On zero-sum game formulation of non zero-sum game

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Abstract

We consider a formulation of a non zero-sum n players game by an $n+1$ players zero-sum game. We suppose the existence of the $n+1$-th player in addition to n players in the main game, and virtual subsidies to the n players which is provided by the $n+1$-th player. Its strategic variable affects only the subsidies, and does not affect choice of strategies by the n players in the main game. His objective function is the opposite of the sum of the payoffs of the n players. We will show 1) The minimax theorem by Sion (Sion(1958)) implies the existence of Nash equilibrium in the n players non zero-sum game. 2) The maximin strategy of each player in $\{1, 2, \dots, n\}$ with the minimax strategy of the $n+1$-th player is equivalent to the Nash equilibrium strategy of the n players non zero-sum game. 3) The existence of Nash equilibrium in the n players non zero-sum game implies Sion's minimax theorem for pairs of each of the n players and the $n+1$-th player.

Keywords: zero-sum game, non zero-sum game, minimax theorem, virtual subsidy

JEL Classification: C72

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1 Introduction

We consider a formulation of a non zero-sum n players game by an $n+1$ players zero-sum game. We suppose the existence of the $n+1$-th player in addition to n players in the main game, and virtual subsidies to the n players which is provided by the $n+1$-th player. Its strategic variable affects only the subsidies, and does not affect choice of strategies by the n players in the main game. His objective function is the opposite of the sum of the payoffs of the n players, then the game with $n+1$ players, n players in the main game and the $n+1$-th player, is a zero-sum game.

We will show the following results.

1. The minimax theorem by Sion (Sion (1958)) implies the existence of Nash equilibrium in the n players non zero-sum game.

2. The maximin strategy of each player in $\{1, 2, \dots, n\}$ with the minimax strategy of the $n+1$-th player is equivalent to the Nash equilibrium strategy of the n players non zero-sum game.

3. The existence of Nash equilibrium in the n players non zero-sum game implies Sion's minimax theorem for pairs of each of the n players and the $n+1$-th player.

2 The model and the minimax theorem

There are n players Player $1, 2, \dots, n$ in a non zero-sum game. The strategic variable of Player i is denoted by x_i. The common strategy space of the players is denoted by X, which is a compact set. There exists another player, Player $n+1$. His strategic variable is f, We consider virtual subsidies to each player other than Player $n+1$, $\psi(f)$, which is provided by Player $n+1$ and is equal for any player. It is zero at the equilibrium.

The payoff of Player $i \in \{1, 2, \dots, n\}$ is written as

$$\pi_i(x_1, x_2, \dots, x_n, f) = \varphi_i(x_1, x_2, \dots, x_n) + \psi(f),\ i \in \{1, 2, \dots, n\}.$$

The objective function of Player $n+1$ is

$$\pi_{n+1} = -(\pi_1 + \pi_2 + \dots \pi_n) = -\sum_{i=1}^{n} \varphi_i(x_1, x_2, \dots, x_n) - n\psi(f).$$

The strategy space of Player $n+1$ is denoted by F which is a compact set. Player $n+1$ is not a dummy player because he can determine the value of its strategic variable. We assume

$$\min_{f \in F} \psi(f) = 0.$$

Denote

$$a = \arg\min_{f \in F} \psi(f).$$

We postulate that this is unique. The game with Player 1, 2, ..., n and Player $n+1$ is a zero-sum game because

$$\pi_1(x_1, x_2, \dots, x_n, f) + \pi_2(x_1, x_2, \dots, x_n, f) + \dots + \pi_n(x_1, x_2, \dots, x_n, f)$$
$$+\pi_{n+1}(x_1, x_2, \dots, x_n, f) = 0.$$

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) for a continuous function is stated as follows.

Lemma 1. *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \to \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable. Then*

$$\max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \max_{x\in X} f(x, y).$$

We follow the description of this theorem in Kindler (2005).

Let x_k's for $k \neq i$ be given, then π_i is a function of x_i and f. We can apply Lemma 1 to such a situation, and get the following equation[1]

$$\max_{x_i\in X} \min_{f\in F} \pi_i(x_1, x_2, \dots, x_n, f) = \min_{f\in F} \max_{x_i\in X} \pi_i(x_1, x_2, \dots, x_n, f). \quad (1)$$

We assume that $\arg\max_{x_i\in X} \min_{f\in F} \pi_i(x_1, x_2, \dots, x_n, f)$, $\arg\min_{f\in F} \max_{x_i\in X} \pi_i(x_1, x_2, \dots, x_n, f)$ and so on are unique, that is, single-valued. We also assume that the best responses of players in any situation are unique.

3 The main results

Choice of f by Player $n+1$ has an effect only on the fixed subsidy for each player. The optimal value of f for Player $n+1$, which is equal to a, is determined independently of x_1, x_2, ..., x_n, and the optimal values of the strategic variables for Player 1, 2, ..., n are determined independently of f. We have

$$\pi_i(x_1, x_2, \dots, x_n, f) - \psi(f) = \pi_i(x_1, x_2, \dots, x_n, a) = \varphi_i(x_1, x_2, \dots, x_n),\ i \in \{1, 2, \dots, n\},$$

for any value of f. Thus,

$$\arg\max_{x_i\in X} \pi_i(x_1, x_2, \dots, x_n, f) = \arg\max_{x_i\in X} \pi_i(x_1, x_2, \dots, x_n, a) \text{ for any } f,$$

and

$$\arg\min_{f\in F} \pi_i(x_1, x_2, \dots, x_n, f) = a,\ i \in \{1, 2, \dots, n\}. \quad (2)$$

First we show the following result.

[1]We do not require

$$\max_{x_i\in X} \min_{x_j\in X} \pi_i(x_1, x_2, \dots, x_n, f) = \min_{x_j\in X} \max_{x_i\in X} \pi_i(x_1, x_2, \dots, x_n, f),$$

for $i, j \in \{1, 2, \dots, n\}$.

Theorem 1. 1. *Sion's minimax theorem (Lemma 1) implies the existence of Nash equilibrium in the non zero-sum main game.*

2. *The maximin strategy of each player in $\{1, 2, \dots, n\}$ with the minimax strategy of Player $n+1$ is equivalent to its Nash equilibrium strategy of the non zero-sum main game.*

Proof. Let $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ be the solution of the following equation.

$$\begin{cases} \tilde{x}_1 = \arg\max_{x_1 \in X} \min_{f \in F} \pi_1(x_1, \tilde{x}_2, \dots, \tilde{x}_n, f) \\ \tilde{x}_2 = \arg\max_{x_2 \in X} \min_{f \in F} \pi_2(\tilde{x}_1, x_2, \tilde{x}_3, \dots, \tilde{x}_n, f) \\ \dots \\ \tilde{x}_n = \arg\max_{x_n \in X} \min_{f \in F} \pi_n(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n-1}, x_n, f). \end{cases}$$

Then, we have

$$\max_{x_i \in X} \min_{f \in F} \pi_i(\tilde{x}_1, \dots, x_i, \dots, \tilde{x}_n, f) = \min_{f \in F} \pi_i(\tilde{x}_1, \dots, \tilde{x}_i, \dots, \tilde{x}_n, f) \\ = \min_{f \in F} \max_{x_i \in X} \pi_i(\tilde{x}_1, \dots, x_i, \dots, \tilde{x}_n, f),\ i \in \{1, 2, \dots, n\}. \tag{3}$$

Since

$$\pi_i(\tilde{x}_1, \dots, \tilde{x}_i, \dots, \tilde{x}_n, f) \leq \max_{x_i \in X} \pi_i(\tilde{x}_1, \dots, x_i, \dots, \tilde{x}_n, f),\ i \in \{1, 2, \dots, n\},$$

and

$$\min_{f \in F} \pi_i(\tilde{x}_1, \dots, \tilde{x}_i, \dots, \tilde{x}_n, f) = \min_{f \in F} \max_{x_i \in X} \pi_i(\tilde{x}_1, \dots, x_i, \dots, \tilde{x}_n, f),\ i \in \{1, 2, \dots, n\},$$

we get

$$\arg\min_{f \in F} \pi_i(\tilde{x}_1, \dots, \tilde{x}_i, \dots, \tilde{x}_n, f) = \arg\min_{f \in F} \max_{x_i \in X} \pi_i(\tilde{x}_1, \dots, x_i, \dots, \tilde{x}_n, f), \\ i \in \{1, 2, \dots, n\}. \tag{4}$$

Because the game is zero-sum,

$$\sum_{i=1}^{n} \pi_i(\tilde{x}_1, \dots, x_i, \dots, \tilde{x}_n, f) = -\pi_{n+1}(\tilde{x}_1, \dots, x_i, \dots, \tilde{x}_n, f).$$

Therefore, from (2)

$$\arg\min_{f \in F} \pi_i(\tilde{x}_1, \dots, x_i, \dots, \tilde{x}_n, f) \\ = \arg\max_{f \in F} \pi_{n+1}(\tilde{x}_1, \dots, x_i, \dots, \tilde{x}_n, f) = a,\ i \in \{1, 2, \dots, n\}. \tag{5}$$

From (3), (4) and (5) we obtain

$$\min_{f\in F}\max_{x_i\in X}\pi_i(\tilde{x}_1,\dots,x_i,\dots,\tilde{x}_n,f)=\max_{x_i\in X}\pi_i(\tilde{x}_1,\dots,x_i,\dots,\tilde{x}_n,a) \tag{6}$$
$$=\min_{f\in F}\pi_i(\tilde{x}_1,\dots,\tilde{x}_i,\dots,\tilde{x}_n,f)=\pi_i(\tilde{x}_1,\dots,\tilde{x}_i,\dots,\tilde{x}_n,a),\ i\in\{1,2,\dots,n\}.$$

(5) and (6) mean that $(x_1,x_2,\dots,x_n,f)=(\tilde{x}_1,\tilde{x}_2,\dots,\tilde{x}_n,a)$ is a Nash equilibrium of the zero-sum game with $n+1$ players.

$\tilde{x}_1$, $\tilde{x}_2$, $\dots$, $\tilde{x}_n$ are determined independently of f. Thus,

$$\max_{x_i\in X}\varphi_i(\tilde{x}_1,\dots,x_i,\dots,\tilde{x}_n)=\varphi_i(\tilde{x}_1,\dots,\tilde{x}_i,\dots,\tilde{x}_n),\ i\in\{1,2,\dots,n\}.$$

Therefore, $(\tilde{x}_1,\tilde{x}_2,\dots,\tilde{x}_n)$ is a Nash equilibrium of the non zero-sum game with Player $1,2,\dots,n$. □

Next we show

Theorem 2. *The existence of Nash equilibrium in the n players non zero-sum game implies Sion's minimax theorem for pairs of Player i, $i\in\{1,2,\dots,n\}$ and Player $n+1$.*

Proof. Let $(\tilde{x}_1,\tilde{x}_2,\dots,\tilde{x}_n)$ be a Nash equilibrium of the n players non zero-sum game. Consequently,

$$\varphi_i(\tilde{x}_1,\dots,\tilde{x}_i,\dots,\tilde{x}_n)\geq\varphi_i(\tilde{x}_1,\dots,x_i,\dots,\tilde{x}_n)\text{ for any }x_i,\ i\in\{1,2,\dots,n\}.$$

This is based on the fact that there exists a value of x_i, x_i^*, such that given x_1, x_2, $\dots$, x_n other than x_i,

$$\varphi_i(x_1,\dots,x_i^*,\dots,x_n)\geq\varphi_i(x_1,\dots,x_i,\dots,x_n)\text{ for any }x_i.$$

Thus,

$$\pi_i(x_1,\dots,x_i^*,\dots,x_n,f)\geq\pi_i(x_1,\dots,x_i,\dots,x_n,f)\text{ for any }x_i\text{ and any value of }f,\ i\in\{1,2,\dots,n\},$$

Since

$$\arg\min_{f\in F}\pi_i(x_1,\dots,x_i^*,\dots,x_n,f)=\arg\max_{f\in F}\psi(f)=a,$$

we have

$$\min_{f\in F}\max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,f)\leq\max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,a) \tag{7}$$
$$=\min_{f\in F}\pi_i(x_1,\dots,x_i^*,\dots,x_n,f)\leq\max_{x_i\in X_i}\min_{x_n\in x_n}\pi_i(x_1,\dots,x_i,\dots,x_n,f),\ i\in\{1,2,\dots,n-1\}.$$

On the other hand, since

$$\min_{f\in F}\pi_i(x_1,\dots,x_i,\dots,x_n,f)\leq\pi_i(x_1,\dots,x_i,\dots,x_n,f),$$

we have

$$\max_{x_i\in X_i}\min_{f\in F}\pi_i(x_1,\dots,x_i,\dots,x_n,f)\le \max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,f).$$

This inequality holds for any f. Thus,

$$\max_{x_i\in X_i}\min_{f\in F}\pi_i(x_1,\dots,x_i,\dots,x_n,f)\le \min_{f\in F}\max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,f).$$

With (7), we obtain

$$\max_{x_i\in X_i}\min_{f\in F}\pi_i(x_1,\dots,x_i,\dots,x_n,f)= \min_{f\in F}\max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,f), \tag{8}$$

given $x_1, x_2, \dots, x_n$ other than x_i. (7) and (8) imply

$$\max_{x_i\in X_i}\min_{f\in F}\pi_i(x_1,\dots,x_i,\dots,x_n,f)= \max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,a),$$

$$\min_{f\in F}\max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,f)= \min_{f\in F}\pi_i(x_1,\dots,x_i^*,\dots,x_n,f).$$

From

$$\min_{f\in F}\pi_i(x_1,\dots,x_i,\dots,x_n,f)\le \pi_i(x_1,\dots,x_i,\dots,x_n,a),$$

and

$$\max_{x_i\in X_i}\min_{f\in F}\pi_i(x_1,\dots,x_i,\dots,x_n,f)= \max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,a),$$

we have

$$\arg\max_{x_i\in X_i}\min_{f\in F}\pi_i(x_1,\dots,x_i,\dots,x_n,f)=\arg\max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,a)=x_i^*,\ i\in\{1,2,\dots,n-1\}.$$

We also have

$$\max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,f)\ge \pi_i(x_1,\dots,x_i^*,\dots,x_n,f),$$

and

$$\min_{f\in F}\max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,f)= \min_{f\in F}\pi_i(x_1,\dots,x_i^*,\dots,x_n,f).$$

Therefore, we get

$$\arg\min_{f\in F}\max_{x_i\in X_i}\pi_i(x_1,\dots,x_i,\dots,x_n,f)=\arg\min_{f\in F}\pi_i(x_1,\dots,x_i^*,\dots,x_n,f)=a,\ i\in\{1,2,\dots,n-1\}.$$

Thus, if $(x_1,x_2,\dots,x_n)=(\tilde{x}_1,\tilde{x}_2,\dots,\tilde{x}_n)$,

$$\arg\max_{x_i\in X_i}\min_{f\in F}\pi_i(\tilde{x}_1,\dots,x_i,\dots,\tilde{x}_n,f)=\tilde{x}_i,\ i\in\{1,2,\dots,n\}.$$

$\square$

4 An example

Consider a three firms oligopoly with differentiated goods. There are Firm 1, 2 and 3. Assume that the inverse demand functions are

$$p_1 = a - x_1 - bx_2 - bx_3,$$

$$p_2 = a - bx_1 - x_2 - bx_3,$$

$$p_3 = a - bx_1 - bx_2 - x_3,$$

with $0 < b < 1$. p_1, p_2, p_3 are the prices of the goods of Firm 1, 2, 3. x_1, x_2, x_3 are the outputs of the firms. The cost functions of the firms with the subsidies are

$$c_1(x_1) = c_1x_1 - (f - a)^2,$$

$$c_2(x_2) = c_2x_2 - (f - a)^2,$$

and

$$c_3(x_3) = c_3x_3 - (f - a)^2.$$

f is a non-negative number and a is a positive number. c_1, c_2, c_3 are constant numbers. The profits of the firms are

$$\pi_1 = (a - x_1 - bx_2 - bx_3)x_1 - c_1x_1 + (f - a)^2,$$

$$\pi_2 = (a - bx_1 - x_2 - bx_3)x_2 - c_2x_2 + (f - a)^2,$$

and

$$\pi_3 = (a - bx_1 - bx_2 - x_3)x_3 - c_3x_3 + (f - a)^2.$$

The condition for minimization of π_1 with respect to f is

$$\frac{\partial \pi_1}{\partial f} = 2(f - a) = 0.$$

Thus, $f = a$. Substituting this into π_1,

$$\pi_1|_{f=a} = (a - x_1 - bx_2 - bx_3)x_1 - c_1x_1.$$

The condition for maximization of $\pi_1|_{f=a}$ with respect to x_1 is

$$\frac{\partial\ \pi_1|_{f=a}}{\partial x_1} = a - 2x_1 - bx_2 - bx_3 - c_1 = 0.$$

Thus,

$$\arg\max_{x_1 \in X} \min_{f \in F} \pi_1(x_1, x_2, x_3, f) = \frac{a - c_1 - bx_2 - bx_3}{2}.$$

Similarly, we get

$$\arg\max_{x_2\in X}\min_{f\in F}\pi_2(x_1,x_2,x_3,f)=\frac{a-c_2-bx_1-bx_3}{2},$$

$$\arg\max_{x_3\in X}\min_{f\in F}\pi_3(x_1,x_2,x_3,f)=\frac{a-c_3-bx_1-bx_2}{2}.$$

Solving

$$x_1=\frac{a-c_1-bx_2-bx_3}{2},$$
$$x_2=\frac{a-c_2-bx_1-bx_3}{2},$$
$$x_3=\frac{a-c_3-bx_1-bx_2}{2},$$

we obtain

$$x_1=\frac{(2-b)a+bc_3+bc_2-(2+b)c_1}{2(2-b)(b+1)},$$
$$x_2=\frac{(2-b)a+bc_3+bc_1-(2+b)c_2}{2(2-b)(b+1)},$$
$$x_3=\frac{(2-b)a+bc_1+bc_2-(2+b)c_3}{2(2-b)(b+1)}.$$

They are the same as the equilibrium outputs of the oligopoly with Firm 1, 2 and 3.

In this paper we presented a zero-sum game formulation of a non zero-sum n players game considering the $n+1$-th player and virtual subsidies to the players provided by the $n+1$-th player.

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