Theoretical framework for Measuring Social Deprivation

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Abstract:

This paper tries to look at deprivation from an entirely different standpoint. The approach framed by us is somewhat different from the existing deliberations on deprivation. In the literature, deprivation relates to non-attainments of some desirable attributes. In our view this approach is one-sided. Deprivation, as we understand, is not only the result of non-attainment but accumulation of non-desirable traits that have been a result of such non-attainment. A proper measure of deprivation thus should measure not only the non-attainment factor but also the result of the accumulated, unwanted outcomes. The second perspective in which we differ from the standard analysis is the emphasis on the efficacy of the use of resources in combating the deprivation or ill consequences. The approach should emphasize not only on the availability of welfare improving inputs but to the extent to which these inputs have been transformed into the welfare-improving outcomes.

Theoretical framework for Measuring Social Deprivation

1. Introduction

In this paper we are have tried to look at deprivation from an entirely different standpoint. This approach is somewhat different from the existing deliberations on deprivation. In the literature, deprivation relates to non-attainments of some desirable attributes. For example- poverty measures the non-attainment of a minimum modicum of income that is necessary for decent living. Illiteracy measures the non-attainment of the ability to read and write a crucial element for the modern society. Measures of ill-health give us an idea about the extent to which a person has not been able to achieve a minimum health status that is crucial for fruitful laboring and earning potentiality.

In our view this approach is one-sided. Deprivation, as we understand, is not only the result of non-attainment but accumulation of non-desirable traits that have been a result of such non-attainment. For example-a poor people is not only suffering from an inadequacy of income but also the result that follows from it such as under-nutrition, stunted growth, low resistance towards diseases and so on and so forth. Similarly, illiteracy breeds superstitious belief, wrong choice of actions, un-critical adherence to some belief or faith and excessive dependence on the rumors and so on and so forth. A proper measure of deprivation thus should measure not only the non-attainment factor but also the result of the accumulated, unwanted outcomes.

The second perspective in which we differ from the standard analysis is the emphasis on the efficacy of the use of resources in combating the deprivation or ill consequences. The approach should emphasize not only on the availability of welfare improving inputs but to the extent to which these inputs have been transformed into the welfare-improving outcomes. For example- the supply of quality toilet is not enough. It is also necessary to see whether these are used properly and whether these have made a meaningful dent in open defecation. This issue leads us to the consideration of Human Development within the Input-Output framework in place of the standard treatments of the outcomes only.

In this chapter, we have first discussed these two issues then moved on to build up the axiomatic structure that emanates from our way of looking at the deprivation.

2. Issues in positive and negative outcomes

We first consider the coupling of non-attainment and accumulation of considerable income. In order to formalize our way of thinking, we first use the concept of social welfare. The concept can be used from a divergent point of views (Sen, 1970). However, in our exercise, we are mainly interested whether there is an increase or decrease in deprivation. The social welfare function is thus defined to be as negatively dependence on deprivation. As deprivation falls the social welfare rises and vice-versa.

In short, $SW^T = \int (D^T)$ (1)

With $\int ' < 0$

Here we define SW^T as the traditional social welfare and D^T as the traditional deprivation.

In our approach, however, instead of defining the deprivation straightforward we assume it to be a function of both positive attributes and negative attributes. Positive attributes are desirable attributes (such as the ability to read and write healthy live, sustainable income and so on). Negative attributes are those that are undesirable.(Such as Out of school children, People without assets and so on and so forth).

$$SW^N = \phi\,(A^N_+,\ A^N_-) \tag{2}$$

With $\frac{d\phi}{dA^N_+} > 0, \frac{d\phi}{dA^N_-} < 0$

Where SW^N is the new social welfare, A^N_+ is the positive attribute and A^N_- is the negative attribute.

To put the idea more completely suppose we consider two items in the field of education----the Literacy Rate (LR) - a positive outcome and the Out of School Children (OS) - a negative outcome. Hence our social welfare function in the field of education becomes

$$SW^N = \phi\,(LR, OS) \tag{2.1}$$

Diagrammatically, this will generate an indifference map corresponding to different levels of social welfare.

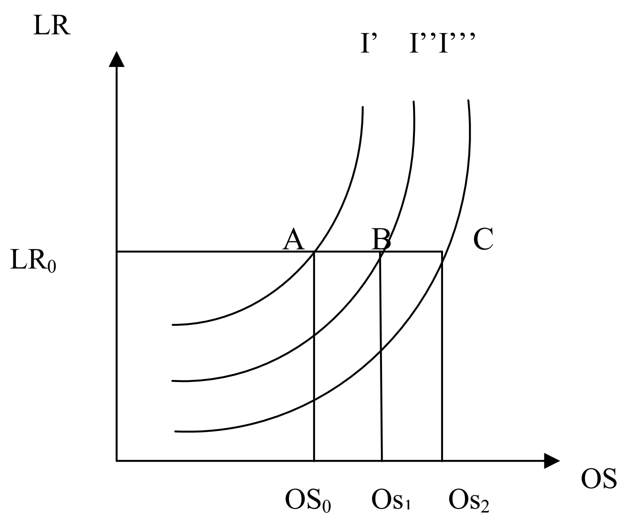

Figure 1

The above diagram (Figure 1) shows upward sloping Indifference Curves (ICs). The reason is simple. If we fix the positive attribute at a level (LR at LR_0) on increase in OS should lead to a lower level of social welfare.

The contrary will happen if we increase the positive attribute while keeping the negative attribute constant.

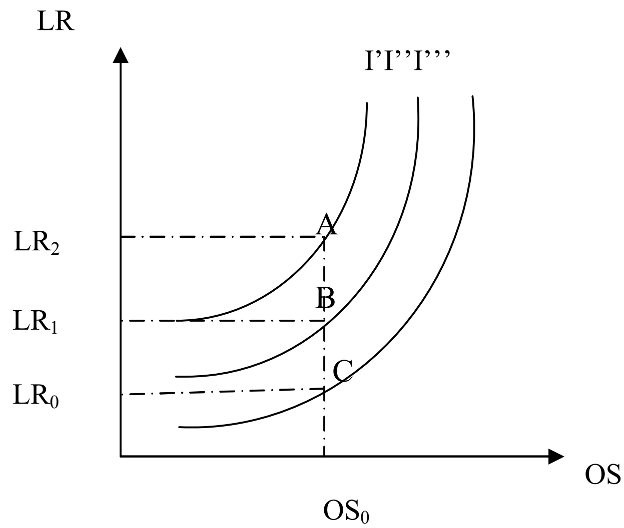

Figure 2

Fixing the negative output at any level (here OS at OS_0) and raising the positive output (LR) will shift the economy to a higher IC.

Our exercise thus makes the comparison across two cross-sectional units problematical.

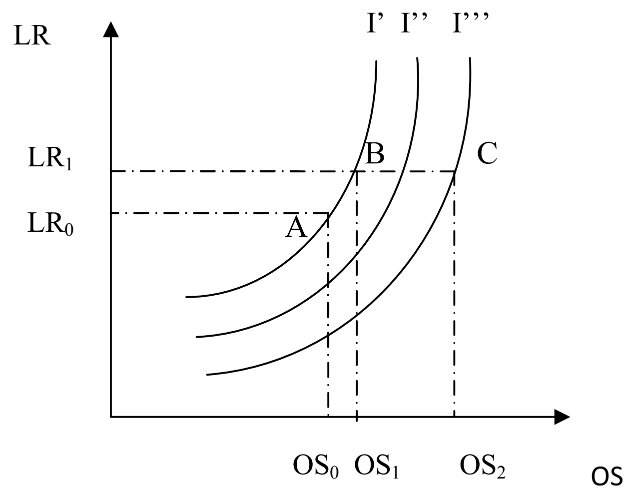

Figure 3

Suppose we are comparing a cross-sectional point A with a cross-sectional point B. At point B the positive output and the negative output both are higher to A. Still, given the shape of the social welfare function, B has the same level of Social welfare as A. Thus the illusion that the higher literacy improves social welfare does not carry in our exercise.

The result is more striking if OS rises to OS_2. In this case, even if LR rises from LR_0 to LR_1 the social welfare falls. The fall in social welfare due to the rise of OS more than offset the rise in social welfare due to the rise of LR.

The above discussion clearly shows that the measurement of deprivation in our exercise inherently depends on the nature and the form of the social welfare function; depending upon this form we may have different measures of deprivation. Even with a similar change in the positive and negative attribute.

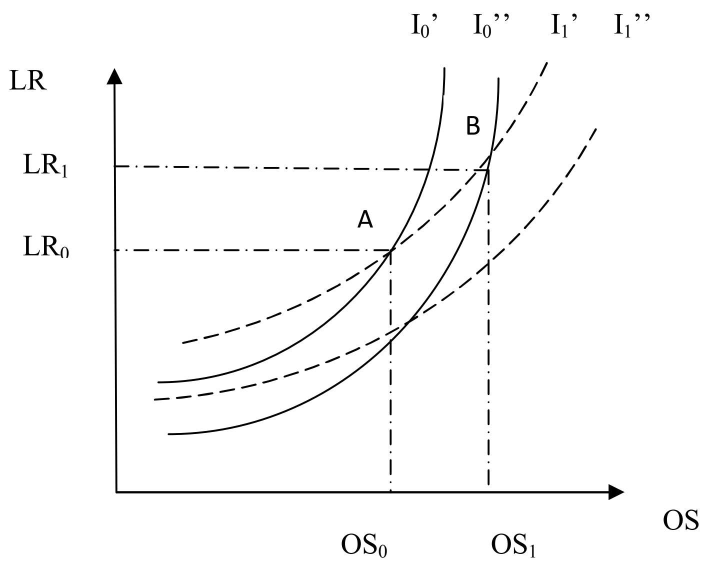

Figure 4

In the above diagram, we consider two social welfare functions----one with the bold lines (I_0', I_0'') and another with the dotted lines (I_1', I_1''). Now consider the points A and B. According to the dotted line social welfare function, there is no change in the social welfare level between A and B. However when we consider the bold line, social welfare level has fallen as the society has moved to a lower indifference curve at B. The nature of the social welfare function measures the extent of deprivation as defined in our approach.

Thus in order to calculate the deprivation following our methodology, we have to specify the social welfare function. This is so because, in our analysis of deprivation, the effect has to be captured by the impact on social welfare. Our next task is to find out a form of social welfare function and analyze the effect of deprivation through the changes in social welfare as generated by this function.

In order to be more objective, we have first defined a set of axioms that seems reasonable for any meaningful analysis of deprivation.[4] The form of the social welfare function is derived from these axioms (in section 3). The form of social welfare function that emanates from our axioms is the following

$$SW^N = \alpha(A^N_+)\,[1-\beta(A^N_-)] \tag{3}$$

Where $\alpha > 0$ and $\beta < 0$

In the next section, we have discussed another feature of our approach – Input-Output analysis.

3. Deprivation in the Input-Output framework.

In order to successfully couple the success and failure of human development within a more generalized index, we have considered both the positive and negative aspects of Human Development in respect to the three dimensions of HDI- Livelihood, Education and Health. Coupling the positive and negative aspects is important. It should be noted that we cannot take the same variable as positive and its un-attainment as negative. For example, we should not only include literacy rate in the positive side and ill-literacy rate in the negative side. In choosing the positive and negative sides, it is important to access the impact on social welfare. For example, increasing literacy rate might be a good indicator of attainment. On the contrary, a rise in drop-out will hurt the social welfare negatively. Literacy rate imposes human capital formation whereas drop-out hurts at the very basis of Human Development. Similar considerations are made for livelihood and health concerns. For each index, we have considered a two-dimensional framework popularized by a number of

[4] Many of the axioms are variance of the standard axioms used in the deprivation literature (Chakravarty, 2000)

authors (Ramirez, Ranis and Stewart (October 1998)). In this approach, each factor has been visualized from both the dimensions resulting in a 2x2 definitional matrix—

Table 1: Definitional Matrix

Virtuous provinces (Good Input, Good outcome)	Lop-sided outcome (Good Input,Bad Outcome)
Lop-sided Input (Bad Input, Good Outcome)	Vicious Provinces (Bad Input, Bad Outcome)

Source: *Author's construction*

In the above table 1, the first cell in the first row indicates the Virtuous situation. Both input and outcome are good here. There is enough infrastructure and is properly used. In the second cell in the first row is the lop-sided Outcome. Here Input is good but the outcome is bad. In this case, Input facilities are not utilized. The first cell in the second row represents lop-sided Input. Probably, private initiatives are more important here. These private initiatives may come from some NGO's or non-profitable institutions. They may also come from the private budget. The last cell in the second row is the worst possible condition. There is neither infrastructure nor utilization.

We now try to formalize the concept using the axiomatic structure.

4. Axiomatisation of the new deprivation measure.

4.1 Positive aspect of Human Development.

In this section, we will discuss the mathematical axiomatization of our new approach. As stated earlier there are three dimensions in the approach- the effects of positive attributes, the effects of negative attributes and the overall social approach

Suppose that the i^{th} component of positive well-being is measurable and is quantified by a number x_i (where i = 1, 2,…………………., k) that remains within the bounds $[m_i, M_i]$ that is defined on a real line R_+^1. For non-emptiness of the interval, we assume that $m_i < M_i$. Let U associates a value U (x_i, m_i, M_i) to each $x_i \in [m_i, M_i]$, where U is a real-valued function and an indicator for i.

It is assumed that U is twice differentiable and continuous. Further, it is assumed that U is independent.

We now suggest some properties for an arbitrary index U

Positive Normalization(PNOM): $U(x_i, m_i, M_i) = 0$, if $x_i = m_i$,

$= 1$, if $x_i = M_i$

Normalization constraint the value of x_i between 0 and 1.

Positive Monotonicity (PMN): Monotonicity assures that for a give m_i and M_i, the index U increases with an increase in x_i.

Positive Translation Invariance (PTI): $U(x_i, m_i, M_i) = U(x_i + a, m_i + a, M_i + a)$, where a is any scalar such that $m_i + a \geq 0$. TI says that the index U is origin independent.

Positive Homogeneity (PHM): For any $a \geq 0$, $U(x_i, m_i, M_i) = U(ax_i, am_i, aM_i)$. HM posits that the index is scale independent.

The lower gain in Indicator at a Higher level of attainment difference (LGI)[5]: Let $x_i \in [m_i, M_i]$ be any attainment level of attribute i. Then for any $\boldsymbol{\Delta}>0$ such that $x_i + \boldsymbol{\Delta} \in [m_i, M_i]$,the magnitude of indicator gain $U(x_i, m_i, M_i) = U(x_i + \boldsymbol{\Delta}, m_i, M_i) - U(x_i, m_i, M_i)$ is a non-increasing function of $\boldsymbol{\Delta}$. LGI indicates that gain in the indicator do not rise with an increase at the higher levels as compared to an equivalent increase at the lower level.

The property PMN nearly constraints the HDI between Zero and unity. This is the standard unitary scale from which the indices are normally based. Monotonicity is simply a statement of the derived nature of HDI. It is PTI and PHM that are most important for defining the shape of HDI.PTI makes the desired measure origin-free. Infact the differencing from minimum is derived from PTI. PHM makes the measure scale-free. This gives the measure in the ratio form.

LGI is important in making the HDI in making the HDI insensitive for further and further addition in one dimension. Suppose a country has a low level of education and health. It now adopts a strategy that raises its income without affecting the health and education. LGI will guarantee that the resulting improvement in the overall index becomes weaker and weaker as the income indicator rises to a higher level. HDI is an overall measure that spans across several dimensions. Thus improvement in one dimension neglecting others should not raise HDI. The LGI property follows from the basic philosophical logic of Human Development.

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[5]This is less stringent than the LI of Chakravarty (2000).

Using these axioms the HDI can be written as[6]:-

$$\textbf{HDI} = \sum_{i=1}^{k}((x_i - m_i)/(M_i - m_i))/k$$

4.2 Negative aspect of Human Development.

Before proceeding to axiomize the negative aspect we have to provide the rationale for it. Any improvement in the component of the HDI makes the society well-off. However, corresponding to each improvement there are vast "wastelands" that are untouched. Consider, for example, literacy rate- an important component of HDI. Overall illiteracy is clearly bad but for a group, the more important concern is the illiteracy of that group. Even if overall illiteracy falls, the group will feel deprived if its literacy rate remains constant (or worst still if it falls). The fact that overall literacy has improved is a little solace to it. Similarly, if both per capita GDP and inequality index of a country rises, the gain in welfare due to an expansion of income is mitigated by the blow of inequality. Instances may be cited in favours of other dimensions of human development as well.

Before proceeding to further discussion, we must differentiate between the two strings of poverty measure-Human Poverty Index (HPI) and Multi-dimensional Poverty Index (MPI). The first measure was fashioned earlier to be improved by MPI later. HPI uses average country wise figures to achieve at an overall figure. On the contrary, MPI is based on micro-level data. It tries to assess the performances of families across various dimensions and use a cut-off point to ascertain both the incidence and intensity of poverty at family levels.

[6]For the theorem and their proof see Chakravarty 2000.

The construction of poverty (or negative) indices in our case requires **pairing**. It is of utmost importance to match each achievement with the corresponding failure- income with poverty, educational endowment with deprivation in education, health endowment with health failures and so on. However such pairing does not mean they should be a mirror image of one another. For positive aspect, we consider the overall achievement. For failure, we follow the Rawlsian principle. We seek to choose the failures of those that are most deprived. This we think would be more appropriate than making the negative side a mere mirror image of the positive side.

A further note is here needed for some elaborations on the technical side. Chakravarty and Mazumdar (2005) provided an axiomatization of the Human Poverty. However, they never tried to link the poverty aspect with the negative side of Human Development. Hence, in spite of apparent similarity, our structure is in a completely in a different focus. These differences crop up in the certain subtleties-the definition of domain and range, the specification of axioms (such as Normalization) and also in the structuring of the final measure. It is these that enable us to move to a "comprehensive measure".

Following the above examples, we have to consider the negative aspect of Human Development.[7]Let us consider $y_i$to be the quantified negative ith component of human well-being that lies within the interval $[N_i, n_i]$that is a subset of real line[8] R+. For the existence of the bound, we assume that $\mathrm{N_i} > \mathrm{n_i}$.

Hence we posit, as in the positive case a real-valued function V that associates a

[7] Since the positive and negative aspects comes in pair , the number of positive elements should be equal to the numbers of negative elements

[8]In order to prevent unnecessary mathematical conundrums we take the absolute value of the negative indicators. However since these are negative indicators, higher the value of negative attributes lower will be the social welfare.

value $V(y_i, N_i, n_i)$ to each y_i € $[N_i, n_i]$.

V is assumed to be twice differentiable and continuous. As in the positive case function, V is assumed in the same form for the entire negative attribute.

The arbitrary index V also poses some properties as in the positive case. These properties are:

Negative Normalization (NNOM): $V(y_i, N_i, n_i) = 0 \quad if\ y_i = N_i$

$$= 1 \ \ if \ \ y_i = n_i \ \ (4)$$

Negative Monotonicity (NMO): Given N_i and n_i, an increase in y decreases V.

Negative Translation Invariance (NTI):

$$V(y_i N_i, n_i) = V(y_i - d, N_i - d, n_i - d), \qquad (5)$$

Where, d is any scalar such that $N_i - d \geq 0$.

Negative Homogeneity (NHM):

For any, $e > 0, V(y_i, N_i, n_i) = V(ey_i, en_i, eN_i)$ (6)

Lower loss in Indicator at Higher level of attainment difference (LLI): Let y_i € $[N_i, n_i]$ be any attainment level of attribute i. Then for any $\alpha > 0$ such that $y_i + \alpha$ € $[n_i, N_i]$ the magnitude of indicator loss $V(y_i + \alpha, N_i + \alpha, n_i + \alpha) - V(y_i, N_i, n_i)$ is a non-decreasing function of α .

Using these axioms negative HDI can be written as:-

NHDI = $\sum_{i=1}^{k}((y_i - n_i)/(N_i - n_i))/k$ (4.4)

Theorem 1: Since the indicator, V is twice differentiable. Then V will satisfy the above properties if V inherits the following form:-

$$V(y_i, N_i, n_i) = \mu((y_i - n_i)/(N_i - n_i)) \qquad (7)$$

Where, μ lies between the bounds 0 and 1 and is defined on a real line R_+^1 and is twice differentiable, decreasing and strictly convex function. Further μ (0) =1 and μ (1) =0.

Proof of Theorem 1: Sincey_i € $[N_i, n_i]$which guarantees $y_i - n_i \geq 0$then by NTI we have

$$V(y_i, N_i, n_i) = V(y_i - n_i, N_i - n_i, n_i - n_i)$$

$$= V(y_i - n_i, N_i - n_i, n_i - n_i)$$

$$= h(y_i - n_i, N_i - n_i) \quad (8)$$

Now by NHM we have $V(y_i, N_i, n_i) = V(ey_i, eN_i, en_i)$

$$= h(e(y_i - n_i), (eN_i - n_i))$$

Putting $e = \frac{1}{N_i - n_i}$ we have:-

$$= h((y_i - n_i/N_i - n_i), (N_i - n_i/N_i - n_i))$$

$$= h((y_i - n_i/N_i - n_i), 1)$$

$$= \eta((y_i - n_i/N_i - n_i)) \quad (9)$$

NMO requires decreasing of η. Further, it is clear that η lies in the range 0 and 1 because in the extreme cases when $y_i = N_i$ and $y_i = n_i$, the index becomes η(1) and η(0).This verifies NMO.

Again,

When $y_i = n_i then, V(y_i, N_i, n_i) = 1$

$$\eta(0) = 1 \quad (10)$$

$and y_i = N_i then, B(y_i, N_i, n_i) = 0$

$$\eta(1) = 0 \quad (11)$$

Twice differentiability of V implies differentiability of η.

Next, let $j > 0$ then given $y_i > z_i$, LLI requires that

$$\eta((y_i + j - n_i)/(N_i - n_i)) - \eta((y_i - n_i)/(N_i - n_i)) > \eta((z_i + j - n_i)/(N_i - n_i)) - \eta((z_i - n_i)/(N_i - n_i)) \qquad \ldots(12)$$

Dividing both sides of (12) by j and letting $j \to 0$ we get

$$\eta'((y_i - n_i)/(N_i - n_i)) > \eta'((z_i - n_i)/(N_i - n_i)) \qquad (13)$$

Where, η' is the derivative of η.The argument given above shows that (12) implies (13).To prove the reverse implication, let us take the definite integral of the left (right)- hand side of (13) from the lower limit $y_i(z_i)$ to the upper limit $y_i + j(z_i + j)$ to get

$$\int_{y_i}^{y_i+j} \eta'((k - n_i)/(N_i - n_i))\, dk > \int_{z_i}^{z_i+j} \eta'((k - n_i)/(N_i - n_i))\, dk \qquad (14)$$

Which is (12).Therefore (12) implies and is implied by (13). Given $y_i > z_i$, for inequality (11) to hold we require increasingness of η', that is strict convexity of η. In other words, we need $\eta''>0$, where η'' the derivative of η', This proves the necessity part of the theorem. The sufficiency is easy to verify.

Theorem 2: Properties NNOM, NMO, NTI, NHM and LLI are independent.

Proof of Theorem 2: (i) Let us consider the indicator of the following form

$$V_1(y_i, N_i, n_i) = 2 - ((y_i - n_i)/(N_i - n_i))^r, \text{ Where, } 0 < r < 1 \qquad (15)$$

Since $V_1(y_i, N_i, n_i) = 2$ if $y_i = n_i$ *and*

$V_1(y_i, N_i, n_i) = 1$ if $y_i = N_i$

The postulate NNOM is violated.

However V_1 satisfies all the remaining postulate,

(ii) Suppose the indicator is of the form

$V_2\ (y_i, N_i, n_i) = [e^{(y_i - n_i)/(N_i - n_i)} - 1]/[e-1]$, (16)

Where, e represents exponential function.

Since

$V_2\ (y_i, N_i, n_i) = 0$ if $y_i = n_i$ *and*

$V_2\ (y_i, N_i, n_i) = 1$ if $y_i = N_i$.

Thus the postulate NNOM is satisfied however V_2 is an increasing function of y_i , NMO is not fulfilled. Although V_2 meets the other four axioms.

(iii) The indicator is given by

$V_3\ (y_i, N_i, n_i) = [e^{-(y_i - N_i)/n_i} - 1]/[e^{-(n_i - N_i)/n_i} - 1]$ (17)

Violates NTI because $e^{-(y_i - N_i)/n_i}$ and $e^{-(n_i - N_i)/n_i}$ do not remain invariant under equal absolute changes in y_i , N_i, n_i. Clearly, V_3 is normalized, monotone, homogeneous of degree zero and strictly convex in y_i .

(iv) Since the indicator

$V_4\ (y_i, N_i, n_i) = [e^{-(y_i - N_i)} - 1]/[e^{-(n_i - N_i)} - 1]$ (18)

is not homogeneous of degree zero in (y_i , N_i, n_i), NHM is violated . However V_4 verifies the postulates NNOM, NMO, NTI and LLI.

(v) The indicator is defined as

$V_5\ (y_i, N_i, n_i) = [e^{-(y_i - N_i)/(n_i - N_i)} - 1]/[e^{-1} - 1]$. (19)

Violates LLI but fulfils NNOM, NMO, NTI and NHM

4.3 Deprivation Oriented (or Net) Human Development.

We have considered both the positive and negative sides of the various components of Human Development. It is now necessary to combine these two aspects into a single measure. The exercise involves a number of interesting aspects. Consider, an index I defined over k dimensions where,I is a scalar index and all the components of I are lying between the same unit lengths.

In Essence, I is a mapping from k unit lines to a scalar or mathematically, $I : [0,1]^k \rightarrow k$

Let us denote a_i be the positive component and b_i' be the negative components such that $b_i' = 1 - b_i$. We have taken b' instead of b in order to avoid inappropriate penalty. Suppose a country has a high per-capita GDP attaining a high 'a' and a high degree of poverty attaining a low 'b'. It is clear that unless 'b' is corrected to take the impact of high poverty, this would give the country a higher net measure than a country with similar per-capita GDP and a low-level of poverty and hence a high 'b'. So, the achievement index

$G((x_1, m_1, M_1),\ldots\ldots\ldots\ldots\ldots,(x_p, m_p, M_p),(y_1, N_1, n_1),\ldots\ldots\ldots\ldots\ldots,(y_{k-p}, N_{k-p}, n_{k-p}))$

$$\begin{bmatrix} where, \quad k - p = p \\ k = 2p \end{bmatrix}$$

Can be written as

$$I\ (a_1, a_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_p, b_1', b_2', \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots b_{k-p}') \tag{20}$$

Assuming that there are p positive component and $k - p$ negative components. We impose certain properties on the lecture of I.

Positive Normalization (PNOM) : For any z € [0,1], $I\ (\underbrace{z, z\ldots\ldots\ldots, z}_{p}, \underbrace{0, 0\ldots\ldots 0\ldots\ldots 0}_{k-p}) = z$

Negative normalization (NNOM): For any z € [0,1], $I\,(\underbrace{0,0\ldots\ldots,0}_{p},\underbrace{z,z\ldots\ldots,z}_{k-p}) = -z$

Consistency in Aggregation (CIA) :$\underline{a}$,$\underline{b'}$,$\underline{c}$€$[0,1]^k$

$I(\,a_1 + c_1,\, a_2 + c_{2,}\ldots\ldots\ldots\ldots, a_p + c_{p,}\, b'_1 + c_{p+1,} b'_2 + c_{p+2,}\ldots\ldots\ldots b'_{k-p} + c_k) = \,I(\,a_1\,,$
$a_2\,,\ldots\ldots\ldots\ldots, a_p, b'_1\,, b'_2\,,\ldots\ldots\ldots\ldots, b'_{k-p}) + I(c_1\,, c_2\,,\ldots\ldots\ldots\ldots, c_k)$

Symmetry (SYM) : For all $\underline{a}$€ $[0,1]^k$, $\underline{b}'$€ $[0,1]^k$, $I(\underline{a}\underline{b}') = I(\underline{a}P\underline{b}'Q)$ where P *and* Q are any $k\ X\ k$ permutation matrices

It can be proved easily that if there is no negative component in the Human Development Index, the indicator function *I* takes the familiar additive form[9]. Extending this we can easily show that the indicator function in our case will turn out to be

Theorem 3

$$I(\,a_1\,, a_2\,,\ldots\ldots\ldots\ldots, a_p, b'_1\,, b'_2,\ldots\ldots\ldots\ldots, b'_{k-p}) = \frac{1}{p}\sum_{i=1}^{p} a_i - \frac{1}{k-p}\sum_{j=p+1}^{k} b'_j \qquad (21)$$

Proof[10]: For any $i, 1 \leq i \leq p, define$

$I(0\,,\ldots\ldots,0,a_i,0,\ldots\ldots,0) = t_i(a_i)$.Now, by CIA we get :

$I(\,a_1\,, a_2\,,\ldots\ldots\ldots\ldots,0) = I(\,a_1\,,0,\ldots\ldots,0) + I(\,0\,, a_2,0,\ldots\ldots,0)$

$= t_1(a_1) + \,t_2(a_2)$

Repeating this procedure we get

$I(\,a_1\,, a_2\,,\ldots\ldots\ldots, a_{p,} 0\ldots\ldots\ldots,0) = \sum_{i=1}^{p} t_i(a_i)$

Now by SYM t_i's are identical say $t_i = t\ for\ all\ i$

$$\text{Therefore,} I(\,a_1\,, a_2\,,\ldots\ldots\ldots, a_p,0,\ldots\ldots\ldots\ldots,0) = \sum_{i=1}^{p} t(a_i) \qquad (22)$$

[9] See Chakravarty (2000).
[10] The idea of the proof is taken from Aczél (1976).

If a_i are all identical, say $a_i = a$ for all i, then

$I(a, a, \ldots\ldots\ldots, a, 0 \ldots\ldots\ldots, 0) = p\ t(a)$ (23)

But by PNOM, in this extreme case

$I(a, a, \ldots\ldots\ldots, a, 0, \ldots\ldots\ldots\ldots\ldots, 0) = a$ (24)

From (23) and (24) it follows that $t(a) = \frac{a_i}{p}$. Substituting this explicit form of t in (22) we get

$I(a_1, a_2, \ldots\ldots\ldots, a_p, 0, \ldots\ldots\ldots\ldots, 0) = \frac{1}{p}\sum_{i=1}^{p} a_i$ (25)

Similarly, for the negativeside, we get

$I(0, 0, \ldots\ldots\ldots\ldots, 0, b_1', b_2', \ldots\ldots\ldots\ldots, b_{k-p}') = -\frac{1}{k-p}\sum_{j=p+1}^{k} b_j'$ (26)

Now,

$I(a_1, a_2, \ldots\ldots\ldots\ldots, a_p, b_1', b_2', \ldots\ldots\ldots\ldots, b_{k-p}') =$

$I(a_1, a_2, \ldots\ldots, a_p, 0, \ldots\ldots 0) + (0, 0, \ldots\ldots\ldots, 0, b_1', b_2', \ldots\ldots, b_{k-p}')$

$= \frac{1}{p}\sum_{i=1}^{p} a_i - \frac{1}{k-p}\sum_{j=p+1}^{k} b_j'$ [from (25) and (26)]

However, in this simply netting procedure, we make no considerations for any weightage to the negative side. Such a weight structure can be introduced by tinkering with the NNOM preposition. Suppose we rewrite the NNOM in the following way

$I(\underbrace{0, 0 \ldots\ldots\ldots, 0}_{P}, \underbrace{b', b', \ldots\ldots\ldots, b'}_{k-p}) = -\theta b'$ (27)

It is quite easy to show the Deprivation Oriented HDI in the following way

$I(a_1, a_2, \ldots\ldots\ldots\ldots, a_p, b'_1, b'_1, \ldots\ldots\ldots\ldots, b'_{k-p}) = \frac{1}{p}\sum_{i=1}^{p} a_i - \frac{1}{k-p}\theta\sum_{j=p+1}^{k} b'_j$ (28)

Now, θ is the society's weight to the negative aspect of Human Development. The weight i.e., attached to θ radically change relative e rankings of the countries in the context of Human Development.

A legitimate argument against the form (21) and (27) is that they may become negative.[11] We follow the cue from Lee (2007) to generate a Deprivation Oriented Human Development measure that lies between zero and unity. To do this we have to define the parameter θ in the following way:

$\theta = -\sum_{i=1}^{p} a_i$ (29)

So we can write (27) as:

$$I\,(0,0,\ldots\ldots\ldots\ldots,0, b', b', \ldots\ldots\ldots\ldots, b') = -b' \sum_{i=1}^{p} a_i \qquad (30)$$

It is easy to prove that the Deprivation Oriented (or Net) Index becomes

$$I\,(a_1, a_2, \ldots\ldots, a_p, b_1', b_2', \ldots\ldots b_{k-p}') = \frac{1}{p}\sum a_i - \left(\sum_{i=1}^{p} a_i\right)\left[\frac{1}{k-p}\sum_{j=1}^{p} b_j'\right] \qquad (31)$$

By simple pairing and noting that $k - p = p$, we can write the equation as

$$I\,(a_1, a_2, \ldots\ldots\ldots\ldots, a_p, b_1', b_2', \ldots\ldots\ldots b_p') = \frac{1}{p}\sum\{a_i\,(1 - b_i')\} \qquad (32)$$

Or, $I\,(a_1, a_2, \ldots\ldots\ldots\ldots, a_p, b_1', b_2', \ldots\ldots\ldots b_p') = \frac{1}{p}\sum_{i=1}^{p}(\Delta a_i)$ (33)

Where, $\Delta a_i = \sum\{a_i\,(1 - b_i')$ (34)

Now, it is easy to see that this index satisfies NNOM.

[11] This argument has been raised by a number of social thinkers in the case of effective literacy rate. (Basu and Lee,2007)

5. Conclusion

We can treat Deprivation Oriented Human development as capturing a true Human Development and "Efficiency Loss" due to the negative impact of b_i'. Thus our measure broadens the concept of HDI to include the "Efficiency Losses" due to the impending negativity i.e. untouched by rapidly improving the positive side.

Let us take some examples in each of the contexts of Livelihood, Education and Health to posit our logic. First, let us evaluate the aspect of Livelihood. Consider a rise in income. It raises human welfare. But according to the UNDP definition if income rises beyond a certain level the additional to human welfare due to a further rise in income is almost negligible. Hence, if the additional income is transferred to the richer section of the populace without touching the poorer people, this will not add to human welfare. Hence there is an "Efficiency Loss" of the rise in the "income component".

Next, we consider in the context of Education. Suppose there is an increase in the Gross enrollment rate. It is a positive achievement in the context of Education. However, it does not say anything about the drop-out rates in the economy. Suppose there is an improvement in the enrolment at the primary level of a country or region. Again, consider a huge percentage of student's drop-out at the primary level. An education index based on enrolment will fail to capture the losses that undergone when a person drops out. Drop-out is a cost both on the individual and society. A society has invested in education to attract enrolment. For drop-out, this investment is a loss. Drop-out itself fails to build upon the human capital they have started to accumulate. In many cases, they regress and loss whatever is gained at the primary level. Too much concentration on enrolment without considering the drop-out again causes an efficiency loss. Attainment somewhere is matched by loss otherwise.

Lastly, we consider in the context of Health. Suppose there is a rise in Health infrastructure facilities. It raises human welfare. However, imagine a situation where the population is sharply divided between “health elites” and health deprived”. The health deprived are seriously lacking in basic amenities of life. The elites, however,enjoy a very good health standard. Suppose the improvement in health infrastructure implies a further rise in their amenities- for example having a fully air-conditioned hospital ward. Such improvement in health infrastructure will not add to the well being of anybody. Hence there is an “Efficiency Loss” of the rise in the “health infrastructure”.

The standard HDI measure fails to address this problem. Our measure is more sensitive in this regard. Hence it is admittedly better than the standard measure.

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