Ergodicity conditions for a double mixed Poisson autoregression

Aknouche, Abdelhakim and Demouche, Nacer

USTHB and Qassim University, USTHB

3 March 2018

Ergodicity conditions for a double mixed Poisson autoregression

Abdelhakim Aknouche*† and Nacer Demouche*

March 3, 2018

Abstract

We propose a double mixed Poisson autoregression in which the intensity, scaled by a unit mean independent and identically distributed (*iid*) mixing process, has different regime specifications according to the state of a finite unobserved *iid* chain. Under some *contraction in mean* conditions, we show that the proposed model is strictly stationary and ergodic with a finite mean. Applications to various count time series models are given.

Keywords: Double mixed Poisson autoregression, negative binomial mixture INGARCH model, ergodicity, weak dependence, contraction in mean.

1. Introduction

Count time series analysis has recently seen an "explosive" interest (see e.g. Davis and Liu, 2016) where numerous models and methods have been introduced. Zhu et *al.* (2010) proposed a Poisson (finite) mixture integer-valued ARCH (Mixture INARCH : MINARCH(q)) model with an independent and identically distributed (*iid*) mixing sequence. In this model, the

*Faculty of Mathematics, University of Science and Technology Houari Boumediene, Algeria.
†Mathematics Department, Qassim University, Saudi Arabia.

conditional distribution is a finite mixture of Poisson distributions where the intensity of each component (or regime) is a linear function of the q lagged observations. The MINARCH model essentially aims at accounting for multimodality of the marginal distribution, a fact that is frequently observed in real applications. It turns out that this model may also represent other well-known characteristics of count time series frequently observed in practice such as conditional overdispersion and asymmetry. To gain in model parsimony, Diop et *al.* (2016) generalized Zhu et *al.*'s (2010) model so as to include lagged values of each regime intensity. The generalization (Mixture INGARCH : MINGARCH(p,q)) was made in the spirit of Haas et *al.* (2004) so that each regime-specific has its own INGARCH dynamic. Specifically, all lagged intensities in each regime are conditioned on the current value of the regime process. This device, called by Aknouche and Rabehi (2010) the *present mixture*, avoids the model to have the well-known path dependence problem (see e.g. Haas et *al.* 2004 in the mixture real-valued GARCH case). In particular, it easily allows to estimate the parameters using the EM algorithm. Zhu et *al.* (2010) and Diop et *al.* (2016) studied the properties in mean and the autocovariance structure of their models. However, some important path-properties such as strict stationarity and ergodicity, which are essential for asymptotic inference, have not been considered.

In this paper we study the ergodicity of a double mixed Poisson autoregression that generalizes Zhu et *al.*'s (2010) model in three directions: i) Firstly, the conditional distribution of the proposed model is a superposition of two mixtures of Poisson distributions. The first mixture allows for finitely many regime specifications for the intensity. It is described by a finite-valued *iid* sequence called the regime process. The second mixture, which is a scaling factor of the intensity, controls the conditional distribution of each regime (Poisson, negative binomial, Poisson-inverse Gaussian...). ii) Secondly, the model permits the inclusion of lagged values of the intensity in each regime which are rather driven by the lagged values (in the respective order) of the regime sequence (see Example 2.4 below). Our specification is then different from the one of Diop et *al.* (2016) and is characterized by the path dependence of the intensity. iii) Thirdly, the intensity of each regime is a general function of its lagged values and of the observations. In particular, infinite linear or nonlinear INARCH(∞) representations are allowed.

Thus the model we propose is quite general and appears to have a great potential flexibility compared to the Poisson mixture case at the cost of just a few additional parameters. From the statistical point of view, while this model has the path dependence problem which makes the maximum likelihood estimation infeasible, it may in principle be estimated by other quite comparable estimation methods such as the generalized method of moment and Bayesian MCMC methods (see Francq and Zakoïan, 2008 and Bauwens et *al.*, 2010 for similar real-valued mixture GARCH cases).

Under some *contraction in mean* conditions we show that the proposed model admits a strictly stationary and ergodic solution with a finite mean. In some cases, the sufficient conditions are also necessary for ergodicity. Our analysis follows the weak dependence approach by Doukhan and Wintenberger (2008) and Doukhan et *al.* (2012); see also Aknouche et *al.* (2018).

The rest of this note is outlined as follows. Section 2 defines the model and gives some important examples. Section 3 proposes contraction in mean conditions for ergodicity of two important subclasses: the double mixed generalized INARCH(∞) model and the double mixed generalized INGARCH$(1,1)$ model. Applications to certain important subclasses of count time series models are considered. Section 4 concludes while proofs of the main results are postponed to Section 5.

2. Double mixed Poisson autoregression: structure and examples

Consider an *iid* sequence of unobservable random variables, $\{\Delta_t, t \in \mathbb{Z}\}$, valued in the finite set $\{1, ..., K\}$ ($K \in \mathbb{N}^* = \{1, 2, ...\}$) with probability mass function $P(\Delta_t = k) = \pi(k)$, where $\pi(k) \geq 0$ and $\sum_{k=1}^{K} \pi(k) = 1$. The values taken by Δ_t are called regimes or components whereas the probabilities $(\pi(k))_{1 \leq k \leq K}$ are referred to as the mixing proportions. Assume also that for all $1 \leq k \leq K$, $\{Z_t(k), t \in \mathbb{Z}\}$ is an *iid* sequence of positive random variables with unit mean and variance $\sigma^2(k) \geq 0$. In contrast with the regime variable Δ_t which should be finite, the mixing variables $(Z_t(k))_{1 \leq k \leq K}$ may be discrete or absolutely continuous, although they are frequently taken to be absolutely continuous.

Let $\{N_t(.), t \in \mathbb{Z}\}$ be an independent sequence of homogeneous Poisson processes with unit intensity. An integer-valued stochastic process $\{Y_t, t \in \mathbb{Z}\}$ is said to be a *double mixed Poisson autoregressions* with an independent regime switching if it is a solution to the following equation

$$Y_t = N_t(Z_t(\Delta_t)\lambda_t), \quad \lambda_t = f_{\Delta_t}(Y_{t-1}, ..., Y_{t-q}, \lambda_{t-1}, ..., \lambda_{t-p}; \theta(\Delta_t)), \quad t \in \mathbb{Z}, \tag{2.1}$$

where $p, q \in \mathbb{N} \cup \{\infty\}$ and $\{\theta(1), ..., \theta(K)\}$ is a set of real parameter vectors with $\theta(k) \in \Theta_k \subset \mathbb{R}^{m_k}$ ($m_k \in \mathbb{N}^*$). The function $f_k : \mathbb{N} \times (0, \infty) \times \Theta_k \to (0, \infty)$ is measurable and positive real-valued ($1 \leq k \leq K$). It is assumed that $\{N_t(.), t \in \mathbb{Z}\}$, $\{\Delta_t, t \in \mathbb{Z}\}$ and $\{Z_t(k), t \in \mathbb{Z}\}$ ($1 \leq k \leq K$) are independent. Two particular cases of the orders in (2.1) are emphasized. The first one is the infinite generalized INARCH(∞) form for which $p = 0$ and $q = \infty$, i.e.

$$Y_t = N_t(Z_t(\Delta_t)\lambda_t), \quad \lambda_t = f_{\Delta_t}(Y_{t-1}, Y_{t-2}, ...; \theta(\Delta_t)), \quad t \in \mathbb{Z}, \tag{2.2}$$

and the second one is the generalized INGARCH$(1, 1)$ specification corresponding to $p = q = 1$, i.e.

$$Y_t = N_t(Z_t(\Delta_t)\lambda_t), \quad \lambda_t = f_{\Delta_t}(Y_{t-1}, \lambda_{t-1}; \theta(\Delta_t)), \quad t \in \mathbb{Z}. \tag{2.3}$$

The term "generalized" is introduced in order to point out the general functional form of $(f_k)_{1 \leq k \leq K}$. Letting $\mathcal{F}_t$ be the σ-algebra generated by $\{(Y_t, \Delta_t), (Y_{t-1}, \Delta_{t-1}), ...\}$, model (2.1) may be written in the following conditional distribution form

$$\begin{aligned} P(Y_t = y_t / \mathcal{F}_{t-1}) &= \sum_{k=1}^{K} \pi(k) \int_{-\infty}^{+\infty} e^{-z(k)\lambda_t(k)} \frac{(z(k)\lambda_t(k))^{y_t}}{y_t!} dF_{Z_t(k)}(z(k)), \quad y_t \in \mathbb{N}, & (2.4a) \\ \lambda_t(k) &:= f_k(Y_{t-1}, ..., Y_{t-q}, \lambda_{t-1}, ..., \lambda_{t-p}; \theta(k)), & (2.4b) \end{aligned}$$

where $F_{Z_t(k)}(.)$ is the cumulative distribution of $Z_t(k)$.

It turns out that model (2.4) consists of a "composition" of two mixtures of Poisson distributions with intensities $(\lambda_t(k))_{1\leq k\leq K}$ satisfying K specific-regime generalized INGARCH representations. This is why model (2.1) is called *double mixed Poisson autoregression.* The first mixture, driven by Δ_t, governs the intensity λ_t while allowing for regime switching. The second one, materialized by $Z_t(k)$, is a scaling factor for the k-th component intensity and is designed to control the distribution of that component. As will be seen, the distribution of $Z_t(k)$ does not influence neither the conditional mean of the model (cf. $(2.5a)$) nor the ergodicity conditions for the model (cf. (3.1) and (3.5)). In contrast with the one-regime Poisson autoregression (e.g. Doukhan et *al.*, 2012; Davis and Liu, 2016), λ_t in (2.1), which may also be written as $\lambda_t(\Delta_t)$, is not $\mathcal{F}_{t-1}$-measurable. In fact, provided that Δ_t is non-degenerate, equation (2.1) is a subclass of *parameter-driven* models in the sense of Cox (1981).

Under the properties given above, the conditional mean and conditional variance of model (2.1) are given as follows:

$$\mathrm{E}(Y_t/\mathcal{F}_{t-1}) = \sum_{k=1}^{K} \pi(k)\,\lambda_t(k), \tag{2.5a}$$

$$\mathrm{Var}(Y_t/\mathcal{F}_{t-1}) = \sum_{k=1}^{K} \pi(k)\left(\lambda_t(k)+\sigma^2(k)\,\lambda_t^2(k)\right)+\sum_{k=1}^{K}\pi(k)\,\lambda_t^2(k)-\left(\sum_{k=1}^{K}\pi(k)\,\lambda_t(k)\right)^2 \tag{2.5b}$$

where $\lambda_t(k)$ is given by $(2.4b)$. Note that model (2.1) is quite general because of the wide range of possible conditional distributions of Y_t given $\mathcal{F}_{t-1}$. These distributions can be given explicitly for some specific laws of $(Z_t(k))_{1\leq k\leq K}$ and Δ_t. For example, when both Δ_t and $(Z_t(k))_{1\leq k\leq K}$ are degenerate at 1, model (2.1) is just the Poisson autoregression (e.g. Doukhan et *al.*, 2012). When Δ_t is degenerate at 1 (i.e. $K=1$), $Z_t(k)$ is simply written as Z_t and model (2.1) reduces to the mixed Poisson autoregression proposed by Christou and Fokianos (2014). Other notable particular cases of (2.1) are given as follows.

Example 2.1 (Poisson (finite) mixture autoregression) When $Z_t(k)$ is degenerate at 1 for all k, the conditional distribution of model (2.1) reduces to a (finite) mixture of Poisson distributions (see Zhu et *al.*, 2010 for the particular mixture INARCH(q) model), i.e. $Y_t/\mathcal{F}_{t-1} \sim \sum_{k=1}^{K}\pi(k)\,\mathcal{P}(\lambda_t(k))$, where $\lambda_t(k)$ is given by $(2.4b)$ and $\mathcal{P}(\lambda)$ stands for the Poisson distribution with parameter $\lambda > 0$. The conditional mean and conditional variance of Y_t are given by (2.5) while taking $\sigma^2(k) = 0$ for all k. This model also allows for conditional overdispersion provided that $K > 1$. $\square$

Example 2.2 (Negative binomial mixture autoregression) When $Z_t(k) \sim \mathcal{G}(\sigma^{-2}(k), \sigma^{-2}(k))$ ($\sigma^2(k) > 0$, $1 \leq k \leq K$), the conditional distribution of model (2.1) reduces to a finite mixture of negative binomial distributions, i.e. $Y_t/\mathcal{F}_{t-1} \sim \sum_{k=1}^{K} \pi(k) \mathcal{NB}\left(\sigma^{-2}(k), \frac{\sigma^{-2}(k)}{\sigma^{-2}(k)+\lambda_t(k)}\right)$, where $\mathcal{NB}(r,p)$ and $\mathcal{G}(a,b)$ denote respectively the negative binomial distribution with parameters $r > 0$ and $p \in (0,1)$, and the Gamma distribution with shape $a > 0$ and rate $b > 0$. The conditional mean and conditional variance of Y_t are given by (2.5), so this model is conditionally overdispersed even when $K = 1$. $\square$

In view of (2.5), it turns out that in the general case where both $Z_t(k)$ and Δ_t are non-degenerate, model (2.1) allows for conditional overdispersion with an order of magnitude greater than the one obtained by both the mixed Poisson autoregression (Christou and Fokianos, 2014) and the Poisson mixture autoregression (Example 2.1). This shows the great flexibility of model (2.1).

Other well-known conditional distributions of Y_t may be obtained from the specification of the distribution of the mixing variable $Z_t(k)$. For instance, if $Z_t(k)$ is distributed as an inverse-Gaussian then $Y_t/\mathcal{F}_{t-1}$ follows a finite mixture of Poisson-inverse Gaussian (cf. Dean et *al.*, 1989). Moreover, if the distribution of $Z_t(k)$ is log-normal then the conditional distribution of Y_t is a mixture of Poisson-log-normal (cf. Hind, 1992). Beside the wide range of allowed conditional distributions, the generality of model (2.1) also stems from the general form of the regime functional forms $(f_k)_{1 \leq k \leq K}$ which may be linear or nonlinear. Some important cases of these forms are as follows.

Example 2.3 (Double mixed Poisson INARCH(∞)) When $(p,q) = (0,\infty)$ and f_k ($1 \leq k \leq K$) are linear in $Y_{t-1}, Y_{t-2}...$, we get the following double mixture infinite INARCH(∞) specification

$$\lambda_t = \alpha_0(\Delta_t) + \sum_{j=1}^{\infty} \alpha_j(\Delta_t) Y_{t-j}, \quad t \in \mathbb{Z}, \tag{2.6a}$$

where $\alpha_0(k) > 0$ and $\alpha_j(k) \geq 0$ ($1 \leq k \leq K$).

$i)$ When $Z(k)$ is degenerate at 1 for all $1 \leq k \leq K$, model (2.6a) is just the Poisson mixture INARCH(∞) model which is an infinite-order version of the Poisson mixture INARCH(q) model

$$\lambda_t = \alpha_0(\Delta_t) + \sum_{j=1}^{q} \alpha_j(\Delta_t) Y_{t-j}, \quad t \in \mathbb{Z}, \tag{2.6b}$$

introduced by Zhu et *al.* (2010).

$ii)$ When $Z_t(k) \sim G(\sigma^{-2}(k), \sigma^{-2}(k))$ $(\sigma^2(k) > 0,\ 1 \leq k \leq K)$ we call the resulting model negative binomial mixture INARCH(∞). $\square$

Example 2.4 (Double mixed Poisson INGARCH$(1,1)$ model) A leading example of (2.3) is the double mixed Poisson INGARCH$(1,1)$ model given by the linear forms: $f_k = f$ $(1 \leq k \leq K)$ with $p = q = 1$ and $f(y, \lambda; \theta(k)) = \omega(k) + \alpha(k) y + \beta(k) \lambda$, $(1 \leq k \leq K)$, i.e.

$$Y_t = N_t(Z_t(\Delta_t) \lambda_t(\Delta_t)), \quad \lambda_t = \omega(\Delta_t) + \alpha(\Delta_t) Y_{t-1} + \beta(\Delta_t) \lambda_{t-1}, \quad t \in \mathbb{Z}, \tag{2.7}$$

where $\theta(k) = (\omega(k), \alpha(k), \beta(k))' \in (0, \infty)^3$ for $1 \leq k \leq K$.

$i)$ When $Z(k)$ is degenerate at 1 for all $1 \leq k \leq K$, model (2.7) reduces to a Poisson mixture INGARCH$(1,1)$ model. As emphasized in the introduction, this model is different from the Poisson mixture INGARCH proposed by Diop et *al.* (2016) which in the case $p = q = 1$ has the following specification

$$Y_t = N_t(Z_t(\Delta_t) \lambda_t(\Delta_t)), \quad \lambda_t(\Delta_t) = \omega(\Delta_t) + \alpha(\Delta_t) Y_{t-1} + \beta(\Delta_t) \lambda_{t-1}(\Delta_t), \quad t \in \mathbb{Z}. \tag{2.8}$$

The difference between (2.7) and (2.8) is due to the term, $\beta(\Delta_t) \lambda_{t-1} = \beta(\Delta_t) \lambda_{t-1}(\Delta_{t-1})$, in (2.7) which is different from the "present mixture" term, $\beta(\Delta_t) \lambda_{t-1}(\Delta_t)$, in Diop et *al.*'s (2016) model (see also Aknouche and Rabehi, 2010).

$ii)$ When $Z_t(k) \sim G(\sigma^{-2}(k),\ \sigma^{-2}(k))$ $(\sigma^2(k) > 0,\ 1 \leq k \leq K)$ we call the resulting model, negative binomial mixture INGARCH$(1,1)$. The latter is a finite mixture extension of the negative binomial $INGARCH(1,1)$ model (e.g. Zhu, 2011; Christou and Fokianos, 2014). Note that unless $K = 1$, model (2.7) is not a particular case of (2.6a). Indeed, by successive substitution in (2.7) under the requirement $\sum_{k=1}^{K} \pi(k) \log \beta(k) < 0$, we obtain the following INARCH(∞) form

$$\lambda_t = \alpha_0(\underline{\Delta}_{t,\infty}) + \alpha_1(\underline{\Delta}_{t,1}) Y_{t-1} + \alpha_2(\underline{\Delta}_{t,2}) Y_{t-1} + ..., \quad t \in \mathbb{Z}, \tag{2.9}$$

where $\alpha_0\left(\underline{\Delta}_{t,\infty}\right) = \sum_{j=0}^{\infty}\prod_{i=0}^{j-1}\beta\left(\Delta_{t-i}\right)\omega\left(\Delta_{t-j}\right)$, $\alpha_j\left(\underline{\Delta}_{t,j}\right) = \prod_{i=0}^{j-1}\beta\left(\Delta_{t-i}\right)\alpha\left(\Delta_{t-j}\right)$ $(j \in \mathbb{N}^*)$, $\underline{\Delta}_{t,j} = (\Delta_t, ..., \Delta_{t-j+1})'$ and $\underline{\Delta}_{t,\infty} = (\Delta_t, \Delta_{t-1}, ...)'$. The main difference between (2.9) and (2.6a) is that $\left\{\underline{\Delta}_{t,j}, t \in \mathbb{Z}, j \in \mathbb{N}\right\}$ is not *iid*, so (2.9) is not a particular case of (2.6a). $\square$

3. Ergodicity conditions

This Section proposes sufficient and/or necessary conditions on the functions $f_1, ..., f_K$ such that equation (2.1) with $(p,q) = (1,1)$ or $(p,q) = (0,\infty)$ admits a strictly stationary, ergodic and weakly dependent solution having a finite mean (see Dedecker and Prieur, 2004 for the definition of weak dependence).

3.1. Double mixed Poisson generalized $INARCH(\infty)$

For model (2.2), consider the following "*contraction in mean*" assumption:

A1 *For all* $k \in \{1, ..., K\}$ and $\mathbf{y} = (y_1, y_2, ...)$, $\mathbf{y}' = (y_1', y_2' ...) \in \mathbb{N}^\infty$,

$$\left|f_k\left(\mathbf{y};\theta(k)\right) - f_k\left(\mathbf{y}';\theta(k)\right)\right| \leq \sum_{i=1}^{\infty}\kappa_i(k)\left|y_i - y_i'\right|, \tag{3.1a}$$

where $\left(\kappa_i(k)\right)_{i\in\mathbb{N},1\leq k\leq K}$ *are non-negative constants satisfying*

$$\sum_{k=1}^{K}\pi(k)\sum_{i=1}^{\infty}\kappa_i(k) < 1. \tag{3.1b}$$

Condition (3.1a) means that the functions $f_1, ..., f_K$ are Lipschitz and satisfy the *contraction in mean* (3.1b). It is interesting to note that in the case $K > 1$, it is not necessary for all regime functional forms $f_k\left(\mathbf{y};\theta(k)\right)$ to be contracting. For the linear infinite INARCH(∞) form (cf. Example 2.3), **A1** results in

$$\sum_{k=1}^{K}\pi(k)\sum_{i=1}^{\infty}\alpha_i(k) < 1. \tag{3.2a}$$

Considering the finite-order mixture linear INARCH(q) model, **A1** reduces to

$$\sum_{k=1}^{K}\pi(k)\sum_{i=1}^{q}\alpha_i(k) < 1, \tag{3.2b}$$

which is the same stationarity in mean condition given by Zhu et *al.* (2010) for the Poisson mixture INARCH(q) case. Letting $\zeta_t = (N_t, \Delta_t, Z_t(\Delta_t))$ and $F(Y_{t-1}, Y_{t-2}...,; \zeta_t) = N_t(Z_t(\Delta_t) f_{\Delta_t}(Y_{t-1}, Y_{t-2}, ...; \theta(\Delta_t)))$, model (2.2) may be written as the following infinite chain (cf. Doukhan and Wintemberger, 2008)

$$Y_t = F(Y_{t-1}, Y_{t-2}, ...; \zeta_t), \quad t \in \mathbb{Z}, \tag{3.3}$$

where $\{\zeta_t, t \in \mathbb{Z}\}$ is *iid*. The following result establishes the ergodicity of model (2.2).

Theorem 3.1 *Under* (3.1)*, equation* (2.2) *admits a strictly stationary, ergodic and weakly dependent solution* $\{Y_t, t \in \mathbb{Z}\}$ *having a finite mean. Moreover, this solution is unique and is given by the following causal scheme*

$$X_t = H(\zeta_t, \zeta_{t-1}, ...), \quad t \in \mathbb{Z}, \tag{3.4}$$

for some measurable function $H : (\mathbb{N} \times (0, \infty) \times \{1, ..., K\})^{\mathbb{N}} \to \mathbb{N}$.

For the double mixed Poisson INARCH(∞) (2.6a), Theorem 3.1 simplifies as follows.

Corollary 3.1 *Under* (3.2a) *(resp.* (3.2b)*) the double mixed Poisson* INARCH(∞) *process (resp.* INARCH(q)*) given by* (2.6a) *(resp. given by* (2.6b)*) is ergodic, weakly dependent and has a finite mean.*

It easy to show that condition (3.2a) is also necessary for ergodicity of model (2.6a).

3.2. Double mixed Poisson generalized $INGARCH(1,1)$

For model (2.3), consider the following conditions:

A2 The functions $f_1, ..., f_K$ are Lipschitz, i.e., *for all* $k \in \{1, ..., K\}$, y, $y' \in \mathbb{N}$ *and* λ, $\lambda' \in (0, \infty)$,

$$|f_k(y, \lambda; \theta(k)) - f_k(y', \lambda'; \theta(k))| \leq \kappa(k)|y - y'| + \tau(k)|\lambda - \lambda'|, \tag{3.5a}$$

where $(\kappa(k))_{1\leq k\leq K}$ *and* $(\tau(k))_{1\leq k\leq K}$ *are positive constants satisfying one of the following contraction-type conditions*

$$\max_{1\leq k\leq K}\left(\tfrac{\kappa(k)+\tau(k)}{\kappa(k)}\right)\sum_{k=1}^{K}\pi(k)\kappa(k) \quad < \quad 1. \tag{3.5b}$$

$$\frac{1}{\min\limits_{1\leq k\leq K}\left(\frac{\tau(k)}{\kappa(k)}\right)}\sum_{k=1}^{K}\pi(k)(\kappa(k)+\tau(k))\tfrac{\tau(k)}{\kappa(k)} \quad < \quad 1. \tag{3.5c}$$

When $K = 1$, each one of (3.5b) and (3.5c) reduces to the standard contraction condition $\kappa(k) + \tau(k) < 1$ (Christou and Fokianos, 2014). For the double mixed Poisson INGARCH(1, 1) model of Example 2.4, the functions $f_1, ..., f_K$ being linear, conditions (3.5b) and (3.5b) reduce respectively to

$$\max_{1 \le k \le K} \left(\tfrac{\alpha(k)+\beta(k)}{\alpha(k)}\right) \sum_{k=1}^{K} \pi(k)\,\alpha(k) \quad < \quad 1, \tag{3.6a}$$

$$\frac{1}{\min\limits_{1 \le k \le K}\left(\frac{\beta(k)}{\alpha(k)}\right)} \sum_{k=1}^{K} \pi(k)\left(\alpha(k) + \beta(k)\right) \tfrac{\beta(k)}{\alpha(k)} \quad < \quad 1, \tag{3.6b}$$

where in the case $p = 0$, (3.6a) is the same as the stationarity in mean condition given by Zhu et *al.* (2010). Letting $X_t = (Y_t, \lambda_t)$, $\zeta_t = (N_t, \Delta_t, Z_t(\Delta_t))$ and

$$\mathbf{F}(X_{t-1}; \zeta_t) = \left(N_t\left(Z_t(\Delta_t) f_{\Delta_t}(X_{t-1}; \theta(\Delta_t))\right), f_{\Delta_t}(X_{t-1}; \theta(\Delta_t))\right),$$

model (2.3) may be written as the following Markov chain

$$X_t = \mathbf{F}(X_{t-1}; \zeta_t), \qquad t \in \mathbb{Z}. \tag{3.9}$$

Theorem 3.2 *Under* (3.5a) and (3.5b) or (3.5c), *equation* (2.3) *admits a strictly stationary, ergodic and weakly dependent solution* $\{(Y_t, \lambda_t), t \in \mathbb{Z}\}$ *having a finite mean. Moreover, this solution is unique and is given by the following causal scheme*

$$X_t = \mathbf{H}\left(\zeta_t, \zeta_{t-1}, ...\right), \qquad t \in \mathbb{Z}, \tag{3.10}$$

for some measurable function $\mathbf{H} : (\mathbb{N} \times (0, \infty) \times \{1, ..., K\})^{\mathbb{N}} \to \mathbb{N} \times (0, \infty)$.

Unless $K = 1$, it appears that conditions (3.5a) and (3.5b) are not necessary for ergodicity of model (2.3). Now for Example 2.4, Theorem 3.2 simplifies as follows.

Corollary 3.2 i) *Under* (3.6a) *or* (3.6b) *the double mixed Poisson* INGARCH(1, 1) *model* (2.7) *with linear regime intensity is ergodic and weakly dependent with a finite mean.*

4. Conclusion

This paper proposed a double mixed Poisson autoregression in which the intensity is scaled by a unit mean *iid* mixing process, while having different regime specifications according to the state of a finite unobservable *iid* chain. This model may account for multimodality of the marginal distribution and the persistence in intensity which are often observed in applications. Under the contraction in mean conditions (3.1) and (3.5) we have shown that models (2.2) and (2.3), respectively, are strictly stationary and ergodic.

It is interesting to study the ergodicity of the general model (2.1) in the case where the regime sequence $\{\Delta_t, t \in \mathbb{Z}\}$ is a stationary and ergodic Markov chain, leading to a *Markov switching* mixed Poisson autoregression. However, the approach by Doukhan and Wintenberger (2008) we followed in this paper is no longer applicable since in equations (3.3) and (3.9) the sequence $\{\zeta_t, t \in \mathbb{Z}\}$ is non longer *iid*.

5. Proofs

Proof of Theorem 3.1 The proof is based on checking condition (3.1) of Doukhan and Wintenberger (2008). Set $\mathbf{y} = (y_1, y_2, ...)$ and $\mathbf{y}' = (y_1', y_2', ...) \in \mathbb{R}^\infty$. In view of (3.3), the Liptchitz property (3.1a), the fact that $E(Z_t(k)) = 1$ for all k, the Poisson property of the process $N_t(.)$, and the independence of the processes $\{N_t(.), t \in \mathbb{Z}\}$, $\{\Delta_t, t \in \mathbb{Z}\}$ and $\{Z_t(k), t \in \mathbb{Z}\}$, it follows that

$$\begin{aligned}
E\left|F(\mathbf{y},\zeta_t) - F(\mathbf{y}',\zeta_t)\right| &= E\left(E\left|N_t\left(Z_t(\Delta_t) f_{\Delta_t}(\mathbf{y};\theta(\Delta_t))\right) - N_t\left(Z_t(\Delta_t) f_{\Delta_t}(\mathbf{y}';\theta(\Delta_t))\right)\right| / \Delta_t\right) \\
&= \sum_{k=1}^{K} \pi(k) E\left(\left|Z_t(k) f_k(\mathbf{y};\theta(k)) - Z_t(k) f_k(\mathbf{y}';\theta(k))\right|\right) \\
&\leq \sum_{k=1}^{K} \pi(k) \sum_{i=1}^{\infty} \kappa_i(k) \left|y_i - y_i'\right|. \qquad (5.1)
\end{aligned}$$

In view of (5.1) and (3.1b), it follows that condition (3.1) of Doukhan and Wintenberger (2008) is satisfied. By Theorem 3.1 of Doukhan and Wintenberger (2008), there exists a unique causal solution of (3.3) which is strictly stationary, ergodic, weakly dependent, having a finite mean and whose expression is given by (3.4).

Proof of Theorem 3.2 For all $x = (y, \lambda)' \in \mathbb{R}^2$ and $\epsilon > 0$, let $\|.\|_\epsilon$ be a norm on $\mathbb{R}^2$ defined by $\|x\|_\epsilon = |y| + \epsilon|\lambda|$. In view of (3.5$a$) and (3.9), and using the same arguments in the proof of Theorem 3.1, it follows that

$$\begin{aligned}
E\left\|\mathbf{F}\left(x,\zeta_t\right)-\mathbf{F}\left(x',\zeta_t\right)\right\|_\epsilon &= (1+\epsilon)\sum_{k=1}^{K}\pi\left(k\right)\left|f_k\left(y,\lambda;\theta\left(k\right)\right)-f_k\left(y',\lambda';\theta\left(k\right)\right)\right| \\
&\leq (1+\epsilon)\sum_{k=1}^{K}\pi\left(k\right)\left[\kappa\left(k\right)\left|y-y'\right|+\tau\left(k\right)\left|\lambda-\lambda'\right|\right] \\
&\leq \sum_{k=1}^{K}\pi\left(k\right)(1+\epsilon)\max\left(\kappa\left(k\right),\tfrac{\tau(k)}{\epsilon}\right)\left\|x-x'\right\|_\epsilon . \qquad (5.2)
\end{aligned}$$

Taking $\epsilon=\max\limits_{1\leq k\leq K}\left(\frac{\tau(k)}{\kappa(k)}\right)$ we have $(1+\epsilon)\max\left(\kappa\left(k\right),\frac{\tau(k)}{\epsilon}\right)=\max\limits_{1\leq k\leq K}\left(1+\frac{\tau(k)}{\kappa(k)}\right)\kappa\left(k\right)$, $1\leq k\leq K$, so inequality (5.2) becomes

$$E\left\|F\left(x,\zeta_t\right)-F\left(x',\zeta_t\right)\right\|_\epsilon\leq\max_{1\leq k\leq K}\left(1+\tfrac{\tau(k)}{\kappa(k)}\right)\sum_{k=1}^{K}\pi\left(k\right)\kappa\left(k\right)\left\|x-x'\right\|_\epsilon . \qquad (5.3a)$$

Similarly, if we take $\epsilon=\min\limits_{1\leq k\leq K}\left(\frac{\tau(k)}{\kappa(k)}\right)$ then $\max\left(\kappa\left(k\right),\frac{\tau(k)}{\min\limits_{1\leq k\leq K}\left(\frac{\tau(k)}{\kappa(k)}\right)}\right)=\frac{\tau(k)}{\min\limits_{1\leq k\leq K}\left(\frac{\tau(k)}{\kappa(k)}\right)}$, $1\leq k\leq K$. Hence, $(1+\epsilon)\max\left(\kappa\left(k\right),\frac{\tau(k)}{\epsilon}\right)\leq\frac{\tau(k)}{\epsilon\kappa(k)}\left(\kappa\left(k\right)+\tau\left(k\right)\right)$, so inequality (5.2) becomes

$$E\left\|F\left(x,\zeta_t\right)-F\left(x',\zeta_t\right)\right\|_\epsilon\leq\frac{1}{\min\limits_{1\leq k\leq K}\left(\frac{\tau(k)}{\kappa(k)}\right)}\sum_{k=1}^{K}\pi\left(k\right)\tfrac{\tau(k)}{\kappa(k)}\left(\kappa\left(k\right)+\tau\left(k\right)\right)\left\|x-x'\right\|_\epsilon . \qquad (5.3b)$$

In view of (5.3a) and (5.3b), it follows that under (3.5a) and (3.5b) or (3.5c), condition (3.1) of Doukhan and Wintenberger (2008) is satisfied, so the conclusion follows for their Theorem 3.1.

References

[1] Aknouche, A.; Rabehi, N. (2010). On an independent and identically distributed mixture bilinear time series model. *Journal of Time Series Analysis*, **31**, 113-131.

[2] Aknouche, A., Bentarzi, W.; Demouche, N. (2018). On periodic ergodicity of a general periodic mixed Poisson autoregression. *Statistics & Probability Letters*, **134**, 15-21.

[3] Bauwens, L., Preminger, A.; Rombouts, J. (2010). Theory and inference for a Markov Switching GARCH model. *Econometrics Journal*, **13**, 218-244.

[4] Cox, D.R. (1981). Statistical analysis of time series: some recent developments. *Scandinavian Journal of Statistics*, **8**, 93-115.

[5] Christou, V.; Fokianos, K. (2014). Quasi-likelihood inference for negative binomial time series models. *Journal of Time Series Analysis*, **35**, 55-78.

[6] Davis, R.A.; Liu, H. (2016). Theory and inference for a class of observation-driven models with application to time series of counts. *Statistica Sinica*, **26**, 1673-1707.

[7] Dean, C., Lawless, J.F.; Willmot, G.E. (1989). A mixed Poisson-Inverse-Gaussian regression model. *The Canadian Journal of Statistics*, **17**, 171-181.

[8] Dedecker, J.; Prieur, C. (2004). Coupling for τ-dependent sequences and applications. *Journal of Theoretical Probability*, **17**, 861-885.

[9] Diop, M.L., Diop, A.; Diongue, A.K. (2016). A mixture integer-valued $GARCH$ model. *REVSTAT - Statistical Journal*, **14**, 245-271.

[10] Doukhan, P., Fokianos, K.; Tjøstheim, D. (2012). On weak dependence conditions for Poisson autoregressions. *Statistics and Probability Letters*, **82**, 942-948.

[11] Doukhan, P.; Wintenberger, O. (2008). Weakly dependent chains with infinite memory. *Stochastic Processes and their Applications*, **118**, 1997-2013.

[12] Francq, C.; Zakoian, J.M. (2008). Deriving the autocovariances of powers of Markov-switching GARCH models, with applications to statistical inference. *Computational Statistics & Data Analysis*, **52**, 3027-3046.

[13] Haas, M., Mittnik, S.; Paollela, M.S. (2004). A new approach to Markov-Switching GARCH models. *Journal of Financial Econometrics*, **4**, 493-530.

[14] Hind, J. (1982). Compound Poisson regression models. *Proceedings of the International Conference on Generalised Linear Models*, pp 109-121.

[15] Zhu, F. (2011). A negative binomial integer-valued $GARCH$ model. *Journal of Time Series,* **32**, 54-67.

[16] Zhu, F., Li, Q.; Wang, D. (2010). A mixture integer-valued $ARCH$ model. *Journal of Statistical Planning and Inference*, **140**, 2025-2036.