Classical labour values – properties of economic reproduction

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Abstract

We attempt to clarify the meaning of labour value, a concept that originated in classical political economy. Using a modern formalism, we show that labour values are understood as a field property, or equivalently a characteristic accounting property, of economic reproduction. The applicability of the concept is discussed and its relation to productivity, employment, surplus labour and unproductive activities are demonstrated.

1 Introduction

What is the basis of economic value? This question repeatedly crops up in practical political discourse, cf. [Bacon and Eltis, 1978, Mazzucato, 2018]. In the classical approach to political economy, as well as in the early labour movement, the answer was clear: social labour forms the basis of economic value, cf. [Smith, 1776, Ricardo, 1817, Marx, 1867]. That is, the real cost of commodities is measured in units of social labour required to reproduce them. From this starting point, it was possible to study the organization of production, productivity of an economy, extraction of an economic surplus, distribution of output between classes, etc.

It has often been assumed that this notion of 'labour value' is only applicable to market-based economies. However, if we view labour value as the social cost of economic reproduction, its application is more general. As we aim to clarify in this paper, labour values are a property of any economic system that can redeploy labour across a range of production processes. This includes capitalist market economies, planned economies and mixed state-regulated economies. We show that economic reproduction gives rise to a field of labour requirements as well as a characteristic accounting structure, using the formalism of [Schwartz, 1961, Pasinetti, 1979]. Both are material properties that yield equivalent definitions of labour value. We proceed to show that this generalized conception addresses central questions that concerned classical political economy and the early labour movement.

2 Labour value as a field property

> Not only the labour applied immediately to commodities affect their value, but the *labour also which is bestowed* on the complements, tools, and buildings, with which much labour is assisted.
> [Ricardo, 1817, ch. 1, emph. added]

We consider an interconnected economic system that is capable of reproducing itself. At a given point in time, it produces n distinct types of outputs, each of which is labeled as $1, 2, \ldots, n$. The average technical conditions of production are described by an $n \times n$ input-output coefficient matrix $\mathbf{A}$ and $1 \times n$ vector of direct labour coefficients $\boldsymbol{\ell}$. An example of $\mathbf{A}$ and $\boldsymbol{\ell}$ is illustrated below. Note that both quantities can be estimated in real economies using data from national accounts.[1]

Example 1 (Simple economy). Consider an economy with $n = 3$ outputs: corn, iron and sugar indexed as $\{1, 2, 3\}$. The average technical requirements of production are described by:[2]

$$\mathbf{A} = \begin{bmatrix} 0.02 & 0 & 0.01 \\ 0.2 & 0.5 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\ell} = \begin{bmatrix} 0.7 & 0.6 & 0.3 \end{bmatrix}$$

Each column represents the input requirements for corn, iron and sugar, respectively. Take corn as an example: to reproduce one unit of corn, the economy uses up an average of 0.02 units of corn, 0.2 units of iron, and 0 units of sugar in the process. It also requires 0.7 units of labour. The inputs, iron and corn, in turn require their own inputs, and so on. This leads to a series of intermediate, coexisting input requirements necessary for the net output of corn. Each set of inputs necessitate a certain amount of labour. Let the vector $\mathbf{e}_1 = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$ represent one unit of corn. Then the labour requirements of its inputs are illustrated in Figure 1.

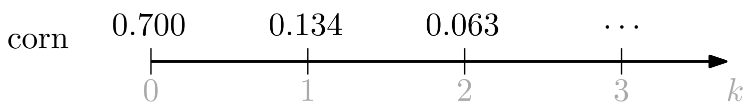


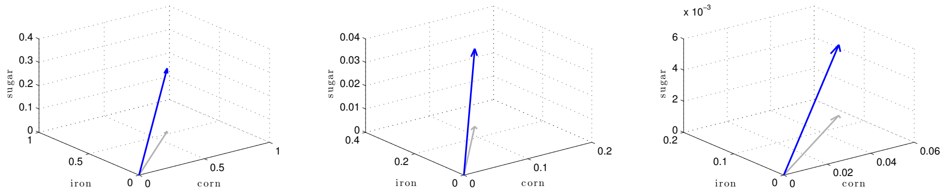
Figure 1: Reproduction of a unit of corn necessitates sets of intermediate inputs indexed by k. By convention, let $k = 0$ denote the immediate input of labour, i.e., $\boldsymbol{\ell}\mathbf{e}_1 = 0.700$ units. Then $k = 1$ corresponds to the necessary inputs, which require $\boldsymbol{\ell}\mathbf{A}\mathbf{e}_1 = 0.134$ units of direct labour. Similarly, $k = 2$ corresponds to their inputs $\boldsymbol{\ell}\mathbf{A}^2\mathbf{e}_1 = 0.063$, and so on. In a viable economy, the series progresses with ever-decreasing amounts of labour.

The matrix $\mathbf{A}$ transforms outputs into their corresponding input requirements. For instance, let one unit of output-type i be represented by an $n \times 1$ vector $\mathbf{e}_i$, which contains a 1 at element i and all remaining elements are 0. Then $\mathbf{A}\mathbf{e}_i$ represents the input requirements for one unit of output i, which corresponds to $\boldsymbol{\ell}\mathbf{A}\mathbf{e}_i$ units of direct labour. Multiplying by $\mathbf{A}$ again further transforms these inputs into their input requirements. The operation can be repeated indefinitely to specify all necessary sets of inputs, where the kth input set requires $\boldsymbol{\ell}\mathbf{A}^k\mathbf{e}_i$ units of direct labour. The requirements for each output type is then summarized by the $1 \times n$ vector $\boldsymbol{\lambda}(k) = \boldsymbol{\ell}\mathbf{A}^k$ for the kth input set. Thus the economic system gives rise to a vector field across all sets of inputs,

$$\boldsymbol{\lambda}(0),\ \boldsymbol{\lambda}(1),\ \boldsymbol{\lambda}(2),\ \ldots,$$

that represents the coexisting human requirements of economic reproduction at a given point in time, cf. [Mirowski, 1989] and [Wright, 2015, ch. 6].

Example 2 (Vector field in simple economy)*.* We illustrate the labour requirements for three input sets, $\boldsymbol{\lambda}(0)$, $\boldsymbol{\lambda}(1)$ and $\boldsymbol{\lambda}(2)$, in the simple economy in Figure 2.



(a) $\boldsymbol{\lambda}(0)$ (b) $\boldsymbol{\lambda}(1)$ (c) $\boldsymbol{\lambda}(2)$

Figure 2: Vector field of coexisting labour requirements for each output: corn, iron, and sugar. $\boldsymbol{\lambda}(0)$ denotes their direct labour requirements, $\boldsymbol{\lambda}(1)$ are the labour requirements of their inputs, and $\boldsymbol{\lambda}(2)$ are the requirements for the subsequent inputs. Note that the magnitudes of these requirements decrease for each set of inputs $k = 0, 1, 2, \ldots$.

> We see then that that which determines the magnitude of the value of any article is the amount of labour socially necessary, or the labour time socially *necessary for its production*.
> [Marx, 1867, ch. 1, emph. added]

Definition 1 (Field property)**.** Unit labour values are defined by integrating the field of labour requirements across all sets of inputs:

$$\boxed{\mathbf{v} \;=\; \boldsymbol{\lambda}(0) \;+\; \boldsymbol{\lambda}(1) \;+\; \boldsymbol{\lambda}(2) \;+\; \cdots\;,} \tag{1}$$

where $\boldsymbol{\lambda}(k)$ is the vector of labour requirements of the kth set of intermediary inputs. The sum in (1) converges for an economy capable of reproducing itself.

Labour value is therefore not a intrinsic property of products, rather it is a field property that reproducible goods and services acquire from the economic system.[3]

Example 3 (Classical labour values)*.* In the simple economy considered above, the unit labour values equal

$$\begin{aligned}\mathbf{v} &= \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \\ &= \begin{bmatrix} 0.959 & 1.200 & 0.344 \end{bmatrix},\end{aligned}$$

which corresponds to corn, iron and sugar, respectively. Thus a unit of iron requires in total nearly four times as much labour than a unit of sugar.

> The values of commodities are directly as the times of labour employed in their production, and are *inversely* as the productive powers of the labour employed.
> [Marx, 1865, sec. IV, emph. added]

Technical and organizational changes in the economy alter the average production requirements, and therefore the vector field.[4] Let $\dot{\mathbf{v}} = \frac{d}{dt}\mathbf{v}$ denote the change in labour values per unit of time. This quantity has profound effects on both production and employment.

Result 1 (Productivity)**.** *Suppose the labour value of output i is reduced at the relative rate $\rho_i \equiv -\dot{v}_i/v_i$. Then, for a fixed level of employment, the net output of i can grow at the relative rate ρ_i. Thus labour values are (inverse) measures of total productivity in the economy.*

Example 4 (Labour value and total productivity growth)*.* Suppose the simple economy produces a net output of 100 units of corn. The labour value of corn can be lowered by decreasing the direct labour input and/or by decreasing the amount of coexisting inputs required. Thus technical improvements in the production of iron affect the labour value of corn. Suppose its unit value v_1 decrease by the rate $\rho_1 = 5\%$ per annum. Then the net output of corn can increase exponentially as shown in Figure 3.

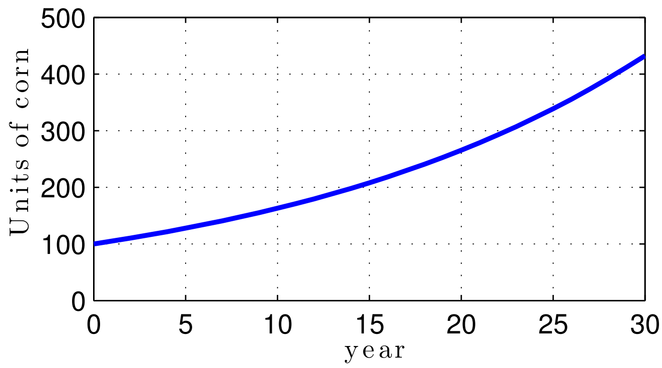


Figure 3: Growth of capacity to produce corn as labour value of corn decreases by $\rho_1 = 5\%$ per annum. Starting with an annual output of 100 units of corn, the amount increases fourfold within 30 years.

Result 2 (Employment)**.** *Suppose the final demand for output i grows at the relative growth rate d_i. Then the total demand for labour changes by the rate $d_i - \rho_i$. Thus labour values are employment multipliers in the economy.*[5]

We see that economies with institutions that progressively lower the labour values of the outputs are capable of increasing material living standards and/or leisure time exponentially. At the same time, economies that lack coordination between technical change and changes in consumption and investment demands can give rise to both persistent unemployment and chronic labour shortages. When $d_i < \rho_i$, the total demand for labour declines exponentially and must be compensated by increased demand among other outputs to prevent the rise of unemployment.

3 Labour value as an eigenstructure property

> The real price of everything, *what everything really costs* to the man who wants to acquire it, is the toil and trouble of acquiring it.
> [Smith, 1776, book 1, ch. 5, emph. added]

The net product is the collection of goods and services available for consumption and investments, after deducting the intermediate inputs consumed in the production process. The net product is represented by an $n \times 1$ vector $\mathbf{n}$. Next, we consider the costs of replacing the net outputs of the economy and let a $1 \times n$ vector $\mathbf{v}$ represent their unit replacements costs that will be defined below.[6] By definition, the total cost to replace the net product is $\mathbf{vn}$.

One part of $\mathbf{n}$ is consumed by the workforce and its dependents, the remainder is a surplus product consisting of investment goods, luxuries, and so on. Thus the total replacement cost can be divided in two parts

$$\mathbf{vn} = W + S, \tag{2}$$

where W and S correspond to the costs to replace the consumption bundle of the workforce and the surplus outputs, respectively. The surplus share of the replacement costs is

$$\boxed{\sigma \equiv \frac{S}{\mathbf{vn}},}$$

and is bounded between 0% and 100%.[7]

Next, we specify the cost structure of W. Let $\boldsymbol{\omega}$ denote the $n \times 1$ real wage rate vector, where the ith element is average amount of output i consumed per unit of employed labour.[8] Then the consumption bundle of the workforce is derived as follows. The net product $\mathbf{n}$ requires a total employment of $\boldsymbol{\ell}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{n}$ units of labour.[9] Thus $\boldsymbol{\omega}(\boldsymbol{\ell}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{n})$ corresponds to the consumtion bundle of the workforce and it follows that its replacement cost is

$$W = \mathbf{vLn},$$

where $\mathbf{L} = \boldsymbol{\omega}\boldsymbol{\ell}(\mathbf{I} - \mathbf{A})^{-1}$ is a matrix of integrated wage goods requirements. That is, the ijth element of $\mathbf{L}$ corresponds to the total amount of wage goods i directly and indirectly required in the production of a unit of output j.

Example 5 (Real-wage requirement matrix). Suppose the real wage rate in the simple economy consists only of corn:

$$\boldsymbol{\omega} = \begin{bmatrix} 0.6 \\ 0 \\ 0 \end{bmatrix} \qquad \text{then} \qquad \mathbf{L} = \begin{bmatrix} 0.58 & 0.72 & 0.21 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus to replace a unit of corn, sugar or steel requires a total of 0.58, 0.72 or 0.21 units of corn for the workforce.

The total replacement cost of the surplus product can always be written as $S = \sigma\mathbf{vn}$. Using this expression along with the expression for W, the identity (2) equals

$$\mathbf{vn} = \mathbf{vLn} + \sigma\mathbf{vn} \tag{3}$$

which holds irrespective of the composition of the net product $\mathbf{n}$. Then unit replacement costs $\mathbf{v}$ naturally arise from the accounting identity as an eigenstructure property of economic reproduction.[10]

Definition 2 (Eigenstructure property)**.** Unit replacement costs $\mathbf{v}$ are defined by the accounting identity

$$\boxed{\mathbf{v} = \mathbf{vL} + \sigma\mathbf{v},} \tag{4}$$

i.e., $\mathbf{v}$ corresponds to the nonnegative left eigenvector of $\mathbf{L}$. Thus the replacement cost of a unit is decomposed into the cost required to replace all wage goods for labour employed in the economy plus the costs to meet the surplus.[11]

> The value of a commodity [...] depends on the relative quantity of labour which is necessary for its production, and *not* on the greater or less compensation which is paid for that labour.
> [Ricardo, 1817, ch. 1, emph. added]

Result 3 (Equivalence)**.** *Labour values* (1) *and replacement costs* (4) *are equivalent. The replacement costs therefore correspond to units of labour required to replace the net product at current technical conditions of production. The costs are invariant to changes in real wages or the distribution of the net product.*[12]

Result 4 (Rate of surplus labour)**.** *The surplus share* σ *equals the fraction of work in the economy to replace the surplus product, aka. 'rate of surplus labour'. The rate is given directly by the structure of* $\mathbf{L}$*, which has a unique nonzero eigenvalue* $1-\sigma$*.*[13]

Example 6 (Rate of surplus labour). Consider the real-wage requirement matrix

$$\mathbf{L} = \begin{bmatrix} 0.58 & 0.72 & 0.21 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in the simple economy. By computing the eigenvalue of $\mathbf{L}$, we obtain the following rate of surplus labour $\sigma = 42\%$. Thus 42% of the work in the economy is materialized in the form of surplus outputs.

4 Applicability of concept

Using (1), or equivalently (4), there exists a set of unit labour values $\mathbf{v}$ that is associated one to one with the set of products. These labour values represent the reproduction cost, in terms of labour, of each product. From these definitions, it appears that $\mathbf{v}$ is an economic property that is invariant to the social institutions under which production takes place. We present here a few remarks on the applicability of the concept.

> Every child knows [...] that the masses of products corresponding to the different needs required different and *quantitatively determined masses of the total labour of society*. That this necessity of the distribution of social labor in definite proportions *cannot* possibly be done away with by a particular form of social production but can only change the mode of its appearance, is self-evident. No natural laws can be done away with. What can change in historically different circumstances is *only the form* in which these laws assert themselves.
> [Marx, 1868, emph. added]

Using our notation, $\mathbf{n}$ represents the 'masses' of different products and the corresponding elements of $\mathbf{v}$ quantitatively determine the masses of total labour of society required. The definition of $\mathbf{v}$ assumes i) a viable economic system that is ii) capable of training and redeploying its finite amount of available labour time across different producition processes. The former assumption corresponds to the maximum eigenvalue of $\mathbf{A}$ being less than unity, while the latter assumption enables the existence of a vector $\boldsymbol{\ell}$ that represents labour inputs across different production processes in commensurable units of time.

It is in the continual process of training and redeployment of manpower across production that an economic system renders concrete work tasks as an expenditure of a commensurable abstract labour resource, quantified in units of time.[14] This would include a range of self-reproducing economic systems. Did, for instance, value and the disposition of labour time matter to the slave lords of antiquity? According to Cato, it appears that they did:

> When [the master of a farmstead] has learned the condition of the farm, what work has been accomplished and what remains to be done, let him call in his overseer the next day and inquire of him what part of the work has been completed, what has been left undone; whether what has been finished was done betimes, and whether it is possible to complete the rest; and what was the yield of wine, grain, and all other products. Having gone into this, he should make a calculation of the labourers and the time consumed. [Hooper and Ash, 1935, p.9].

In the slave plantations described above by Cato, the disposition of labour is self evident and 'natural', it is not obscured by monetary indirection. But it is still labour in the abstract, albeit of a given group of slaves, being distributed between concrete tasks: meadow clearing, faggot bundling, road-work, etc.

The necessity to take into account the usage of labour time, whether that be the time of slaves, wage labourers, citizens of a socialist commonwealth, is a natural necessity that could not be abolished, only change its historical form. By contrast, in economies with institutions that prevent the redeployment of workers across tasks, e.g. rigid forms of caste hierarchies, there can be no general labour resource quantifiable in commensurable units.

4.1 Capitalist market economies

In a capitalist economy, the necessity to distribute labour appears as simply expenditures of money on wages to top-level managers in decentralized firms. So the wage budget allocated to different branches of a firm provides an indirect representation of the needed allocation of labour.

As one descends the management hierarchy, the simple monetary view of things becomes insufficient. The subsidiary managers have to allocate specific people to specific tasks just as the slave overseer had to. By contrast, as one moves further away from the production process, the representation of labour becomes increasingly obscure and monetary. Indeed, when the products of the economy are allocated between agents as commodities, the monetary calculations are based on market prices which randomly fluctuate from one transaction to the other. The relation between market prices of commodities and their labour values is necessarily a statistical one, see [Farjoun and Machover, 1983].[15] To an individual, money appears to be freely disposable between different products, but in reality such choices are limited by macroeconomic constraints set by $\mathbf{v}$, which represent real costs irrespective of random market prices.

Nevertheless, firms in a capitalist market economy do solve labour allocation problems via decentralized monetary calculations. The feasibility of this monetary accounting mechanism rests on the fact that human labour is flexible and can be redirected, either within the firm or on the employment market, between activities. In capitalism, the redeployment of labour between concrete tasks across the production system occurs through the transfer, hiring and firing of workers within and across decentralized firms. This allows single scalar measure like money to function as a system of social accounting.

4.2 Planned economies

Planned economies too have to grapple with the finite nature of their labour supplies, and the need to expend effort for any worthwhile effect. This implies that they too will have to have social forms in which this necessity will be expressed. The necessity for the labour force to be allocated in a manner jointly determined by the matrix **A** and the net output **n** took, in the planned Soviet-socialist economies, the form of the directive plan. This plan involved drawing up material and labour balances for the overall economy. We know that Soviet-socialist economies continued to use monetary calculations, which, to a greater or lesser degree of adequacy, allowed indirect calculations to be done on social labour requirements. While monetary calculation and allocation in capitalist market economies redeploys a certain amount of labour via the recreation of a pool of unemployed, the Soviet-socialist economies did not develop the kind of labour time accounting, planning and regulation that would be required to carry out reallocations of labour within a fully employed workforce.

In capitalist war economies, production, by and large, still took place in privately owned firms. There were state munitions factories like the Royal Arsenal or the Oak Ridge and Los Alamos atomic weapons plants, but these were exceptions. The state directed labour, by conscripting it into the army, and by conscripting women and men in key trades into essential war work. It also rationed the supply of key materials, fuels, and foodstuffs. Firms were subject to negotiated direction to produce only munitions, or restricted ranges of 'utility' products [Edgerton, 2011]. Money was still used to pay for the munitions delivered, and to pay workers. Buying food required both money and ration cards. Money alone was not enough either for the consumer or for firms. In peace, money as the universal ration constrains everything. Shortage of it constrains the working class consumers and uncertainty about future revenue constrains even those firms who have good cash reserves. Because the constraint on production comes via market exchange in price units rather than directly in units of products, peace-time capitalist market economies typically operate somewhat below full capacity. In war, national survival dictates that every available resource be put to use. The economy operates at the limits of its physical resources in materials, people and machines.

The state as primary purchaser has to look not just at the projected costs of ships, aircraft, etc., that it is ordering, but at all sorts of material constraints. In deciding what type of destroyers to order, the Navy first took into account the requirements of their admirals for the ships to carry guns of different types, torpedoes and anti-submarine weapons: all technical not financial issues. They then had to take into account the number of slip yards in the country able to build ships of different sizes, the delivery schedules for different kinds of projected weapons and ships machinery, the availability of metals and alloys of different weights and strengths. They then had to ask whether the demands on skilled labour would require the cancellation or postponement of other orders.[16] Money was a relatively secondary concern. The availability of state credit, that, at least within the domestic economy was effectively unlimited, removed money as a constraining resource [Keynes, 2010]. The same point about money applied *a fortiori* to the socialist economies. Money was never a constraint for them. Labour, plus available plant and equipment, however, were.

4.3 Fully automated economies?

Choosing to evaluate the replacement cost of products in terms of labour time reflects not merely a concern for human beings but also the fact that humans possess a capacity to be allocated across a wide range of concrete tasks. This general capacity is then realized in economies that redeploy labour and gives rise to the abstract representation $\boldsymbol{\ell}$ of direct labour requirements. However, it may be objected that some future society may have at its disposal a race of robots, so skilled and dexterous, so intelligent and adaptable, that these beings may come to supplant us in our toils. Would that not invalidate our system of calculation of values?

Not at all, it would merely substitute the time of these general purpose robots for our own time. The equations of value would still apply, but with this simple proviso, that the labour input time is to be understood as redeployable general robot time. The real-wage requirement matrix $\mathbf{L}$ would at that point be translated into the 'robot construction and maintainence' requirement matrix, for these robots too will need energy, will require repair and will absorb the effort of other robots in their initial construction.

Humans, in this hypothetical society, would be in the position of slave-owning ancients: idlers depending on the surplus labour of others.

5 Extraction of surplus labour

> The specific economic form, in which unpaid surplus-labour is pumped out of direct producers, determines the relationship of rulers and ruled, as it *grows directly out of production itself* and, in turn, reacts upon it as a determining element.
> [Marx, 1894, ch. 47, emph. added]

The surplus product is by definition the residual of the net product after deducting the outputs consumed by the workforce, i.e., $\mathbf{s} = \mathbf{n} - \mathbf{L}\mathbf{n}$. The relations between the products of the economy are in general not symmetric: some outputs may enter directly or indirectly as inputs to all goods and services, while other outputs may not. This implies certain consequences in the production of $\mathbf{s}$ which we deduce below.

Definition 3 (Basic and nonbasic outputs)**.** Basic outputs are directly and indirectly required in the production of all outputs, while the nonbasic outputs are not. More formally, let $\widetilde{\mathbf{A}} = \mathbf{A} + \boldsymbol{\omega}\boldsymbol{\ell}$ denote the augmented input-output matrix, where the order of the outputs is arranged to form a upper-block triangular structure:

$$\widetilde{\mathbf{A}} = \begin{bmatrix} \widetilde{\mathbf{A}}_b & \widetilde{\mathbf{A}}_{bu} \\ \mathbf{0} & \widetilde{\mathbf{A}}_u \end{bmatrix} \tag{5}$$

The upper-left block corresponds to outputs indexed $i = 1, \dots, b$, which we denote as 'basic'. The remaining outputs are 'nonbasic'.[17]

Example 7 (Basics and nonbasics). For the simple economy we have

$$\begin{aligned}\widetilde{\mathbf{A}} &= \begin{bmatrix} 0.02 & 0 & 0.01 \\ 0.2 & 0.5 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} + \begin{bmatrix} 0.6 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 & 0.3 \end{bmatrix} \\ &= \begin{bmatrix} 0.44 & 0.36 & 0.19 \\ 0.2 & 0.5 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}\end{aligned} \tag{6}$$

which is upper block-triangular as in (5). Therefore corn and iron ($i = 1, 2$) are basic outputs, while sugar is a nonbasic output.

The production of basic outputs forms a self-reproducing sector of the economy which is critical in determining the rate of surplus labour σ.

Result 5 (Determinants of the rate)**.** *The rate of surplus labour is determined by productivity in and the workers' consumption from the basic sector of the economy. That is,*

$$\sigma = 1 - \mathbf{v}_b \boldsymbol{\omega}_b,$$

where $\mathbf{v}_b$ and $\boldsymbol{\omega}_b$ are the labour values of basic outputs and corresponding real wage rates, respectively. Luxuries and other nonbasic outputs do not affect σ.[18]

In other words, the rate σ increases when the real wage rate decreases and/or the unit labour values in the basic sector decrease. The real wage rate $\boldsymbol{\omega}_b$ can be decreased by extending the number of working hours without compensation, while the lowering of labour values $\mathbf{v}_b$ require technical changes in the basic sector.[19]

Result 6 (Dependence on surplus labour)**.** *Production of nonbasic outputs is predicated on the extraction of surplus labour. More formally, if the rate of surplus labour is $\sigma = 0$ then the production of nonbasics outputs equals* $\mathbf{0}$.[20]

> A man *grows rich* by employing a multitude of manufacturers: he *grows poor* by maintaining a multitude of menial servants.
> [Smith, 1776, book II, ch. III, emph. added]

This remark may merely seem to apply to an individual employer but in fact generalizes into a macroeconomic property: Production of luxuries and other nonbasic outputs drains the surplus in the basic sector. In modern capitalist economies, this includes the arms industry and finance sector.

Result 7 (Drain on the basic sector)**.** *The surplus of basic outputs is impeded by the production of nonbasic outputs. More formally, let $\mathbf{n}_b$ and $\mathbf{n}_u$ denote the net products from the basic and nonbasic sectors, respectively. If the labour required to sustain the nonbasic sector is redeployed to expand the basic sector, the surplus product in the latter can be increased by*

$$\frac{\mathbf{v}_u \mathbf{n}_u}{\mathbf{v}_b \mathbf{n}_b} (\mathbf{I} - \mathbf{L}_b)\mathbf{n}_b + \mathbf{L}_{bu}\mathbf{n}_u \;\geq\; \mathbf{0}, \tag{7}$$

where $\mathbf{L}_b$ and $\mathbf{L}_{bu}$ are the top blocks in $\mathbf{L} = \begin{bmatrix} \mathbf{L}_b & \mathbf{L}_{bu} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\mathbf{v}_u$ are the labour values of the nonbasic outputs.[21] *Thus* (7) *represents a drain on the surplus capacity of the basic sector by the production of nonbasic outputs.*

Example 8 (Redeployment to basic sector). Suppose the net product $\mathbf{n}$ of the simple economy consists of 100 units of corn, 10 units of iron and 50 units of sugar. Then the surplus product equals

$$\mathbf{s} = \mathbf{n} - \mathbf{Ln} = \begin{bmatrix} 100 \\ 10 \\ 50 \end{bmatrix} - \begin{bmatrix} 0.58 & 0.72 & 0.21 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 10 \\ 50 \end{bmatrix} = \begin{bmatrix} 24.93 \\ 10.00 \\ 50.00 \end{bmatrix}.$$

Suppose the total labour devoted to sustain the nonbasic sugar is redeployed to expand net output of basic corn and iron uniformly. Then using (7), the new surplus product is

$$\mathbf{s}' = \begin{bmatrix} 40.87 \\ 11.59 \\ 0 \end{bmatrix}.$$

That is an increase of surplus corn and iron by +64% and +16%, respectively.

Notes

[1]See product-product input-output matrices from national statistics agencies.

[2]The example is adapted from [Wright, 2017] but is designed to resemble the structure of reproduction schemes considered in [Marx, 1885] where Departments IIa, I and IIb correspond to ‘corn’, ‘iron’ and ‘sugar’, resepectively.

[3]Thus classical labour values can be understood as a field theory of value rather than a substance theory of value, contrary to the characterization in [Mirowski, 1989]. Consider a set of products described by the $n \times 1$ vector $\mathbf{s}$. The total labour value of this set is a scalar obtained by an integration over the field: $(\boldsymbol{\lambda}(0) + \boldsymbol{\lambda}(1) + \boldsymbol{\lambda}(2) + \cdots)\mathbf{s} = \mathbf{vs}$.

[4]“The value of a commodity would therefore remain constant, if the labour time required for its production also remained constant. But the latter changes with every variation in the productiveness of labour. This productiveness is determined by various circumstances, amongst others, by the average amount of skill of the workmen, the state of science, and the degree of its practical application, the social organisation of production, the extent and capabilities of the means of production, and by physical conditions.” [Marx, 1867, ch. 1]

[5]The total employment requirement for producing n_i units of output i is $L_i = v_i n_i$. Therefore the relative change of employment is given by the identity $\dot{L}_i/L_i = -\rho_i + d_i$, where $d_i = \dot{n}_i/n_i$. If the actual employment is fixed, then the left-hand side is 0 and correspondingly $d_i = \rho_i$.

[6]The costs could be measured in some arbitrary monetary units.

[7]More specifically, $\sigma \in [0, 1)$. Note that Marx’s unbounded ‘rate of surplus value’ $\frac{S}{W} = \frac{\sigma}{1-\sigma} \in [0, \infty)$ is a mere transformation of σ.

[8]The real-wage rate vector $\boldsymbol{\omega}$ can be estimated from national accounts data using the inputs to the household sector and the total wage bill.

[9]Let $\mathbf{q}$ denote the $n \times 1$ vector of gross outputs. Then the inputs used up in production equals $\mathbf{Aq}$. By definition, the net product equals $\mathbf{n} = \mathbf{q} - \mathbf{Aq}$, so that $\mathbf{q} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{n}$.

[10]From this point of view, classical labour values also account for the required surplus labour in the economy. See [Wright, 2015] for a different point of view. The cost structure of the surplus is accounted differently in classical labour values than the ‘super-integrated’ labour values defined in [Wright, 2015].

[11]An equivalent definition is $\mathbf{v} = (1 + \rho)\mathbf{vL}$, where $\rho = \frac{\sigma}{1-\sigma} = \frac{S}{W}$ equals Marx’s unbounded ‘rate of surplus value’. Contrary to the misleading presentation in [Samuelson, 1970], neither definition depends on a defining a ‘subsistence wage’.

[12]Note that neither definition (1) nor (4) are interpreted via a simultaneous valuation of ‘inputs and outputs’ in production.

[13]Since an eigenvector of $\mathbf{L}$ satisfies $\lambda\mathbf{v} = \mathbf{vL}$, it follows from (4) that $\lambda = 1 - \sigma$. The eigenvalue λ is obtained as the solution to $\det(\lambda\mathbf{I} - \mathbf{L}) = 0$. Using the matrix determinant

lemma, this is equivalent to $(1 - \boldsymbol{\ell}(\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\omega}\lambda^{-1})\lambda^n = 0$ or $\lambda = \boldsymbol{\ell}(\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\omega}$. Next, postmultiply (4) by $(\mathbf{I} - \mathbf{A})$ and rearrange to obtain

$$\mathbf{v}\Big[(\mathbf{I} - \mathbf{A}) - \frac{1}{1-\sigma}\boldsymbol{\omega}\boldsymbol{\ell}\Big] = \mathbf{0}.$$

Inserting $1 - \sigma = \boldsymbol{\ell}(\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\omega}$ in the denominator of this expression, yields a nontrivial solution $\mathbf{v} = \boldsymbol{\ell}(\mathbf{I} - \mathbf{A})^{-1}$. Using the series expansion $(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k$, we have $\mathbf{v} = \sum_{k=0}^{\infty} \boldsymbol{\lambda}(k)$ which equals (1).

[14]Classical labour values therefore differ radically from the concept of 'value' developed by the so-called value-form school. In the latter conception, there can be no abstract labour measured in hours nor can it be measured before the act of market exchange [Heinrich, 2012, pp.50, 55, 65].

[15]Since market prices are randomly fluctuating quantities they do not form a 'dual system' with repect to labour values.

[16][Friedman and Baker, 2009] gives several examples of scheduling constraints on new gun mountings, and slip sizes affecting UK destroyer construction plans in WWII. [Friedman, 2015] gives the example of construction of the Admiral class capital ships being postponed due to there not being enough shipbuilding labour to both build them and destroyers in 1917. For large scale shipbuilding programmes, even in peace, similar forward planning of physical constraints has to be done by the state [Arena et al., 2005].

[17]An equivalent definition, which does not require rearranging the sectors, is that output i is basic if $\mathbf{e}_i^\top(\widetilde{\mathbf{A}}^1 + \widetilde{\mathbf{A}}^2 + \cdots + \widetilde{\mathbf{A}}^n) > \mathbf{0}$. The concept is a slight generalization of Sraffas 'basic goods' and includes the production of the workers' consumption bundle. Note that the outputs that are basic and nonbasic may change over time as the structure of the economy changes, see [Cockshott and Zachariah, 2006].

[18]Using the inverse of the upper block triangular matrix $(\mathbf{I} - \mathbf{A})$, we have that

$$\begin{aligned}
\mathbf{v} &= \boldsymbol{\ell}(\mathbf{I} - \mathbf{A})^{-1} \\
&= \begin{bmatrix} \boldsymbol{\ell}_b & \boldsymbol{\ell}_u \end{bmatrix} \begin{bmatrix} (\mathbf{I} - \mathbf{A}_b)^{-1} & (\mathbf{I} - \mathbf{A}_b)^{-1}\mathbf{A}_{bu}(\mathbf{I} - \mathbf{A}_u)^{-1} \\ \mathbf{0} & (\mathbf{I} - \mathbf{A}_u)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{\ell}_b(\mathbf{I} - \mathbf{A}_b)^{-1} & \boldsymbol{\ell}_b(\mathbf{I} - \mathbf{A}_b)^{-1}\mathbf{A}_{bu}(\mathbf{I} - \mathbf{A}_u)^{-1} + \boldsymbol{\ell}_u(\mathbf{I} - \mathbf{A}_u)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{v}_b & \mathbf{v}_u \end{bmatrix}.
\end{aligned} \tag{8}$$

Then it follows that $\sigma = 1 - \boldsymbol{\ell}(\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\omega} = 1 - \mathbf{v}_b\boldsymbol{\omega}_b$. It is seen that the theory of surplus value is only completed in the analysis of input-output relations [Marx, 1885, pt. III] rather than the presentation in [Marx, 1867].

[19]This corresponds to distinction between 'absolute' and 'relative' surplus value described in [Marx, 1867]. Note that the nonbasic sector therefore cannot produce 'relative' surplus value, see [Cockshott and Zachariah, 2006].

[20]Using (4), we have $\mathbf{vs} = \mathbf{vn} - \mathbf{vLn} = \sigma\mathbf{vn} = 0$, when $\sigma = 0$. Since $\mathbf{v} > \mathbf{0}$ and $\mathbf{s} \geq \mathbf{0}$ it follows that $\mathbf{s} = \mathbf{0}$. For the real-wage requirement matrix, we have that

$$\begin{aligned}
\mathbf{L} &= \boldsymbol{\omega}\boldsymbol{\ell}(\mathbf{I} - \mathbf{A})^{-1} \\
&= \begin{bmatrix} \boldsymbol{\omega}_b \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\ell}_b(\mathbf{I} - \mathbf{A}_b)^{-1} & \boldsymbol{\ell}_b(\mathbf{I} - \mathbf{A}_b)^{-1}\mathbf{A}_{bu}(\mathbf{I} - \mathbf{A}_u)^{-1} + \boldsymbol{\ell}_u(\mathbf{I} - \mathbf{A}_u)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{L}_b & \mathbf{L}_{bu} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}
\end{aligned} \tag{9}$$

and it follows from the definition $\mathbf{s} = \mathbf{n} - \mathbf{Ln}$ that the net production of nonbasics are surplus products, that is, $\mathbf{n}_u = \mathbf{s}_u$. Using $\mathbf{q} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{n}$, we have that gross production in the nonbasic sector is $\mathbf{q}_u = (\mathbf{I} - \mathbf{A}_u)^{-1}\mathbf{n}_u = (\mathbf{I} - \mathbf{A}_u)^{-1}\mathbf{s}_u = \mathbf{0}$.

[21]Consider redeploying the resources devoted to support the nonbasic sector to the basic sector alone. Let the net product before and after the change be $\mathbf{n}$ and $\mathbf{n}'$, respectively, where total employment remains the same, i.e. $\mathbf{vn}' = \mathbf{vn}$. Suppose the redeployment is such that the net product in the basic sector is increased uniformly by a factor α, i.e.,

$$\mathbf{n}' = \begin{bmatrix} (1+\alpha)\mathbf{n}_b \\ \mathbf{0} \end{bmatrix}.$$

Then it follows that the factor is $\alpha = \frac{\mathbf{v}_u\mathbf{n}_u}{\mathbf{v}_b\mathbf{n}_b}$. The resulting change in the surplus product of the economy is

$$\begin{aligned}\boldsymbol{\Delta} &= \mathbf{s}' - \mathbf{s} \\ &= (\mathbf{I} - \mathbf{L})\mathbf{n}' - (\mathbf{I} - \mathbf{L})\mathbf{n} \\ &= (\mathbf{I} - \mathbf{L}) \begin{bmatrix} \alpha \mathbf{n}_b \\ -\mathbf{n}_u \end{bmatrix} \\ &= \begin{bmatrix} \alpha(\mathbf{I} - \mathbf{L}_b)\mathbf{n}_b + \mathbf{L}_{bu}\mathbf{n}_u \\ -\mathbf{n}_u \end{bmatrix},\end{aligned} \tag{10}$$

where the top rows correspond to the basic sector.

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