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Sion's mini-max theorem and Nash equilibrium in a multi-players game with two groups which is zero-sum and symmetric in each group*

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Abstract

We consider the relation between Sion's minimax theorem for a continuous function and Nash equilibrium in a multi-players game with two groups which is zero-sum and symmetric in each group. We will show the following results.

1. The existence of Nash equilibrium which is symmetric in each group implies a modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy for players in each group.
2. A modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy for players in each group implies the existence of Nash equilibrium which is symmetric in each group.

Thus, they are equivalent. An example of such a game is a relative profit maximization game in each group under oligopoly with two groups such that firms in each group have the same cost functions and maximize their relative profits in each group, and the demand functions are symmetric for the firms in each group.

Keywords: multi-players zero-sum game, two groups, Nash equilibrium, Sion's minimax theorem

JEL Classification: C72

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1 Introduction

We consider the relation between Sion's minimax theorem for a continuous function and the existence of Nash equilibrium in a multi-players game with two groups which is zero-sum and symmetric in each group. There are n players. Players $1, 2, \ldots, m$ are in one group, and Players $m+1, m+2, \ldots, n$ are in the other group. We assume $n \geq 4$ and $2 \leq m \leq n-2$. Thus, each group has at least two players. Players $1, 2, \ldots, m$ have the same payoff functions and strategy spaces, and they play a game which is zero-sum in this group, that is, the sum of the payoffs of Players $1, 2, \ldots, m$ is zero. Similarly, Players $m+1, m+2, \ldots, n$ have the same payoff functions and strategy spaces, and they play a game which is zero-sum in this group, that is, the sum of the payoffs of Players $m+1, m+2, \ldots, n$ is zero.

We will show the following results.

1. The existence of Nash equilibrium which is symmetric in each group implies a modified version of Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy for players in each group.

2. A modified version of Sion's minimax theorem for players with the coincidence of the maximin strategy and the minimax strategy in each group implies the existence of Nash equilibrium which is symmetric in each group.

Thus, they are equivalent.

An example of such a game is a relative profit maximization game in each group under oligopoly with two groups such that firms in each group have the same cost functions and maximize their relative profits in each group, and demand functions are symmetric for the firms in each group. Assume that there are six firms, A, B, C, D, E and F. Let $\bar{\pi}_A$, $\bar{\pi}_B$, $\bar{\pi}_C$, $\bar{\pi}_D$, $\bar{\pi}_E$ and $\bar{\pi}_F$ be the absolute profits of, respectively, Firms A, B, C, D, E and F. Firms A, B and E have the same cost function, and the demand functions are symmetric for them. Firms C, D and F have the same cost function, and the demand functions are symmetric for them. However, the firms in different groups have different cost functions, and the demand functions are not symmetric for firms in different groups.

The relative profits of Firms A, B and E are

$$\pi_A = \bar{\pi}_A - \frac{1}{2}(\bar{\pi}_B + \bar{\pi}_E),$$

$$\pi_B = \bar{\pi}_B - \frac{1}{2}(\bar{\pi}_A + \bar{\pi}_E),$$

$$\pi_E = \bar{\pi}_E - \frac{1}{2}(\bar{\pi}_A + \bar{\pi}_B).$$

The relative profits of Firms C, D and F are

$$\pi_C = \bar{\pi}_C - \frac{1}{2}(\bar{\pi}_D + \bar{\pi}_F),$$

$$\pi_D = \bar{\pi}_D - \frac{1}{2}(\bar{\pi}_C + \bar{\pi}_F),$$

$$\pi_F = \bar{\pi}_F - \frac{1}{2}(\bar{\pi}_C + \bar{\pi}_D).$$

We see

$$\pi_A + \pi_B + \pi_E = 0,$$

$$\pi_C + \pi_D + \pi_F = 0.$$

Firms A, B, C, D, E and F maximize, respectively, π_A, π_B, π_C, π_D, π_E and π_F. Thus, the relative profit maximization game in each group is a zero-sum game[1]. In Section 4 we present an example of relative profit maximization in each group under oligopoly with two groups.

2 The model and Sion's minimax theorem

Consider a multi-players game with two groups which is zero-sum and symmetric in each group. Our analysis can be easily extended to a case with more than two groups. However, since notation is very complicated, we will present arguments of a two groups case. There are n players. Players $1, 2, \ldots, m$ are in one group, and Players $m+1, m+2, \ldots, n$ are in the other group. We assume $n \geq 4$ and $2 \leq m \leq n-2$. Thus, each group has at least two players. Players $1, 2, \ldots, m$ have the same payoff functions and strategy spaces, and they play a game which is zero-sum in this group, that is, the sum of the payoffs of Players $1, 2, \ldots, m$ is zero. Similarly, Players $m+1, m+2, \ldots, n$ have the same payoff functions and strategy spaces, and they play a game which is zero-sum in this group, that is, the sum of the payoffs of Players $m+1, m+2, \ldots, n$ is zero. The strategic variables for the players are $s_1, s_2, \ldots, s_n$, and $(s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$. $S_1, S_2, \ldots, S_n$ are convex and compact sets in linear topological spaces.

The payoff function of each player is $u_i(s_1, s_2, \ldots, s_n)$, $i = 1, 2, \ldots, n$. We assume

> u_i's for $i = 1, 2, \ldots, n$ are continuous real-valued functions on $S_1 \times S_2 \times \cdots \times S_n$, quasi-concave on S_i for each $s_j \in S_j$, $j \neq i$, and quasi-convex on S_j for $j \neq i$ for each $s_i \in S_i$.

Since the game is zero-sum in each group, we have

$$u_1(s_1, s_2, \ldots, s_n) + u_2(s_1, s_2, \ldots, s_n) + \ldots, u_m(s_1, s_2, \ldots, s_n) = 0, \tag{1}$$

$$u_{m+1}(s_1, s_2, \ldots, s_n) + u_{m+2}(s_1, s_2, \ldots, s_n) + \ldots, u_n(s_1, s_2, \ldots, s_n) = 0, \tag{2}$$

for given $(s_1, s_2, \ldots, s_n)$.

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) for a continuous function is stated as follows.

Lemma 1. *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \to \mathbb{R}$ be a function, that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable. Then*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

We follow the description of this theorem in Kindler (2005).

Let s_h's for $h \neq i, j$; $i, j \in \{1, 2, \ldots, m\}$ be given; then, $u_i(s_1, s_2, \ldots, s_n)$ is a function of s_i and s_j. We can apply Lemma 1 to such a situation, and get the following equation.

$$\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_1, s_2, \ldots, s_n) = \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_1, s_2, \ldots, s_n). \tag{3}$$

[1] About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997)

By symmetry

$$\max_{s_j \in S_j} \min_{s_i \in S_i} u_j(s_1, s_2, \ldots, s_n) = \min_{s_i \in S_i} \max_{s_j \in S_j} u_j(s_1, s_2, \ldots, s_n).$$

Similarly, let s_h's for $h \neq k, l;\ k, l \in \{m+1, m+2, \ldots, n\}$ be given; then we obtain

$$\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s_1, s_2, \ldots, s_n) = \min_{s_l \in S_l} \max_{s_k \in S_k} u_i(s_1, s_2, \ldots, s_n). \tag{4}$$

By symmetry

$$\max_{s_l \in S_l} \min_{s_k \in S_k} u_l(s_1, s_2, \ldots, s_n) = \min_{s_k \in S_k} \max_{s_l \in S_l} u_j(s_1, s_2, \ldots, s_n).$$

We assume that $\arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_1, s_2, \ldots, s_n)$, $\arg\min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_1, s_2, \ldots, s_n)$ and so on are unique, that is, single-valued. By the maximum theorem they are continuous in s_h's, $h \neq i, j$ or in s_h's, $h \neq k, l$. Also, throughout this paper we assume that the maximin strategy and the minimax strategy of players in any situation are unique, and the best responses of players in any situation are unique.

Let us consider a point such that $s_i = s$ for $i \in \{1, 2, \ldots, m\}$ and $s_k = s'$ for $k \in \{m+1, m+2, \ldots, n\}$, and consider the following function.

$$\begin{pmatrix} s \\ s' \end{pmatrix} \to \begin{pmatrix} \arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s, \ldots, s, s', \ldots, s') \\ \arg\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s, \ldots, s, s_k, s_l, s', \ldots, s') \end{pmatrix},$$

for $i \in \{1, 2, \ldots, m\}$, $k \in \{m+1, m+2, \ldots, n\}$. Since u_i and u_k are continuous, $S_i = S_j$ is compact and $S_k = S_l$ is compact, these functions are also continuous. Thus, there exists a fixed point of (s, s'). Denote it by $(\tilde{s}, \hat{s})$. It satisfies

$$\tilde{s} = \arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}),\ \ i \in \{1, 2, \ldots, m\}, \tag{5}$$

$$\hat{s} = \arg\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(\tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}),\ \ k \in \{m+1, m+2, \ldots, n\}. \tag{6}$$

Now we assume

Assumption 1. *About $\tilde{s}$ and $\hat{s}$ which satisfy (5) and (6),*

$$\arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}) = \arg\min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}),$$

$$\arg\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(\tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}) = \arg\min_{s_l \in S_l} \max_{s_k \in S_k} u_k(\tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}),$$

for $i \in \{1, 2, \ldots, m\}$, $k \in \{m+1, m+2, \ldots, n\}$, that is, the maximin strategy and the minimax strategy coincide.

Based on Assumption 1 we present a modified version of Sion's minimax theorem.

Lemma 2 (Modified version of Sion's minimax theorem)**.** *Let $j \neq i,\ i, j \in \{1, 2, \ldots, m\}$, and S_i and S_j be non-void convex and compact subsets of two linear topological spaces, and Let $l \neq k,\ k, l \in \{m+1, m+2, \ldots, n\}$, and S_k and S_l be non-void convex and compact subsets of two linear topological spaces. Let $u_i : S_i \times S_j \to \mathbb{R}$ given the strategies of all other players and $u_k : S_k \times S_l \to \mathbb{R}$ given the strategies of all other players be functions that is continuous on $S_1 \times S_2 \times \cdots \times S_n$, quasi-concave on S_i (or S_k) and quasi-convex on S_j (or S_l). Then, there exist $\tilde{s}$ and $\hat{s}$ which satisfy (3), (4), (5), (6) and Assumption 1.*

As we will show in the Appendix, without Assumption 1 we may have a Nash equilibrium which is asymmetric in each group.

3 The main results

Consider a Nash equilibrium which is symmetric in each group. Let s_i^*'s and s_k^*'s be the values of s_i's for $i \in \{1, 2, \ldots, m\}$ and s_k's for $k \in \{m+1, m+2, \ldots, n\}$ which, respectively, maximize u_i's and u_k's, that is,

$$u_i(s_1^*, s_2^*, \ldots, s_i^*, \ldots, s_n^*) \geq u_i(s_1^*, s_2^*, \ldots, s_i, \ldots, s_n^*) \text{ for any } s_i \in S_i,$$

and

$$u_k(s_1^*, s_2^*, \ldots, s_k^*, \ldots, s_n^*) \geq u_k(s_1^*, s_2^*, \ldots, s_k, \ldots, s_n^*) \text{ for any } s_k \in S_k,$$

If the Nash equilibrium is symmetric in each group, s_1^*'s for all $i \in \{1, 2, \ldots, m\}$ are equal, and s_k^*'s for all $k \in \{m+1, m+2, \ldots, n\}$ are equal.

Notations of strategy choice by players are as follows.

> $(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**})$ is a vector of strategy choice by players such that Players $1, \ldots, m$ other than i choose s^* and Players $m+1, \ldots, n$ choose s^{**}. $(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**})$ is a vector such that Players $1, \ldots, m$ choose s^* and Players $m+1, \ldots, n$ other than k choose s^{**}. $(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**})$ is a vector such that Players $1, \ldots, m$ other than i and j choose s^* and Players $m+1, \ldots, n$ choose s^{**}. $(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**})$ is a vector such that Players $1, \ldots, m$ choose s^* and Players $m+1, \ldots, n$ other than k and l choose s^{**}.
>
> $(s_i, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s})$ is a vector of strategy choice by players such that Players $1, \ldots, m$ other than i choose $\tilde{s}$ and Players $m+1, \ldots, n$ choose $\hat{s}$. $(\tilde{s}, \ldots, \tilde{s}, s_k, \hat{s}, \ldots, \hat{s})$ is a vector such that Players $1, \ldots, m$ choose $\tilde{s}$ and Players $m+1, \ldots, n$ other than k choose $\hat{s}$. $(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s})$ is a vector such that Players $1, \ldots, m$ other than i and j choose $\tilde{s}$ and Players $m+1, \ldots, n$ choose $\hat{s}$. $(\tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s})$ is a vector such that Players $1, \ldots, m$ choose $\tilde{s}$ and Players $m+1, \ldots, n$ other than k and l choose $\hat{s}$.
>
> The same applies to other similar notations.

We show the following theorem.

Theorem 1. *The existence of Nash equilibrium which is symmetric in each group implies Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy.*

Proof. Let $(s_1, \ldots, s_m, s_{m+1}, \ldots, s_n) = (s^*, \ldots, s^*, s^{**}, \ldots, s^{**})$ be a Nash equilibrium which is symmetric in each group. Since the game is zero-sum in each group.

$$u_i(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) + \sum_{j=1, j\neq i}^{m} u_j(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = 0,$$

and

$$u_k(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}) + \sum_{l=m+1, k\neq k}^{n} u_l(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}) = 0$$

imply

$$u_i(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = -(m-1)u_j(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

and

$$u_k(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}) = -(n-m-1)u_l(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}).$$

These equations hold for any s_i and s_k. Therefore,

$$\arg\max_{s_i \in S_i} u_i(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \arg\min_{s_i \in S_i} u_j(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

$$\arg\max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}) = \arg\min_{s_k \in S_k} u_l(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}).$$

By the assumption of uniqueness of the best responses, they are unique. By symmetry for each group

$$\arg\max_{s_i \in S_i} u_i(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \arg\min_{s_j \in S_j} u_i(s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

$$\arg\max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}) = \arg\min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_l, s^{**}, \ldots, s^{**}).$$

Therefore,

$$u_i(s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \min_{s_j \in S_j} u_i(s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) \leq u_i(s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

$$u_k(s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_l, s^{**}, \ldots, s^{**}) \leq u_k(s^*, \ldots, s^*, s_l, s^{**}, \ldots, s^{**}).$$

We get

$$\max_{s_i \in S_i} u_i(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = u_i(s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \min_{s_j \in S_j} u_i(s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

$$\max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}) = u_k(s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_l, s^{**}, \ldots, s^{**}),$$

They mean

$$\begin{aligned} &\min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) \leq \max_{s_i \in S_i} u_i(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) \\ &= \min_{s_j \in S_j} u_i(s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) \leq \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}). \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\min_{s_l \in S_l} \max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}) \leq \max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}) \\ &= \min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_l, s^{**}, \ldots, s^{**}) \leq \max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}). \end{aligned} \tag{8}$$

On the other hand, since

$$\min_{s_j \in S_j} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) \leq u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

$$\min_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}) \leq u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}),$$

we have

$$\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) \leq \max_{s_i \in S_i} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

$$\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}) \leq \max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}).$$

These inequalities hold for any s_j and s_l. Thus,

$$\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) \leq \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

$$\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}) \leq \min_{s_l \in S_l} \max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}),$$

With (7) and (8), we obtain

$$\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}), \tag{9}$$

$$\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}) = \min_{s_D \in S_D} \max_{s_C \in S_C} u_C(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}). \tag{10}$$

From

$$\min_{s_j \in S_j} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) \leq u_i(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

$$\min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}) \leq u_k(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}),$$

$$\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \max_{s_i \in S_i} u_i(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

and

$$\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}) = \max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}),$$

we have

$$\arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \arg\max_{s_i \in S_i} u_i(s_i, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = s^*,$$

$$\arg\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}) = \arg\max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s^{**}, \ldots, s^{**}) = s^{**}.$$

From

$$\max_{s_i \in S_i} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) \geq u_i(s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

$$\max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}) \geq u_k(s^*, \ldots, s^*, s_l, s^{**}, \ldots, s^{**}),$$

$$\min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \min_{s_j \in S_j} u_i(s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}),$$

and

$$\min_{s_l \in S_l} \max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}) = \min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_l, s^{**}, \ldots, s^{**}),$$

we get

$$\arg\min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = \arg\min_{s_j \in S_j} u_i(s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = s^*,$$

$$\arg\min_{s_l \in S_l} \max_{s_k \in S_k} u_k(s^*, \ldots, s^*, s_k, s_l, s^{**}, \ldots, s^{**}) = \arg\min_{s_l \in S_l} u_k(s^*, \ldots, s^*, s_l, s^{**}, \ldots, s^{**}) = s^{**}.$$

Therefore,

$$\begin{aligned} &\arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) \\ &= \arg\min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, s^*, \ldots, s^*, s^{**}, \ldots, s^{**}) = s^*, \end{aligned} \tag{11}$$

$$\arg\max_{s_k\in S_k}\min_{s_l\in S_l} u_k(s^*,\ldots,s^*,s_k,s_l,s^{**},\ldots,s^{**}) \tag{12}$$
$$=\arg\min_{s_l\in S_l}\max_{s_k\in S_k} u_k(s^*,\ldots,s^*,s_k,s_l,s^{**},\ldots,s^{**}) = s^{**}.$$

□

Next we show the following theorem.

Theorem 2. *Sion's minimax theorem with the coincidence of the maximin strategy and the minimax strategy implies the existence of a Nash equilibrium which is symmetric in each group.*

Proof. We denote a state such that Players $1, 2, \ldots, m$ choose $\tilde{s}$, and Players $m+1, m+2, \ldots, n$ choose $\hat{s}$ by $(\tilde{s},\ldots,\tilde{s},\hat{s}\ldots,\hat{s})$.

Let $\tilde{s}$ and $\hat{s}$ be the values of s_i's for $i\in\{1,2,\ldots,m\}$ and s_k's for $k\in\{m+1,m+2,\ldots,n\}$ such that

$$\tilde{s}=\arg\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i,s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s})=\arg\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i,s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s}),$$

$$\hat{s}=\arg\max_{s_k\in S_k}\min_{s_l\in S_l} u_k(\tilde{s},\ldots,\tilde{s},s_k,s_l,\hat{s},\ldots,\hat{s})=\arg\min_{s_l\in S_l}\max_{s_k\in S_k} u_k(\tilde{s},\ldots,\tilde{s},s_k,s_l,\hat{s},\ldots,\hat{s}),$$

$$\max_{s_i\in S_i}\min_{s_j\in S_j} u_i(s_i,s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s})=\min_{s_j\in S_j} u_i(s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s})$$
$$=\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i,s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s})=\max_{s_i\in S_i} u_i(s_i,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s}),$$

and

$$\max_{s_k\in S_k}\min_{s_l\in S_l} u_k(\tilde{s},\ldots,\tilde{s},s_k,s_l,\hat{s},\ldots,\hat{s})=\min_{s_l\in S_l} u_k(\tilde{s},\ldots,\tilde{s},s_l,\hat{s},\ldots,\hat{s})$$
$$=\min_{s_l\in S_l}\max_{s_k\in S_k} u_k(\tilde{s},\ldots,\tilde{s},s_k,s_l,\hat{s},\ldots,\hat{s})=\max_{s_k\in S_k} u_k(\tilde{s},\ldots,\tilde{s},s_k,\hat{s},\ldots,\hat{s}).$$

Since

$$u_i(s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s})\le\max_{s_i\in S_i} u_i(s_i,s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s}),$$
$$\min_{s_j\in S_j} u_i(s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s})=\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i,s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s}),$$

we get

$$\arg\min_{s_j\in S_j} u_i(s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s})=\arg\min_{s_j\in S_j}\max_{s_i\in S_i} u_i(s_i,s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s})=\tilde{s}.$$

Similarly, from

$$u_k(\tilde{s},\ldots,\tilde{s},s_l,\hat{s},\ldots,\hat{s})\le\max_{s_k\in S_k} u_k(\tilde{s},\ldots,\tilde{s},s_k,s_l,\hat{s},\ldots,\hat{s}),$$
$$\min_{s_l\in S_l} u_k(\tilde{s},\ldots,\tilde{s},s_l,\hat{s},\ldots,\hat{s})=\min_{s_l\in S_l}\max_{s_k\in S_k} u_k(\tilde{s},\ldots,\tilde{s},s_k,s_l,\hat{s},\ldots,\hat{s}),$$

we get

$$\arg\min_{s_l\in S_l} u_k(\tilde{s},\ldots,\tilde{s},s_l,\hat{s},\ldots,\hat{s})=\arg\min_{s_l\in S_l}\max_{s_k\in S_k} u_k(\tilde{s},\ldots,\tilde{s},s_k,s_l,\hat{s},\ldots,\hat{s})=\hat{s}.$$

Since

$$u_i(s_i,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s})\ge\min_{s_j\in S_j} u_i(s_i,s_j,\tilde{s},\ldots,\tilde{s},\hat{s},\ldots,\hat{s}),$$

and

$$\max_{s_i \in S_i} u_i(s_i, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}) = \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}),$$

we obtain

$$\arg\max_{s_i \in S_i} u_i(s_i, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}) = \arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}) = \tilde{s}.$$

Similarly, from

$$u_k(\tilde{s}, \ldots, \tilde{s}, s_k, \hat{s}, \ldots, \hat{s}) \geq \min_{s_l \in S_l} u_k(\tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}),$$

and

$$\max_{s_k \in S_k} u_k(\tilde{s}, \ldots, \tilde{s}, s_k, \hat{s}, \ldots, \hat{s}) = \max_{s_k \in S_k} \min_{s_l \in S_l} u_k(\tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}),$$

we obtain

$$\arg\max_{s_k \in S_k} u_k(\tilde{s}, \ldots, \tilde{s}, s_k, \hat{s}, \ldots, \hat{s}) = \arg\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(\tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}) = \hat{s}.$$

Therefore,

$$u_i(s_j, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}) \geq u_i(\tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}) \geq u_i(s_i, \tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}),$$

$$u_k(\tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}) \geq u_k(\tilde{s}, \ldots, \tilde{s}, \hat{s}, \ldots, \hat{s}) \geq u_k(\tilde{s}, \ldots, \tilde{s}, s_k, \hat{s}, \ldots, \hat{s}).$$

Thus, $(\tilde{s}, \ldots, \tilde{s}, \hat{s} \ldots, \hat{s})$ is a Nash equilibrium which is symmetric in each group. □

4 Example of relative profit maximization in each group of six-firms oligopoly

Consider a six-players game. The players are A, B, C, D, E and F. Suppose that the payoff functions of Players A, B and E are symmetric, and those of Players C, D and F are symmetric. The payoff functions of the players are

$$\begin{aligned}
\pi_A =& (a - x_A - x_B - x_E - bx_C - bx_D - bx_F)x_A - c_A x_A \\
& - \frac{1}{2}[(a - x_A - x_B - x_E - bx_C - bx_D - bx_F)x_B - c_A x_B \\
& + (a - x_A - x_B - x_E - bx_C - bx_D - bx_F)x_E - c_A x_E],
\end{aligned}$$

$$\begin{aligned}
\pi_B =& (a - x_A - x_B - x_E - bx_C - bx_D - bx_F)x_B - c_A x_B \\
& - \frac{1}{2}[(a - x_A - x_B - x_E - bx_C - bx_D - bx_F)x_A - c_A x_A \\
& + (a - x_A - x_B - x_E - bx_C - bx_D - bx_F)x_E - c_A x_E],
\end{aligned}$$

$$\begin{aligned}
\pi_E =& (a - x_A - x_B - x_E - bx_C - bx_D - bx_F)x_B - c_A x_B \\
& - \frac{1}{2}[(a - x_A - x_B - x_E - bx_C - bx_D - bx_F)x_B - c_A x_B \\
& + (a - x_A - x_B - x_E - bx_C - bx_D)x_B - c_A x_B],
\end{aligned}$$

$$
\begin{aligned}
\pi_C =&(a - x_C - x_D - x_F - bx_E - bx_A - bx_B)x_C - c_C x_C \\
&- \frac{1}{2}[(a - x_C - x_D - x_F - bx_E - bx_A - bx_B)x_D - c_C x_D \\
&+ (a - x_C - x_D - x_F - bx_E - bx_A - bx_B)x_F - c_C x_F],
\end{aligned}
$$

$$
\begin{aligned}
\pi_D =&(a - x_C - x_D - x_F - bx_E - bx_A - bx_B)x_D - c_C x_D \\
&- \frac{1}{2}[(a - x_C - x_D - x_F - bx_E - bx_A - bx_B)x_C - c_C x_C \\
&+ (a - x_C - x_D - x_F - bx_E - bx_A - bx_B)x_F - c_C x_F],
\end{aligned}
$$

$$
\begin{aligned}
\pi_F =&(a - x_C - x_D - x_F - bx_E - bx_A - bx_B)x_F - c_C x_F \\
&- \frac{1}{2}[(a - x_C - x_D - x_F - bx_E - bx_A - bx_B)x_C - c_C x_C \\
&+ (a - x_C - x_D - x_F - bx_E - bx_A - bx_B)x_D - c_C x_D].
\end{aligned}
$$

This is a model of relative profit maximization in each group in a six firms oligopoly with two groups. x_A, x_B, x_C, x_D, x_E and x_F are the outputs of the firms, and p_A, p_B, p_C, p_D, p_E and p_F are the prices of their goods. The demand functions are symmetric for Firms A, B and E, and they have the same cost functions, also the demand functions are symmetric for Firms C, D and F, and they have the same cost functions. However, the demand function for Firm A (or B or E) is not symmetric for Firm C (or D or F), and the demand function for Firm C (or D or F) is not symmetric for Firm A (or B or E). Firm A's (or Firm B's or Firm E's) cost function is different from the cost function of Firm C (or Firm D or Firm F). The cost functions of the firms are linear and there is no fixed cost.

We assume that Firm A (or B or E) maximizes its profit relatively to the profit of Firm B and E (or A and E, or B and E), and Firm C (or D or F) maximizes its profit relatively to the profit of Firm D and F (or C and F, or D and F). Note that

$$
\pi_A + \pi_B + \pi_E = 0, \;\; \pi_C + \pi_D + \pi_F = 0.
$$

Thus, this is a model of zero-sum game in each group with two groups.

Under the assumption of Cournot type behavior, the equilibrium outputs are

$$
\begin{aligned}
x_A &= \frac{bc_C - c_A - ab + a}{3(1-b)(1+b)}, \\
x_B &= \frac{bc_C - c_A - ab + a}{3(1-b)(1+b)}, \\
x_C &= \frac{bc_A - c_C - ab + a}{3(1-b)(1+b)}, \\
x_D &= \frac{bc_A - c_C - ab + a}{3(1-b)(1+b)}, \\
x_E &= \frac{bc_C - c_A - ab + a}{3(1-b)(1+b)}, \\
x_F &= \frac{bc_A - c_C - ab + a}{3(1-b)(1+b)}.
\end{aligned}
$$

The equilibrium prices of the goods are

$$p_A = c_A,$$
$$p_B = c_A,$$
$$p_C = c_C,$$
$$p_D = c_C,$$
$$p_E = c_A,$$
$$p_C = c_C.$$

Therefore, the prices of the goods are equal to the marginal costs in each group.

The maximin and minimax strategies between Firms A and B are

$$\arg\max_{x_A}\min_{x_B}\pi_A,\ \ \arg\min_{x_B}\max_{x_A}\pi_A.$$

Similarly, we can define the maximin and minimax strategies between Firms A and E, those between Firms B and E, Firms E and A, Firms E and B.

Those between Firm C and D are

$$\arg\max_{x_C}\min_{x_D}\pi_C,\ \ \arg\min_{x_D}\max_{x_C}\pi_C.$$

Similarly, we can define the maximin and minimax strategies between Firms C and F, those between Firms D and F, Firms F and C, Firms F and D.

In our example, under the assumption that $x_E = x_A$, we obtain

$$\arg\max_{x_A}\min_{x_B}\pi_A = \frac{bc_C - c_A - ab + a}{3(1-b)(1+b)},$$
$$\arg\min_{x_B}\max_{x_A}\pi_A = \frac{bc_C - c_A - ab + a}{3(1-b)(1+b)},$$
$$\arg\max_{x_C}\min_{x_D}\pi_C = \frac{bc_A - c_C - ab + a}{3(1-b)(1+b)},$$
$$\arg\min_{x_D}\max_{x_C}\pi_C = \frac{bc_A - c_C - ab + a}{3(1-b)(1+b)},$$
$$\arg\max_{x_E}\max_{x_A}\pi_E = \frac{bc_C - c_A - ab + a}{3(1-b)(1+b)},$$
$$\arg\min_{x_F}\max_{x_C}\pi_C = \frac{bc_A - c_C - ab + a}{3(1-b)(1+b)},$$

and so on. They are the same as Nash equilibrium strategies.

5 Concluding Remark

We have analyzed the relation between Sion's minimax theorem for a continuous function and Nash equilibrium in a multi-players game with two groups which is zero-sum and symmetric in each group. Our analysis can be easily extended to a case with more than two groups.

Appendix: Note on the case where Assumption 1 is not assumed.

Let $(\tilde{s}, \hat{s}, s^1, s^2)$ be the solution (fixed point) of the following equations.

$$\tilde{s} = \arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}),$$

$$s^1 = \arg\min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}),$$

$$\hat{s} = \arg\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}).$$

and

$$s^2 = \arg\min_{s_l \in S_l} \max_{s_k \in S_k} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}),$$

with $s_j = s^1$ and $s_l = s^2$. By (3) and (4)

$$\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = \min_{s_j \in S_j} u_i(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s})$$
$$= \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = \max_{s_i \in S_i} u_i(s_i, s^1, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}),$$

and

$$\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}) = \min_{s_l \in S_l} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s})$$
$$= \min_{s_l \in S_l} \max_{s_k \in S_k} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}) = \max_{s_k \in S_k} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s^2, \hat{s}, \ldots, \hat{s}).$$

Since

$$\max_{s_i \in S_i} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) \geq u_i(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}),$$

$$\min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = \min_{s_j \in S_j} u_i(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}),$$

we have

$$\arg\min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = \arg\min_{s_j \in S_j} u_i(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = s^1.$$

Similarly, from

$$\max_{s_k \in S_k} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}) \geq u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}),$$

$$\min_{s_l \in S_l} \max_{s_k \in S_k} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}) = \min_{s_l \in S_l} u_i(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}),$$

we have

$$\arg\min_{s_l \in S_l} \max_{s_k \in S_k} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}) = \arg\min_{s_l \in S_l} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}) = s^2.$$

Since

$$\min_{s_j \in S_j} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) \leq u_i(s_i, s^1, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}),$$

$$\min_{s_l \in S_l} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}) \leq u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s^2, \hat{s}, \ldots, \hat{s}),$$

$$\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = \max_{s_i \in S_i} u_i(s_i, s^1, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}),$$

and

$$\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}) = \max_{s_k \in S_k} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s^2, \hat{s}, \ldots, \hat{s}),$$

we have

$$\arg\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = \arg\max_{s_i \in S_i} u_i(s_i, s^1, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = \tilde{s}, \tag{13}$$

$$\arg\max_{s_k \in S_k} \min_{s_l \in S_l} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s_l, \hat{s}, \ldots, \hat{s}) = \arg\max_{s_k \in S_k} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_k, s^2, \hat{s}, \ldots, \hat{s}) = \hat{s}. \tag{14}$$

Because the game is zero-sum in each group,

$$\sum_{i=1, i \neq j}^{m} u_i(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) + u_j(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = 0,$$

$$\sum_{k=m+1, k \neq l}^{n} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s} + u_l(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}) = 0.$$

By symmetry for each group

$$(m-1)u_i(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) + u_j(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = 0,$$

$$(n-m-1)u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s} + u_l(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}) = 0.$$

Thus,

$$(m-1)u_i(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = -u_j(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}),$$

$$(n-m-1)u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}) = -u_l(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}).$$

They mean

$$\arg\min_{s_j \in S_j} u_i(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = \arg\max_{s_j \in S_j} u_j(s_j, \tilde{s}, \ldots, \tilde{s}, s^2, \hat{s}, \ldots, \hat{s}) = s^1, \tag{15}$$

$$\arg\min_{s_l \in S_l} u_k(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}) = \arg\max_{s_l \in S_l} u_l(s^1, \tilde{s}, \ldots, \tilde{s}, s_l, \hat{s}, \ldots, \hat{s}) = s^2. \tag{16}$$

Therefore, if $s^1 \neq \tilde{s}$ or $s^2 \neq \hat{s}$, there may exist a Nash equilibrium denoted as follows;

$$(\tilde{s}, \ldots, s^1, \ldots, \tilde{s}, \hat{s}, \ldots, s^2, \ldots \hat{s}),$$

We may have $s^1 = \tilde{s}$ or $s^2 = \hat{s}$.

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