\in\mathcal{U}} \sup_{f\in\mathfrak{F}_{all}} \int \mathbf{1}(T_w(z) \geq C'(\alpha))h^*(U'\Sigma^{-1/2}(f)(z-\mu_0))\det(\Sigma^{-1/2}(f))d\lambda_{\mathbb{R}^n}(z) \leq \alpha$$

for some $\mu_0 \in \mathfrak{M}_0$, since the l.h.s. in the preceding display is easily seen to be an upper bound for the l.h.s. of (64) in view of $G(\mathfrak{M}_0)$-invariance of T_w (cf. Remark 5.17). However, this will often lead to a quite conservative choice for $C'(\alpha)$. A similar remark applies to the more general result indicated by the lemma given below.

We next show that the above reasoning based on Lemma E.2 can actually be extended to even larger classes of distributions, including cases where $\mathbf{z}$ need not have a density. Consider again the linear model (1) with all the assumptions as in Section 2, except that $\mathbf{U}$ is now assumed to be distributed as $\sigma\Sigma^{1/2}U\mathbf{z}$ where U is an orthogonal matrix and $\mathbf{z}$ has a distribution ζ with the following properties: the pair (ζ, U) belongs to a set $\mathcal{E} \subseteq Z \times \mathcal{U}$, where Z is a set of distributions with the property that (i) no $\zeta \in Z$ has an atom at the origin and (ii) that the distribution of the random vector $\mathbf{z}/\|\mathbf{z}\|$ under each $\zeta \in Z$ has a density $\bar{h}_\zeta$ w.r.t. the uniform distribution $v_{S^{n-1}}$ on the Borel-sets of the unit sphere S^{n-1}.[23] Let $Q_{\mu,\sigma^2\Sigma,\zeta,U}$ denote the distribution of $\mathbf{Y}$ resulting from the preceding assumptions. Then we have the following lemma.

Lemma E.5. *Suppose the assumptions in the preceding paragraph are satisfied. Furthermore, assume that there is a $v_{S^{n-1}}$-integrable envelope $\bar{h}^*$ for $\{\bar{h}_\zeta : \zeta \in Z\}$ (i.e., $\bar{h}_\zeta(s) \leq \bar{h}^*(s)$ holds $v_{S^{n-1}}$-a.e. for every $\zeta \in Z$ and $\int_{S^{n-1}} \bar{h}^*(s)dv_{S^{n-1}}(s) < \infty$). Let W_m be a sequence of $G(\mathfrak{M}_0)$-*

---

[23]Without a further assumption such as that $\mathbf{z}$ has mean zero and unit covariance matrix under each $\zeta \in Z$ the interpretation of the parameters in the model is somewhat obscure and they are not guaranteed to be identified.

invariant rejection regions such that

$$\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} P_{\mu_0, \sigma^2 \Sigma}(W_m) \to 0 \quad \text{for} \ m \to \infty. \tag{66}$$

Then

$$\sup_{(\zeta, U) \in \mathcal{E}} \sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{\Sigma \in \mathfrak{C}} Q_{\mu_0, \sigma^2 \Sigma, \zeta, U}(W_m) \to 0 \quad \text{for} \ m \to \infty. \tag{67}$$

Proof: Let $Q_{\mu_{0,m}, \sigma_m^2 \Sigma_m, \zeta_m, U_m}$ with $\mu_{0,m} \in \mathfrak{M}_0$, $0 < \sigma_m^2 < \infty$, $\Sigma_m \in \mathfrak{C}$, and $(\zeta_m, U_m) \in \mathcal{E}$ be a sequence such that $Q_{\mu_{0,m}, \sigma_m^2 \Sigma_m, \zeta_m, U_m}(W_m)$ differs from the multiple supremum in (67) only by a null sequence. Then we have $P_{\mu_{0,m}, \sigma_m^2 \Sigma_m}(W_m) \to 0$ as $m \to \infty$ as a consequence of (66). Furthermore,

$$
\begin{aligned}
P_{\mu_{0,m}, \sigma_m^2 \Sigma_m}(W_m) &= \Pr(\mu_{0,m} + \sigma_m \Sigma_m^{1/2} \mathbf{G} \in W_m) = \Pr(\mu_{0,m} + \sigma_m \Sigma_m^{1/2} \mathbf{G} / \|\mathbf{G}\| \in W_m) \quad (68) \\
&= \Pr(U_m' \mathbf{G} / \|\mathbf{G}\| \in B_m) = \Pr(\mathbf{G} / \|\mathbf{G}\| \in B_m) = \int_{S^{n-1}} \mathbf{1}_{B_m}(s) dv_{S^{n-1}}(s)
\end{aligned}
$$

where $B_m = U_m' \sigma_m^{-1} \Sigma_m^{-1/2}(W_m - \mu_{0,m})$, where we have used $G(\mathfrak{M}_0)$-invariance in the second step, and where $\mathbf{G}$ is a standard Gaussian $n \times 1$ random vector. Similarly,

$$
\begin{aligned}
Q_{\mu_{0,m}, \sigma_m^2 \Sigma_m, \zeta_m, U_m}(W_m) &= \Pr(\mu_{0,m} + \sigma_m \Sigma_m^{1/2} U_m \mathbf{z}_m \in W_m) \\
&= \Pr(\mu_{0,m} + \sigma_m \Sigma_m^{1/2} U_m \mathbf{z}_m / \|\mathbf{z}_m\| \in W_m) \\
&= \Pr(\mathbf{z}_m / \|\mathbf{z}_m\| \in B_m) = \int_{S^{n-1}} \mathbf{1}_{B_m}(s) \bar{h}_{\zeta_m}(s) dv_{S^{n-1}}(s),
\end{aligned}
$$

where $\mathbf{z}_m$ is a random vector with distribution ζ_m. Let m' be an arbitrary subsequence of m. Since a fortiori the integral in (68) converges to zero along the subsequence m', there exists a subsequence m'' of m' and a $v_{S^{n-1}}$-null set $A_1 \subseteq S^{n-1}$ such that $\mathbf{1}_{B_{m''}}(s) \to 0$ for every $s \in S^{n-1} \backslash A_1$ (e.g., Theorem 3.12 in Rudin (1987)). Since the envelope $\bar{h}^*$ is $v_{S^{n-1}}$-integrable, it is finite everywhere, except possibly on a $v_{S^{n-1}}$-null set $A_2 \subseteq S^{n-1}$. By assumption, $0 \leq \bar{h}_{\zeta_{m''}}(s) \leq \bar{h}^*(s)$ holds for all m'' outside of a $v_{S^{n-1}}$-null set $A_3 \subseteq S^{n-1}$. Set $A = A_1 \cup A_2 \cup A_3$. But then $\mathbf{1}_{B_{m''}}(s) \bar{h}_{\zeta_{m''}}(s) \to 0$ holds for every s outside the $v_{S^{n-1}}$-null set A. Furthermore, $\left| \mathbf{1}_{B_{m''}}(s) \bar{h}_{\zeta_{m''}}(s) \right| \leq \bar{h}^*(s)$ holds for every m'' and for every $s \notin A$. But then the Dominated Convergence Theorem gives

$$\Pr(\mathbf{z}_{m''} / \|\mathbf{z}_{m''}\| \in B_{m''}) = \int_{S^{n-1}} \mathbf{1}_{B_{m''}}(s) \bar{h}_{\zeta_{m''}}(s) dv_{S^{n-1}}(s) \to 0.$$

Since the subsequence m' was arbitrary, we conclude that $Q_{\mu_{0,m}, \sigma_m^2 \Sigma_m, \zeta_m, U_m}(W_m)$, and hence (67), converges to zero. $\blacksquare$

Since all rejection regions in the size control results in this paper are $G(\mathfrak{M}_0)$-invariant, it is now obvious how this lemma can be used to transfer the size control results for the Gaussian case to the setup considered here. We abstain from spelling out the details.

Remark E.6. (i) Restricted to $G(\mathfrak{M}_0)$-invariant rejection regions, Lemma E.2 is indeed a special case of Lemma E.5. This follows since Lemma D.1 in Preinerstorfer and Pötscher (2017) applied to $h \in H$ shows that $\mathbf{z}/\|\mathbf{z}\|$ has a density $\bar{h}$; furthermore, applying that lemma once again, but now to h^* (which is – up to a normalization – a probability density), produces an $\upsilon_{S^{n-1}}$-integrable envelope $\bar{h}^*$ for the collection of densities $\bar{h}$.

(ii) Similar to Remark E.1 one can also allow for cross-restrictions between $(\beta, \sigma^2, \Sigma)$ and (h, U) $((\zeta, U)$, respectively) here.

(iii) The exact size control results can also be generalized beyond Gaussianity and elliptical symmetry under appropriate assumptions, but we do not discuss this here.

F   Appendix: Description of the algorithms

Subsequently, for a symmetric and positive definite matrix Σ we denote by $\mathrm{chol}(\Sigma)$ the (unique) lower triangular matrix that satisfies $\Sigma = \mathrm{chol}(\Sigma)\mathrm{chol}(\Sigma)'$.

Algorithm 1 Numerical approximation of $C_\Diamond(\alpha, p) = \sup_{\rho \in (-1,1)^p} F_\rho^{-1}(1 - \alpha)$.

1: **Input** Positive integers $M_0 \geq M_1 \geq M_2$, $N_0 \leq N_1 \leq N_2$.

2: **Stage 0: Initial value search**

3: Generate a pseudorandom sample $Z_1, \ldots, Z_{N_0}$ from P_{0, I_n}.

4: **for** $j = 1$ **to** $j = M_0$ **do**

5:     Obtain a candidate $\rho_j \in (-1, 1)^p$.

6:     Compute $\tilde{F}_j^{-1}(1 - \alpha)$ where $\tilde{F}_j(x) = N_0^{-1} \sum_{i=1}^{N_0} \mathbf{1}_{(-\infty, x]}(T_w(\mu_0 + \text{chol}(\Sigma(f_{\rho_j}))Z_i))$ for $x \in \mathbb{R}$.

7: **end for**

8: Rank the candidates ρ_j according to the value (from largest to smallest) of the corresponding quantities $\tilde{F}_j^{-1}(1 - \alpha)$ to obtain $\rho_{1:M_0}, \ldots, \rho_{M_1:M_0}$, the initial values for the next stage.

9: **Stage 1: Coarse localized optimizations**

10: **for** $j = 1$ **to** $j = M_1$ **do**

11:     Generate a pseudorandom sample $Z_1, \ldots, Z_{N_1}$ from P_{0, I_n}.

12:     Let $\bar{F}_{j,\rho}(x) = N_1^{-1} \sum_{i=1}^{N_1} \mathbf{1}_{(-\infty, x]}(T_w(\mu_0 + \text{chol}(\Sigma(f_\rho))Z_i))$ for $x \in \mathbb{R}$ and $\rho \in (-1, 1)^p$.

13:     Obtain ρ_j^* by running a numerical optimization algorithm for the problem $\sup_{\rho \in (-1,1)^p} \bar{F}_{j,\rho}^{-1}(1 - \alpha)$ initialized at $\rho_{j:M_0}$.

14: **end for**

15: Rank the obtained numbers ρ_j^* according to the value (from largest to smallest) of the corresponding $\bar{F}_{j,\rho_j^*}^{-1}(1 - \alpha)$ to obtain $\rho_{1:M_1}^*, \ldots, \rho_{M_2:M_1}^*$, the initial values for the next stage.

16: **Stage 2: Refined localized optimization**

17: **for** $j = 1$ **to** $j = M_2$ **do**

18:     Generate a pseudorandom sample $Z_1, \ldots, Z_{N_2}$ from P_{0, I_n}.

19:     Let $\bar{\bar{F}}_{j,\rho}(x) = N_2^{-1} \sum_{i=1}^{N_2} \mathbf{1}_{(-\infty, x]}(T_w(\mu_0 + \text{chol}(\Sigma(f_\rho))Z_i))$ for $x \in \mathbb{R}$ and $\rho \in (-1, 1)^p$.

20:     Obtain ρ_j^{**} by running a numerical optimization algorithm for the problem $\sup_{\rho \in (-1,1)^p} \bar{\bar{F}}_{j,\rho}^{-1}(1 - \alpha)$ initialized at $\rho_{j:M_1}^*$.

21: **end for**

22: **Return** $\max_{j=1, \ldots, M_2} \bar{\bar{F}}_{j,\rho_j^{**}}^{-1}(1 - \alpha)$.

Remark F.1. *(Other test statistics)* Algorithm 1 has been formulated for the test statistic T_w, but clearly works as it stands also for any other test statistic T (upon replacing T_w by T) provided T satisfies the conditions given in Remark 4.2 in Section 4.

Remark F.2. *(Generation of candidates in Stage 0 of Algorithm 1)* In Stage 0 of Algorithm 1 candidates $\rho_j \in (-1,1)^p$ need to be obtained, from which the best ones (in the sense of giving the highest values of the (numerically approximated) objective function) are then used in Stage 1 as starting values for a numerical optimization procedure. The main purpose of Stage 0 is to decrease the risk of running into a local, but not global, optimum. Different approaches can be used to generate these candidates: If the autoregressive order p is relatively small, one could use an equally spaced grid of starting values. Since the dimension of the feasible set is growing linearly in p, this is not feasible for moderate to large autoregressive orders, and in particular not feasible for the case where $\mathfrak{F} = \mathfrak{F}_{\mathrm{AR}(n-1)}$ (which is equivalent to using $\mathfrak{F} = \mathfrak{F}_{\mathrm{all}}$) for typical sample sizes n. In such cases, one can generate the candidates by drawing partial autocorrelation coefficients from a distribution that induces a uniform distribution on the stationarity region of $\mathrm{AR}(p)$ coefficients as described in Jones (1987). One could also think of many variants of this approach that are designed to more thoroughly exhaust subsets of the feasible set corresponding to lower-dimensional autoregressive coefficient vectors. One variant is as follows: One generates starting values on $(-1,1)^{p_1}, \ldots, (-1,1)^{p_l}$, respectively, for $1 \leq p_1 < p_2 < \ldots < p_l = p$ following the method described in Jones (1987), and then converts the vector of partial autocorrelation coefficients in $(-1,1)^{p_i}$ for $i = 1, \ldots, l-1$ into a vector of partial autocorrelation coefficients in $(-1,1)^p$ by setting the remaining $p - p_i$ coefficients to 0.

Remark F.3. *(Numerical optimization in Stages 1 and 2 of Algorithm 1)* Since numerical computation of derivatives would be computationally intensive, we use derivative-free optimization methods as, e.g., variants of the Nelder and Mead (1965) algorithm in Stages 1 and 2.

Remark F.4. *(Critical values if $\mathfrak{F}$ is a subset of $\mathfrak{F}_{\mathrm{AR}(p)}$ described by restrictions on the partial autocorrelation coefficients)* Suppose it is desired to solve the problem (18) with $\mathfrak{F}$ a subset of $\mathfrak{F}_{\mathrm{AR}(p)}$ described by restrictions on the partial autocorrelation coefficients such as $\rho \in (-1+\varepsilon, 1-\varepsilon)^p$ for some $0 < \varepsilon < 1$. Algorithm 1 can easily be modified to accommodate such a situation, by choosing candidates in Stage 0 from the set $(-1+\varepsilon, 1-\varepsilon)^p$ (e.g., by suitably modified versions of the procedures discussed in Remark F.2), and by solving the optimization problems in Stages 1 and 2 over the set $(-1+\varepsilon, 1-\varepsilon)^p$ instead of $(-1,1)^p$. The so-obtained critical value then, of course, additionally also depends on ε.

Remark F.5. *(Computation of Cholesky factorization)* As is well known, $\mathrm{chol}(\Sigma(f_\rho))$ can be efficiently obtained from the partial autocorrelation coefficients through a variant of the Durbin-Levinson recursion (e.g., by combining Porat (1994) Table 6.2 on p. 159 and Theorem 2.13 in the same reference).

Algorithm 2 Numerical approximation of $\sup_{\mu_0 \in \mathfrak{M}_0} \sup_{0 < \sigma^2 < \infty} \sup_{f \in \mathfrak{F}_{\text{AR(p)}}} P_{\mu_0, \sigma^2 \Sigma(f)} (T_w \geq C)$.

1: **Input** A real number C and positive integers $M_0 \geq M_1 \geq M_2$, $N_0 \leq N_1 \leq N_2$.
2: **Stage 0: Initial value search**
3: Generate a pseudorandom sample $Z_1, \ldots, Z_{N_0}$ from P_{0, I_n}.
4: **for** $j = 1$ to $j = M_0$ **do**
5:      Obtain a candidate $\rho_j \in (-1, 1)^p$.
6:      Compute $\tilde{p}_j = N_0^{-1} \sum_{i=1}^{N_0} \mathbf{1}_{[C, \infty)}(T_w(\mu_0 + \text{chol}(\Sigma(f_{\rho_j}))Z_i))$.
7: **end for**
8: Rank the candidates ρ_j according to the value (from largest to smallest) of the corresponding quantities $\tilde{p}_j$ to obtain $\rho_{1:M_0}, \ldots, \rho_{M_1:M_0}$, the initial values for the next stage.
9: **Stage 1: Coarse localized optimizations**
10: **for** $j = 1$ to $j = M_1$ **do**
11:      Generate a pseudorandom sample $Z_1, \ldots, Z_{N_1}$ from P_{0, I_n}.
12:      Let $\bar{p}_{j,\rho} = N_1^{-1} \sum_{i=1}^{N_1} \mathbf{1}_{[C, \infty)}(T_w(\mu_0 + \text{chol}(\Sigma(f_\rho))Z_i))$ for $\rho \in (-1, 1)^p$.
13:      Obtain ρ_j^* by running a numerical optimization algorithm for the problem $\sup_{\rho \in (-1,1)^p} \bar{p}_{j,\rho}$ initialized at $\rho_{j:M_0}$.
14: **end for**
15: Rank the obtained numbers ρ_j^* according to the value (from largest to smallest) of the corresponding $\bar{p}_{j,\rho_j^*}$ to obtain $\rho_{1:M_1}^*, \ldots, \rho_{M_2:M_1}^*$, the initial values for the next stage.
16: **Stage 2: Refined localized optimization**
17: **for** $j = 1$ to $j = M_2$ **do**
18:      Generate a pseudorandom sample $Z_1, \ldots, Z_{N_2}$ from P_{0, I_n}.
19:      Let $\bar{\bar{p}}_{j,\rho} = N_2^{-1} \sum_{i=1}^{N_2} \mathbf{1}_{[C, \infty)}(T_w(\mu_0 + \text{chol}(\Sigma(f_\rho))Z_i))$ for $\rho \in (-1, 1)^p$.
20:      Obtain ρ_j^{**} by running a numerical optimization algorithm for the problem $\sup_{\rho \in (-1,1)^p} \bar{\bar{p}}_{j,\rho}$ initialized at $\rho_{j:M_1}^*$.
21: **end for**
22: **Return** $\max_{j=1,\ldots,M_2} \bar{\bar{p}}_{j,\rho_j^{**}}$.

Remark F.6. Remarks analogous to the ones given after Algorithm 1 also apply to Algorithm 2.

G   Appendix: Numerically checking Condition (11) for the test problems in Section 4.2

First observe that for the hypothesis considered we have that $\mathfrak{M}_0^{lin}$ is the span of e_+ and $(1, 2, 3, \ldots, 100)'$, and hence coincides with the span of $(E_{100,0}(0), E_{100,1}(0))$. It follows that $\rho(0, \mathfrak{M}_0^{lin}) = 2$, since $\text{span}(E_{100,2}(0)) = \text{span}((1, 2^2, 3^2, \ldots, 100^2)')$ is not contained in $\mathfrak{M}_0^{lin}$. We now show that $\rho(\gamma, \mathfrak{M}_0^{lin}) = 0$ for every $\gamma \in (0, \pi]$: Note that $\text{rank}(E_{100,0}(0), E_{100,1}(0), E_{100,0}(\gamma)) = \min(100, 4) = 4$ if $0 < \gamma < \pi$ and equals $\min(100, 3) = 3$ if $\gamma = \pi$, because of Lemma C.1 in Appendix C. But this implies that the span of $E_{100,0}(\gamma)$ is not contained in $\mathfrak{M}_0^{lin}$ whenever $\gamma \in (0, \pi]$, establishing the claim.

Second, we now verify that condition (11) holds for each of the 128 design matrices considered. That, is we show that $\text{span}(E_{100,\rho(\gamma,\mathfrak{M}_0^{lin})}(\gamma)) \not\subseteq \mathsf{B}$ for every $\gamma \in [0, \pi]$ where $\rho(\gamma, \mathfrak{M}_0^{lin}) = 2$ for
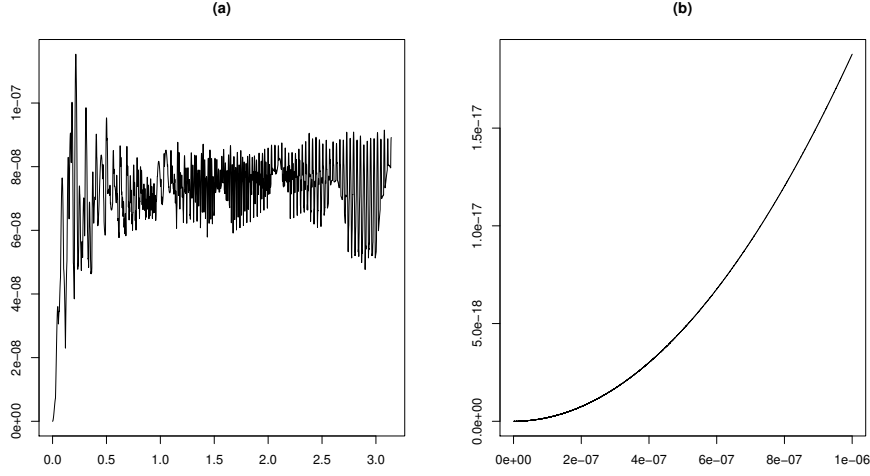
Figure 3: (a) The point-wise minimum over all 128 regression models of the function defined in (69) on the interval $[0, \pi]$. (b) The point-wise minimum over all 128 regression models of the function defined in (69) on the interval $[0, 10^{-6}]$.

$\gamma = 0$ and $\rho(\gamma, \mathfrak{M}_0^{lin}) = 0$ for $\gamma \neq 0$. Consider first the case where $\gamma = 0$: Since $\mathrm{span}(E_{100,2}(0)) = \mathrm{span}((1, 2^2, 3^2, \ldots, 100^2)')$ it suffices to show that $B((1, 2^2, 3^2, \ldots, 100^2)')$ is nonzero where B is given in (8). Since $\mathrm{rank}(R(X'X)^{-1}X') = q = 1$ holds, it is in turn sufficient to show that each coordinate of the residual vector obtained from regressing $(1, 2^2, 3^2, \ldots, 100^2)'$ onto the design matrix X is nonzero. For each of the 128 design matrices considered this has been numerically confirmed. Next, to check condition (11) for $\gamma \in (0, \pi]$ it suffices to verify that for each of the 128 cases $c(\gamma) \notin \mathsf{B}$ for every $\gamma \in (0, \pi]$ where $c(\gamma) = (\cos(\gamma), \cos(2\gamma), \ldots, \cos(100\gamma))'$ is the first column of $E_{100,0}(\gamma)$. Since $c(\gamma) \in \mathsf{B}$ is equivalent to $B(c(\gamma)) = 0$ (as $q = 1$ holds), we compute for each of the 128 cases the function

$$\gamma \mapsto \|B(c(\gamma))\|_\infty \quad \text{where} \quad c(\gamma) = (\cos(\gamma), \cos(2\gamma), \ldots, \cos(100\gamma))' \quad (69)$$

(a) on a grid of 100 000 equally spaced points in $[0, \pi]$ and (b) also on a grid of 100 000 equally spaced points in $[0, 10^{-6}]$ to get a more refined resolution in this region (note that the function in the preceding display has a root at 0 in each of the 128 cases since $\rho(0, \mathfrak{M}_0^{lin}) > 0$). Then we plot for each value γ in these grids the smallest of the 128 norms in (69) corresponding to the 128 cases considered. These plots are shown in Figures 3(a) and 3(b), respectively. The figures suggest that $\|B(c(\gamma))\| \neq 0$ holds for $\gamma \in (0, \pi]$ for each of the 128 cases, implying that for each of these cases we have $\mathrm{span}(E_{100,0}(\gamma)) \notin \mathsf{B}$ for $\gamma \in (0, \pi]$.

H   Appendix: Settings for Algorithms 1 and 2 used in Section 4.2

Here we describe the settings for Algorithms 1 and 2 used in the computations for Section 4.2. The actual computations were performed using the implementations of Algorithms 1 and 2 provided in the R-package **acrt** (Preinerstorfer (2016)) through the functions critical.value and size, respectively. Initial values in Stage 0 are generated as follows, cf. Remark F.2 in Appendix F:

1. For $p \in \{1, 2\}$ the initial values are a pseudorandom sample of size 5000 in $(-1, 1)^p$ drawn according to the distribution generating a uniform distribution on the stationarity region of $\mathrm{AR}(p)$ coefficients following Jones (1987).

2. For $p \in \{5, 10, 25, 50, 99\}$ we proceed as follows: For every $p_i \in \{2, 5, 10, 25, 50, 99\}$ that does not exceed p, we generate pseudorandom samples in $(-1, 1)^{p_i}$ of size 5000, each according to the distribution that generates a uniform distribution on the stationarity region of $\mathrm{AR}(p_i)$ parameters. Then $\mathrm{AR}(p)$ partial autocorrelation coefficients are obtained from these pseudorandom samples in $(-1, 1)^{p_i}$ by appending $p - p_i$ zeros.

Furthermore we use $M_1 = 10$, $M_2 = 2$ and $N_0 = 1000$, $N_1 = 10\,000$, $N_2 = 50\,000$. Note that the number of replications in the Monte-Carlo algorithms of 10 000 in the first stage, and of 50 000 in the second stage are (at least) of the same order of magnitude as the number of replications used in contemporary simulation studies concerning rejection probabilities of autocorrelation robust tests (e.g., Sun et al. (2011) use 10 000 replications). [Of course, in a particular application, where one needs to determine the critical value for *one* model and for *one* parameter p only, one could choose parameters M_i and N_i that provide an even higher level of accuracy.] The optimization algorithm we employ in Stages 1 and 2 is a Nelder-Mead algorithm with default settings concerning the reflection, contraction, and expansion coefficients as implemented in the optim function in R. The relative convergence tolerance was set equal to $N_1^{-1/2}$ and to $N_2^{-1/2}$ in Stages 1 and 2, respectively. Furthermore, the maximal number of iterations in Stages 1 and 2 were set equal to $20n$ and $30n$, respectively (recall $n = 100$ here). Since the optim function supplies an implementation of the Nelder-Mead algorithm that optimizes functions from a Euclidean space to the real numbers, we rephrased our optimization problems as unrestricted optimization problems over $\mathbb{R}^p$ using the function $(2/\pi) \arctan$. For the i.i.d. case (i.e., $p = 0$) the problem considerably simplifies as noted earlier, since the distribution of the test statistic under the null does then not depend on any nuisance parameter; in this case the maximal rejection probabilities and the $1 - \alpha$ quantiles were in each scenario obtained from a Monte Carlo sample of size 50 000.

I   Appendix: A random-walk-based critical value

As suggested by a referee, we compute – in the context of the null hypotheses and the 128 models considered in Section 4.2 – the critical values for $|t_w|$ that result from the presumption that the errors follow a Gaussian random walk and where we use the same weights as in Section 4.2. Since we may set $\beta_1 = \beta_2 = \beta_3 = 0$ and the innovation variance $\sigma^2 = 1$ by invariance, the computation of such a critical value reduces to determining the $(1-\alpha)$-quantile of the distribution of $|t_w|$ under a fixed Gaussian distribution, which can easily be achieved by Monte-Carlo. For $\alpha = 0.05$ and for each of the 128 models considered in Section 4.2 we have computed these critical values from 10 000 Monte-Carlo samples. We show a scatter-plot of these random-walked-based critical values versus the critical values that control size over $\mathfrak{F}_{AR(1)}$ (which have been obtained in Section 4.2) in Figure 4, whereas in Figure 5 we plot these versus the critical values that control size over $\mathfrak{F}_{AR(2)}$ (which also have been obtained in Section 4.2). We can draw the following conclusions:

1. The random-walk-based critical values are – for the majority of the 128 models considered – roughly of the same magnitude as the critical values that guarantee size control over $\mathfrak{F}_{AR(1)}$, although for some models they are too small.

2. Compared to the critical values guaranteeing size control over $\mathfrak{F}_{AR(2)}$ the random-walk-based critical values are way too small, and hence will not control size over $\mathfrak{F}_{AR(2)}$; and a fortiori not over $\mathfrak{F}_{AR(p)}$ with $p \geq 2$ or $\mathfrak{F}_{all}$.

As a consequence, the random-walk-based critical values are certainly no substitute for the size-controlling critical values whenever one wants to allow for correlation structures richer than stationary AR(1). If one is willing to only maintain stationary AR(1) correlations, the first conclusion above may lead one to believe that the random-walk-based critical values roughly deliver size control over $\mathfrak{F}_{AR(1)}$. However, this is not true either in general. Note that Conclusion 1 above is based only on computations involving the 128 models (regressors) considered. It does *not* generalize to other models (regressors) as is easily seen by the following example where size control over $\mathfrak{F}_{AR(1)}$ is possible, but where the random-walk-based critical value is much too small. To this end consider a model with only one regressor given by

$$X = (1+\varepsilon, 1+\varepsilon, 1-\varepsilon, 1-\varepsilon, 1+\varepsilon, 1+\varepsilon, \dots, 1+\varepsilon, 1+\varepsilon)', \tag{70}$$

where $n = 100$ and where we vary ε from 0.01 to 0.20 in steps of size 0.01. The null hypothesis is that the coefficient of the regressor is equal to zero and again the corresponding test statistic $|t_w|$ with the same weights as before is used. In Figure 6 we present the random-walk-based critical values (computed from 10 000 Monte-Carlo samples) as well as the critical values that control size over $\mathfrak{F}_{AR(1)}$ (computed via Algorithm 1) as a function of ε.[24] It is apparent from that figure that the random-walk-based critical values are way to small. As an additional observation we note that for the location model (i.e., $\varepsilon = 0$ in (70)) the random-walk-based critical value can be

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[24]The settings used here for Algorithm 1 are similar as in Appendix H with $p = 1$, but with $M_0 = 100$, $M_1 = M_2 = 1$, and $N_0 = N_1 = 1\,000$, $N_2 = 10\,000$.
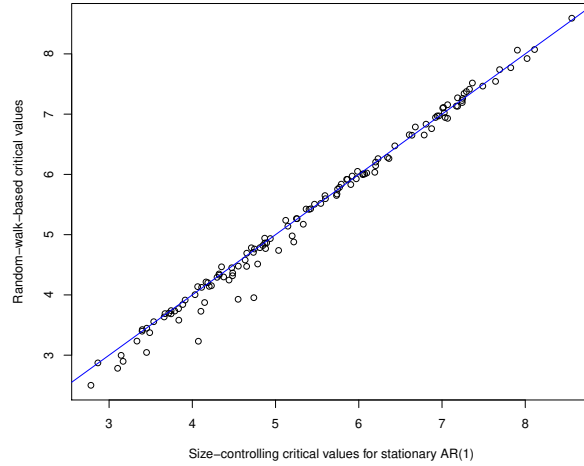
Figure 4: Scatterplot of random-walk-based critical values versus critical values controlling size over $\mathfrak{F}_{AR(1)}$, i.e., over the class of stationary AR(1) processes. The line represents the graph of the identity function.

computed to be 9.6. However, as discussed earlier, for the location model with *stationary* AR(1) errors no critical value exists that leads to size control for $|t_w|$ (since size is equal to 1 for every choice of critical value). Hence the random-walk-based critical value is completely misleading in this model.

J   Appendix: Covariance models corresponding to starting-value solutions of autoregressive models

A referee has asked what happens if instead of the covariance model $\mathfrak{C}(\mathfrak{F}_{AR(1)})$ derived from stationary autoregressive processes of order 1 one considers the covariance model $\mathfrak{C}^0_{AR(1)}$ generated by $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ where $\mathbf{u}_t = \rho \mathbf{u}_{t-1} + \varepsilon_t$ for $t = 1, \ldots, n$ with starting-value $\mathbf{u}_0 = 0$, the innovations ε_t are distributed independently as $N(0, \sigma_\varepsilon^2)$, $0 < \sigma_\varepsilon^2 < \infty$, and $\rho \in (-1, 1)$.[25] More precisely, $\mathfrak{C}^0_{AR(1)}$ consists of the $n \times n$ matrices with (i, j)-th entry given by

$$\rho^{\max(i,j)-\min(i,j)}(1 - \rho^{2\min(i,j)})/(1 - \rho^2).$$

It is easy to see that the covariance model $\mathfrak{C}^0_{AR(1)}$ is norm bounded and has no singular limit points (and the same is true for higher-order analoga) and hence one would, given the results in Preinerstorfer and Pötscher (2016), intuitively expect that size control for test statistics like T_w

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[25]This model has a problematic aspect to it in that it assigns a special meaning to the time point $t = 0$ which is hardly justifiable; cf. the discussion in Section 3.2.2 of Preinerstorfer and Pötscher (2016).
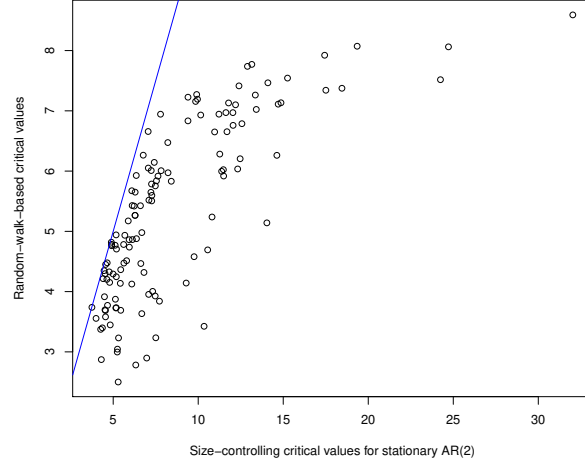
Figure 5: Scatterplot of random-walk-based critical values versus critical values controlling size over $\mathfrak{F}_{\mathrm{AR}(2)}$, i.e., over the class of stationary AR(2) processes. The line represents the graph of the identity function.
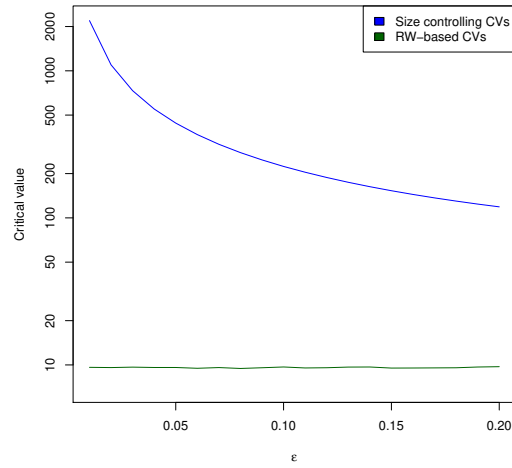
Figure 6: Lower curve shows random-walk-based critical values as a function of ε. Upper curve shows critical values controlling size over $\mathfrak{F}_{\mathrm{AR}(1)}$, i.e., over the class of stationary AR(1) processes, as a function of ε.
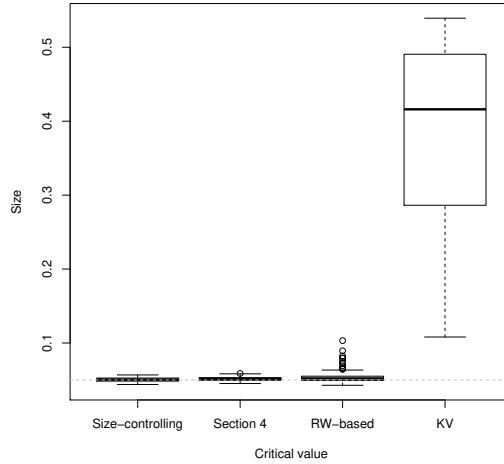
Figure 7: The boxplots labeled "Size-controlling", "Section 4", "RW-based", and "KV", respectively, are each the boxplots of the 128 sizes corresponding to the critical values c_{1i}, c_{2i}, c_{i3}, and 2.260568, respectively.

(or equivalently $|t_w|$) is *always* possible. This is indeed the case and follows from Remark 3.7(ii), cf. also Remark 5.7. However, this does not mean that standard critical values suggested in the literature based on asymptotic considerations will come close to providing size control over the model $\mathfrak{C}^0_{\mathrm{AR}(1)}$ (or over higher-order analoga). In fact, as we shall show below, such critical values will often be much too small and will lead to considerable size distortions.

We first consider again the 128 models from Section 4.2 with the only difference that we now assume that the errors follow the starting-value solution $\mathbf{u}_t$ described in the preceding paragraph. For the test statistic $|t_w|$ (with the same weights as in Section 4.2) and for each of the 128 models we then numerically computed the critical value c_{1i}, $i = 1, \ldots, 128$, that guarantees size control (at $\alpha = 0.05$) over the covariance model $\mathfrak{C}^0_{\mathrm{AR}(1)}$ by a suitable variant of Algorithm 1. We then consider the following four tests for each of the 128 models: (i) Reject if $|t_w| \geq c_{1i}$, (ii) reject if $|t_w| \geq c_{2i}$, where c_{2i} is the critical value computed in Section 4.2 (i.e., the critical value that would control size if the errors were stationary AR(1)-processes), (iii) $|t_w| \geq c_{3i}$ where c_{3i} is the random-walk-based critical value computed in Appendix I, (iv) $|t_w| \geq 2.260568$ (Kiefer-Vogelsang critical value). For each of the four tests we then computed the size of the test (using $\mathfrak{C}^0_{\mathrm{AR}(1)}$ as the underlying covariance model of course). Note that by construction of c_{1i} the size of the test in (i) should be 0.05. Since c_{1i} has been determined only numerically, we recomputed the size also in this case. Also observe that, by construction, the random-walk-based critical value c_{3i} – ignoring numerical error – can not be larger than the size-controlling critical value c_{1i}. The results are shown in Figure 7 in the form of boxplots, each boxplot representing the size of one of the tests over the 128 models.

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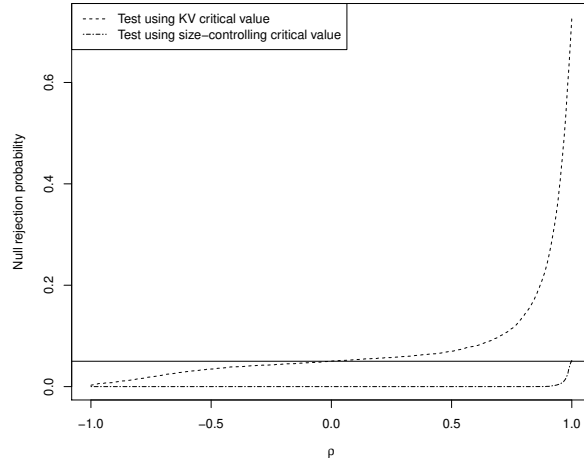
Figure 8: Null-rejection probabilities in location model and for $\mathfrak{C}^0_{\mathrm{AR}(1)}$. The horizontal line corresponds to $\alpha = 0.05$.

It transpires that c_{1i} indeed controls size (as it should by construction) and that the critical values c_{2i} (computed to deliver size control over the related covariance model $\mathfrak{C}(\mathfrak{F}_{\mathrm{AR}(1)})$) pretty much work also under the covariance model considered here. The random-walk-based critical values do a reasonable job in most, but not all of the 128 cases. Finally, the Kiefer-Vogelsang critical value is seen to be way too small and leads to quite dramatic size distortions.

We also considered the location model with the errors following the process as described in the first paragraph. The null hypothesis is that the location parameter is zero and the test statistic considered is the corresponding test statistic $|t_w|$ (with the same weights as before). While no size-controlling critical value exists for this problem if the underlying covariance model is $\mathfrak{C}(\mathfrak{F}_{\mathrm{AR}(1)})$ as already noted earlier, this is different if the covariance model $\mathfrak{C}^0_{\mathrm{AR}(1)}$ is maintained as is done here (since under $\mathfrak{C}^0_{\mathrm{AR}(1)}$ size control is always possible as noted above). We computed this size-controlling value $c_{1,loc}$ (corresponding to $\alpha = 0.05$) by a suitable variant of Algorithm 1. We then computed the null-rejection probabilities of the tests (i) $|t_w| \geq c_{1,loc}$ and (ii) $|t_w| \geq 2.260568$ (Kiefer-Vogelsang critical value) as a function of ρ (note that these probabilities do not depend on σ_ε^2). The graphs of these two functions are given in Figure 8. By construction of $c_{1,loc}$, the corresponding null-rejection probabilities never exceed $\alpha = 0.05$, and reach that value at the right endpoint of the parameter interval for ρ. The Kiefer-Vogelsang critical values again lead to substantial size distortions.

Finally, critical values such as c_{1i}, c_{2i}, or c_{3i} defined above (and a fortiori the Kiefer-Vogelsang critical value) will typically be too small to guarantee size control once we pass from AR(1)-models to higher order models.

K   Appendix: Comments on stochastic regressors

The assumption of nonstochastic regressors made in the present paper may be considered restrictive for some applications and has been criticized by referees. It should be noted, however, that the results obtained under this assumption are quite strong, in that we are able to obtain exact finite-sample size results and, in particular, size guarantee results, which is in contrast to merely (pointwise) asymptotic results in the literature that carry no size guarantee (in fact, are often plagued by considerable size distortions). Furthermore, as already mentioned in Section 2, the case where X is random and independent of $\mathbf{U}$ can easily be accommodated by our theory through conditioning on X. Of course, the resulting conditional size control results then a fortiori imply unconditional size control.[26] Conditioning on X in such a scenario makes perfect sense, as one can argue that values of X other than the observed ones should be irrelevant. For a more detailed discussion of arguments supporting conditional inference see, e.g., Robinson (1979). Again, this form of strict exogeneity (i.e., independence of X and $\mathbf{U}$) may be considered restrictive for some applications.

We now turn to the case where X is random, with X and $\mathbf{U}$ being dependent. While our current theory is then not applicable as it stands, we show by means of numerical examples that the critical values obtained from a naive application of the algorithm proposed in Section 4 (acting as if X and $\mathbf{U}$ were independent) – while not guaranteed to deliver size control – lead to tests that have better size properties than tests that are based on standard critical values suggested in the literature. And this is so despite the fact that the asymptotic theory used in the literature to justify the latter critical values *is* applicable to cases where X and $\mathbf{U}$ are dependent. Of course, the subsequent discussion is based only on a limited Monte-Carlo study and the case of stochastic regressors needs further study which is in progress.

In a first step we shall now formulate a suitable framework for investigating size properties of OLS-based tests allowing for dependence between X and $\mathbf{U}$. To this end consider the linear regression model $y_t = a + b x_t + u_t$ where x_t follows a stationary Gaussian process and u_t is given by the stationary solution of the model $u_t = \rho u_{t-1} + \varepsilon_t$ with ρ varying in $(-1, 1)$ and Gaussian white noise ε_t with positive variance σ_ε^2. We furthermore assume that x_t and ε_t (and thus x_t and u_t) are jointly Gaussian, ergodic, and stationary and that x_t has positive variance. The probability law of the process $(x_t, \varepsilon_t/\sigma_\varepsilon)$ may also depend on some (possibly infinite dimensional) parameter η, say, where we assume that η and ρ are variation-free. [We work with $(x_t, \varepsilon_t/\sigma_\varepsilon)$ instead of (x_t, ε_t) here, because this conveniently eliminates the dependence on σ_ε^2.] Autocorrelation robust test statistics such as T_w are based on the ordinary least squares estimator $(\hat{a}, \hat{b})$. In order for such inference to be meaningful, we need to ensure at a minimum that this estimator is consistent for (a, b). By the very notion of consistency this means that $(\hat{a}, \hat{b})$ has to converge to (a, b) in probability regardless of what the true values of a, b, and of the nuisance parameters ρ, σ_ε^2, and η are. But as is easy to see, this means that x_t and u_t have to be uncorrelated whatever the

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[26]The implied critical values will – by construction – typically depend on the observed values of X, and thus will be "random" critical values when viewed from an unconditional viewpoint.

value of $\rho \in (-1, 1)$ (and of the other nuisance parameters) is. This uncorrelatedness condition on x_t and u_t imposes a restriction on the structure of x_t as we show in the following proposition. Recall that $u_t = \sum_{j=0}^{\infty} \rho^j \varepsilon_{t-j}$ holds, and that the uncorrelatedness condition just mentioned can equivalently be expressed as uncorrelatedness of x_t and $\sum_{j=0}^{\infty} \rho^j (\varepsilon_{t-j}/\sigma_\varepsilon)$ (which has the advantage of making the problem free of σ_ε^2). Let P_η denote the probability law of the process $(x_t, \varepsilon_t/\sigma_\varepsilon)$ and let E_η denote the corresponding expectation operator.

Proposition K.1. *Suppose $E_\eta(x_t \sum_{j=0}^{\infty} \rho^j (\varepsilon_{t-j}/\sigma_\varepsilon)) = 0$ holds for every $\rho \in (-1, 1)$ and some (every, respectively) η. Then x_t is independent of $(\varepsilon_t/\sigma_\varepsilon, \varepsilon_{t-1}/\sigma_\varepsilon, \varepsilon_{t-2}/\sigma_\varepsilon, \ldots)$ under P_η for this (every, respectively) η.*

Proof: By the Gaussianity assumption it suffices to show that $E_\eta(x_t(\varepsilon_{t-j}/\sigma_\varepsilon)) = 0$ for every $j \geq 0$ and some (every, respectively) η. The assumption can be written as $\sum_{j=0}^{\infty} \rho^j E_\eta(x_t \varepsilon_{t-j}/\sigma_\varepsilon) = 0$ for some (every, respectively) η, the sum being guaranteed to converge absolutely for $\rho \in (-1, 1)$, since the coefficients $E_\eta(x_t \varepsilon_{t-j}/\sigma_\varepsilon)$ are a bounded sequence for given η in view of Cauchy-Schwartz. Also note that the coefficients do not depend on ρ by the variation-freeness assumption. Hence by the identity theorem for analytic functions the claim follows. ∎

Before we proceed with the numerical examples we note the following upshot of the preceding result: In the context of the model considered here, we either must be willing to assume that x_t is dependent on some future innovations ε_{t+i}, $i > 0$ (in which case X and $\mathbf{U}$ will typically be dependent), or otherwise arrive at independence of X and $\mathbf{U}$ (in which case our theory can be applied via conditioning).

The numerical examples are now constructed in such a way that the processes (x_t) and (u_t) (and in particular X and $\mathbf{U}$) are indeed dependent in order to generate a scenario that is *unfavorable* to our procedure in the sense that applying our *theory* via conditioning is not feasible. In light of the proposition this means that we need to let x_t depend on future innovations (since x_t must be independent of current and past innovations to justify ordinary least squares estimation in light of the preceding proposition). It should be noted, however, that the so-constructed examples are favorable to the Kiefer-Vogelsang approach in the sense that the asymptotic theory developed in those papers applies.

Let now x_t be given as $x_t = \gamma(\varepsilon_{t+1}/\sigma_\varepsilon) + (1 - \gamma)z_t$ with $0 \leq \gamma \leq 1$ and with z_t the stationary solution of $z_t = \delta z_{t-1} + \tilde{\varepsilon}_t$, where $(\tilde{\varepsilon}_t)$ is Gaussian white noise that is independent of the process (ε_t) and where $\delta \in (-1, 1)$. The variance of $\tilde{\varepsilon}_t$ is denoted by $\sigma_{\tilde{\varepsilon}}^2$ and is assumed to be positive. Note that we have $\eta = (\gamma, \delta, \sigma_{\tilde{\varepsilon}}^2)$. We are interested in testing the null hypothesis $b = 0$ versus the alternative $b \neq 0$ in the above model at the 0.05 significance level. [In fact, by invariance considerations it follows that the results are the same as for testing the null hypothesis $b = b_*$.] We use the test-statistic T_w (or equivalently the corresponding root t_w). As in Section 4, we use sample size $n = 100$ and the Bartlett weights $w(j, n) = (1 - |j|/M_n)\mathbf{1}_{(-1,1)}(j/M_n)$ with $M_n = n/10$ (i.e., bandwidth parameter M_n equal to 10). For a range of choices for the parameters γ, δ, and ρ (and setting $\sigma_{\tilde{\varepsilon}}^2 = 1 - \delta^2$, $\sigma_\varepsilon^2 = 1 - \rho^2$) we obtain a Monte-Carlo sample for $(x_1, \ldots, x_n)'$ and $\mathbf{U} = (u_1, \ldots, u_n)'$ from which we then build X in an obvious way. We proceed

to computing the dependent variable $\mathbf{Y}$ as $X(0,0)' + \mathbf{U}$ (observe that also setting $a = 0$ has no effect on the null-rejection probabilities in view of invariance considerations). We then do two things: First, we compute the test statistic t_w from the data $\mathbf{Y}$ and X and compare its absolute value to the Kiefer and Vogelsang (2005) critical value 2.260568 (corresponding to a nominal significance level of 0.05). By repeating this for 1 000 Monte-Carlo samples we obtain a Monte-Carlo estimate for the null-rejection probability when using this Kiefer-Vogelsang critical value. We report these in the right panel of the tables given below. Second, for a given Monte-Carlo sample, we use the matrix X and employ our Algorithm 1 (with $p = 1$ and tuning parameters $M_0 = 20$, $M_1 = M_2 = 1$, $N_0 = N_1 = 1\,000$, $N_2 = 10\,000$, cf. Appendix F) to compute critical values for $|t_w|$ that would result in size control over $\mathfrak{F}_{\mathrm{AR}(1)}$ if the regressors matrix were nonrandom and equal to the given X. This results in a critical value $c(X)$. We then compare the test statistic $|t_w|$ computed from the given Monte-Carlo sample with $c(X)$ and record whether we reject or not. Repeating over the Monte-Carlo samples this gives us a Monte-Carlo estimate of the null-rejection probability of the so defined procedure. We report these in the left panel of the tables given below.

$\gamma = 0$												
Method	Alg						KV					
$\rho \setminus \delta$	0	0.5	0.75	0.9	0.95	0.99	0	0.5	0.75	0.9	0.95	0.99
0	0.04	0.04	0.02	0.01	0.00	0.00	0.06	0.07	0.07	0.07	0.07	0.07
0.5	0.04	0.04	0.02	0.01	0.01	0.00	0.05	0.08	0.07	0.09	0.11	0.11
0.75	0.04	0.04	0.03	0.02	0.01	0.01	0.05	0.09	0.11	0.15	0.16	0.17
0.9	0.03	0.04	0.05	0.03	0.03	0.02	0.05	0.10	0.15	0.20	0.26	0.24
0.95	0.03	0.04	0.04	0.03	0.03	0.03	0.05	0.09	0.15	0.24	0.28	0.33
0.99	0.02	0.03	0.03	0.04	0.05	0.04	0.03	0.09	0.15	0.28	0.37	0.41
$\gamma = \frac{1}{6}$												
Method	Alg						KV					
$\rho \setminus \delta$	0	0.5	0.75	0.9	0.95	0.99	0	0.5	0.75	0.9	0.95	0.99
0	0.04	0.03	0.02	0.01	0.00	0.00	0.06	0.04	0.07	0.08	0.09	0.07
0.5	0.04	0.03	0.03	0.01	0.01	0.01	0.06	0.06	0.09	0.10	0.11	0.10
0.75	0.04	0.04	0.04	0.02	0.01	0.01	0.06	0.07	0.11	0.15	0.16	0.17
0.9	0.04	0.05	0.06	0.03	0.03	0.02	0.06	0.09	0.15	0.18	0.25	0.27
0.95	0.04	0.03	0.04	0.04	0.04	0.03	0.06	0.09	0.14	0.25	0.31	0.34
0.99	0.03	0.04	0.04	0.04	0.06	0.06	0.05	0.08	0.14	0.26	0.35	0.44

$\gamma = \frac{1}{3}$												
Method			Alg							KV		
$\rho \setminus \delta$	0	0.5	0.75	0.9	0.95	0.99	0	0.5	0.75	0.9	0.95	0.99
0	0.05	0.04	0.02	0.01	0.01	0.01	0.06	0.07	0.06	0.06	0.07	0.06
0.5	0.05	0.06	0.02	0.01	0.02	0.02	0.08	0.07	0.08	0.09	0.10	0.11
0.75	0.06	0.05	0.04	0.02	0.01	0.02	0.07	0.09	0.12	0.14	0.12	0.14
0.9	0.06	0.06	0.06	0.05	0.04	0.05	0.08	0.11	0.13	0.22	0.23	0.24
0.95	0.06	0.04	0.06	0.05	0.05	0.08	0.09	0.08	0.15	0.26	0.28	0.30
0.99	0.07	0.05	0.05	0.06	0.07	0.09	0.10	0.10	0.17	0.25	0.37	0.39

$\gamma = \frac{1}{2}$												
Method			Alg							KV		
$\rho \setminus \delta$	0	0.5	0.75	0.9	0.95	0.99	0	0.5	0.75	0.9	0.95	0.99
0	0.04	0.04	0.03	0.02	0.02	0.02	0.06	0.07	0.07	0.06	0.07	0.06
0.5	0.06	0.05	0.05	0.04	0.03	0.04	0.07	0.08	0.09	0.11	0.10	0.10
0.75	0.07	0.08	0.07	0.06	0.06	0.08	0.09	0.12	0.11	0.12	0.15	0.16
0.9	0.09	0.07	0.08	0.07	0.07	0.11	0.11	0.09	0.14	0.21	0.21	0.22
0.95	0.11	0.11	0.08	0.09	0.07	0.13	0.14	0.14	0.15	0.22	0.23	0.26
0.99	0.15	0.12	0.11	0.10	0.11	0.18	0.18	0.16	0.19	0.28	0.32	0.38

$\gamma = \frac{2}{3}$												
Method			Alg							KV		
$\rho \setminus \delta$	0	0.5	0.75	0.9	0.95	0.99	0	0.5	0.75	0.9	0.95	0.99
0	0.04	0.04	0.04	0.02	0.04	0.03	0.05	0.06	0.05	0.06	0.07	0.06
0.5	0.07	0.06	0.06	0.05	0.05	0.06	0.08	0.08	0.08	0.08	0.09	0.10
0.75	0.09	0.09	0.10	0.10	0.09	0.11	0.11	0.12	0.12	0.14	0.15	0.14
0.9	0.14	0.13	0.14	0.15	0.14	0.17	0.17	0.15	0.17	0.20	0.20	0.23
0.95	0.19	0.15	0.14	0.16	0.17	0.22	0.21	0.17	0.19	0.24	0.26	0.28
0.99	0.26	0.19	0.18	0.18	0.21	0.27	0.29	0.22	0.25	0.27	0.33	0.37

$\gamma = \frac{5}{6}$												
Method			Alg							KV		
$\rho \setminus \delta$	0	0.5	0.75	0.9	0.95	0.99	0	0.5	0.75	0.9	0.95	0.99
0	0.04	0.04	0.05	0.04	0.04	0.04	0.06	0.06	0.07	0.06	0.05	0.06
0.5	0.07	0.07	0.08	0.06	0.07	0.07	0.09	0.09	0.09	0.08	0.10	0.08
0.75	0.12	0.12	0.12	0.12	0.11	0.11	0.14	0.13	0.13	0.14	0.13	0.13
0.9	0.17	0.18	0.18	0.20	0.17	0.18	0.19	0.20	0.19	0.23	0.19	0.20
0.95	0.22	0.21	0.18	0.21	0.23	0.21	0.23	0.23	0.19	0.24	0.26	0.23
0.99	0.29	0.29	0.29	0.32	0.30	0.30	0.32	0.33	0.34	0.36	0.35	0.35

$\gamma = 1$												
Method			Alg							KV		
$\rho \setminus \delta$	0	0.5	0.75	0.9	0.95	0.99	0	0.5	0.75	0.9	0.95	0.99
0	0.07	0.05	0.06	0.05	0.05	0.05	0.09	0.07	0.07	0.06	0.07	0.06
0.5	0.07	0.07	0.08	0.07	0.09	0.07	0.09	0.08	0.10	0.08	0.11	0.08
0.75	0.12	0.11	0.12	0.12	0.10	0.11	0.14	0.13	0.14	0.13	0.12	0.13
0.9	0.20	0.19	0.17	0.17	0.17	0.19	0.21	0.21	0.18	0.19	0.19	0.21
0.95	0.25	0.21	0.24	0.24	0.23	0.22	0.27	0.23	0.26	0.25	0.25	0.23
0.99	0.33	0.32	0.32	0.31	0.30	0.32	0.36	0.35	0.35	0.34	0.35	0.36

The results in the preceding tables clearly show that our method based on Algorithm 1, which presently has no theoretical justification in the example considered (except in the case where $\gamma = 0$), typically performs better in terms of null-rejection probabilities than the competitor method based on the asymptotic theory developed in Kiefer and Vogelsang (2005), often by a considerable margin. In particular, the range of parameters γ, ρ, δ, for which the null-rejection probabilities of our method do not exceed the prescribed significance level of 0.05, is considerably larger than the corresponding range for the competitor method (and includes cases with $\gamma \neq 0$); for the other cases, where the null-rejection probabilities of our method exceed 0.05, the null-rejection probabilities of the competitor method are even larger, sometimes considerably. [Note that in case $\gamma = 1$ the results do not depend on the value of δ, the variation in the rows of these subtables hence only reflecting Monte-Carlo uncertainty.] A fortiori the same conclusion applies if the Kiefer-Vogelsang critical value is replaced by a smaller critical value (as, e.g., the standard normal critical value 1.96 suggested by standard bandwidth-to-zero asymptotics).

L   Appendix: Tables

In the following we present tables for the numerical results underlying Figure 1 in Section 4.2. For a detailed description of the FRED-MD database, including description of the variables and the transformations applied to each time series to achieve stationarity, we refer the reader to McCracken and Ng (2016).

Table 1: Sizes of the t-type tests corresponding to T_w using the Kiefer-Vogelsang critical value and obtained by an application of Algorithm 2.

Regressor	i.i.d.	AR(1)	AR(2)	AR(5)	AR(10)	AR(25)	AR(50)	AR(99)
RPI	0.11	0.40	0.97	0.97	0.96	0.94	0.94	0.92
W875RX1	0.11	0.41	0.96	0.99	0.97	0.95	0.92	0.91
DPCERA3M086SBEA	0.11	0.45	0.97	0.94	0.95	0.93	0.94	0.94
CMRMTSPLx	0.19	0.46	0.95	0.99	0.98	0.88	0.89	0.90
RETAILx	0.15	0.24	0.52	0.53	0.53	0.53	0.53	0.53
INDPRO	0.15	0.25	0.63	0.64	0.64	0.64	0.64	0.64

Table 1 continues on the next page.

IPFPNSS	0.18	0.32	0.58	0.59	0.60	0.59	0.60	0.60
IPFINAL	0.16	0.28	0.60	0.60	0.60	0.60	0.60	0.60
IPCONGD	0.13	0.37	0.96	0.90	0.90	0.95	0.96	0.94
IPDCONGD	0.17	0.32	0.68	0.69	0.69	0.80	0.83	0.87
IPNCONGD	0.08	0.38	0.97	0.99	0.94	0.92	0.92	0.98
IPBUSEQ	0.12	0.23	0.57	0.57	0.57	0.57	0.58	0.57
IPMAT	0.12	0.22	0.68	0.68	0.68	0.68	0.68	0.68
IPDMAT	0.13	0.23	0.61	0.63	0.63	0.62	0.63	0.63
IPNMAT	0.16	0.22	0.68	0.69	0.69	0.69	0.69	0.69
IPMANSICS	0.18	0.29	0.61	0.62	0.62	0.62	0.62	0.62
IPB51222S	0.07	0.14	0.65	0.69	0.69	0.69	0.69	0.69
IPFUELS	0.07	0.11	0.78	0.78	0.78	0.78	0.78	0.78
CUMFNS	0.13	0.23	0.59	0.60	0.60	0.60	0.60	0.60
HWI	0.12	0.42	0.95	0.99	0.77	0.83	0.82	0.90
HWIURATIO	0.12	0.50	0.98	0.98	0.97	0.95	0.96	0.95
CLF16OV	0.09	0.49	0.97	0.93	0.94	0.94	0.96	0.94
CE16OV	0.12	0.52	0.97	0.94	0.94	0.95	0.96	0.96
UNRATE	0.14	0.52	0.97	0.95	0.96	0.96	0.95	0.97
UEMPMEAN	0.09	0.53	0.98	0.95	0.96	0.97	0.97	0.96
UEMPLT5	0.08	0.37	0.96	0.98	0.86	0.91	0.90	0.93
UEMP5TO14	0.14	0.26	0.75	0.76	0.76	0.76	0.76	0.77
UEMP15OV	0.12	0.52	0.97	0.96	0.95	0.95	0.96	0.96
UEMP15T26	0.16	0.32	0.64	0.65	0.65	0.65	0.65	0.65
UEMP27OV	0.11	0.53	0.98	0.95	0.97	0.95	0.95	0.98
CLAIMSx	0.17	0.28	0.71	0.71	0.71	0.71	0.71	0.71
PAYEMS	0.13	0.52	0.97	0.95	0.95	0.95	0.95	0.98
USGOOD	0.14	0.53	0.97	0.94	0.99	0.97	0.96	0.96
CES1021000001	0.09	0.43	0.94	0.87	0.88	0.91	0.87	0.91
USCONS	0.11	0.52	0.97	0.95	0.96	0.95	0.98	0.96
MANEMP	0.15	0.53	0.97	0.94	0.95	0.93	0.95	0.96
DMANEMP	0.15	0.51	0.97	0.99	0.99	0.93	0.93	0.95
NDMANEMP	0.13	0.50	0.97	0.95	0.97	0.97	0.96	0.98
SRVPRD	0.12	0.51	0.97	0.94	0.98	0.94	0.95	0.96
USTPU	0.13	0.52	0.97	0.96	0.92	0.94	0.95	0.96
USWTRADE	0.14	0.53	0.97	0.95	0.95	0.96	0.95	0.95
USTRADE	0.13	0.50	0.97	0.95	0.94	0.93	0.96	0.95
USFIRE	0.11	0.52	0.98	0.95	0.95	0.96	0.95	0.96
USGOVT	0.08	0.47	0.97	0.93	0.94	0.93	0.93	0.96
CES0600000007	0.09	0.41	0.96	0.89	0.89	0.89	0.89	0.88
AWOTMAN	0.10	0.38	0.95	0.99	0.89	0.87	0.91	0.88
AWHMAN	0.10	0.40	0.95	0.89	0.90	0.90	0.93	0.94
HOUST	0.11	0.51	0.97	0.94	1.00	0.97	0.96	0.95
HOUSTNE	0.11	0.44	0.97	0.96	0.97	0.96	0.95	0.96
HOUSTMW	0.07	0.16	0.52	0.62	0.63	0.63	0.63	0.62
HOUSTS	0.09	0.47	0.97	0.91	0.98	0.93	0.92	0.95
HOUSTW	0.08	0.47	0.97	0.94	0.92	0.95	0.94	0.93
PERMIT	0.13	0.46	0.96	0.99	0.93	0.92	0.94	0.93
PERMITNE	0.15	0.29	0.68	0.80	0.81	0.79	0.81	0.80
PERMITMW	0.08	0.36	0.94	0.85	0.88	0.86	0.89	0.87

Table 1 continues on the next page.

PERMITS	0.11	0.45	0.95	0.99	0.82	0.90	0.92	0.88
PERMITW	0.10	0.47	0.96	0.91	0.90	0.92	0.92	0.94
ACOGNO	0.08	0.33	0.61	0.62	0.62	0.62	0.62	0.64
AMDMNOx	0.10	0.21	0.61	0.62	0.61	0.62	0.62	0.62
ANDENOx	0.08	0.99	1.00	1.00	1.00	1.00	1.00	1.00
AMDMUOx	0.11	0.46	0.93	0.86	0.92	0.83	0.91	0.95
BUSINVx	0.14	0.39	0.63	0.64	0.64	0.63	0.64	0.64
ISRATIOx	0.12	0.46	0.97	0.99	0.94	0.93	0.96	0.96
M1SL	0.10	0.41	0.70	0.71	0.71	0.71	0.71	0.71
M2SL	0.11	0.50	0.97	0.94	1.00	0.93	0.95	0.96
M2REAL	0.11	0.48	0.97	0.93	0.93	0.94	0.95	0.97
AMBSL	0.08	0.19	0.67	0.67	0.67	0.67	0.67	0.67
TOTRESNS	0.08	0.21	0.66	0.68	0.68	0.68	0.68	0.68
NONBORRES	0.09	0.36	0.96	0.98	0.91	0.95	0.95	0.96
BUSLOANS	0.10	0.52	0.98	0.96	0.96	0.96	0.96	0.96
REALLN	0.09	0.51	0.97	0.96	0.96	0.96	0.96	0.96
NONREVSL	0.11	0.49	0.97	0.99	0.94	0.97	0.95	0.95
CONSPI	0.09	0.43	0.96	0.91	0.90	0.92	0.93	0.93
S.P.500	0.11	0.32	0.94	0.99	0.71	0.84	0.83	0.68
S.P..indust	0.10	0.31	0.61	0.62	0.62	0.62	0.65	0.63
S.P.div.yield	0.14	0.39	0.96	0.99	0.91	0.87	0.88	0.93
S.P.PE.ratio	0.21	0.38	0.71	0.71	0.71	0.72	0.72	0.72
FEDFUNDS	0.22	0.46	0.96	0.91	0.92	0.88	0.93	0.96
CP3Mx	0.17	0.47	0.97	0.93	0.94	0.92	0.93	0.95
TB3MS	0.22	0.45	0.97	0.94	0.92	0.92	0.94	0.94
TB6MS	0.18	0.47	0.97	0.99	0.92	0.94	0.95	0.94
GS1	0.16	0.46	0.97	0.93	0.93	0.94	0.97	0.96
GS5	0.08	0.47	0.98	0.95	0.95	0.94	0.96	0.95
GS10	0.07	0.23	0.52	0.53	0.53	0.53	0.53	0.52
AAA	0.08	0.41	0.95	0.91	0.97	0.91	0.93	0.95
BAA	0.10	0.50	0.98	0.95	0.96	0.97	0.96	0.96
COMPAPFFx	0.19	0.42	0.96	0.91	0.91	0.89	0.92	0.96
TB3SMFFM	0.22	0.39	0.95	0.86	0.86	0.87	0.89	0.90
TB6SMFFM	0.12	0.17	0.69	0.76	0.76	0.76	0.76	0.76
T1YFFM	0.09	0.50	0.98	0.95	0.95	0.95	0.95	0.96
T5YFFM	0.08	0.30	0.69	0.71	0.71	0.71	0.71	0.70
T10YFFM	0.07	0.17	0.68	0.69	0.69	0.69	0.69	0.69
AAAFFM	0.08	0.14	0.60	0.61	0.62	0.61	0.61	0.61
BAAFFM	0.12	0.18	0.66	0.66	0.66	0.66	0.66	0.66
TWEXMMTH	0.08	0.51	0.98	0.95	0.95	0.96	0.95	0.97
EXSZUSx	0.09	0.47	0.96	0.93	0.94	0.95	0.95	0.96
EXJPUSx	0.09	0.53	0.98	0.96	0.97	0.95	0.97	0.97
EXUSUKx	0.11	0.16	0.72	0.71	0.72	0.72	0.72	0.72
EXCAUSx	0.09	0.52	0.98	0.99	0.97	0.96	0.96	0.96
WPSFD49207	0.08	0.46	0.97	0.95	0.99	0.95	0.93	0.95
WPSFD49502	0.08	0.47	0.97	0.94	0.97	0.94	0.96	0.96
WPSID61	0.08	0.44	0.96	0.91	0.95	1.00	0.93	0.96
WPSID62	0.08	0.34	0.85	0.62	0.63	0.63	0.63	0.64
OILPRICEx	0.08	0.37	0.96	0.99	0.88	0.88	0.90	0.90

PPICMM	0.11	0.22	0.57	0.58	0.58	0.58	0.58	0.58
CPIAUCSL	0.08	0.25	0.57	0.58	0.58	0.58	0.58	0.58
CPIAPPSL	0.08	0.41	0.95	0.99	0.80	0.83	0.83	0.88
CPITRNSL	0.08	0.47	0.98	0.94	0.94	0.97	0.96	0.95
CPIMEDSL	0.10	0.26	0.68	0.69	0.69	0.69	0.69	0.69
CUSR0000SAC	0.08	0.47	0.97	0.94	0.96	0.96	0.94	0.95
CUUR0000SAD	0.08	0.54	0.98	0.96	0.96	0.97	0.96	0.96
CUSR0000SAS	0.09	0.52	0.98	0.95	0.96	0.95	0.96	0.96
CPIULFSL	0.08	0.24	0.55	0.57	0.57	0.57	0.56	0.56
CUUR0000SA0L2	0.07	0.42	0.96	0.91	0.93	0.91	0.92	0.92
CUSR0000SA0L5	0.08	0.25	0.58	0.58	0.59	0.58	0.58	0.59
PCEPI	0.08	0.26	0.58	0.59	0.59	0.59	0.59	0.59
DDURRG3M086SBEA	0.08	0.52	0.97	0.95	0.96	0.96	0.96	0.96
DNDGRG3M086SBEA	0.08	0.47	0.96	0.95	0.96	0.94	0.96	0.95
DSERRG3M086SBEA	0.11	0.52	0.97	0.96	0.96	0.96	0.97	0.96
CES0600000008	0.09	0.28	0.68	0.70	0.70	0.70	0.70	0.70
CES2000000008	0.08	0.29	0.74	0.74	0.74	0.74	0.74	0.74
CES3000000008	0.08	0.31	0.72	0.72	0.72	0.72	0.72	0.72
UMCSENTx	0.07	0.24	0.61	0.63	0.63	0.63	0.63	0.62
MZMSL	0.10	0.48	0.97	0.95	0.95	0.97	0.95	0.95
DTCOLNVHFNM	0.13	0.33	0.66	0.66	0.66	0.66	0.66	0.66
DTCTHFNM	0.15	0.49	0.93	0.88	0.85	0.87	0.90	0.95
INVEST	0.08	0.29	0.90	0.72	0.69	0.69	0.78	0.74
VXOCLSx	0.12	0.27	0.94	0.99	0.87	0.85	0.91	0.89

Table 2: Critical Values guaranteeing size ≤ 0.05 for the t-type tests corresponding to T_w obtained by an application of Algorithm 1.

Regressor	i.i.d.	AR(1)	AR(2)	AR(5)	AR(10)	AR(25)	AR(50)	AR(99)
RPI	2.81	5.26	6.29	6.68	8.13	8.27	8.29	8.37
W875RX1	2.83	5.37	6.60	6.95	6.97	7.69	7.91	7.94
DPCERA3M086SBEA	2.84	6.34	11.27	11.21	11.17	12.56	12.53	12.59
CMRMTSPLx	3.65	6.61	7.06	8.61	8.44	8.58	8.65	8.67
RETAILx	3.21	4.32	4.46	4.89	4.90	5.97	5.95	6.02
INDPRO	3.24	4.44	5.18	5.34	5.90	6.07	6.29	6.13
IPFPNSS	3.61	5.33	5.88	6.18	6.51	8.03	7.92	8.02
IPFINAL	3.34	4.79	5.79	5.83	6.63	7.09	7.29	7.26
IPCONGD	3.04	4.81	4.91	6.09	6.61	6.56	6.69	6.71
IPDCONGD	3.36	4.65	5.63	6.40	7.10	7.11	7.15	6.94
IPNCONGD	2.60	4.87	5.94	7.08	7.21	7.38	7.49	7.48
IPBUSEQ	2.96	4.23	4.79	4.81	4.95	5.43	5.92	5.82
IPMAT	2.96	4.06	5.42	5.44	5.48	6.22	6.24	6.21
IPDMAT	3.08	4.15	5.13	5.43	6.00	6.13	6.14	6.06
IPNMAT	3.37	4.07	5.31	5.35	6.45	6.55	6.44	6.46
IPMANSICS	3.56	5.04	5.95	6.27	6.78	7.79	7.61	7.76
IPB51222S	2.46	3.17	6.97	7.11	7.84	7.73	7.87	8.34
IPFUELS	2.53	2.78	5.29	7.18	8.50	7.52	8.23	8.31
CUMFNS	3.02	4.10	5.16	5.13	6.61	6.80	6.69	6.68

Table 2 continues on the next page.

HWI	3.01	5.74	6.09	6.24	6.65	7.08	7.09	7.06
HWIURATIO	2.92	7.01	12.20	12.47	12.55	12.58	12.51	12.55
CLF16OV	2.66	7.33	12.40	12.45	12.56	17.39	18.71	16.62
CE16OV	2.96	7.17	14.87	15.46	15.26	15.52	15.42	15.34
UNRATE	3.10	6.96	11.63	11.84	11.71	11.77	12.10	11.94
UEMPMEAN	2.65	8.56	32.03	33.13	32.98	32.75	33.15	33.13
UEMPLT5	2.50	3.75	3.75	5.62	6.27	6.26	5.87	6.08
UEMP5TO14	3.15	4.55	7.47	7.46	7.49	9.73	9.64	9.64
UEMP15OV	3.00	7.25	13.36	13.84	13.90	14.01	13.82	14.11
UEMP15T26	3.33	5.20	6.68	6.75	8.25	8.60	8.34	8.61
UEMP27OV	2.85	7.64	15.25	15.86	15.82	15.88	15.87	15.86
CLAIMSx	3.51	4.74	7.09	7.15	7.16	8.48	8.53	8.56
PAYEMS	2.99	7.03	13.41	13.85	13.92	14.84	14.04	13.87
USGOOD	3.07	7.19	11.79	11.98	12.11	12.03	12.19	12.24
CES1021000001	2.67	6.05	11.38	11.65	11.63	11.65	12.50	12.50
USCONS	2.90	7.26	17.51	18.48	18.39	18.37	18.36	18.47
MANEMP	3.25	7.19	9.91	10.11	10.05	10.00	10.29	10.25
DMANEMP	3.28	7.24	9.40	9.40	9.46	9.98	9.95	10.01
NDMANEMP	3.03	6.88	12.05	12.38	12.35	12.32	12.58	12.58
SRVPRD	2.92	7.02	14.70	15.24	15.09	15.21	15.40	15.75
USTPU	3.03	6.94	12.04	12.50	12.48	12.41	12.52	12.53
USWTRADE	3.13	7.69	12.89	13.30	13.33	13.15	13.36	13.45
USTRADE	3.05	6.79	11.69	11.68	12.06	12.08	12.03	12.33
USFIRE	2.88	7.29	18.45	19.43	18.99	19.46	19.61	19.80
USGOVT	2.61	5.54	7.10	7.15	7.19	7.84	7.91	7.94
CES0600000007	2.65	4.87	5.17	5.28	5.84	6.88	6.85	6.81
AWOTMAN	2.78	4.38	4.52	4.86	4.92	4.93	5.47	5.46
AWHMAN	2.76	4.55	4.65	4.89	5.03	5.48	5.48	5.53
HOUST	2.88	6.64	10.96	11.27	11.20	12.11	13.56	12.27
HOUSTNE	2.92	4.75	4.91	7.34	7.32	7.99	8.06	7.76
HOUSTMW	2.44	2.87	4.30	4.79	6.29	6.08	6.30	6.29
HOUSTS	2.64	4.94	5.68	8.66	8.90	8.94	8.73	8.92
HOUSTW	2.57	4.89	6.13	8.22	8.94	8.93	9.22	9.38
PERMIT	3.07	6.20	12.46	12.68	12.56	12.50	12.58	12.76
PERMITNE	3.44	4.03	7.32	7.35	8.21	7.76	8.01	8.07
PERMITMW	2.61	4.32	4.77	5.91	6.69	6.51	6.13	6.58
PERMITS	2.90	6.23	14.62	14.86	14.86	14.87	14.78	14.84
PERMITW	2.78	6.19	12.32	12.40	12.46	14.85	14.42	14.82
ACOGNO	2.56	4.30	5.00	5.00	4.99	5.88	5.79	5.80
AMDMNOx	2.79	3.79	5.18	5.21	6.12	6.42	6.22	6.44
ANDENOx	2.62	3.40	4.36	5.61	6.15	6.10	5.94	6.06
AMDMUOx	2.87	6.68	12.57	12.98	12.90	12.98	13.02	13.47
BUSINVx	3.07	6.05	7.24	7.31	7.39	7.40	7.75	7.65
ISRATIOx	2.89	4.85	4.89	4.97	4.92	5.71	5.86	5.69
M1SL	2.75	6.10	11.48	11.55	11.60	11.55	11.61	11.72
M2SL	2.89	7.04	7.78	8.84	8.83	10.32	10.28	10.56
M2REAL	2.84	5.97	6.36	6.31	7.23	7.05	7.05	7.08
AMBSL	2.58	3.66	6.68	6.73	6.73	7.15	7.47	7.49
TOTRESNS	2.60	3.88	7.72	7.75	7.75	8.40	8.34	8.07

Table 2 continues on the next page.

NONBORRES	2.66	5.12	10.81	10.99	11.03	11.02	12.67	11.42
BUSLOANS	2.72	7.83	13.15	13.16	13.43	13.75	13.64	13.65
REALLN	2.65	7.49	14.10	14.13	14.13	16.08	15.86	16.02
NONREVSL	2.86	7.07	10.15	10.18	10.18	10.10	10.73	10.79
CONSPI	2.72	5.42	6.12	6.67	7.94	8.25	8.17	8.10
S.P.500	2.85	4.48	6.81	6.83	6.84	7.22	7.11	7.03
S.P..indust	2.79	4.35	6.62	6.80	6.84	6.84	6.93	6.95
S.P.div.yield	3.17	4.88	5.12	5.23	6.63	6.70	6.62	6.63
S.P.PE.ratio	3.75	5.22	6.37	6.39	6.70	7.24	7.18	7.20
FEDFUNDS	3.84	5.73	6.29	6.95	7.04	7.06	7.02	7.14
CP3Mx	3.41	5.99	7.06	7.09	7.06	7.48	7.55	7.53
TB3MS	3.83	5.77	7.25	7.28	8.00	8.12	8.42	7.84
TB6MS	3.52	6.07	7.82	7.76	7.94	8.08	7.97	8.13
GS1	3.37	5.92	8.22	8.36	8.18	8.28	8.34	8.35
GS5	2.57	5.90	8.42	9.12	8.87	10.15	10.24	10.18
GS10	2.48	3.49	4.26	4.32	5.54	5.59	5.57	5.54
AAA	2.57	4.73	5.19	5.27	5.56	6.10	6.08	6.05
BAA	2.80	6.43	8.21	8.29	9.22	9.36	9.44	9.69
COMPAPFFx	3.56	4.47	4.56	5.32	5.67	5.68	5.86	5.49
TB3SMFFM	3.82	4.49	5.44	6.80	7.27	7.11	6.81	6.86
TB6SMFFM	2.97	3.10	6.33	6.71	7.41	7.39	7.27	8.07
T1YFFM	2.65	5.60	7.27	10.46	11.02	11.24	11.96	10.96
T5YFFM	2.52	4.20	9.29	9.36	9.38	10.03	9.55	9.68
T10YFFM	2.49	3.40	10.35	10.46	10.34	10.46	10.45	10.35
AAAFFM	2.55	3.14	5.24	5.35	5.28	5.61	6.01	6.16
BAAFFM	2.94	3.45	5.25	5.24	5.33	5.90	5.87	5.93
TWEXMMTH	2.62	6.81	9.40	9.37	10.61	11.56	11.52	11.56
EXSZUSx	2.69	6.20	7.42	7.48	7.47	8.03	7.99	8.05
EXJPUSx	2.66	8.02	17.44	17.67	17.65	17.70	18.56	17.69
EXUSUKx	2.84	3.33	7.50	7.59	8.03	8.26	8.19	8.20
EXCAUSx	2.67	7.07	9.85	9.83	11.44	11.86	11.36	11.11
WPSFD49207	2.58	5.79	7.54	7.55	7.57	8.62	8.62	8.57
WPSFD49502	2.57	5.86	7.63	8.50	8.41	8.74	8.66	8.66
WPSID61	2.58	5.25	6.28	6.28	6.34	6.61	7.34	7.36
WPSID62	2.60	4.19	4.63	4.61	5.25	5.42	5.67	5.68
OILPRICEx	2.57	4.16	4.41	5.49	5.89	6.16	6.12	5.89
PPICMM	2.89	3.84	4.55	4.65	5.25	5.81	5.77	5.73
CPIAUCSL	2.58	3.72	4.52	4.81	5.10	5.57	5.48	5.58
CPIAPPSL	2.59	5.85	11.49	11.61	11.63	11.61	11.64	11.62
CPITRNSL	2.57	5.40	6.23	6.23	6.93	7.09	7.11	7.11
CPIMEDSL	2.72	4.11	6.09	6.13	6.12	6.49	6.89	7.09
CUSR0000SAC	2.52	5.75	7.48	8.10	8.03	8.38	8.25	8.24
CUUR0000SAD	2.62	8.11	19.35	19.41	19.48	19.43	19.48	20.10
CUSR0000SAS	2.68	7.91	24.72	25.02	24.85	25.76	25.95	26.74
CPIULFSL	2.57	3.54	3.99	4.03	5.11	5.24	5.16	5.03
CUUR0000SA0L2	2.52	4.71	5.61	6.73	7.07	7.12	7.16	7.30
CUSR0000SA0L5	2.56	3.75	4.54	5.40	4.59	5.47	5.42	5.42
PCEPI	2.56	3.83	4.67	5.26	5.33	5.52	5.37	5.42
DDURRG3M086SBEA	2.52	6.92	11.22	11.39	11.37	11.30	11.39	11.54

Table 2 continues on the next page.

Regressor	i.i.d.	AR(1)	AR(2)	AR(5)	AR(10)	AR(25)	AR(50)	AR(99)
DNDGRG3M086SBEA	2.54	5.59	7.22	7.83	7.85	8.23	8.23	8.17
DSERRG3M086SBEA	2.83	7.37	24.25	24.92	24.89	24.36	24.94	26.66
CES0600000008	2.64	4.65	10.56	11.00	11.08	11.10	11.07	11.38
CES2000000008	2.52	4.63	9.76	9.91	9.90	9.97	12.22	10.34
CES3000000008	2.56	5.15	14.04	14.28	14.22	14.19	14.16	14.75
UMCSENTx	2.53	3.67	5.44	5.46	5.69	6.21	6.11	6.28
MZMSL	2.81	6.36	6.77	6.96	8.63	8.81	8.62	8.57
DTCOLNVHFNM	3.04	5.47	7.27	7.26	7.43	8.93	9.00	8.98
DTCTHFNM	3.26	7.24	9.96	10.07	10.14	10.66	10.83	10.82
INVEST	2.53	3.91	4.49	4.61	4.98	5.40	5.45	5.46
VXOCLSx	2.90	3.45	4.83	4.86	4.87	5.34	5.30	5.38

Table 3: Critical Values guaranteeing size ≤ 0.05 for the t-type tests corresponding to $T_{E,W}$ obtained by an application of Algorithm 1.

Regressor	i.i.d.	AR(1)	AR(2)	AR(5)	AR(10)	AR(25)	AR(50)	AR(99)
RPI	2.24	4.23	5.20	5.27	7.18	7.36	7.29	7.14
W875RX1	2.28	4.08	4.95	5.26	6.07	6.10	6.09	6.04
DPCERA3M086SBEA	2.28	5.21	7.61	7.54	7.53	10.61	10.51	10.71
CMRMTSPLx	2.22	3.58	4.43	4.48	4.49	4.48	4.57	4.50
RETAILx	2.19	3.23	3.81	3.87	3.85	3.88	3.93	3.99
INDPRO	2.23	3.16	3.81	3.89	3.88	3.97	3.99	3.98
IPFPNSS	2.24	3.19	3.60	3.64	3.61	3.63	3.66	3.67
IPFINAL	2.20	3.08	3.42	3.45	3.49	3.73	3.79	3.77
IPCONGD	2.20	3.74	4.58	4.58	5.08	5.12	5.21	5.21
IPDCONGD	2.16	3.15	3.77	3.77	3.84	3.95	3.99	3.97
IPNCONGD	2.21	4.27	5.36	5.30	6.49	6.80	6.77	6.62
IPBUSEQ	2.24	3.46	4.18	4.20	4.21	4.62	4.59	4.60
IPMAT	2.24	3.27	4.22	4.33	4.33	4.60	4.67	4.62
IPDMAT	2.27	3.17	3.55	3.66	3.67	4.15	4.19	4.16
IPNMAT	2.15	2.54	4.07	4.06	4.25	4.29	4.32	4.34
IPMANSICS	2.23	2.94	3.27	3.28	3.32	3.33	3.38	3.40
IPB51222S	2.08	2.44	3.59	3.61	4.64	4.66	4.67	4.69
IPFUELS	2.10	2.50	5.55	5.57	5.58	5.61	5.73	5.64
CUMFNS	2.25	3.06	3.70	3.78	3.80	4.22	4.25	4.26
HWI	2.29	4.28	5.72	5.83	5.70	5.82	5.84	5.98
HWIURATIO	2.34	5.60	10.47	10.66	10.66	10.64	10.81	10.78
CLF16OV	2.29	5.69	9.47	9.26	9.28	11.80	12.00	12.29
CE16OV	2.33	5.77	11.70	11.69	11.78	11.85	11.84	12.23
UNRATE	2.30	5.18	9.48	9.78	9.75	9.77	9.64	9.78
UEMPMEAN	2.41	7.79	29.75	31.05	31.16	31.12	31.09	31.05
UEMPLT5	2.08	2.83	3.00	3.66	3.66	3.70	3.77	3.75
UEMP5TO14	2.15	2.56	3.24	3.28	3.37	3.59	3.66	3.66
UEMP15OV	2.35	5.53	11.44	11.97	11.89	11.67	11.82	11.88
UEMP15T26	2.25	3.23	3.53	3.54	3.62	3.79	3.81	3.83
UEMP27OV	2.38	6.11	14.43	15.28	15.24	15.17	14.98	15.24
CLAIMSx	2.18	2.61	3.06	3.04	3.04	3.33	3.36	3.35
PAYEMS	2.34	5.71	11.68	12.07	11.88	11.84	11.97	12.11

Table 3 continues on the next page.

USGOOD	2.32	5.29	9.52	9.88	9.78	9.93	9.99	9.92
CES1021000001	2.38	5.22	10.61	10.69	10.84	10.83	10.82	10.92
USCONS	2.36	6.20	16.16	17.26	17.25	17.27	16.92	17.07
MANEMP	2.31	4.75	7.32	7.54	7.55	7.50	7.53	7.59
DMANEMP	2.31	4.54	6.66	6.86	6.84	6.86	6.88	6.94
NDMANEMP	2.32	5.55	10.60	10.78	10.65	10.71	10.65	11.03
SRVPRD	2.34	5.97	13.18	13.26	13.50	13.50	13.69	13.69
USTPU	2.32	5.43	9.96	10.21	10.04	10.22	10.24	10.38
USWTRADE	2.33	5.21	8.66	8.92	8.88	8.92	9.12	9.14
USTRADE	2.32	5.46	9.90	10.04	9.97	10.01	10.20	10.28
USFIRE	2.34	6.43	16.43	16.32	16.61	16.85	17.20	17.68
USGOVT	2.33	6.25	11.95	12.00	12.01	11.99	11.98	12.48
CES0600000007	2.27	4.20	4.56	4.56	5.15	5.33	5.39	5.37
AWOTMAN	2.31	3.94	4.12	4.21	4.62	5.02	5.06	5.05
AWHMAN	2.29	3.87	4.01	4.00	4.00	4.64	4.64	4.62
HOUST	2.17	4.38	6.78	6.91	6.94	6.96	6.94	6.91
HOUSTNE	2.07	3.38	4.44	4.42	4.50	4.51	5.16	4.75
HOUSTMW	2.06	3.03	3.48	3.54	3.92	3.88	3.98	4.01
HOUSTS	2.14	4.09	5.72	5.83	5.72	5.82	7.04	5.96
HOUSTW	2.14	4.25	6.41	6.53	6.41	6.94	7.78	6.86
PERMIT	2.19	4.29	7.17	7.21	7.20	7.18	7.31	7.22
PERMITNE	2.06	3.04	3.97	4.01	4.01	4.08	4.06	5.00
PERMITMW	2.16	3.75	4.61	4.60	4.67	5.89	4.82	4.92
PERMITS	2.20	4.29	6.61	6.69	6.80	6.78	6.89	6.81
PERMITW	2.16	4.32	6.78	6.96	7.00	7.20	7.45	7.47
ACOGNO	2.32	4.50	6.15	6.14	6.15	6.47	6.66	6.61
AMDMNOx	2.18	3.14	3.40	3.48	3.51	4.09	4.09	4.09
ANDENOx	2.15	3.75	4.57	4.65	4.60	4.87	4.89	4.86
AMDMUOx	2.36	4.93	9.68	9.83	9.83	9.81	9.84	9.84
BUSINVx	2.34	4.19	5.45	5.49	5.62	5.60	5.65	5.62
ISRATIOx	2.32	4.34	4.74	4.74	5.10	5.16	5.28	5.23
M1SL	2.31	4.13	5.12	5.10	5.12	5.62	5.71	5.69
M2SL	2.34	5.03	5.89	5.98	6.75	7.15	7.22	7.16
M2REAL	2.30	4.49	4.83	4.85	4.87	5.65	5.69	5.69
AMBSL	2.25	3.30	4.39	4.40	5.20	5.28	5.30	5.31
TOTRESNS	2.25	3.40	4.51	4.57	5.16	5.29	5.26	5.29
NONBORRES	2.31	4.19	5.07	5.38	5.42	6.81	6.87	7.00
BUSLOANS	2.42	6.73	14.66	14.81	14.85	14.86	14.77	15.11
REALLN	2.40	6.50	11.81	11.89	11.86	12.55	12.63	12.73
NONREVSL	2.33	5.75	10.07	9.95	10.23	10.26	10.26	10.34
CONSPI	2.25	4.48	5.27	5.98	5.28	6.42	6.40	6.32
S.P.500	2.18	3.53	4.36	4.42	4.42	4.48	4.91	4.88
S.P..indust	2.22	3.46	4.30	4.30	4.34	4.78	4.84	4.77
S.P.div.yield	2.30	3.11	3.81	3.87	3.87	3.88	3.98	4.00
S.P.PE.ratio	2.28	3.06	3.59	3.63	3.65	4.16	4.19	4.16
FEDFUNDS	2.20	3.77	5.06	5.04	5.08	5.33	5.33	5.38
CP3Mx	2.27	4.32	6.17	6.66	6.19	6.59	6.62	6.63
TB3MS	2.18	3.73	5.08	5.13	5.13	5.38	5.44	5.44
TB6MS	2.23	4.22	6.25	6.18	6.11	6.53	6.62	6.59

Table 3 continues on the next page.

GS1	2.24	4.64	7.32	7.29	7.31	7.56	7.46	7.66
GS5	2.25	4.64	5.41	5.47	5.62	5.70	5.72	5.86
GS10	2.21	3.33	3.82	3.83	4.25	4.34	4.43	4.36
AAA	2.25	3.71	3.86	3.84	3.97	5.11	5.17	5.12
BAA	2.28	4.44	5.08	5.85	6.66	6.61	6.64	6.72
COMPAPFFx	2.18	3.95	5.19	5.21	5.16	5.25	5.35	5.36
TB3SMFFM	2.12	3.09	3.72	3.72	3.73	3.80	3.77	3.79
TB6SMFFM	2.09	2.98	3.02	3.13	3.64	3.75	3.81	3.78
T1YFFM	2.22	5.43	7.25	7.52	7.53	7.65	7.63	7.60
T5YFFM	2.27	3.74	8.02	8.04	8.05	8.05	8.04	8.09
T10YFFM	2.24	3.23	6.37	6.42	6.41	7.18	7.09	7.07
AAAFFM	2.21	2.90	4.16	4.17	4.18	5.99	5.96	5.94
BAAFFM	2.18	2.75	3.54	3.66	4.37	4.42	4.42	4.35
TWEXMMTH	2.36	5.72	7.62	7.67	7.68	8.27	8.30	8.17
EXSZUSx	2.31	4.82	5.41	5.99	6.06	6.18	6.16	6.18
EXJPUSx	2.39	6.16	11.80	11.88	11.91	11.88	11.89	13.36
EXUSUKx	2.14	2.85	3.81	3.85	3.92	4.55	4.46	4.55
EXCAUSx	2.38	6.25	9.42	9.50	9.50	9.93	9.93	9.91
WPSFD49207	2.35	5.43	8.39	8.43	8.44	8.75	8.63	8.75
WPSFD49502	2.34	5.52	8.33	8.43	8.42	8.83	8.89	8.87
WPSID61	2.35	5.17	6.96	6.98	6.98	7.77	7.77	7.83
WPSID62	2.31	4.37	5.40	5.40	6.28	6.39	6.30	6.29
OILPRICEx	2.28	4.31	5.38	5.44	5.44	5.91	6.08	6.09
PPICMM	2.23	3.02	3.44	3.53	3.90	4.04	4.03	4.03
CPIAUCSL	2.27	4.12	5.96	6.01	6.01	6.39	6.15	6.13
CPIAPPSL	2.35	5.35	8.58	8.66	8.67	8.66	8.85	8.78
CPITRNSL	2.34	5.48	8.04	8.21	8.21	8.35	8.38	8.32
CPIMEDSL	2.21	3.51	4.68	4.67	4.70	5.09	5.14	5.13
CUSR0000SAC	2.36	5.69	9.44	9.55	9.49	9.57	9.54	9.55
CUUR0000SAD	2.35	6.43	10.58	10.73	10.72	10.90	10.93	11.38
CUSR0000SAS	2.38	7.33	24.37	24.85	24.62	25.00	25.95	27.12
CPIULFSL	2.24	3.98	5.62	5.68	5.68	6.08	6.06	5.86
CUUR0000SA0L2	2.31	4.93	7.21	7.30	7.29	7.57	7.29	7.53
CUSR0000SA0L5	2.27	4.16	5.96	6.00	6.00	6.34	6.36	6.11
PCEPI	2.29	4.26	6.19	6.26	6.27	6.44	6.45	6.42
DDURRG3M086SBEA	2.31	6.11	9.59	9.66	9.66	9.66	10.04	9.66
DNDGRG3M086SBEA	2.35	5.66	9.13	9.26	9.27	9.26	9.28	9.30
DSERRG3M086SBEA	2.35	6.46	18.82	19.26	19.12	19.04	19.66	20.80
CES0600000008	2.23	4.24	8.65	8.78	8.65	8.77	8.71	9.32
CES2000000008	2.19	4.31	8.97	9.23	9.19	9.20	10.28	10.31
CES3000000008	2.29	4.78	12.70	12.96	13.03	13.01	13.02	13.63
UMCSENTx	2.10	2.65	3.43	3.58	4.09	4.14	4.11	4.13
MZMSL	2.39	5.28	6.18	6.27	7.22	6.95	7.11	7.19
DTCOLNVHFNM	2.23	3.47	3.95	3.98	4.02	3.99	4.08	4.12
DTCTHFNM	2.29	4.32	5.17	5.27	5.38	5.52	5.48	5.54
INVEST	2.22	3.34	3.90	4.00	4.43	4.43	4.43	4.39
VXOCLSx	2.16	2.82	2.91	3.59	3.07	3.76	3.80	3.91

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