Player splitting, players merging, the Shapley set value and the Harsanyi set value

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Abstract

We discuss a value, proposed in the context of cost allocation by Shapley (1981) and Dehez (2011) and in general by Radzik (2012). This value, we call it Shapley set value, covers the weighted Shapley values all at once. It is defined on weighted TU-games in the form of two constituent parts, a weight system and a classical TU-game, where the weights and the coalition function may vary at the same time. In addition, similar to the Shapley set value, we introduce the Harsanyi set value. It captures all TU-values from the Harsanyi set, called Harsanyi payoffs. A player splitting and a players merging property enable new axiomatizations. Examples recommend both solution concepts for profit distribution and cost allocation.

Keywords Cost allocation · Profit distribution · Player splitting · Players merging · Shapley set· Harsanyi set

1. Introduction

Coalition functions don't exist in a vacuum and so in many games personal weights are assigned to the players. Shapley (1953a) spoke of "differences in the external characteristics of the players" and suggested the weighted Shapley values (Shapley, 1953a). The family of all such TU-values is also known as the Shapley set. Each TU-value of this set incorporates a fixed given weight system.

But often, the weights can depend on the coalition functions as handled in some problems of profit distribution (earnings per share) or cost allocation (see e. g. Moriarity (1975)). If, for example, the weights are the singleton worths of the players, one can apply the formula of a weighted Shapley value. Then she has to change the weighted Shapley value for a new game if the singleton worths are no longer in the same proportion as in the old game. This was the idea of the "Independent Cost Proportional Scheme (ICPS)" in Gangolly (1981). In effect, instead of a weighted Shapley value, a different

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value is used, called proportional Shapley Value (Gangolly, 1981; Besner, 2016; Béal et al., 2017).

In practice, it does not matter to compute the payoff by the formula of the weighted Shapley values. But if someone wants to know which axiomatizations the used value has, for example to argue why a certain value should be selected, she has to identify the correct value. That we have to change a value if the underlying coalition function is altering is not a convincing argument to choose a value for calculating. Also, that we cannot apply the same value in cost games and cost-saving games or in profit games and cooperation benefit games in the same situation seems unnatural.

All these mentioned lacks do not occur by the Shapley set value (Shapley, 1981; Dehez, 2011; Radzik, 2012). At our knowledge, the first ones who gave attention to the fact that a value can be defined on weighted TU-games which consist of a TU-game and a weight system with personally given weights are Shapley (1981) and Dehez (2011)[1]. So they introduced weighted cost games and presented particular axiomatizations of a "weighted value" in the context of cost allocation. Radzik (2012) formulated the value in general. This is not a TU-value in the original sense: it takes into account the coalition function and players' weights at once. Radzik introduced weighted coalition functions, defined weighted TU-games and shaped out a value axiomatized by a large group of not logically independent axioms. Then Radzik called this value "weighted Shapley value" because it satisfies adapted well-known axiomatizations of the weighted Shapley values in Nowak and Radzik (1995). But the axioms are adaptions of axioms for TU-games. The value coincides with weighted Shapley values only on subdomains.

Thus, in some respects, we consider the naming "weighted Shapley value" as not accurate: On the one hand to avoid name conflicts with the weighted Shapley values for TU-games and, in our opinion, not the value is weighted, it is rather that we may be able to regard the coalition functions as weighted. On the other hand, this value, we call it Shapley set value, satisfies different axioms as the weighted Shapley values, particularly in the case of subdomains. So it turns out that the Shapley set value captures, e. g., the proportional Shapley value too.

The weighted Shapley values are obtained by distributing the dividends (Harsanyi, 1959) by weights of a weight system where a player owns for each coalition containing her the same weight. Hammer (1977) and Vasil'ev (1978) introduced the Harsanyi set, a set of TU-values called Harsanyi payoffs. For each player exists for each coalition containing her a possibly different weight. In this system of weights, called sharing system, the weights of the players for each coalition are non-negative, sum up to one and the dividends have to be distributed between the players according to these weights.

Similar to the Shapley set value we introduce the Harsanyi set value that covers all Harsanyi payoffs at once and is defined on sharing TU-games which consist of a TU-game together with a sharing system at a time. This value also coincides on subdomains with values which are not in the Harsanyi set.

In the main part of this article, we propose new axiomatizations of the Shapley set value and the Harsanyi set value. Within these axiomatizations a player splitting property and a players merging property play a decisive role and may help to close the gap using cooperative game theory in profit distribution and cost allocation not only in theory but

[1] Shapley (1981) notes only a comment at an accounting conference that was worked out and proved later by Dehez (2011).

also in practice.

The article is organized as follows. Section 2 contains some preliminaries. In Section 3 we provide motivating examples for profit games and refer shortly to an example for cost games. Section 4 presents axioms for the Shapley set value, gives axiomatizations for this value and reveals, as a side-benefit, a new axiomatization of the Shapley value. In Section 5 we transfer our proceeding with the Shapley set value in the previous section to the Harsanyi set value. Section 6 summarizes the results and gives a conclusion. An appendix (Section 7) provides all the proofs and shows the logical independence of the axioms used for axiomatization.

2. Preliminaries

We denote by $\mathbb{N}$ the natural numbers, by $\mathbb{R}$ the real numbers, by $\mathbb{R}_{++}$ the set of all positive real numbers, and by $\mathbb{Q}_{++}$ the set of all positive rational numbers. Let $\mathfrak{U}$ be a countably infinite set, the universe of all players. We define by $\mathcal{N}$ the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) consisting of a player set $N \in \mathcal{N}$ and a **coalition function** $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. If N is clear, we refer to a TU-game also by v. We call each subset $S \subseteq N$ a **coalition**, $v(S)$ represents the **worth** of coalition S and we denote by Ω^S the set of all non-empty subsets of S. The set of all TU-games with player set N is denoted by $\mathbb{V}(N)$ and, if $v(\{i\}) > 0$ for all $i \in N$ or if $v(\{i\}) < 0$ for all $i \in N$, by $\mathbb{V}_0(N)$. The **restriction** of (N, v) to $N' \in \Omega^N$ is denoted by (N', v).

Let $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$. The **dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\Delta_v(S) := \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), \text{ if } S \in \Omega^N, \text{ and} \\ 0, \text{ if } S = \emptyset. \end{cases}$$

An **unanimity game** (N, u_S), $S \in \Omega^N$, is defined for all $T \subseteq N$ by $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. It is a well-known fact that each $v \in \mathbb{V}(N)$ has a unique presentation

$$v = \sum_{S \in \Omega^N} \Delta_v(S) u_S. \tag{1}$$

The **marginal contribution** $MC_i^v(S)$ of player $i \in N$ to $S \subseteq N \backslash \{i\}$ is defined by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. A player $i \in N$ is called a **null player** in v if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$; players $i, j \in N$, $i \neq j$, are **symmetric** in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$, they are called (mutually) **dependent** (Nowak and Radzik, 1995) in v if $v(S \cup \{i\}) = v(S) = v(S \cup \{j\})$; a coalition $Q \subseteq N$ is called a **partnership** (Kalai and Samet, 1987) in v if $v(S \cup T) = v(S)$ for all $S \subseteq N \backslash Q$ and $T \subsetneq Q$.

We define by $\Lambda^N := \{f : N \to \mathbb{R}_{++}\}$, $\lambda_i := \lambda(i)$ for all $\lambda \in \Lambda^N$, $i \in N$, the collection of all positive **weight systems** on N and by $\Lambda^N_{\mathbb{Q}} := \{f\colon N \to \mathbb{Q}_{++}\}$ the collection of all positive rational weight systems on N. Let $v \in \mathbb{V}(N)$ and $\lambda \in \Lambda^N$. Then $v^\lambda := (\lambda, v)$ is called a **weighted coalition function** and (N, v^λ) or short also v^λ is said to be a

weighted TU-game[2] (**WTU-game**). We denote by $\mathbb{V}^{\Lambda}(N)$ the set of all WTU-games with player set N and by $\mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$ the subset of $\mathbb{V}^{\Lambda}(N)$ where $\lambda \in \Lambda^N_{\mathbb{Q}}$.

The collection Γ^N on N of all **sharing systems** $\gamma \in \Gamma^N$ is defined by

$$\Gamma^N := \left\{ \gamma = (\gamma_{S,i})_{S\in\Omega^N, i\in S} \middle| \sum_{i\in S} \gamma_{S,i} = 1 \text{ and } \gamma_{S,i} \geq 0 \text{ for each } S \in \Omega^N \text{ and all } i \in S \right\}$$

and the collection $\Gamma^N_{\mathbb{Q}}$ on N of all positive rational sharing systems is given by

$$\Gamma^N_{\mathbb{Q}} := \left\{ \gamma = (\gamma_{S,i})_{S\in\Omega^N, i\in S} \middle| \sum_{i\in S} \gamma_{S,i} = 1 \text{ and } \gamma_{S,i} \in \mathbb{Q}_{++} \text{ for each } S \in \Omega^N \text{ and all } i \in S \right\}.$$

Let $v \in \mathbb{V}(N)$ and $\gamma \in \Gamma^N$. Then $v^\gamma := (\gamma, v)$ is called a **sharing coalition function** and (N, v^γ) or short also v^γ is said to be a **sharing TU-game** (**STU-game**). We denote by $\mathbb{V}^{\Gamma}(N)$ the set of all STU-games with player set N, by $\mathbb{V}^{\Gamma_{\mathbb{Q}}}(N)$ the subset of $\mathbb{V}^{\Gamma}(N)$ where $\gamma \in \Gamma^N_{\mathbb{Q}}$ and by $\mathbb{V}^{\Gamma}_0(N)$ the subset of $\mathbb{V}^{\Gamma}(N)$ where $\gamma \in \Gamma^N$ and $v \in \mathbb{V}_0(N)$.

For any $N \in \mathcal{N}$, a **TU-value** φ is an operator that assigns to any $v \in \mathbb{V}(N)$ a payoff vector $\varphi(N, v) \in \mathbb{R}^N$, a **weighted TU-value** (**WTU-value**) φ^Λ is an operator that assigns to any $v^\lambda \in \mathbb{V}^\Lambda(N)$ a payoff vector $\varphi^\Lambda(N, v^\lambda) \in \mathbb{R}^N$ and a **sharing TU-value** (**STU-value**) φ^Γ is an operator that assigns to any $v^\gamma \in \mathbb{V}^\Gamma(N)$ a payoff vector $\varphi^\Gamma(N, v^\gamma) \in \mathbb{R}^N$.

Let $v \in \mathbb{V}(N)$ and $\lambda \in \Lambda$: The (positively) **weighted Shapley Value**[3] Sh^λ (Shapley, 1953a) is defined by

$$Sh^\lambda_i(N, v) := \sum_{S\subseteq N,\, S\ni i} \frac{\lambda_i}{\sum_{j\in S} \lambda_j} \Delta_v(S) \quad \text{for all} \quad i \in N. \tag{2}$$

The **Shapley set** is the set of all weighted Shapley values. A special case of a weighted Shapley value, all weights are equal, is the **Shapley value** Sh (Shapley, 1953b), given by

$$Sh_i(N, v) := \sum_{S\subseteq N,\, S\ni i} \frac{\Delta_v(S)}{|S|} \quad \text{for all} \quad i \in N.$$

The set of the following TU-values is called **Harsanyi set** (Hammer, 1977; Vasil'ev, 1978), also known as **selectope** (Derks, Haller, and Peters, 2000), where the payoffs are obtained by distributing the dividends with the help of a sharing system γ. Each TU-value H^γ, $\gamma \in \Gamma^N$, in this set, titled **Harsanyi payoff**, is defined by

$$H^\gamma_i(N, v) := \sum_{S\subseteq N,\, S\ni i} \gamma_{S,i} \Delta_v(S), \text{ for all } i \in N.$$

Let $v \in \mathbb{V}_0(N)$. The **proportional Shapley Value** Sh^p (Gangolly, 1981; Besner, 2016; Béal et al., 2017) is defined by

$$Sh^p_i(N, v) := \sum_{S\subseteq N,\, S\ni i} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})} \Delta_v(S) \quad \text{for all} \quad i \in N.$$

[2] Radzik (2012) called such a game a "transferable utility weighted game in characteristic function form."
[3] We desist from weights of zero as possible in Kalai and Samet (1987)

Let $v^\lambda \in \mathbb{V}^\Lambda(N)$. As a WTU-value, the **Shapley set value** Sh^Λ (Shapley, 1981/Dehez, 2011; Radzik, 2012)[4] is defined by

$$Sh_i^\Lambda(N, v^\lambda) := \sum_{S \subseteq N,\, S \ni i} \frac{\lambda_i}{\sum_{j \in S} \lambda_j} \Delta_v(S) \ \text{ for all } \ i \in N. \tag{3}$$

Let $v^\gamma \in \mathbb{V}^\Gamma(N)$. As an STU-value, we introduce the **Harsanyi set value** H^Γ defined by

$$H_i^\Gamma(N, v^\gamma) := \sum_{S \subseteq N,\, S \ni i} \gamma_{S,i} \Delta_v(S) \text{ for all } \ i \in N. \tag{4}$$

3. Profit and cost allocation and motivating examples

A cooperative game with a coalition function v commonly represents profits or savings for all possible coalitions of the player set. Thus in the following we denote by v a **profit game**. A profit game is given by a **payout function** p by

$$v(S) := p(S) - \sum_{i \in S} f(\{i\}) \text{ for each } S \in \Omega^N, \tag{5}$$

where $f(\{i\})$ stands for the (financial) **involvement** of the player i and $p(S)$ for the payout to coalition S. Another application for coalition functions are games where the worths of coalitions represent costs. We denote such **cost games** by c. Closely related to cost games are **cost-saving games** d which give the savings obtained by forming coalitions and are defined by

$$d(S) := \sum_{i \in S} c(\{i\}) - c(S) \text{ for each } S \in \Omega^N.$$

In addition, similar to cost-saving games, we introduce **cooperation benefit games** q which are connected to profit games v by

$$q(S) := v(S) - \sum_{i \in S} v(\{i\}) \text{ for each } S \in \Omega^N \tag{6}$$

and present the profit of cooperating towards to be lone fighters.

E. g., Amer et al. (2007) claim a coherent solution should exist for both cost and saving (related) problems, so that all players are indifferent between sharing costs and sharing savings. That means for a player i we should have

$$\varphi_i(c) = c(\{i\}) - \varphi_i(d) \text{ for all } i \in N. \tag{7}$$

In the same sense, we should have that players are indifferent between sharing profits and sharing cooperation benefits

$$\varphi_i(v) = v(\{i\}) + \varphi_i(q) \text{ for all } i \in N. \tag{8}$$

In the following, we give examples of profit sharing and refer to examples of cost allocation where the usage of the Shapley set value, or, if the weights of players are in different proportion to each other in some coalitions, the Harsanyi set value is recommended.

[4] Shapley (1981) and Dehez (2011) denoted this value as "weighted value" and used it in the context of cost games, Radzik (2012) denoted this value as "weighted Shapley value." We also desist from possibly null weights and refer in this respect to Section 6.

3.1. Examples for profit allocation

An entrepreneur wants to bridge a short-term need for finance of 50 million monetary units (MMU) for one year. He is willing to pay 5 percent interest in this year. Additionally, for some reasons, he prefers as few financiers as possible. Thus, the deposits must be multiples of 5 million and the entrepreneur pays a bonus of $100,000$ if the deposit amounts not less then 30 million and a bonus of $500,000$ if the deposit amounts exactly the sum of 50 million. So, we have a payout function p (in MMU) given by

$$p(f) := \begin{cases} 1.05f, & \text{if } 5 \leq f < 30, \\ 1.05f + 0.1, & \text{if } 30 \leq f < 50, \\ 1.05f + 0.5, & \text{if } f = 50, \end{cases} \tag{9}$$

where f is the deposit of a financier under the restriction that $f = 5k$, $k \in \mathbb{N}$.

3.1.1. Scenario 1

Three investors, investors A, B, and C, will cooperate to achieve a share of the bonus too. Therefore, they must occur as one single financier. A wants to make an involvement about 20 million, B and C prefer to invest in each case 16 million. The investors agree, that the investor with the larger investment possibilities waives the not needed share of the optimal deposit (Table 1).

Table 1: Deposits f (MMU)

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
$f(S)$	20	15	15	19+16=35	19+16=35	15+15=30	18+16+16=50

By (9), we establish a payout function p on $N = \{A, B, C\}$ where $p(S)$, $S \subseteq N$, is the payout to coalition S. By (5), we obtain Table 2 for the related profit game v and, by (6), Table 3 for the cooperation benefit game q.

Table 2: Profit game v (MMU)

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
$v(S)$	1	0.75	0.75	1.85	1.85	1.6	3

Table 3: Cooperation benefit game q (MMU)

S	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
$q(S)$	0	0	0	0.1	0.1	0.1	0.5

The three investors wish to use cooperative game theory to share the benefits. The ideal would be a one-point solution that not has to be changed if the investments are altering. B and C propose at first the Shapley value Sh. But A disagrees. A argues that the Shapley value prefers players with lower deposits and points to the following scenario.

3.1.2. Scenario 2

Investor A splits up in two new investors A_1 and A_2. With investors A_1, A_2, B, and C the deposits $\tilde{f}$ should be given as shown in Table 4 where the sum of the deposits of players A_1 and A_2 in coalitions which contain both players equals the deposit of player A in the related coalitions in Scenario 1.

Table 4: Deposits $\tilde{f}$ (MMU)

S	$\{A_1\}$	$\{A_2\}$	$\{B\}$	$\{C\}$	$\{A_1, A_2\}$	$\{A_1, B\}$	$\{A_1, C\}$	$\{A_2, B\}$	$\{A_2, C\}$
$\tilde{f}(S)$	10	10	15	15	10+10=20	10+15=25	10+15=25	10+15=25	10+15=25

S	$\{B, C\}$	$\{A_1, A_2, B\}$	$\{A_1, A_2, C\}$	$\{A_1, B, C\}$	$\{A_2, B, C\}$	$\{A_1, A_2, B, C\}$
$\tilde{f}(S)$	15+15=30	9.5+9.5+16=35	9.5+9.5+16=35	10+15+15=40	10+15+15=40	9+9+16+16=50

We have a new payout function $\tilde{p}$ on $\tilde{N} = \{A_1, A_2, B, C\}$ and obtain the profit game $\tilde{v}$ in Table 5 and the cooperation benefit game $\tilde{q}$ given in Table 6.

Table 5: Profit game $\tilde{v}$ (MMU)

S	$\{A_1\}$	$\{A_2\}$	$\{B\}$	$\{C\}$	$\{A_1, A_2\}$	$\{A_1, B\}$	$\{A_1, C\}$	$\{A_2, B\}$	$\{A_2, C\}$
$\tilde{v}(S)$	0.5	0.5	0.75	0.75	1	1.25	1.25	1.25	1.25

S	$\{B, C\}$	$\{A_1, A_2, B\}$	$\{A_1, A_2, C\}$	$\{A_1, B, C\}$	$\{A_2, B, C\}$	$\{A_1, A_2, B, C\}$
$\tilde{v}(S)$	1.6	1.85	1.85	2.1	2.1	3

Table 6: Cooperation benefit game $\tilde{q}$ (MMU)

S	$\{A_1\}$	$\{A_2\}$	$\{B\}$	$\{C\}$	$\{A_1, A_2\}$	$\{A_1, B\}$	$\{A_1, C\}$	$\{A_2, B\}$	$\{A_2, C\}$
$\tilde{q}(S)$	0	0	0	0	0	0	0	0	0

S	$\{B, C\}$	$\{A_1, A_2, B\}$	$\{A_1, A_2, C\}$	$\{A_1, B, C\}$	$\{A_2, B, C\}$	$\{A_1, A_2, B, C\}$
$\tilde{q}(S)$	0.1	0.1	0.1	0.1	0.1	0.5

Here is a special kind of "dependency":

- The worth of any coalition containing both players A_1 and A_2 in $\tilde{v}$ and $\tilde{q}$ is the same as the worth of the related coalition containing player A in v and q. Players A_1 and A_2 spend together in these coalitions the same investment as player A in the old coalition, the investments of the other players are not changing.
- In each game the marginal contributions of player A_1 or player A_2 to any coalition which does not contain the respective other player are only the singleton worths of these players. So A_1 and A_2 are dependent in $\tilde{q}$.
- Coalitions which are the same in both scenarios have the same worth in v and $\tilde{v}$ and q and $\tilde{q}$ respectively.

In the sum, splitting player A in players A_1 and A_2 has no effect on the other players. A argues that the payoff to not split players should not change in such a case. But this is not true for the Shapley value (Table 7)! Thus, the investors reject the Shapley value. A

Table 7: Profits (MMU)

Investor	A	A_1	A_2	B	C
Shapley value $Sh(N, v)$	1.167	-	-	0.917	0.917
Shapley value $Sh(\tilde{N}, \tilde{v})$	-	0.617	0.617	0.883	0.883
Proportional Shapley value $Sh^p(N, v)$	1.194	-	-	0.903	0.903
Proportional Shapley value $Sh^p(\tilde{N}, \tilde{v})$	-	0.597	0.597	0.903	0.903
Shapley set value $Sh^\Lambda(N, v^\lambda)$	1.194	-	-	0.903	0.903
Shapley set value $Sh^\Lambda(\tilde{N}, \tilde{v}^{\tilde{\lambda}})$	-	0.597	0.597	0.903	0.903
Harsanyi set value $H^\Gamma(N, v^\gamma)$	1.181	-	-	0.910	0.910
Harsanyi set value $H^\Gamma(\tilde{N}, \tilde{v}^{\tilde{\gamma}})$	-	0.590	0.590	0.910	0.910

makes the proposal to use the proportional Shapley value Sh^p that provides the desired result for profit games (Table 7).

Now B opposed that the players should be also indifferent between sharing profits and sharing cooperation benefits (eq. 8) where Sh^p completely fails since the singleton worths are out of the domain. C suggests a weighted Shapley value Sh^λ. The Investors agree that the singleton involvements should be the weights. So they can use formula (2) but they recognize that the weights are not independent of the involvements and so of the coalition function. So they have to change the value if the coalition function is altering which is not the desired approach.

Using the Shapley set value Sh^Λ they achieve the same result as by Sh^p (Table 7) for the profit games and (8) is fulfilled too. But now occurs another problem. Apparently, the investments of the players differ in some coalitions, and so the weights have to be adapted. The investors realize that they must apply a sharing system $\gamma \in \Gamma^N$, given by

$$\gamma_{S,i} := \frac{f_i(S)}{f(S)}, \ S \in \Omega^N,$$

and a sharing system $\tilde{\gamma} \in \Gamma^{\tilde{N}}$, given by

$$\tilde{\gamma}_{S,i} := \frac{\tilde{f}_i(S)}{\tilde{f}(S)}, \ S \in \Omega^{\tilde{N}},$$

where $f_i(S), \tilde{f}_i(S)$ are the respective shares of the player i in the deposit of coalition S. However: if they use a Harsanyi payoff H^γ they face the same problem as before by the weighted Shapley value, the coalition functions and the sharing systems are not independent and they are forced to change the value if the coalition function is altering. Finally, all investors reached a consensus to use the Harsanyi set value H^Γ that meets all requirements (Table 7) and it is easy to verify that also (8) is satisfied.

3.2. Examples for cost allocation

We refer to the examples in Besner (2017). There the Shapley value Sh fails if we regard City 2 after City 1. The proportional Shapley value Sh^p allocates the unsplit players for the cost game in both scenarios the same payoffs. But if we consider (7) as a desirable requirement, Sh^p completely fails because in the related cost-saving games the singletons have a worth of zero. A weighted Shapley value Sh^λ cannot be applied because the weights are the singleton worths from the cost game and depend so on the coalition function. Finally, the Shapley set value Sh^Λ satisfies all requirements we demand. We obtain for the cost game the same payoffs as by Sh^p and it can be easily shown that the players are indifferent between sharing costs and sharing savings. If there, unlike as in these examples, the players' weights differ for some coalitions so that we have to use a sharing system γ, the Harsanyi set value H^Γ is recommended for the usage if γ depends on the coalition function.

4. The Shapley set value

In general, the set of all weighted Shapley values requires for each value that the fixed weights are independent of the coalition function because a TU-value must hold for all coalition functions. Shapley (1981) and Dehez (2011), only in the context of cost games, and Radzik (2012) introduced a value for weighted TU-games which consist of a TU-game and a weight system at once. But there is no reference that the weights related to a coalition function can here depend on it. The weights are not fixed, and the value holds for all coalition functions and all positive weight systems.

Thus, we have, in particular on some subdomains, no problem as long as the weights depend on the coalition function in such a way that they are positive. A naïve reader could think that in the mentioned articles above is just discussed a variety of the weighted Shapley values. This is not the case. We call this value Shapley set value since it coincides with values from the Shapley set on certain subdomains. But also it captures values which are not in the Shapley set. The following subsections are intended to demonstrate the differences. As Aristotle said: "The whole is greater than the sum of its parts."

4.1. Axioms for TU-values

We refer to the following standard axioms for TU-values φ:

Efficiency, E. For all $v \in \mathbb{V}(N)$, we have $\sum_{i\in N}\varphi_i(N,v) = v(N)$.

Null player, N. For all $v \in \mathbb{V}(N)$ and $i \in N$ such that i is a null player in v, we have $\varphi_i(N,v) = 0$.

Linearity, L. For all $v, w \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$, we have
$\varphi(N, \alpha v + w) = \alpha\varphi(N,v) + \varphi(N,w)$.

Additivity, A. For all $v, w \in \mathbb{V}(N)$, we have $\varphi(N,v) + \varphi(N,w) = \varphi(N, v+w)$.

λ-weighted proportionality, WP$^\lambda$ (Nowak and Radzik, 1995)[5]**.** For all $v \in \mathbb{V}(N)$, $i, j \in N$ such that i and j are dependent in v and $\lambda \in \Lambda$, we have

$$\frac{\varphi_i(N,v)}{\lambda_i} = \frac{\varphi_j(N,v)}{\lambda_j}.$$

[5]This is the essential part of the ω-Mutual Dependence axiom in (Nowak and Radzik, 1995).

4.2. Radzik's axioms for WTU-values

Radzik (2012) presented a lot of axioms for WTU-values. One of these axioms requires an additional player set. Radzik introduced for a fixed coalition $Q \in \Omega^N$ a new player set $N^Q := (N\backslash Q) \cup \{Q\}$ where Q is regarded as a single player and so $\{Q\}$ is a singleton. Then he defined for this player set a new WTU-game, related to the old game. To avoid any misunderstanding, in the following definition we denote the coalition Q by q if we treat Q as a player.

Definition 4.1. (Radzik, 2012) *Let $Q \in \Omega^N$, $q := Q$ if Q is regarded as a player, $N^Q := (N\backslash Q) \cup \{q\}$, $(N, v^\lambda) \in \mathbb{V}^\Lambda(N)$. The game $\big(N^Q, (v^\lambda)^Q\big) \in \mathbb{V}^\Lambda(N^Q)$, $(v^\lambda)^Q := (\lambda^Q, v^Q)$, is called a **merged players WTU-game**[6] (**MPW-game**) to (N, v^λ) if for all $S \subseteq N\backslash Q$*

- $\lambda_k^Q = \lambda_k$ *if* $k \in N^Q\backslash\{q\}$ *and* $\lambda_q^Q = \sum_{i\in Q} \lambda_i$,
- $v^Q(S) = v(S)$,
- $v^Q(S \cup \{q\}) = v(S \cup Q)$.

The following axioms for WTU-values φ^Λ come from Radzik (2012):

Efficiency$^\Lambda$, E$^\Lambda$. For all $v^\lambda \in \mathbb{V}^\Lambda(N)$, we have $\sum_{i\in N} \varphi_i^\Lambda(N, v^\lambda) = v(N)$.

Null player$^\Lambda$, N$^\Lambda$. For all $v^\lambda \in \mathbb{V}^\Lambda(N)$ and $i \in N$ such that i is a null player in v, we have $\varphi_i^\Lambda(N, v^\lambda) = 0$.

Additivity$^\Lambda$ (in the coalition function)**, A$^\Lambda$.** For all $v^\lambda, w^\lambda \in \mathbb{V}^\Lambda(N)$, we have $\varphi^\Lambda(N, v^\lambda) + \varphi^\Lambda(N, w^\lambda) = \varphi^\Lambda(N, (v + w)^\lambda)$.

Linearity$^\Lambda$ (in the coalition function)**, L$^\Lambda$.** For all $v^\lambda, w^\lambda \in \mathbb{V}^\Lambda(N)$ and $\alpha \in \mathbb{R}$, we have $\varphi^\Lambda(N, (\alpha v + w)^\lambda) = \alpha\varphi^\Lambda(N, v^\lambda) + \varphi^\Lambda(N, w^\lambda)$.

Weight proportionality$^\Lambda$, WP$_Q^\Lambda$. For all $v^\lambda \in \mathbb{V}^\Lambda(N)$, $Q \in \Omega^N$ such that Q is a partnership in v and $i \in Q$, we have

$$\frac{\varphi_i^\Lambda(N, v^\lambda)}{\lambda_i} = \text{const.}$$

Marginality$^\Lambda$ (in the coalition function)**, M$^\Lambda$.** For all $v^\lambda, w^\lambda \in \mathbb{V}^\Lambda(N)$ and $i \in N$ such that $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N\backslash\{i\}$, we have $\varphi_i^\Lambda(N, v^\lambda) = \varphi_i^\Lambda(N, w^\lambda)$.

Amalgamating payoffs$^\Lambda$, AP$^\Lambda$. For all $v^\lambda \in \mathbb{V}^\Lambda(N)$, $Q \in \Omega^N$ a partnership in v, $q := Q$ if Q is regarded as a player and $\big(N^Q, (v^\lambda)^Q\big) \in \mathbb{V}^\Lambda(N^Q)$ an MPW-game to v^λ, we have

$$\sum_{i\in Q} \varphi_i^\Lambda(N, v^\lambda) = \varphi_q^\Lambda\big(N^Q, (v^\lambda)^Q\big).$$

Continuity$^\Lambda$, C$^\Lambda$. For all $v^\lambda \in \mathbb{V}^\Lambda(N)$ and $\alpha \in \mathbb{R}$, we have $\varphi^\Lambda(N, \alpha v^\lambda)$ is a continuous function of variable α.

Remark 4.2. *Note that **A**$^\Lambda$, **L**$^\Lambda$, and **M**$^\Lambda$ hold only in the coalition function and therefore only on subdomains. The weight systems must always be equal.*

[6]In Radzik (2012) this game is only mentioned as a reduced weighted game.

4.3. Additional axioms for WTU-values

Besner (2017) introduced a corresponding split player game for TU-games where a fixed player is split in two new players. In analogy, we define a split player game for WTU-games. In contrast to the MPW-game, in the split player game the new players are completely independent of the "split" player in the original game apart from the given properties in the following definition.

Definition 4.3. *Let $j \in N$, $N^j := (N \backslash \{j\}) \cup \{k, \ell\}$, $k, \ell \in \mathfrak{U}$, $k, \ell \notin N$, $(N, v^\lambda) \in \mathbb{V}^\Lambda(N)$. The game $\big(N^j, (v^\lambda)^j\big) \in \mathbb{V}^\Lambda(N^j)$, $(v^\lambda)^j := (\lambda^j, v^j)$, is called a* ***split player WTU-game (SPW-game)*** *to (N, v^λ) if for all $S \subseteq N \backslash \{j\}$*

- $\lambda^j_k + \lambda^j_\ell = \lambda_j$ *and* $\lambda^j_i = \lambda_i$ *if* $i \in N \backslash \{j\}$,
- $v^j(S \cup \{m\}) = v(S)$ *if* $m \in \{k, \ell\}$,
- $v^j(S \cup \{k, l\}) = v(S \cup \{j\})$,
- $v^j(S) = v(S)$.

Remark 4.4. *Regard that the players k and ℓ are dependent in v^j.*

In the following we present additional axioms for WTU-values φ^Λ:

Equal-weighted symmetry$^\Lambda$, EWS$^\Lambda$ (Shapley, 1981)**.** For all $v^\lambda \in \mathbb{V}^\Lambda(N)$, $i, j \in N$ such that i and j are symmetric in v and $\lambda_i, \lambda_j \in \Lambda^N$ with $\lambda_i = \lambda_j$, we have
$\varphi^\Lambda_i(N, v^\lambda) = \varphi^\Lambda_j(N, v^\lambda)$.

Null player out$^\Lambda$[7], NO$^\Lambda$. For all $v^\lambda \in \mathbb{V}^\Lambda(N)$ and $j \in N$ such that j is a null player in v, we have $\varphi^\Lambda_i(N, v^\lambda) = \varphi^\Lambda_i(N \backslash \{j\}, v^\lambda)$ for all $i \in N \backslash \{j\}$.

Weighted proportionality$^\Lambda$, WP$^\Lambda$. For all $v^\lambda \in \mathbb{V}^\Lambda(N)$ and $i, j \in N$ such that i and j are dependent in v, we have

$$\frac{\varphi^\Lambda_i(N, v^\lambda)}{\lambda_i} = \frac{\varphi^\Lambda_j(N, v^\lambda)}{\lambda_j}.$$

Weighted standardness$^\Lambda$[8], WS$^\Lambda$. For all $v^\lambda \in \mathbb{V}^\Lambda(N)$, $N = \{i, j\}$, $i \neq j$, we have

$$\varphi^\Lambda_i(N, v^\lambda) = v(\{i\}) + \frac{\lambda_i}{\lambda_i + \lambda_j}\big[v(\{i, j\}) - v(\{i\}) - v(\{j\})\big].$$

Players merging$^\Lambda$, PM$^\Lambda$. For all $v^\lambda \in \mathbb{V}^\Lambda(N)$, $k, \ell \in N$ two dependent players in v, and $\big(N^{\{k,\ell\}}, (v^\lambda)^{\{k,\ell\}}\big) \in \mathbb{V}^\Lambda(N^{\{k,\ell\}})$ an MPW-game to (N, v^λ), we have

$$\varphi^\Lambda_i(N, v^\lambda) = \varphi^\Lambda_i\big(N^{\{k,\ell\}}, (v^\lambda)^{\{k,\ell\}}\big) \text{ for all } i \in N \backslash \{k, \ell\}.$$

Player splitting$^\Lambda$, PS$^\Lambda$. For all $v^\lambda \in \mathbb{V}^\Lambda(N)$, $j \in N$, and $\big(N^j, (v^\lambda)^j\big) \in \mathbb{V}^\Lambda(N^j)$ an SPW-game to (N, v^λ), we have

$$\varphi^\Lambda_i(N, v^\lambda) = \varphi^\Lambda_i\big(N^j, (v^\lambda)^j\big) \text{ for all } i \in N \backslash \{j\}.$$

[7] This axiom is part of the dummy elimination axiom in (Shapley, 1981) and comes as a TU-axiom from (Derks and Haller, 1999).

[8] This axiom comes from the w-proportional property for two-person games in Hart and Mas-Colell (1989).

$\mathbf{PM}^\Lambda$ states that the players' payoffs do not change if some other dependent players merge into one new player who has the same impact to the game as the old merging players, $\mathbf{PS}^\Lambda$ states that the players' payoffs do not change if another player splits into two new dependent players who have the same impact to the new game as the original player in the old one.

Remark 4.5. *It is easy to see that $\boldsymbol{WP}_Q^\Lambda$ is equivalent to $\boldsymbol{WP}^\Lambda$. Hereafter, we use only the shortcut $\boldsymbol{WP}^\Lambda$.*

The players merging and the player splitting property, both pick up the idea in Banker (1981) that splitting up a cost center or merging cost centers should not change the allocation of costs to the remaining cost centers. We can interpret the procedure of amalgamating players in the amalgamating payoffs axiom as a form of adapting the idea in Lehrer (1988) that the new players get together the same payoff as the old split player. Here, if the WTU-value is efficient, Lehrer's idea is satisfied too.

Remark 4.6. *Let $N \in \mathcal{N}$, $v^\lambda \in \mathbb{V}^\Lambda(N)$, $j \in N$, and $\left(N^j, (v^\lambda)^j\right) \in \mathbb{V}^\Lambda(N^j)$ an SPW-game to (N, v^λ). If φ^Λ is a WTU-value which satisfies $\boldsymbol{E}^\Lambda$ and $\boldsymbol{PS}^\Lambda$, we have*

$$\varphi_k^\Lambda\left(N^j, (v^\lambda)^j\right) + \varphi_\ell^\Lambda\left(N^j, (v^\lambda)^j\right) = \varphi_j^\Lambda(N, v^\lambda) \text{ for } k, \ell \in N^j,\ k, \ell \notin N.$$

4.4. Subdomains

Any TU-value ignores weights which potentially are allocated to the players and are not considered within the definition of the value. Thus, we can reformulate each TU-value in a WTU-value:

Remark 4.7. *Let $v^\lambda \in \mathbb{V}^\Lambda(N)$. Each TU-value φ coincides with a WTU-value φ^Λ by $\varphi_i^\Lambda(N, v^\lambda) := \varphi_i(N, v)$ for all $i \in N$, in particular, we have for a weighted Shapley value $Sh^{\lambda'}$, $\lambda' \in \Lambda^N$, $(Sh_i^{\lambda'})^\Lambda(N, v^\lambda) := Sh_i^{\lambda'}(N, v)$ for all $i \in N$.*

At first glance, the Shapley set value looks very similar to a weighted Shapley value. But for one thing, we have only one value for all weight systems λ, whereas the weight system used in a weighted Shapley value is fixed, and secondly, for some subdomains the weights can be correlated to coalition functions. For different subdomains, this value coincides with different values.

Remark 4.8. *Let $\lambda' \in \Lambda^N$ and $\mathbb{V}^{\Lambda_{\lambda'}}(N)$ the set of all WTU-games $(N, v^\lambda) \in \mathbb{V}^\Lambda(N)$ with $\lambda := \lambda'$. Sh^Λ coincides on $\mathbb{V}^{\Lambda_{\lambda'}}(N)$ with the weighted Shapley value $Sh^{\lambda'}$.*

Even by a weighted Shapley value, on some (small) subdomains the weights can be regarded as not independent of the coalition function, e. g. if the weights of the players are in the same proportion to each other as the singleton worths. If the weights are all equal, the special case that a weighted Shapley value coincides with the Shapley value, the weighted Shapley value coincides with the proportional Shapley value on the subset of all coalition functions where all singletons have the same positive worth. But altering the coalition function where the weights depend on it usually leads to a problem: we have to change the weighted Shapley value too. Thus, in general, we require that the weights for a weighted Shapley value are independent of the coalition function.

This is unnecessary for the Shapley set value. On some subdomains this value coincides with non-linear values, so on the subdomain that contains all coalition functions with only positive or negative singleton worths and where the weights of the players are restricted to the singleton worths of the related coalition functions.

Remark 4.9. *For all $v \in \mathbb{V}_0(N)$ let $\lambda^v \in \Lambda^N$ such that $\lambda_i^v := |v(\{i\})|$ for all $i \in N$ and let $\mathbb{V}_0^{\Lambda_{\lambda^v}}(N)$ the set of all WTU-games $(N, v^\lambda) \in \mathbb{V}^\Lambda(N)$ with $\lambda := \lambda^v$ and $v \in \mathbb{V}_0(N)$. Sh^Λ coincides on $\mathbb{V}_0^{\Lambda_{\lambda^v}}(N)$ with the proportional Shapley value Sh^p.*

Casajus (2017) presented a huge class of TU-values φ^ω, $\omega \in \overline{\Omega}$, $\overline{\Omega} := \{f : \mathbb{R} \times \mathfrak{U} \to \mathbb{R}_{++}\}$, given by

$$\varphi_i^\omega(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\omega(v(\{i\}), i)}{\sum_{j \in S} \omega(v(\{j\}), j)} \Delta_v(S) \text{ for all } i \in N \text{ and } v \in \mathbb{V}(N).$$

With the weight functions $\omega \in \overline{\Omega}$ we can generalize the previous two remarks.

Remark 4.10. *Let $\omega \in \overline{\Omega}$, $\lambda^{\omega_v} \in \Lambda^n$ such that $\lambda_i^{\omega_v} := \omega(v(\{i\}), i)$ for all $i \in N$, and all $v \in \mathbb{V}(N)$ such that $v(\{i\})$ is in the domain of ω for all $i \in N$, and let $\mathbb{V}^{\Lambda_{\lambda^{\omega_v}}}(N)$ the set of all WTU-games $(N, v^\lambda) \in \mathbb{V}^\Lambda(N)$ with $\lambda := \lambda^{\omega_v}$ and $v \in \mathbb{V}(N)$ such that $v(\{i\})$ is in the domain of ω for all $i \in N$. Sh^Λ coincides on $\mathbb{V}^{\Lambda_{\lambda^{\omega_v}}}(N)$ with φ^ω.*

4.5. Inherited axiomatizations

In the introducing text section to Radzik (2012, Theorem 3.2) are, inter alia, the following two collections of axioms,

C1: $\mathbf{E}^\Lambda$, $\mathbf{L}^\Lambda$, $\mathbf{N}^\Lambda$, and $\mathbf{WP}^\Lambda$,

C2: $\mathbf{E}^\Lambda$, $\mathbf{MC}^\Lambda$, and $\mathbf{WP}^\Lambda$.

It is easy to see that the Shapley set value inherits analogous axiomatizations from the weighted Shapley values in Nowak and Radzik (1995).

Theorem 4.11. *Sh^Λ is the unique WTU-value which is determined by any of the collections of axioms **C1** and **C2**.*

We omit the proof since the related axiomatizations of the weighted Shapley values in Nowak and Radzik (1995) can be easily adapted and extended to a proof of Theorem 4.11.

Remarks 4.12. *Obviously, the value ϕ_* in Radzik (2012, Theorem 3.1) coincides with the Shapley set value Sh^Λ. But we cannot agree that the value $\phi_0(\lambda, v) = \{\phi_0^N(\lambda, v) : N \in \mathcal{N}\}$ in Radzik (2012, Theorem 3.2) is a weighted Shapley value: From a purely formal point of view, the value $\phi_0(\lambda, v)$ is a WTU-value and not a TU-value. If the notion of the value $\phi_0(\lambda, v)$ is not correct and this value is meant as a TU-value and λ is a fixed weight system, the value does not meet the axioms for WTU-values in **C1** and **C2**. If a corresponding WTU-value to the weighted Shapley values is meant as discussed in Remark 4.7, Remark 4.8, this value coincides with the Shapley set value only on a subdomain. Also, if we regard the set of the WTU-values which coincides with the set of all weighted Shapley values, no value of this set meets, e. g., $\boldsymbol{WP}^\Lambda$ on the whole range of the required domain.*

4.6. Additivity can replace linearity

In Radzik (2012, Remark 4.6) is pointed out that $\mathbf{L}^{\Lambda}$ can be weakened to $\mathbf{A}^{\Lambda}$ if we replace $\mathbf{L}^{\Lambda}$ by $\mathbf{A}^{\Lambda}$ and add $\mathbf{C}^{\Lambda}$. The following theorem and the next corollary are showing that we need only $\mathbf{A}^{\Lambda}$ for the collection **C1** in Theorem 4.11 above and so also in Radzik (2012, Theorem 3.1).

Theorem 4.13. *Let $\lambda \in \Lambda^N$. Sh^{λ} is the unique TU-value that satisfies **E**, **N**, $\boldsymbol{WP}^{\lambda}$, and **A**.*

By Theorem 4.13, Remark 4.8, and since the proof of Theorem 4.13 holds for all $\lambda \in \Lambda$, we have the following corollary.

Corollary 4.14. *Sh^{Λ} is the unique WTU-value that satisfies $\boldsymbol{E}^{\Lambda}$, $\boldsymbol{N}^{\Lambda}$, $\boldsymbol{WP}^{\Lambda}$, and $\boldsymbol{A}^{\Lambda}$.*

4.7. Insertion

We introduce a special case of the λ-weighted proportionality property $\mathbf{WP}^{\lambda}$, all weights are equal. Two dependent players get the same payoff.

Dependency, D. For all $v \in \mathbb{V}(N)$ and $i, j \in N$ such that i and j are dependent in v, we have $\varphi_i(N, v) = \varphi_j(N, v)$.

By Theorem 4.13 we obtain the following corollary for the Shapley value.

Corollary 4.15. *Sh is the unique TU-value that satisfies **E**, **N**, **D**, and **A**.*

This axiomatization is weaker than the well-known axiomatization in Shapley (1953b) by efficiency, symmetry, additivity and the null player property because symmetry implies dependency but not vice versa.

4.8. Player splitting

In our examples, Scenario 2, we can interpret the game $\tilde{q}$ as an SPW-game to game q in Scenario 1. The payoff to not involved players calculated by the Shapley set value Sh^{Λ} is in Scenario 2 the same as in Scenario 1 as required in $\mathbf{PS}^{\Lambda}$. This holds for the Shapley set value in general.

Lemma 4.16. *Sh^{Λ} satisfies $\boldsymbol{PS}^{\Lambda}$.*

The following lemma shows dependence on $\mathbf{EWS}^{\Lambda}$ for efficient values which meet $\mathbf{PS}^{\Lambda}$.

Lemma 4.17. *If a WTU-value φ^{Λ} satisfies $\boldsymbol{E}^{\Lambda}$ and $\boldsymbol{PS}^{\Lambda}$ then φ^{Λ} satisfies also $\boldsymbol{EWS}^{\Lambda}$.*

The next lemma makes use of Lemma 4.17 in the proof.

Lemma 4.18. *Let $v^{\lambda} \in \mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$. If a WTU-value φ^{Λ} satisfies $\boldsymbol{E}^{\Lambda}$ and $\boldsymbol{PS}^{\Lambda}$ then φ^{Λ} satisfies also $\boldsymbol{WP}^{\Lambda}$.*

We obtain by Lemma 4.16, Lemma 4.18, and Corollary 4.14 the following corollary.

Corollary 4.19. *Let $v^\lambda \in \mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$. Sh^Λ is the unique WTU-value that satisfies $\boldsymbol{E}^\Lambda$, $\boldsymbol{N}^\Lambda$, $\boldsymbol{PS}^\Lambda$, and $\boldsymbol{A}^\Lambda$.*

By Lemma 4.18 and Theorem 4.11 with regard to the collection of axioms **C2** we get another corollary.

Corollary 4.20. *Let $v^\lambda \in \mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$. Sh^Λ is the unique WTU-value that satisfies $\boldsymbol{E}^\Lambda$, $\boldsymbol{PS}^\Lambda$, and $\boldsymbol{M}^\Lambda$.*

Remark 4.21. *Lemma 4.18 holds for $v^\lambda \in \mathbb{V}^\Lambda(N)$ if we require continuity of φ^Λ in λ for all $\lambda \in \Lambda^N$ in an additional axiom. Thus also Corollary 4.19 and Corollary 4.20 are valid for $v^\lambda \in \mathbb{V}^\Lambda(N)$ if there is in each case an additional continuity axiom.*

4.9. Players merging

The Shapley set value and the players merging property fit together well.

Lemma 4.22. *Sh^Λ satisfies $\boldsymbol{PM}^\Lambda$.*

Since it is easy to adapt the proof from Lemma 4.16 the proof is omitted. It follows an axiomatization that uses the players merging property.

Theorem 4.23. *Sh^Λ is the unique WTU-value that satisfies $\boldsymbol{NO}^\Lambda$, $\boldsymbol{PM}^\Lambda$, $\boldsymbol{WS}^\Lambda$, and $\boldsymbol{A}^\Lambda$.*

5. The Harsanyi set value

The same considerations as for the weighted Shapley values in the introduction of Section 4 apply similarly for the Harsanyi payoffs H^γ. These values require a fixed sharing system. If the sharing system depends on the coalition function, we can use the Harsanyi set value H^Γ without changing the value for each altered coalition function.

5.1. Axioms for STU-values

To transfer axioms for WTU-values into axioms for STU-values, we need two definitions we already know as definitions in the version for WTU-values.

Definition 5.1. *Let $Q \in \Omega^N$, $q := Q$ if Q is regarded as a player, $N^Q := (N \backslash Q) \cup \{q\}$, $(N, v^\gamma) \in \mathbb{V}^\Gamma(N)$. The game $\left(N^Q, (v^\gamma)^Q\right) \in \mathbb{V}^\Gamma(N^Q)$, $(v^\gamma)^Q := (\gamma^Q, v^Q)$, is called a **merged players STU-game (MPS-game)** to (N, v^γ) if for all $S \subseteq N \backslash Q$*

- $\gamma^Q_{S,k} = \gamma_{S,k}$, $k \in S$,
- $\gamma^Q_{S\cup\{q\},k} = \begin{cases} \gamma_{S\cup Q,k}, & \text{if } k \in S, \\ \sum_{i\in Q} \gamma_{S\cup Q,i} & \text{if } k = q, \end{cases}$
- $v^Q(S) = v(S)$,

- $v^Q(S \cup \{q\}) = v(S \cup Q)$

Definition 5.2. *Let $j \in N$, $N^j := (N \backslash \{j\}) \cup \{k, \ell\}$, $k, \ell \in \mathfrak{U}$, $k, \ell \notin N$, $(N, v^\gamma) \in \mathbb{V}^\Gamma(N)$. The game $\big(N^j, (v^\gamma)^j\big) \in \mathbb{V}^\Gamma(N^j)$, $(v^\gamma)^j := (\gamma^j, v^j)$, is called a* ***split player STU-game (SPS-game)*** *to (N, v^γ) if for all $S \subseteq N \backslash \{j\}$, $i \in S$, $m \in \{k, \ell\}$,*

- $\gamma^j_{S,i} = \gamma_{S,i}$, $\gamma^j_{S\cup\{k,\ell\},k} + \gamma^j_{S\cup\{k,\ell\},\ell} = \gamma_{S\cup\{j\},j}$ *and* $\gamma^j_{S\cup\{k,\ell\},i} = \gamma_{S\cup\{j\},i}$,
- $v^j(S \cup \{m\}) = v(S)$,
- $v^j(S \cup \{k, l\}) = v(S \cup \{j\})$,
- $v^j(S) = v(S)$.

Remark 5.3. *In definition 5.2 the shares of all players from a coalition $(S \cup \{m\})$ which contains only one split player are arbitrary within the domain of a sharing system.*

Remark 5.4. *In definition 5.2 the players k and ℓ are dependent in v^j.*

Our axioms for STU-values φ^Γ are simple adaptions of the related axioms for WTU-values.

Efficiency$^\Gamma$, E$^\Gamma$. For all $v^\gamma \in \mathbb{V}^\Gamma(N)$, we have $\sum_{i\in N} \varphi^\Gamma_i(N, v^\gamma) = v(N)$.

Null player out$^\Gamma$, NO$^\Gamma$. For all $v^\gamma \in \mathbb{V}^\Gamma(N)$ and $j \in N$ such that j is a null player in v, we have $\varphi^\Gamma_i(N, v^\gamma) = \varphi^\Gamma_i(N\backslash\{j\}, v^\gamma)$ for all $i \in N\backslash\{j\}$.

Additivity$^\Gamma$(in the coalition function), **A$^\Gamma$.** For all $v^\gamma, w^\gamma \in \mathbb{V}^\Gamma(N)$, we have $\varphi^\Gamma(N, v^\gamma) + \varphi^\Gamma(N, w^\gamma) = \varphi^\Gamma(N, (v + w)^\gamma)$.

Weighted standardness$^\Gamma$, WS$^\Gamma$. For all $v^\gamma \in \mathbb{V}^\Gamma(N)$, $N = \{i, j\}, i \neq j$, we have

$$\varphi^\Gamma_i(N, v^\gamma) = v(\{i\}) + \frac{\gamma_{N,i}}{\gamma_{N,i} + \gamma_{N,j}}\big[v(\{i, j\}) - v(\{i\}) - v(\{j\})\big].$$

Players merging$^\Gamma$, PM$^\Gamma$. For all $v^\gamma \in \mathbb{V}^\Gamma(N)$, $k, \ell \in N$ two dependent players in v, and $\big(N^{\{k,\ell\}}, (v^\gamma)^{\{k,\ell\}}\big) \in \mathbb{V}^\Gamma(N^{\{k,\ell\}})$ an MPS-game to (N, v^γ), we have

$$\varphi^\Gamma_i(N, v^\gamma) = \varphi^\Gamma_i\big(N^{\{k,\ell\}}, (v^\gamma)^{\{k,\ell\}}\big) \text{ for all } i \in N\backslash\{k, \ell\}.$$

Player splitting$^\Gamma$, PS$^\Gamma$. For all $v^\gamma \in \mathbb{V}^\Gamma(N)$, $j \in N$, and $\big(N^j, (v^\gamma)^j\big) \in \mathbb{V}^\Gamma(N^j)$ an SPS-game to (N, v^γ), we have

$$\varphi^\Gamma_i(N, v^\gamma) = \varphi^\Gamma_i\big(N^j, (v^\gamma)^j\big) \text{ for all } i \in N\backslash\{j\}.$$

5.2. Subdomains

Similar to the WTU-values we can reformulate each TU-value also in an STU-value:

Remark 5.5. *Let $v^\gamma \in \mathbb{V}^\Gamma(N)$. Each TU-value φ coincides with an STU-value φ^Γ by $\varphi^\Gamma_i(N, v^\gamma) := \varphi_i(N, v)$ for all $i \in N$, in particular, we have for a Harsanyi payoff H^{γ^k}, $\gamma^k \in \Gamma^N$, $(H^{\gamma^k}_i)^\Gamma(N, v^\gamma) := H^{\gamma^k}_i(N, v)$ for all $i \in N$.*

The Harsanyi set value looks very similar to a Harsanyi payoff too. But for one thing, we have only one value for all sharing systems γ, whereas the sharing system used in a Harsanyi payoff is fixed, and secondly, for some subdomains the sharing systems γ can be correlated to the coalition functions. On different subdomains, this value coincides with different values.

Remark 5.6. *Let $\gamma' \in \Gamma^N$ and $\mathbb{V}^{\Gamma_{\gamma'}}(N)$ the set of all STU-games $(N, v^\gamma) \in \mathbb{V}^\Gamma(N)$ with $\gamma := \gamma'$. H^Γ coincides on $\mathbb{V}^{\Gamma_{\gamma'}}(N)$ with the Harsanyi payoff $H^{\gamma'}$.*

Remark 5.7. *Let $\lambda \in \Lambda^N$ and $\gamma^\lambda \in \Gamma^N$ such that*

$$\gamma^\lambda_{T,i} := \frac{\lambda_i}{\sum_{j\in T}\lambda_j}, T \in \Omega^N,\ i \in T,$$

and let $\mathbb{V}^{\Gamma_{\gamma^\lambda}}(N)$ the set of all STU-games $(N, v^\gamma) \in \mathbb{V}^\Gamma(N)$ with $\gamma := \gamma^\lambda$. H^Γ coincides on $\mathbb{V}^{\Gamma_{\gamma^\lambda}}(N)$ with the weighted Shapley value Sh^λ.

Remark 5.8. *For all $\lambda \in \Lambda^N$ let $\gamma^\lambda \in \Gamma^N$ such that*

$$\gamma^\lambda_{T,i} := \frac{\lambda_i}{\sum_{j\in T}\lambda_j}, T \in \Omega^N,\ i \in T,$$

$\Gamma_\Lambda(N) := \{\gamma^\lambda : \lambda \in \Lambda^N\}$, and let $\mathbb{V}^{\Gamma_\Lambda}(N)$ the set of all STU-games $(N, v^\gamma) \in \mathbb{V}^\Gamma(N)$ with $\gamma \in \Gamma_\Lambda(N)$. H^Γ coincides on $\mathbb{V}^{\Gamma_\Lambda}(N)$ with the Shapley set value Sh^Λ.

Thus, the Harsanyi set value coincides with non-linear values on some subdomains too.

Remark 5.9. *Let $v \in \mathbb{V}_0(N)$ and $\gamma^v \in \Gamma^N$ such that*

$$\gamma^v_{T,i} := \frac{v(\{i\})}{\sum_{j\in T} v(\{j\})}, T \in \Omega^N,\ i \in T,$$

and let $\mathbb{V}_0^{\Gamma_{\gamma^v}}(N)$ the set of all STU-games $(N, v^\gamma) \in \mathbb{V}_0^\Gamma(N)$ with $\gamma := \gamma^v$. H^Γ coincides on $\mathbb{V}_0^{\Gamma_{\gamma^v}}(N)$ with the proportional Shapley value Sh^p.

5.3. Axiomatizations

Lemma 5.10. *H^Γ satisfies $\boldsymbol{E}^\Gamma$, $\boldsymbol{NO}^\Gamma$, $\boldsymbol{PS}^\Gamma$, $\boldsymbol{PM}^\Gamma$, $\boldsymbol{WS}^\Gamma$, and $\boldsymbol{A}^\Gamma$.*

In contrary to Corollary 4.19, the next theorem needs the null player out property instead of the null player property.

Theorem 5.11. *Let $v^\gamma \in \mathbb{V}^{\Gamma_\mathbb{Q}}(N)$. H^Γ is the unique WTU-value that satisfies $\boldsymbol{E}^\Gamma$, $\boldsymbol{NO}^\Gamma$, $\boldsymbol{PS}^\Gamma$, and $\boldsymbol{A}^\Gamma$.*

Remark 5.12. *Similar to Remark 4.21, Theorem 5.11 holds for $v^\gamma \in \mathbb{V}^\Gamma(N)$ if we require continuity of φ^Γ in γ for all $\gamma \in \Gamma^N$ in an additional axiom.*

Our last axiomatization can be transferred from Theorem 4.23 one to one.

Theorem 5.13. *H^Γ is the unique WTU-value that satisfies $\boldsymbol{NO}^\Gamma$, $\boldsymbol{PM}^\Gamma$, $\boldsymbol{WS}^\Gamma$, and $\boldsymbol{A}^\Gamma$.*

The proof is omitted because it can be transmitted one to one from the proof of Theorem 4.23.

6. Conclusion and discussion

In this article, we have carved out the differences between the weighted Shapley values and the Shapley set value and between the Harsanyi payoffs and the Harsanyi set value. The introduced player splitting and players merging properties lead to convincing axiomatizations, particularly in the context of profit and cost allocation. Nevertheless, all presented axiomatizations hold only on the given domains and using subdomains would lead to other axiomatizations in general.

It should be mentioned and is easy to proof, besides the axiomatizations in Radzik (2012) and Shapley (1981)/Dehez (2011), that the Shapley set value, e. g., also satisfies adaptions for WTU-values from the well-known axiomatizations of the weighted Shapley values given in Myerson (1980)/Hart and Mas-Colell (1989) by efficiency and weighted balanced contributions and Hart and Mas-Colell (1989) by consistency and weighted standardness.

The amalgamating payoffs property allows an interesting axiomatization too. Whereas in the axiomatization in Radzik (2012, Theorem 3.1) $\mathbf{AP}^{\Lambda}$ is redundant, it is clear, applying ideas from the proof of Theorem 4.23, that $\mathbf{Sh}^{\Lambda}$ ($\mathbf{H}^{\Gamma}$) can be axiomatized by $\mathbf{E}^{\Lambda}$, $\mathbf{NO}^{\Lambda}$, $\mathbf{AP}^{\Lambda}$, $\mathbf{WS}^{\Lambda}$, and $\mathbf{A}^{\Lambda}$ ($\mathbf{E}^{\Gamma}$, $\mathbf{NO}^{\Gamma}$, $\mathbf{AP}^{\Gamma}$, $\mathbf{WS}^{\Gamma}$, and $\mathbf{A}^{\Gamma}$). Here $\mathbf{AP}^{\Lambda}$ is not redundant. Look for this, e. g., to the WTU-value φ, defined by

$$\varphi_i(N, v^\lambda) = \sum_{\substack{S\subseteq N,\\ S\ni i,\, |S|\leq 2}} \frac{\lambda_i}{\sum_{j\in S}\lambda_j}\Delta_v(S) + \sum_{\substack{S\subseteq N,\\ S\ni i,\, |S|\geq 3}} \frac{\Delta_v(S)}{|S|} \text{ for all } i \in N,$$

which meets $\mathbf{E}^{\Lambda}$, $\mathbf{NO}^{\Lambda}$, $\mathbf{WS}^{\Lambda}$, and $\mathbf{A}^{\Lambda}$ and does not match $\mathbf{AP}^{\Lambda}$. But it is still an open question if all axioms are logically independent.

We desisted from weights of zero for the Shapley set value. If one wants to allow zero-weights, we recommend the Harsanyi set value. In accordance with Kalai and Samet (1987), one has to specify the weights for coalitions where all players have originally a weight of zero how to share in this case the dividend of these coalitions. The sharing weights of the players for all other coalitions can be handled by the given proportion of the original weights. So we have also valid axiomatizations (for the Harsanyi set value) in contrast to the axiomatizations for the Shapley set value as stressed out in the concluding remarks in Dehez (2011).

7. Appendix

The following remark is used in some proofs.

Remark 7.1. *Players $i, j \in N$, $i \neq j$, are dependent in $v \in \mathbb{V}(N)$, iff $\Delta_v(S \cup \{k\}) = 0$, $k \in \{i, j\}$, for all $S \subseteq N \backslash \{i, j\}$.*

7.1. Proof of Theorem 4.13

Let $\lambda \in \Lambda^N$ and $v \in \mathbb{V}(N)$. It is well-known that Sh^λ satisfies all axioms from Theorem 4.13. So, due to property (1) and $\mathbf{A}$, it is sufficient to show that φ is uniquely defined on games $v_S := \Delta_v(S) \cdot u_S$, $S \in \Omega^N$.

Let $S \in \Omega^N$ arbitrary and φ a value that satisfies all axioms from Theorem 4.13. All players $j \in N \backslash S$ are null players in v_S and so φ is unique on v_S for all $j \in N \backslash S$ by **N**. Thus, if S is a singleton, φ is unique on v_S by **E**. Let now $|S| \geq 2$. By Remark 7.1 all $i \in S$ are dependent in v_S and therefore, by **E** and **WP**, φ is also unique on v_S for all $i \in N$. □

7.2. Proof of Lemma 4.16

Let $v^\lambda \in \mathbb{V}^\Lambda(N)$, $j \in N$, and $\left(N^j, (v^\lambda)^j\right)$ an SPW-game to (N, v^λ). We point out that we have for all $S \in \Omega^{N \backslash \{j\}}$, $\Delta_{v^j}(S) = \Delta_v(S)$, $\Delta_{v^j}(S \cup \{k, l\}) = \Delta_v(S \cup \{j\})$, and, by Remark 7.1, $\Delta_{v^j}(S \cup \{k\}) = \Delta_{v^j}(S \cup \{\ell\}) = 0$. Then we get for all $i \in N \backslash \{j\}$

$$
\begin{aligned}
Sh_i^\Lambda(N, v^\lambda) &\underset{(3)}{=} \sum_{R \subseteq N,\, R \ni i} \frac{\lambda_i}{\sum_{m \in R} \lambda_m} \Delta_v(R) \\
&= \sum_{S \subseteq N \backslash \{j\},\, S \ni i} \frac{\lambda_i}{\sum_{m \in S} \lambda_m} \Delta_v(S) + \sum_{S \subseteq N \backslash \{j\},\, S \ni i} \frac{\lambda_i}{\sum_{m \in S \cup \{j\}} \lambda_m} \Delta_v(S \cup \{j\}) \\
&= \sum_{\substack{S \subseteq N^j \backslash \{k, \ell\}, \\ S \ni i}} \frac{\lambda_i^j}{\sum_{m \in S} \lambda_m^j} \Delta_{v^j}(S) + \sum_{\substack{S \subseteq N^j \backslash \{k, \ell\}, \\ S \ni i}} \frac{\lambda_i^j}{\sum_{m \in S \cup \{k, \ell\}} \lambda_m^j} \Delta_{v^j}(S \cup \{k, \ell\}) \\
&= \sum_{R \subseteq N^j,\, R \ni i} \frac{\lambda_i^j}{\sum_{m \in R} \lambda_m^j} \Delta_{v^j}(R) = Sh_i^\Lambda\left(N^j, (v^\lambda)^j\right).
\end{aligned}
$$

□

7.3. Proof of Lemma 4.17

Let $N = \{1, 2, ..., n\} \in \mathcal{N}$, $n \geq 2$, $v^\lambda \in \mathbb{V}^\Lambda(N)$, φ^Λ a WTU-value that satisfies $\mathbf{E}^\Lambda$ and $\mathbf{PS}^\Lambda$ and, w.l.o.g., player 1 and player 2 symmetric in v with $\lambda_1 = \lambda_2$. We split player 1, according to $\mathbf{PS}^\Lambda$, into two new players, player $n+1$ and player $n+2$, $N^1 := \{2, 3, ..., n, n+1, n+2\}$, and obtain

$$\varphi_2^\Lambda\left(N^1, (v^\lambda)^1\right) = \varphi_2^\Lambda(N, v^\lambda), \tag{10}$$

and, if we split player 2, according to $\mathbf{PS}^\Lambda$, into the same players as before, player $n+1$ and player $n+2$, instead, $N^2 := \{1, 3, 4, ..., n, n+1, n+2\}$, we have

$$\varphi_1\left(N^2, (v^\lambda)^2\right) = \varphi_1(N, v^\lambda), \tag{11}$$

where we choose $\lambda_{n+1}^2 := \lambda_{n+1}^1 := \lambda_{n+2}^2 := \lambda_{n+2}^1$.

In the same manner, we split now in the game $\left(N^1, (v^\lambda)^1\right)$ player 2 into two new players, player $n+3$ and player $n+4$, and analogous in the game $\left(N^2, (v^\lambda)^2\right)$ player 1 into the same players as before, player $n+3$ and player $n+4$, and choose $\lambda_{n+3}^{21} := \lambda_{n+3}^{12} := \lambda_{n+4}^{21} := \lambda_{n+4}^{12}$.

We have $N^{12} = N^{21} = \{3, 4, ..., n, n+1, n+2, n+3, n+4\}$ and $(v^\lambda)^{12} = (v^\lambda)^{21}$ and obtain by $\mathbf{E}^\Lambda$, according to remark 4.6,

$$
\begin{aligned}
\varphi_{n+3}\left(N^{12}, (v^\lambda)^{12}\right) + \varphi_{n+4}\left(N^{12}, (v^\lambda)^{12}\right) &= \varphi_2\left(N^1, (v^\lambda)^1\right) \underset{(10)}{=} \varphi_2(N, v^\lambda), \\
\varphi_{n+3}\left(N^{21}, (v^\lambda)^{21}\right) + \varphi_{n+4}\left(N^{21}, (v^\lambda)^{21}\right) &= \varphi_1\left(N^2, (v^\lambda)^2\right) \underset{(11)}{=} \varphi_1(N, v^\lambda).
\end{aligned}
$$

Hence follows $\varphi_1(N, v^\lambda) = \varphi_2(N, v^\lambda)$ and $\mathbf{EWS}^\Lambda$ is shown. $\square$

7.4. Proof of Lemma 4.18

Let $N \in \mathcal{N}$, $|N| \geq 2$, $v^\lambda \in \mathbb{V}^{\Lambda_\mathbb{Q}}(N)$, φ^Λ a WTU-value that satisfies $\mathbf{E}^\Lambda$ and $\mathbf{PS}^\Lambda$ and therefore, by Lemma 4.17, also $\mathbf{EWS}^\Lambda$ and $i, j \in N$ such that i and j are dependent in v. Due to $\lambda_i, \lambda_j \in \mathbb{Q}_{++}$ the weights λ_k, $k \in \{i, j\}$, can be written as a fraction

$$\lambda_k = \frac{p_k}{q_k} \quad \text{with} \quad p_k, q_k \in \mathbb{N}.$$

We choose a main denominator q of these two fractions by $q := q_i q_j$. With $z_i := p_i q_j$ and $z_j := p_j q_i$ we get

$$\lambda_i = \frac{z_i}{q} \quad \text{and} \quad \lambda_j = \frac{z_j}{q}. \tag{12}$$

Applying $\mathbf{PS}^\Lambda$ (repeatedly) to (N, v^λ) and the two players i, j we can obtain the WTU-game $(N', (v^\lambda)')$ where each player k, $k \in \{i, j\}$, is split in z_k players k_1 to $k_{(z_k)}$, such that $N' = (N \backslash \{i, j\}) \cup \{i_m : 1 \leq m \leq z_i\} \cup \{j_m : 1 \leq m \leq z_j\}$ and each player $k_m \in N' \backslash (N \backslash \{i, j\})$, $1 \leq m \leq z_k$, get a singleton worth $v'(\{k_m\}) := 0$ for $k \in \{i, j\}$, with a weight $\lambda_{k_m} := \frac{1}{q}$ where $\left|N' \backslash (N \backslash \{i, j\})\right| = z_1 + z_2$.

All players $\ell \in N' \backslash (N \backslash \{i, j\})$ are symmetric in v' and have the same weights. Hence follows by $\mathbf{EWS}^\Lambda$ and $\mathbf{E}^\Lambda$

$$\varphi_\ell(N', (v^\lambda)') = \frac{\varphi_i(N, v^\lambda) + \varphi_j(N, v^\lambda)}{z_i + z_j} \text{ for } \ell \in N' \backslash (N \backslash \{i, j\}). \tag{13}$$

We get

$$\varphi_k(N, v^\lambda) \underset{(13)}{=} \sum_{1 \leq m \leq z_k} \varphi_{k_m}(N', (v^\lambda)') = \frac{z_k}{z_i + z_j} \left[\varphi_i(N, v^\lambda) + \varphi_j(N, v^\lambda)\right] \text{ for } k \in \{i, j\}.$$

It follows

$$\varphi_i(N, v) = \frac{z_i}{z_j} \varphi_j(N, v^\lambda) \underset{(12)}{=} \frac{\lambda_i}{\lambda_j} \varphi_j(N, v)$$

and $\mathbf{WP}^\Lambda$ is shown. $\square$

7.5. Proof of Theorem 4.23

Let $v^\lambda \in \mathbb{V}^\Lambda(N)$. By (3), Lemma 4.22, and corollary 4.14, it is clear that Sh^Λ satisfies all axioms from Theorem 4.23. Thus, we have only to show uniqueness.

Let φ a value that satisfies all axioms from Theorem 4.23. If $|N| = 2$, φ is unique on v^λ by $\mathbf{WS}^\Lambda$. Let now be one player $i \in N$ a null player. Then, by $\mathbf{NO}^\Lambda$, φ is unique on $(\{j\}, v^\lambda)$ for the other player $j \in N$, but then also for $|N| = 1$ in general.

Let now $|N| \geq 3$. Due to property (1) and $\mathbf{A}^\Lambda$, it is sufficient to show that φ is uniquely defined on games v_S^λ with $v_S := \Delta_v(S) \cdot u_S$, $S \in \Omega^N$.

Let $S \in \Omega^N, S \neq N$. By Remark 7.1, all $j \in S$ are dependent in v_S and all $k \in N\backslash S$ are null players in v_S. If we delete, by $\mathbf{NO}^\Lambda$, all but one null player $k \in N\backslash S$ and merge all players $j \in S$ according to $\mathbf{PM}^\Lambda$, step by step, we get, by $\mathbf{WS}^\Lambda$, that φ is unique on v_S^λ for the null player k and since k was arbitrary, that φ is unique on v_S^λ for all null players $k \in N\backslash S$.

Now we delete all null players $k \in N\backslash S$ or we have $S = N$. By $\mathbf{NO}^\Lambda$, we have $Sh_j^\Lambda(S, v^\lambda) = Sh_j^\Lambda(N, v^\lambda)$ for all $j \in S$. If $|S| = 1$, φ is unique on v_S^λ as shown above. Let now $|S| \geq 2$ and $i \in S$. We merge all players $j \in S\backslash\{i\}$ by $\mathbf{PM}^\Lambda$, step by step, and obtain $\varphi_i^\Lambda(N, v_S^\lambda) = \varphi_i^\Lambda\big(S^{S\backslash\{i\}}, (v_S^\lambda)^{S\backslash\{i\}}\big)$. By $\mathbf{WS}^\Lambda$, $\varphi_i^\Lambda\big(S^{S\backslash\{i\}}, (v_S^\lambda)^{S\backslash\{i\}}\big)$ is unique on $(v_S^\lambda)^{S\backslash\{i\}}$. Therefore $\varphi_i^\Lambda(N, v_S^\lambda)$ is unique on v_S^λ. Since $i \in S$ was arbitrary, φ^Λ is unique for all $i \in S$ and Theorem 4.23 is shown. □

7.6. Proof of Lemma 5.10

Obviously, by (4), H^Γ satisfies $\mathbf{E}^\Gamma$, $\mathbf{NO}^\Gamma$, $\mathbf{WS}^\Gamma$, and $\mathbf{A}^\Gamma$.

We show that H^Γ meets $\mathbf{PS}^\Gamma$. Let $v^\gamma \in \mathbb{V}^\Gamma(N)$, $j \in N$, and $\big(N^j, (v^\gamma)^j\big)$ an SPS-game to (N, v^γ). We point out that we have for all $S \in \Omega^{N\backslash\{j\}}$, $\Delta_{v^j}(S) = \Delta_v(S)$, $\Delta_{v^j}(S \cup \{k, l\}) = \Delta_v(S \cup \{j\})$, and, by Remark 7.1, $\Delta_{v^j}(S \cup \{k\}) = \Delta_{v^j}(S \cup \{\ell\}) = 0$. We obtain

$$\begin{aligned} H_i^\Gamma(N, v^\gamma) &\underset{(4)}{=} \sum_{R\subseteq N,\, R\ni i} \gamma_{R,i}\Delta_v(R) = \sum_{S\subseteq N\backslash\{j\},\, S\ni i} \gamma_{S,i}\Delta_v(S) + \sum_{S\subseteq N\backslash\{j\},\, S\ni i} \gamma_{S,i}\Delta_v(S\cup\{j\}) \\ &= \sum_{S\subseteq N^j\backslash\{k,\ell\},\, S\ni i} \gamma_{S,i}^j\Delta_{v^j}(S) + \sum_{S\subseteq N^j\backslash\{k,\ell\},\, S\ni i} \gamma_{S\cup\{k,\ell\},i}^j\Delta_{v^j}(S\cup\{k,\ell\}) \\ &= \sum_{R\subseteq N^j,\, R\ni i} \gamma_{R,i}^j\Delta_{v^j}(R) = H_i^\Gamma\big(N^j, (v^\gamma)^j\big) \text{ for all } i \in N\backslash\{j\}. \end{aligned}$$

Since it is easy to adapt this part of the proof to show that H^Γ meets $\mathbf{PM}^\Gamma$ too we have omitted this adaption. □

7.7. Proof of Theorem 5.11

Let $v^\gamma \in \mathbb{V}^{\Gamma_\mathbb{Q}}(N)$. By Lemma 5.10, due to property (1), and $\mathbf{A}^\Gamma$, it is sufficient to show that φ is uniquely defined on games (N, v_S^γ), $v_S := \Delta_v(S)\cdot u_S$, $S \in \Omega^N$.

Let $S \in \Omega^N$ arbitrary and φ a value that satisfies all axioms from Theorem 5.11. It is obvious that $\mathbf{E}^\Gamma$ and $\mathbf{NO}^\Gamma$ imply together $\mathbf{N}^\Gamma$. Thus φ also satisfies $\mathbf{N}^\Gamma$. All players $j \in N\backslash S$ are null players in v_S and so φ is unique on v_S for all $j \in N\backslash S$ by $\mathbf{N}^\Gamma$. So, by $\mathbf{NO}^\Gamma$, it is sufficient to show that φ is unique on games (S, v_S^γ).

Let $\gamma' \in \Gamma_\mathbb{Q}^N$ such that

$$\gamma'_{T,i} := \frac{\gamma_{S,i}}{\sum_{j\in T}\gamma_{S,j}}, T \in \Omega^S, i \in T.$$

We show that

$$\varphi_i^\Gamma(S, v_S^{\gamma'}) = \varphi_i^\Gamma(S, v_S^\gamma) \text{ for all } i \in S. \tag{14}$$

If $|S| \leq 2$, we have $\gamma' = \gamma$ and (14) is satisfied.

Let now $|S| \geq 3$, $i \in S$, $T \in \mathcal{N}$, $T := \{i, t\}$, $t \in \mathfrak{U}$, $t \notin N$, $(T, \tilde{v}^{\tilde{\gamma}}) \in \mathbb{V}^{\Gamma_{\mathbb{Q}}}(T)$, $\tilde{\gamma}_{T,k} := \gamma_{S,k}$, $k \in T$, $\tilde{v}(\{i,q\}) := \Delta_v(S)$, and all other coalitions are not active in $\tilde{v}$. Accordingly to $\mathbf{PS}^\Gamma$, we split, possibly repeatedly, the player t into all other players $j \in S\backslash\{i\}$. By remark 5.3, we finally can obtain both games $(S, v_S^{\gamma'})$ and (S, v_S^{γ}) and obtain by $\mathbf{PS}^\Gamma$, $\varphi_i^\Gamma(S, v_S^{\gamma'}) = \varphi_i^\Gamma(S, v_S^{\gamma})$ for all $i \in S$ since i was arbitrary as desired.

But then coincides φ^Γ on (S, v_S^γ) also with a WTU-value φ^Λ on $(S, v_S^\lambda) \in \mathbb{V}^\Lambda(S)$ with $\lambda_i := \gamma_{S,i}$ for all $i \in S$ which satisfies $\mathbf{E}^\Lambda$, $\mathbf{N}^\Lambda$, $\mathbf{PS}^\Lambda$, and $\mathbf{A}^\Lambda$. Therefore, by Corollary 4.19, φ^Γ is unique on v_S and Theorem 5.11 is shown. □

7.8. Logical independence

Finally, we want to show the independence of the axioms used in the axiomatizations.

Remark 7.2. *Let $v^\lambda \in \mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$. The axioms in Corollary 4.20 are logically independent:*

- ***$\boldsymbol{E}^\Lambda$**: The WTU-value φ, defined by $\varphi_i(N, v^\lambda) := 0$ for all $i \in N$, satisfies **$\boldsymbol{PS}^\Lambda$** and **$\boldsymbol{M}^\Lambda$** but not **$\boldsymbol{E}^\Lambda$**.*
- ***$\boldsymbol{PS}^\Lambda$**: The WTU-value φ, defined by*

$$\varphi_i(N, v^\lambda) := \sum_{S \subseteq N,\, S \ni i} \frac{1}{|S|} \Delta_v(S) \text{ for all } i \in N,$$

  *satisfies **$\boldsymbol{E}^\Lambda$** and **$\boldsymbol{M}^\Lambda$** but not **$\boldsymbol{PS}^\Lambda$**.*
- ***$\boldsymbol{M}^\Lambda$**: The WTU-value φ, defined by*

$$\varphi_i(N, v^\lambda) := \frac{\lambda_i}{\sum_{j \in N} \lambda_j} v(N) \text{ for all } i \in N,$$

  *satisfies **$\boldsymbol{E}^\Lambda$** and **$\boldsymbol{PS}^\Lambda$** but not **$\boldsymbol{M}^\Lambda$**.*

Remark 7.3. *Let $v^\lambda \in \mathbb{V}^{\Lambda_{\mathbb{Q}}}(N)$. The axioms in Corollary 4.19 are logically independent:*

- ***$\boldsymbol{E}^\Lambda$**: The WTU-value φ, defined by $\varphi_i(N, v^\lambda) := 0$ for all $i \in N$, satisfies **$\boldsymbol{N}^\Lambda\backslash\boldsymbol{NO}^\Lambda$**, **$\boldsymbol{PS}^\Lambda$**, and **$\boldsymbol{A}^\Lambda$** but not **$\boldsymbol{E}^\Lambda$**.*
- ***$\boldsymbol{N}^\Lambda\backslash\boldsymbol{NO}^\Lambda$**: The WTU-value φ, defined by*

$$\varphi_i(N, v^\lambda) = \frac{\lambda_i}{\sum_{j \in N} \lambda_j} v(N) \text{ for all } i \in N,$$

  *satisfies **$\boldsymbol{E}^\Lambda$**, **$\boldsymbol{PS}^\Lambda$**, and **$\boldsymbol{A}^\Lambda$** but not **$\boldsymbol{N}^\Lambda\backslash\boldsymbol{NO}^\Lambda$**.*
- ***$\boldsymbol{PS}^\Lambda$**: The WTU-value φ, defined by*

$$\varphi_i(N, v^\lambda) = \sum_{S \subseteq N,\, S \ni i} \frac{1}{|S|} \Delta_v(S) \text{ for all } i \in N,$$

  *satisfies **$\boldsymbol{E}^\Lambda$**, **$\boldsymbol{N}^\Lambda\backslash\boldsymbol{NO}^\Lambda$**, and **$\boldsymbol{A}^\Lambda$** but not **$\boldsymbol{PS}^\Lambda$**.*

- $\boldsymbol{A}^\Lambda$: *The WTU-value φ, defined for all $i \in N$ by*

$$\varphi_i(N, v^\lambda) = \begin{cases} 0, & \text{if } i \text{ is a null player in } v, \\ \dfrac{\lambda_i}{\sum_{\substack{j \in N, \\ j \text{ is no null player in } v}} \lambda_j} v(N), & \text{otherwise,} \end{cases}$$

satisfies $\boldsymbol{E}^\Lambda$, $\boldsymbol{N}^\Lambda \backslash \boldsymbol{NO}^\Lambda$, and $\boldsymbol{PS}^\Lambda$ but not $\boldsymbol{A}^\Lambda$.

Remark 7.4. *The axioms in Theorem 4.23 are logically independent:*

- $\boldsymbol{NO}^\Lambda$: *The WTU-value φ, defined for all $i \in N$ by*

$$\varphi_i(N, v^\lambda) := \begin{cases} 0, & \text{if } |N| = 1, \\ \displaystyle\sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{\lambda_i}{\sum_{j \in S} \lambda_j} \Delta_v(S), & \text{otherwise,} \end{cases}$$

satisfies $\boldsymbol{PM}^\Lambda$, $\boldsymbol{WS}^\Lambda$, and $\boldsymbol{A}^\Lambda$ but not $\boldsymbol{NO}^\Lambda$.

- $\boldsymbol{PM}^\Lambda$: *The WTU-value φ, defined by*

$$\varphi_i(N, v^\lambda) = \sum_{\substack{S \subseteq N, \\ S \ni i, |S| \leq 2}} \frac{\lambda_i}{\sum_{j \in S} \lambda_j} \Delta_v(S) \text{ for all } i \in N,$$

satisfies $\boldsymbol{NO}^\Lambda$, $\boldsymbol{WS}^\Lambda$, and $\boldsymbol{A}^\Lambda$ but not $\boldsymbol{PM}^\Lambda$.

- $\boldsymbol{WS}^\Lambda$: *The WTU-value φ, defined by $\varphi_i(N, v^\lambda) := 0$ for all $i \in N$, satisfies $\boldsymbol{NO}^\Lambda$, $\boldsymbol{PM}^\Lambda$, and $\boldsymbol{A}^\Lambda$ but not $\boldsymbol{WS}^\Lambda$.*

- $\boldsymbol{A}^\Lambda$: *Let $R := \{i \in N : i \text{ is no null player in } v\}$. The WTU-value φ, defined for all $i \in N$ by*

$$\varphi_i(N, v^\lambda) = \begin{cases} v(\{i\}), & \text{if } i \in N \backslash R \text{ or } |N| = 1 \text{ or } |R| = 1, \\ v(\{i\}) + \dfrac{\lambda_i}{\sum_{j \in R} \lambda_j} \Delta_v(R), & \text{otherwise,} \end{cases}$$

satisfies $\boldsymbol{NO}^\Lambda$, $\boldsymbol{PM}^\Lambda$, and $\boldsymbol{WS}^\Lambda$ but not $\boldsymbol{A}^\Lambda$.

It is easy to transfer our considerations for logical independence above to STU-games and STU-values. We obtain the following remark.

Remark 7.5. *The axioms in Theorem 5.11 and in Theorem 5.13 are logically independent.*

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