Stackelberg type dynamic zero-sum game with leader and follower

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Abstract

We consider a Stackelberg type dynamic two-players zero-sum game. One of two players is the leader and the other player is the follower. The game is a two-stages game. In the first stage the leader determines the value of its strategic variable. In the second stage the follower determines the value of its strategic variable given the value of the leader's strategic variable. On the other hand, in the static game two players simultaneously determine the values of their strategic variable. We will show that Sion's minimax theorem (Sion(1958)) implies that at the sub-game perfect equilibrium of the Stackelberg type dynamic zero-sum game with a leader and a follower the roles of leader and follower are irrelevant to the payoffs of players, and that the Stackelberg equilbria of the dynamic game are equivalent to the equilibrium of the static game.

Keywords: zero-sum game, Stackelberg, dynamic zero-sum game

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1 Introduction

We consider a Stackelberg type dynamic two-players zero-sum game[1]. One of two players is the leader and the other player is the follower. The game is a two-stages game as follows;

1. In the first stage the leader determines the value of its strategic variable.

2. In the second stage the follower determines the value of its strategic variable given the value of the leader's strategic variable.

On the other hand, in the static game two players simultaneously determine the values of their strategic variable. We will show the following result.

> Sion's minimax theorem (Sion (1958)) implies that at the equilibrium of the Stackelberg type dynamic zero-sum game with a leader and a follower the roles of leader and follower are irrelevant to the payoffs of players, and that the Stackelberg equilibria of the dynamic game are equivalent to the equilibrium of the static game.

In an example we show that in a duopoly, in which firms maximize their relative profits, the Stackelberg equilibrium is equivalent to the Cournot equilibrium.

2 Stackelberg type dynamic zero-sum game

There is a two-players and two-stages game. Players are Player A and Player B. The strategic variables of Players A and B are, respectively, s_A and s_B. The set of strategic variables are, respectively, S_A and S_B, which are convex and compact sets of linear topological spaces. The structure of the game is as follows.

1. The first stage

   Player A determines the value of s_A.

2. The first stage

   Player B determines the value of s_B given the value of s_A.

Thus, the game is a Stackelberg type dynamic game. Player A is the leader and Player B is the follower. We investigate a sub-game perfect equilibrium of this game. We call this game Game G_A. Similarly, we call a game, in which Player B is the leader and Player A is the follower, Game G_B. They are dynamic games with a leader and a follower.

On the other hand, there is a static game in which two players simultaneously determine the values of their strategic variables.

[1]This paper is a generalization of Tanaka (2014) in which only a Stackelberg duopoly with a linear demand function is analyzed.

The payoffs of Players A and B are denoted by $u_A(s_A, s_B)$ and $u_B(s_A, s_B)$. u_A is continuous and quasi-concave in s_A and continuous and quasi-convex in s_B. u_B is continuous and quasi-concave in s_B and continuous and quasi-convex in s_A. We assume

$$u_B(s_A, s_B) = -u_A(s_A, s_B).$$

Therefore, the game is a zero-sum game.

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) for a continuous function is stated as follows.

Lemma 1 (Sion's minimax theorem)**.** *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \to \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable. Then*

$$\max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \max_{x\in X} f(x, y).$$

We follow the description of this theorem in Kindler (2005).

Applying this theorem to our situation, we have

$$\max_{s_A\in S_A} \min_{s_B\in S_B} u_A(s_A, s_B) = \min_{s_B\in S_B} \max_{s_A\in S_A} u_A(s_A, s_B). \quad (1)$$

We show the following theorem

Theorem 1. *Sion's minimax theorem (Lemma 1) implies that at the sub-game perfect equilibria of Game G_A and Game G_B the roles of leader and follower are irrelevant to the payoffs of players, and that the Stackelberg equilibria of the dynamic game are equivalent to the equilibrium of the static game.*

Proof. 1. Consider Game G_A. Since $u_A(s_A, s_B) = -u_B(s_A, s_B)$,

$$\min_{s_B\in S_B} u_A(s_A, s_B) = \min_{s_B\in S_B}(-u_B(s_A, s_B)) = -\max_{s_B\in S_B} u_B(s_A, s_B).$$

Denote

$$\arg\min_{s_B\in S_B} u_A(s_A, s_B) = \arg\max_{s_B\in S_B} u_B(s_A, s_B), \quad (2)$$

given s_A by $s_B(s_A)$, then

$$\min_{s_B\in S_B} u_A(s_A, s_B) = -\max_{s_B\in S_B} u_B(s_A, s_B) = u_A(s_A, s_B(s_A)).$$

We assume that $s_B(s_A)$ is unique given s_A. The equilibrium strategy of Player A is defined by

$$\arg\max_{s_A\in S_A} u_A(s_A, s_B(s_A)).$$

We assume that this is unique. Denote it by s_A^*, then $(s_A^*, s_B(s_A^*))$ is the Stackelberg equilibrium of Game G_A. Since $s_B(s_A)$ satisfies (2), we get

$$s_A^* = \arg\max_{s_A\in S_A} u_A(s_A, s_B(s_A)) = \arg\max_{s_A\in S_A} \min_{s_B\in S_B} u_A(s_A, s_B),$$

and

$$u_A(s_A^*, s_B(s_A^*)) = \max_{s_A \in S_A} u_A(s_A, s_B(s_A)) = \max_{s_A \in S_A} \min_{s_B \in S_B} u_A(s_A, s_B) \tag{3}$$
$$= \min_{s_B \in S_B} u_A(s_A^*, s_B) = -\max_{s_B \in S_B} u_B(s_A^*, s_B) = -u_B(s_A^*, s_B(s_A^*)).$$

Sion's minimax theorem implies the existence of s_A^* and $s_B(s_A^*)$.

Similarly, we denote the equilibrium of Game G_B by $(s_A(s_B^*), s_B^*)$, then

$$s_B^* = \arg\max_{s_B \in S_B} u_B(s_A(s_B), s_B) = \arg\max_{s_B \in S_B} \min_{s_A \in S_A} u_B(s_A, s_B),$$

and

$$u_B(s_A(s_B^*), s_B^*) = \max_{s_B \in S_B} u_B(s_A(s_B), s_B) = \max_{s_B \in S_B} \min_{s_A \in S_A} u_B(s_A, s_B) \tag{4}$$
$$= -\min_{s_B \in S_B} \max_{s_A \in S_A} u_A(s_A, s_B) = -\min_{s_B \in S_B} u_A(s_A(s_B), s_B) = -u_A(s_A(s_B^*), s_B^*)$$
$$= \min_{s_A \in S_A} u_B(s_A, s_B^*) = -\max_{s_A \in S_A} u_A(s_A, s_B^*),$$

where

$$s_A(s_B) = \arg\max_{s_A \in S_A} u_A(s_A, s_B) = \arg\min_{s_A \in S_A} u_B(s_A, s_B).$$

Sion's minimax theorem implies the existence of $s_A(s_B^*)$ and s_B^*.

2. By (1), (3) and (4), we get

$$u_A(s_A^*, s_B(s_A^*)) = u_A(s_A(s_B^*), s_B^*).$$

Similarly,

$$u_B(s_A(s_B^*), s_B^*) = u_B(s_A^*, s_B(s_A^*)).$$

Therefore, the payoffs of Players A and B when Player A is the leader, and their payoffs when Player B is the leader are equal, that is, the roles of leader and follower are irrelevant to the payoffs of the players.

3. Again (1), (3) and (4) mean

$$\min_{s_B \in S_B} u_A(s_A^*, s_B) = \max_{s_A \in S_A} u_A(s_A, s_B^*).$$

Thus,

$$u_A(s_A^*, s_B^*) \geq \min_{s_B \in S_B} u_A(s_A^*, s_B) = \max_{s_A \in S_A} u_A(s_A, s_B^*) \geq u_A(s_A^*, s_B^*).$$

Therefore, we have

$$\max_{s_A \in S_A} u_A(s_A, s_B^*) = u_A(s_A^*, s_B^*), \tag{5}$$

and

$$\min_{s_B \in S_B} u_A(s_A^*, s_B) = u_A(s_A^*, s_B^*). \tag{6}$$

Since the game is zero-sum, (6) means

$$\max_{s_B \in S_B} u_B(s_A^*, s_B) = u_B(s_A^*, s_B^*). \tag{7}$$

From (5) and (7), (s_A^*, s_B^*) is the equilibrium of the static game.

Thus,

$$s_A^* = s_A(s_B^*),$$

and

$$s_B^* = s_B(s_A^*).$$

□

An example: relative profit maximization in a Stackelberg duopoly Consider a Stackelberg duopoly with a homogeneous good. There are two firms, Firm A and Firm B. The outputs of Firms A and B are x_A and x_B, The price of the good is denoted by p. The inverse demand function is

$$p = a - x_A - x_B,\ a > 0.$$

The cost functions of Firms A and B are $c_A x_A$ and $c_B x_B$, where c_A and c_B are positive constants. The relative profit of Firm A is

$$\varphi_A = p x_A - c_A x_A - (p x_B - c_B x_B).$$

The relative profit of Firm B is

$$\varphi_B = p x_B - c_B x_B - (p x_A - c_A x_A).$$

The firms maximize their relative profits. We see

$$\varphi_A + \varphi_B = 0.$$

Thus, the game is a zero-sum game. When Firm A (or B) is the leader, in the first stage of the game Firm A (or B) determines x_A (or x_B) , and in the second stage Firm B (or A) determines x_B (or x_A) given x_A (or x_B) . We can show that at the Stackelberg equilibrium when Firm A is the leader, at the Stackelberg equilibrium when Firm B is the leader and at the Cournot equilibrium the outputs of Firms A and B are

$$x_A = \frac{a - c_A}{2},\ x_B = \frac{a - c_B}{2}.$$

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