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Abstract

Under two-stage budgeting, the consumer allocates first his income to groups of goods and then for each group the expenditure to the goods that belong to the group. This paper derives expressions that relate the price and income elasticities and the elasticities of substitution of the demand for groups to the corresponding elasticities of the demand for groups and those of the within-group demand for goods. In particular, it is shown that the elasticity of substitution between two goods from different groups is equal to the elasticity of substitution between the groups, modified for within-group income effects.

1. Introduction

Under two-stage budgeting the consumer allocates first his income to groups of goods and then allocates for each group the expenditure to the goods that belong to the group.[1] Two-stage budgeting is an attractive way to model demand and supply in terms of aggregates of goods. The idea of two-stage budgeting underlies, implicitly or explicitly, many empirical studies of consumer behavior; see e.g. Deaton (1975, chapter 6) and Blackorby, Boyce and Russell (1978) for explicit use of two-stage budgeting. It has also been used in analyses of trade (e.g., Armington, 1969) and of price formation (Zeelenberg, 1986).

Here I will analyze two-stage budgeting under two additional constraints. First, it is required that the two-stage procedure is consistent, i.e. that the procedure in two stages gives the same demand functions as the one-stage procedure where the demand functions for the goods are determined directly. Second, it is required that the allocations in the first stage can be carried out with knowledge only of a price index for each group. It is well known that these constraints impose restrictions on the functional forms of preferences, and thereby on those of the demand functions (Gorman, 1959, or Green, 1964, chapter 3).

The purpose of this paper is to derive expressions for the income, price and substitution elasticities under two-stage budgeting, and to relate them to the corresponding elasticities of the demand for groups and the within-good demand for goods. It will appear that these formulae are relatively simple and that they can be used with any specific demand systems for the first and the second stage, provided these satisfy the constraints of consistency and the existence of price indexes. For example, for the two stages one can specify demand systems that cannot be solved in a closed form for the demand functions of the goods. The paper is an extension of Zeelenberg (1986, appendix A), where homogeneous two-stage budgeting is analysed.

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[1] See Deaton and Muellbauer (1980, § 5.1) for an introduction to two-stage budgeting.

In § 2 the conditions for two-stage budgeting are formally presented. § 3 derives the formulae for the elasticities.

2. Two-stage budgeting

Let there be N groups and n_G goods in group G (G = 1, 2, ..., N). I assume that the preferences of the consumer can be represented by a well-behaved[2] utility function:

$$u(q) = u(q_{11}, q_{12}, \ldots, q_{1n_1}; q_{21}, q_{22}, \ldots, q_{2n_2}; \ldots$$
$$\ldots; q_{N1}, q_{N2}, \ldots, q_{Nn_N}),$$

where q_{Gi} (G = 1, 2, ..., N; i = 1, 2, ..., n_G) is the quantity of the i-th good in group G. I will write $i \in G$ if good i belongs to group G. The consumer's allocation problem is to maximize the utility function subject to the budget constraint:

$$\left.\begin{array}{l} \max\ u(q) \\ \text{subject to} \displaystyle\sum_{G=1}^{N} \sum_{i \in G} p_{Gi} q_{Gi} = y \end{array}\right\} \qquad (2.1)$$

where p_{Gi} is the price of good i and y is the total, given, budget ('income').

Under two-stage budgeting, the consumer first allocates his income to the groups and then allocates for each group the group expenditure to the goods that belong to the group. It can be shown that, if two-stage budgeting is to be possible, preferences must be separable in the groups; i.e. the utility function can be written as

$$u(q) = U[u_1(q_1), u_2(q_2), \ldots, u_N(q_N)],$$

where $q_G = (q_{G1}, q_{G2}, \ldots, q_{Gn_G})$ is the vector with the quantities of group G, and U and u_G (G = 1, 2, ..., N) are well-behaved utility functions. I

[2] By well-behaved I mean: twice continuously-differentiable, strictly quasi-concave, and increasing in the quantities.

assume that this separability condition holds. The function U is called the macro-utility function and the functions u_G are called the subutility functions.

To carry out the allocation in the first stage one needs for each group a price index that is a function of the prices of the group only; the optimal group expenditures y_G are to be functions of income y and the price indices P_H $(H = 1, 2, \ldots, N)$:

$$y_G = h_G(y, P_1, P_2, \ldots, P_N),$$

where h_G is linearly homogeneous in the prices P_H and income y. A quantity index for each group can be defined by

$$Q_G = \frac{y_G}{P_G} = f_G(y, P_1, P_2, \ldots, P_N), \tag{2.2}$$

where f_G is homogeneous of degree zero in the price indices P_H and income y.

Gorman (1959) has shown that there exist price indices which are functions of only the group prices if and only if one of the following conditions holds:

1. there are two goods;
2. the macro-utility function can be written as

$$\begin{aligned} U = {} & u_1(q_1) + u_2(q_2) + \ldots + u_d(q_d) \\ & + f[u_{d+1}(q_{d+1}), u_{d+2}(q_{d+2}), \ldots, u_N(q_N)], \end{aligned}$$

   where $0 \leq d \leq N$; u_G is homothetic for $G = d + 1, d + 2, \ldots, N$; and the indirect utility function of group G $(G = 1, 2, \ldots, d)$ can be written as

$$\psi(p_G, y_G) = F_G\left[\frac{y_G}{b_G(p_G)}\right] + a_G(p_G), \quad G \in \text{Add}, \tag{2.3}$$

   with F_G monotonically increasing, b_G linearly homogeneous, and a_G

homogeneous of degree zero.

I will analyze the case with more than 2 goods; I assume that condition 2 holds. Thus either the macro-utility function is additive and the subutility functions have the Gorman generalized polar form (2.3) or the subutility functions are homothetic. I will write $G \in \text{Add}$ if $1 \leq G \leq d$, and $G \in \text{Hom}$ if $d + 1 \leq G \leq N$. The group expenditure functions for $G \in \text{Hom}$ can be written as

$$e_G = \theta_G(u_G) b_G(p_G), \qquad G \in \text{Hom}, \tag{2.4}$$

where θ_G is monotonously increasing and b_G is linearly homogeneous in the prices. Thus the group indirect utility functions can be written in a form comparable to (2.3):

$$\psi_G(y_G, p_G) = F_G\left[\frac{y_G}{b_G(p_G)}\right], \qquad G \in \text{Hom}, \tag{2.5}$$

with $F_G = \theta_G^{-1}$. Now define the price indices

$$P_G(p_G) = b_G(p_G). \tag{2.6}$$

Making the substitution $Q_G = y_G / P_G$ we can write the consumer's allocation problem (2.1) as[3]

$$\left.\begin{aligned} &\max \sum_{G=1}^{d} F_G(Q_G) + f[F_{d+1}(Q_{d+1}), F_{d+2}(Q_{d+2}), \ldots, F_N(Q_N)] + \sum_{G=1}^{d} a_G(p_G) \\ &\text{subject to } \sum_{G=1}^{d} P_G Q_G = y. \end{aligned}\right\} \tag{2.7}$$

Solution of (2.7) gives the optimal group quantities Q_G and the optimal group expenditures $y_G = P_G Q_G$. Note that the terms $a_G(p_G)$ are independent of the Q_G and are thus irrelevant to the solution of (2.7).

The allocation in the second stage consists simply of maximizing the

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[3] See Deaton and Muellbauer (1980, § 5.2).

subutility functions u_G subject to the constraint that total expenditure on group G equals the expenditure on the group determined in the first stage:

$$\max \; u_G(q_G)$$

$$\text{subject to} \quad \sum_{i=1}^{n_G} p_{Gi} q_{Gi} = y_G,$$

where y_G is determined in the first stage. The solution of this maximization gives demand functions q_{Gi} that are functions of the prices p_{Gi} and the group budget y_G:

$$q_{Gi} = f_{Gi}(y_G, p_G), \quad G = 1, 2, \ldots, N, \tag{2.8}$$

where $p_G = (p_{G1}, p_{G2}, \ldots, p_{Gn_G})$ is the vector with the prices of the goods belonging to group G.

3. Elasticities under two-stage budgeting

3.1. Income elasticities

The income elasticity of good Gi is from (2.8)

$$\frac{\partial \log q_{Gi}}{\partial \log y} = \frac{\partial \log f_{Gi}}{\partial \log y_G} \frac{\partial \log y_G}{\partial \log y} = \frac{\partial \log f_{Gi}}{\partial \log y_G} \frac{\partial \log Q_G}{\partial \log y}.$$

Thus in elasticity notation

$$\eta_{Gi} = \eta_i^G \eta_G, \tag{3.1}$$

where $\eta_i^G = \partial \log f_{Gi} / \partial \log y_G$ is the within-group[4] income elasticity of good Gi and $\eta_G = \partial \log Q_G / \partial \log y$ is the income elasticity of the demand for group G. Note that in deriving (3.1) we have not yet used the specific additive / homothetic preferences of § 2. The subutility functions for $G \in$ Hom are homothetic, and thus the within-group income elasticities are equal

[4] Within-group variables are denoted by a superscript that indicates the group.

to 1:

$$\eta^G_i = 1, \quad G \in \text{Hom}.$$

Therefore

$$\eta_{Gi} = \eta_G, \quad G \in \text{Hom}. \tag{3.2}$$

Thus all goods within a homothetic group have the same income elasticity.

3.2. Price elasticities

To obtain the price elasticities we differentiate (2.8) logarithmically:

$$\frac{\partial \log q_{Gi}}{\partial \log p_{Hj}} = \frac{\partial \log f_{Gi}}{\partial \log y_G}\frac{\partial \log y_G}{\partial \log p_{Hj}} + \frac{\partial \log f_{Gi}}{\partial \log p_{Hj}}\delta_{GH}, \tag{3.3}$$

where δ_{GH} is the Kronecker delta ($\delta_{GH} = 1$ if $G = H$ and $\delta_{GH} = 0$ if $G \neq H$). Using (2.3) we get

$$\frac{\partial \log y_G}{\partial \log p_{Hj}} = \left[\frac{\partial \log Q_G}{\partial \log P_H} + \delta_{GH}\right]\frac{\partial \log P_H}{\partial \log p_{Hj}}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\varepsilon_{Gi,Hj} = \eta^G_i(\varepsilon_{GH} + \delta_{GH})\frac{\partial \log P_H}{\partial \log p_{Hj}} + \varepsilon^G_{ij}\delta_{GH}, \tag{3.5}$$

where $\varepsilon_{Gi,Hj} = \partial \log q_{Gi} / \partial \log p_{Hj}$ is the elasticity of demand for good Gi with respect to the price of good Hj, $\varepsilon_{GH} = \partial \log Q_G / \partial \log P_H$ is the elasticity of demand for group G with respect to the price of good H, and $\varepsilon^G_{ij} = \partial \log f_{Gi} / \partial \log p_{Gj}$ is the within-group elasticity of demand for good Gi with respect to the price of good Gj. Note that in deriving (3.5) we have not yet used the specific additive / homothetic preferences of § 2. It is shown in the Appendix that under the preferences of § 2, the elasticity of the price index P_H with respect to the price of good Hj is

$$\frac{\partial \log P_H}{\partial \log p_{Hj}} = w_j^H \frac{1 + \varphi\eta_H\eta_j^H}{1 + \varphi\eta_H}, \tag{3.6}$$

where $w_j^H = p_{Hj}q_{Hj} / y_H$ is the within-group budget share of good Hj and φ is the inverse of the income elasticity of the marginal utility of income (the 'income flexibility' or minus the 'overall elasticity of substitution'; see Sato, 1972). From (3.5) and (3.6) we have

$$\varepsilon_{Gi,Hj} = \eta_i^G(\varepsilon_{GH} + \delta_{GH})w_j^H \frac{1 + \varphi\eta_H\eta_j^H}{1 + \varphi\eta_H} + \varepsilon_{ij}^G\delta_{GH}. \tag{3.7}$$

Note that for $H \in$ Hom there holds $\eta_{Hi} = \eta_H$ [see equation (3.2)] and therefore (3.6) reduces to

$$\frac{\partial \log P_H}{\partial \log p_{Hj}} = w_j^H, \quad H \in \text{Hom}.$$

So for homothetic groups the price indices are Divisia price indices and (3.7) reduces to

$$\varepsilon_{Gi,Hj} = \eta_i^G(\varepsilon_{GH} + \delta_{GH})w_j^H + \varepsilon_{ij}^G\delta_{GH}, \quad G \in \text{Hom}.$$

3.3. Compensated elasticities

The three Slutsky equations for goods, groups, and within-group goods are respectively

$$\varepsilon_{Gi,Hj}^* = \varepsilon_{Gi,Hj} + \eta_{Gi}w_{Hj},$$

$$\varepsilon_{GH}^* = \varepsilon_{GH} + \eta_G w_H,$$

$$\varepsilon_{ij}^{*G} = \varepsilon_{ij}^G + \eta_i^G w_j^G,$$

where an asterisk denotes a compensated elasticity. Using (3.7) and the three Slutsky equations, one easily shows that

$$\varepsilon^*_{Gi,Hj} = \varepsilon^{*G}_{ij}\delta_{GH} + \eta^G_i w^H_j \frac{1 + \varphi\eta_H\eta^H_j}{1 + \varphi\eta_H}[\varepsilon^*_{GH}(1 + \varphi\eta_H\eta^H_j) + \varphi\eta_H(\delta_{GH} - \eta_G w_H)(\eta^H_j - 1)]. \quad (3.8)$$

To reduce (3.8) further, we distinguish $H \in$ Hom and $H \in$ Add. For $H \in$ Hom there holds $\eta^H_j = 1$; and thus we have from (3.8)

$$\varepsilon^*_{Gi,Hj} = \varepsilon^{*G}_{ij}\delta_{GH} + \eta^G_i \varepsilon^*_{GH} w^H_j, \qquad H \in \text{Hom}.$$

For $H \in$ Add there holds $\varepsilon^*_{GH} = \varphi\eta_H(\delta_{GH} - \eta_G w_H)$ (see Deaton and Muellbauer 1980, p. 138, Eq. 3.5); thus we have from (3.8)

$$\varepsilon^*_{Gi,Hj} = \varepsilon^{*G}_{ij}\delta_{GH} + \eta^G_i \varepsilon^*_{GH} w^H_j \eta^H_j, \qquad H \in \text{Add}.$$

Therefore the general formula is:

$$\varepsilon^*_{Gi,Hj} = \varepsilon^{*G}_{ij}\delta_{GH} + \eta^G_i \varepsilon^*_{GH} w^H_j \eta^H_j. \quad (3.9)$$

3.4. Elasticities of substitution

There are three Hicks-Allen elasticities of substitution: the elasticity of substitution between goods:

$$\sigma_{Gi,Hj} = \frac{\varepsilon^*_{Gi,Hj}}{w_{Hj}},$$

the elasticity of substitution between groups:

$$\sigma_{GH} = \frac{\varepsilon^*_{GH}}{w_H},$$

and the within-group elasticity of substitution between goods:

$$\sigma^G_{ij} = \frac{\varepsilon^{*G}_{ij}}{w^G_j}.$$

Using these three definitions and (3.9) we get

$$\sigma_{Gi,Hj} = \frac{1}{w_G}\sigma^G_{ij}\delta_{GH} + \eta^G_i\sigma_{GH}\eta^H_j. \tag{3.10}$$

The first term on the right-hand side of (3.10) represents substitution within the group, and the second term represents substitution between groups. In particular for G ≠ H there holds

$$\sigma_{Gi,Hj} = \eta^G_i\sigma_{GH}\eta^H_j; \tag{3.11}$$

i.e. the elasticity of substitution between two goods from different groups is equal to the elasticity of substitution between the groups, modified for within-group income effects.

Appendix. The elasticity of the price index

A.1. Proof of equation (3.6)

This Appendix gives the proof of (3.6), i.e.

$$\frac{\partial \log P_G}{\partial \log p_{Gi}} = w^G_i\frac{1 + \varphi\eta_G\eta^G_i}{1 + \varphi\eta_G}.$$

For $G \in$ Hom there holds $P_G = e_G / \theta_G(p_G)$ [see equation 2.4)], with e_G the expenditure function of group G; thus by Shephard's Lemma

$$\frac{\partial \log P_G}{\partial \log p_{Gi}} = \frac{p_{Gi}q_{Gi}}{y_G} = w^G_i, \qquad G \in \text{Hom}.$$

Since $\eta^G_i = 1$ for $G \in$ Hom, this proves (3.6) for $G \in$ Hom.

For $G \in$ Add the proof consists of two parts. First I will derive an expression for the income flexibility φ in terms of the indirect utility functions ψ_G. The second part consists of deriving expressions for the within-group budget shares and the within-group marginal budget shares.

The first-order conditions for the maximization problem (2.7) are

$$F_G' = \lambda P_G, \qquad G \in \text{Add},$$

$$\frac{\partial f}{\partial Q_G} = \lambda P_G, \qquad G \in \text{Hom},$$

where a prime denotes a derivative, and λ is the Lagrange-multiplier (the 'marginal utility of income'). Differentiation of the conditions for $G \in$ Add with respect to y gives

$$F_G'' \frac{\partial Q_G}{\partial y} = P_G \frac{\partial \lambda}{\partial y}, \qquad G \in \text{Add}.$$

In elasticity notation this can be written as

$$\frac{Q_G}{\lambda} \frac{F_G''}{P_G} = \frac{\omega}{\eta_G}, \qquad G \in \text{Add},$$

where $\omega = \partial \log \lambda / \partial \log y$ is the income elasticity of the marginal utility of income. Since $\lambda P_G = F_G'$, this can be written as

$$\frac{F_G'}{F_G''} = \varphi \eta_G Q_G, \qquad G \in \text{Add}, \tag{A.1}$$

where $\varphi = 1 / \omega$ is the income flexibility.

Application of Roy's Identity to the indirect utility function for group G gives the quantity of good Gi:

$$q_{Gi} = -\frac{\partial \psi_G / \partial p_{Gi}}{\partial \psi_G / \partial y_G} = \frac{y_G}{b_G} \frac{\partial b_G}{\partial p_{Gi}} - \frac{b_G}{F_G'} \frac{\partial a_G}{\partial p_{Gi}} \tag{A.2}$$

It follows from (A.2) that the within-group budget share of good Gi is

$$w_i^G = \frac{p_{Gi} q_{Gi}}{y_G} = \frac{\partial \log b_G}{\partial \log p_{Gi}} - \frac{p_{Gi} b_G}{F_G' y_G} \frac{\partial a_G}{\partial p_{Gj}}. \tag{A.3}$$

It also follows from (A.2) that the within-group marginal budget shares are

$$\mu_i^G = \eta_i^G w_i^G = p_{Gi}\frac{\partial q_{Gi}}{\partial y_G} = \frac{\partial \log b_G}{\partial \log p_{Gi}} + \frac{F_G''}{(F_G')^2} p_{Gi}\frac{\partial a_G}{\partial p_{Gi}}. \tag{A.4}$$

Using (A.1), (A.3), and (A.4) one easily shows that

$$\frac{\partial \log b_G}{\partial \log p_{Gi}} = w_i^G \frac{1 + \varphi\eta_G\eta_i^G}{1 + \varphi\eta_G}, \quad G \in \text{Add},$$

which proves (3.6) since $P_G = b_G$ [see equation (2.6)].

A.2. A special case

A special case arises when the group preferences have the Gorman polar form

$$\psi_G(y_G, p_G) = \frac{y_G - d_G(p_G)}{b_G(p_G)},$$

where d_G is linearly homogeneous in the prices p_G; this form corresponds to (2.4) with $F_G' = 1$, $F_G'' = 0$, and $a_G = -d_G / b_G$. Such a form occurs for example if there exists for each good base quantities q_{Gi}^0 and the sub-utility functions are linearly homogeneous in the excess quantities $q_{Gi} - q_{Gi}^0$ (Keller, 1976); then $d_G(p_G) = \Sigma_{i\in G} p_{Gi} q_{Gi}^0$.

Then from (A.4) we have

$$\frac{\partial \log b_G}{\partial \log p_{Gi}} = \mu_i.$$

Thus the weights in the price index are equal to the marginal budget shares; i.e. the price indices are Frisch price indices. On the other hand we have from (A.3)

$$\frac{\partial \log b_G}{\partial \log p_{Gi}} = \frac{p_{Gi}(q_{Gi} - \partial d_G/\partial p_{Gi})}{y_G - d_G(p_G)}.$$

If $d_G(p_G)$ is interpreted as base expenditure and $\partial d_G/\partial p_{Gi}$ as the base quantity for good Gi, then one can say that the marginal budget shares are equal to the shares of the excess quantities in excess income (the 'excess budget shares').

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