Assessing the Entropies of the Feigenbaum Strange Attractor and the SP-500 Index as Factors Driving the Production of Information in Market Economies.

Dominique, C-Rene

Laval University (ret.)

6 November 2018

Assessing the Entropies of the Feigenbaum Strange Attractor and the S&P-500 Index as Factors Driving the Production of Information in Market Economies.

C-René Dominique

SUMMARY: This note investigates two strange attractors, namely, the Feigenbaum attractor that arises in unimodal maps and the attractor of the S&P-500 Index in relation to their ability to produce market information.

KEYWORDS: Strange attractors, Fractional dimensions, Frequencies, SDIC, Information, Entropy

1-Introduction

The world is presently living in the age of information. Consequently, concepts and narratives are now cast in the language of information, which is considered to be the dominant paradigm. In general, the dominant paradigm shines light on the preceding one because it was able to answer extant unanswered questions and/or pathologies. In a way, we are arguing in this note that the concept of information will eventually answer questions that are now beyond the purview of neo-classical economic analysis; the inherent inaccuracy in economic forecasts is just one such example.

It is clear that market economies produces a vast amount of information. In that connection, think of the information necessary to build a computer compared to that of making a slide rule, or the difference in bits of information needed to produce a word processor and the Olivetti type writer, and millions of similar examples from automobiles, aircrafts, telephones, telecommunication, etc. Economists have examined the so-called ‘market information’ as the basic information that agents are supposed to have so as to trade efficiently, but have neglected the concept of information as factor driving growth or as one of the most valuable assets of an economy and society. Moreover, if the latter kind of information were properly assessed it would have prevented the propagation of erroneous concepts such as the Efficient Market Hypothesis or the idea that accurate long term forecasts of market movements were at hand. The purpose of this note is to bring forth the information produced by the market itself as it evolves in time by appealing to chaos theory. Indeed, even casual observations suggest the existence of chaoticity in market economies and, it so happens that Statistical physics associates chaoticity to information production. In other words Chaos Theory is able to provide valuable quantitative and qualitative insights to better understand unpredictability and to defang other mysteries displayed by chaotic systems. For example, quantitative measurements based on chaos theory may be used to decide a priori whether a time series or a portion thereof is predictable or not.

We will examine the place and extent of information production in two areas. Those who engage in studies of market shares, for example, would probably be interested to know whether or not the Feigenbaum attractor (produced by unimodal maps) is able to produce information, since a cursory view inclines one to think otherwise. We will next examine the S&P-500 Index, which is strongly suspected to be chaotic. But beforehand, we will discuss a few generalities that are necessary to make the argument tractable.

1.1 Generalities

It is well-known that some deterministic dynamical systems may exhibit erratic or even chaotic behavior. But at the same time analyses of these systems are hampered by the lack of knowledge of their equations of motion. This difficulty has prompted analysts to find alternate approaches (in particular the study of scalar time series) that permit analysts to estimate important properties of these systems. For example, dissipative systems possess what Berliner (1992) calls 'dissipativeness'; which means that their phase space volume contracts and converges on singular sets of Lebesgue measure zero by the time evolution. The trajectories of such systems finally settle on a subset of $\mathbb{R}^n$, called the *attractor*. The dimension of the attractor is a measure of the information necessary to specify a point on the attractor, and there are many dimensions of interest. We will make use of three of the generalized Renyi's (1970, 1995) dimensions of order q in our examination of unimodal maps, obtained by partitioning the phase space of the system under study into boxes of size e_i to which one attaches probability p_i that the orbits will visit box i for a given number of non-empty boxes. That approach, albeit in slightly modified form, will guide us in our analysis of the S&P-500 Index.

The first three dimensions of order q we need are D_0, D_1 and D_2, respectively. D_0 or the Hausdorff dimension, is the fractal dimension or a *quantity-like dimension* that can be regarded as a measure of the way orbits of a dynamical system fill the phase space under the action of the flow φ (t) (Medio, 1992). If it is a non-integer then orbits tend to fill up less than an integer subspace of the phase space; that indicator gives an idea of the geometry of the attractor. D_1 is the information dimension, a probabilistic measure which is the ratio of the Shannon and Weaver (1949) information entropy and the logarithm of e_i. We interpret D_1 as the *loss of information* at the critical point of the system. In fact, D_1 can also be viewed as the level of "disorder in dynamic systems (i.e., Kolmogorov-Sinai, 1958 entropy) or "metric" or Shannon entropy (as the lack of information). D_2 is the correlation dimension which is also a *probabilistic* measure that accounts for the frequency with which orbits visit different parts of the attractor. It results when $q = 2$ in the Renyi's framework and it is the equivalent of the value of the embedding dimension obtained from the procedure recommended by Grassberger and Procaccia (1983) to compute the correlation dimension[1].

We hasten to add here that the computation of D_2 is difficult; that follows from the many components of an m-dimensional vector X (in a continuous ($\dot{X}$) or the discrete (X_{t+1} cases) that must be evaluated. However, Takens (1981) has shown that it is possible to reconstruct the attractor from a single component of the system such as a time series. This is not the place to review the whole procedure, except to say that to estimate D_2 the phase space must be reconstructed with embedding vectors d, selected from a d-

[1] The theoretical importance of D_2 lies in its close relation to the concept of correlation as it is determined by the correlation function of the fractal set.

dimensional data vectors from d measurements spaced equidistant in time in which one embeds the time series and compute D_2 of the d-dimensional point set. If the data are from random systems, then as d is increased, D_2 will increase in step. On the other hand, if the system is deterministic, D_2 will converge to the D_2 of the attractor. There exists, however, many pitfalls such as noise, drift, autocorrelation, etc. that tend to affect the estimate. But, contrary to the other dimensions, D_2 is more robust to time changes and dimensionality. In fact, it has been demonstrated (Medio, 1992) that the embedding dimension, regardless of dimensionality, is confined between 2 and 3. So, in this study, we will make use of the value obtained from Peters (1991) from the S&P-500 Index.

The attractor of interest is called 'Strange' i. e., it is topologically the product of a Cantor set of points lying in a portion of the invariant set and, as already indicated, it has a non-integer fractal dimension. We should also note that the concept of entropy that is related to information can also be used to classify dynamical systems. For example, the entropy (whether the Kolmogorov-Sinai's, metric or Shannon's) of a stable system is zero; it is infinite in a random system; and it is finite and positive in a chaotic system.

1.2 Models of Markets and Complexities

The modern market is obviously a complex system. With physical production, investment, consumption, transport, financialization, advertising, risk-taking, etc., cast in the form of pairwise (or multi) relations between agents, the modern market indeed exhibits characteristics of complex multidimensional systems (loosely defined). There seems to be many reasons for that complexity. The modern economy is a multitude of *two-way relations* between agents. These relations form a dynamic process out of which emerge a number of complex characteristics. That is, with feedbacks, the process becomes nonlinear. As it may grow or decay, it is therefore non ergodic. It may receive random external shocks, while legislative constraints delimit its phase space. And as a dissipative system, it has, in this case, a *non-hyperbolic* attractor that becomes more and more complicated as its parameter set undergoes changes. This kind of processes is usually described as:

$$\dot{X} = \mathbf{f}\,(\mathbf{x}\,(t);\, \Omega_0), \quad \mathbf{x}\ \varepsilon\ \mathbb{R}^n,\ \Omega\ \varepsilon\ \mathbb{R}, \qquad (1)$$

where **x** is a set of coordinates describing the system; **f** describes the non-linear time evolution in which the matching of buyers and sellers by not being one-to-one causes **f** to be non-invertible; $\mathbb{R}$ stands for the real line, and; finally, Ω acts as a control parameter.

We hasten to add that our discussion of this type is going to be brief, i. e., just what is needed for the present purpose. The system described by (1) is what is called an Iterative Function System (IFS). It can indeed display an incredible arrays of surprises that can increase our understanding of the kind of irregular dynamic behavior often encountered in physical, chemical and other natural phenomena. It suffices, therefore, to point out that below Ω_0, the system described by (1) produces cycles Γ- (2×2^k), k = 0,1, 2, 3….Thus, asymptotic motion is produced by an attractor that becomes more and more complicated as Ω increases, leading eventually to turbulence at Ω_0. Furthermore, as (1) is dissipative with a non-hyperbolic attractor, the phase space volume is stretched and contracted by the time evolution of the system. In fact, there is stretching in some directions and contractions in others such that in the end initial volume undergoes a net contraction. In some IFSs like (1), the motion may even become unstable. That instability is usually represented by exponential separation in the direction of stretching. If that occurs, the attractor is said to be "*strange*".

In such a strange attractor, the average rate of new information S (.) is measured as:

$$S(.) \leq \Sigma \text{ Positive Lyapunov exponents.} \qquad (2)$$

The many surprises encapsulated in chaotic strange attractors are beyond the scope of this note, but the interested reader is referred to Shaw (1981), Eckmann and Ruelle (1986), Catsigeras and Enrich (1997), Kutnetsov and Osbaldestin (2002), Robledo and Moyano (2009), among others. Nonetheless, it is known that the IFS given in (1) has a few characteristics that distinguish it from the so-called Feigenbaum Attractor. That is, at the critical value of the parameters, it exhibits both periodic and aperiodic motions and, more importantly, it exhibits sensitivity to initial conditions (SDIC), where information is produced. SDIC simply means that two initial conditions that are different but indistinguishable at time t_0, measured at a certain experimental precision, evolve into a distinguishable state at time t_n. Therefore, for (1) to be qualified as a chaotic attractor it must satisfy three conditions, namely, non-linearity, fractality property, and the possession of SDIC. The S&P-500 Index appears to satisfy all three, hence we will treat it as a typical chaotic IFS. To distinguish it further, we will henceforth refer to system (1) as a Strange Attractor or (SA) tout court.

The other type of IFS which is of interest here is the unimodal map, described as:

$$X_{t+1} = \mathbf{f}(\mathbf{x_t}\,; \Omega_0), \quad \mathbf{x}\ \varepsilon\ \mathbb{R}^n, \Omega\ \varepsilon\ \mathbb{R}, \qquad (3)$$

Here, our argument will follow the logistic parabola whenever possible. As in (1), (3) is dissipative with a non-hyperbolic attractor, the phase space volume is stretched and contracted by the time evolution of the system. As in (1), there is stretching in some directions and contractions in others such that in the end initial volume is reduced to a set of Lebesgue measure zero. At the critical value of **Ω**, the so-called Feigenbaum (1978) attractor emerges. The Feigenbaum attractor is also a Cantor set of dust concentrated on a portion of the stable manifold. We will argue that such an attractor produces Shannon entropy, because here uncertainty and information becomes the two sides of the same coin. The more uncertainty there is, the more information can be gained by removing the uncertainty.

Thus, the discrete Shannon information entropy rate at Ω_0 is:

$$S(.) = -\Sigma_i^n\, p_i \log(p_i). \qquad (4)$$

To see how this comes about, let us return to (3). As long as **Ω** is below Ω_0, the system produces cycles 2 periodic motion. As such, no new information can be gained since the extant information is now known with certainty. However, similar to (1), as **Ω** is increased up to the critical value Ω_0, the attractor becomes a Feigenbaum attractor (or FA) with a non-integer fractal dimension. This results from a massive phase change, and a sudden surge in Shannon entropy. What evidence is there for a surge in entropy? Simply because orbits become *aperiodic*. And this is not all. Another distinctive feature of FA is that orbits eventually lend on the visited area of the stable manifold. Put differently, each set of initial conditions generate an orbit that will eventually lend on a different and unpredictable speck of dust on the stable manifold. Thus, at Ω_0 one can attempt to evaluate the entropy and to consider its consequence in terms of market information. We will have more to say on this topic below. In the meantime, however, let us focus on one of the best ways to model an FA.

1.3- The Generalized Dimension of Order q and Entropy Assessment

The evaluation of the entropy in (3) is immensely simplified if we begin with a brief examination of multifractals (i. e. a generalization of the fractal concept where, contrary to an ordinary fractal, a single exponent is not enough to describe its dynamics) on fractal support. Suppose we begin with a line from which a number of open intervals have been removed, leaving line segments of length e_i (i = 1, 2, 3…). Next, one associates a weight or a probability p_i to each segment. Iterating the process of removal and assignment of probabilities continues until one arrives at the limit to an attractor consisting of a Cantor set with probabilities attached to the line segments. This is what is meant by a prototypical multifractal on a fractal support. According to Schroeder (2009), assigning a probability p_i to each line segment e_i allows the modeling of a fractal growth process in which different segments correspond to the different sites in which growth will take place. Thus, the probabilities p_i represent the different growth rates at these sites in diffusion limited-aggregation.

In applying that procedure to an FA, the e_i converge on the different values a dynamic variable can take; this is called the support of the attractor. On the other hand, the probabilities p_i model the frequencies with which the segments are visited under the action of the flow. This is the reason why multifractals are described by two scaling exponents, i. e., one for the supporting fractal, and the other for the probabilities. Following Mandelbrot (1974, 2003), we select a generalized generator, line segments e_i, and probabilities p_i. Next, we introduce two exponents. The exponent τ goes to segments e_i and the exponent q goes to the probabilities p_i. It is easily seen that many values of τ and q would satisfy the equation $\Sigma\, p_i^q\, e_i^{\tau} = 1$. To simplify, we then consider a self-similar fractal of dim D_q. It can be obtained directly from e_i and p_i of the generator as $\Sigma\, p_i^q\, e_i^{(1-q)\, D_q} = 1$. For q = 1, τ (q) = 0, $D_q = -\, d\tau/dq$. We then have[2],

$$D_1 = \Sigma\, p_i \log_{10}(p_i) \,/\, \Sigma\, p_i \log_{10}(e_i). \qquad (5)$$

For q =1, $D_q = \lim_{e\to 0} S(.) / \log_{10}(e_i)$, where S (.) is the *information entropy*.

For q =2,

---

[2] The parameters q and τ are real. For a given q, an approximate value of $\tau = \tau$ (q) can be found that guarantees an asymptotic equality. Equally for a given τ, one can select a value of q = q (τ). Then, the relation of τ and q is used to calculate the generalized dimensions and the f (α), the singularity spectrum.

$$D_2 = \lim_{e\to 0} (1/ (q-1) \log_{10} (\Sigma p_i^q) / \log_{10} (e_i). \qquad (6)$$

It is worth repeating what happens to the FA at Ω_0 (= 3.569956…). That is, iterates are aperiodic but they converge eventually on the stable manifold. That observation, apply demonstrated by Schroeder (2009), shows that the FA is well modeled by a generator with two intervals with lengths e_1 and e_2 and $p_1 = p_2 = ½$. We will have more to say about this in the section following.

2-The Scaling of the S&P-500

For the S&P-500 Index, we use the Grand Microsoft Excel series of closing prices of the S&P-500 from January 3rd 1961 to February 28th, sampled at daily intervals, and expressed as a mixed fractional Brownian motion (Mandelbrot and van Ness, 1968), indexed by the Hurst (1951) exponent. This automatically captures the needed changes in the frequency of the process, which in turn reflect the unavoidable reflexivity of human agents on markets. However, the computation of the Hurst exponent seems to vary with series lengths, sampling intervals, time periods, and the method employed (Greene and Fielitz, 1997; Kaplan and Jay, 1993; Bayraktar, Poor, and Sicard, 2004; Alvarez-Ramirez and Alvarez, *et al*., 2008). Variations in H, for whatever reason, affect both the Hausdorff dimension (D_0) and power spectra of the Fourier transform; D_0 in turn affects the other Renyi's dimensions; although D_2 shows less sensitivity to these factors than the others. For example, the variation in D_2 of the S&P-500 computed by Peters (1991) and that of the Dow Jones Industrial Average computed by Monte (1994) some 3 years later is only 0.16 (= 2.33 – 2.17). To simplify therefore, in Table 2 we keep the D_2 obtained by Peters fixed and observe the volatility in the other dimensions; minor error here will not affect our conclusions.

As already indicated, the computation of the embedding dimension is not a simple affair (see Monte (1994) and Gluzman and Yukalov (1998)), but once at hand, it remains fairly stable. Incidentally, the reader should also be made aware that Table 2 reports Peters' D_2 in topological dimension zero[3].

Table 1 shows the first three Renyi's dimensions for the logistic parabola computed from different methods; the table also shows how concordant these are in the absence of the H exponent. The logistic parabola is a prototypical strange attractor which clearly shows that D_1 falls between D_0 and D_2 as one would expect. This fact reflects the multifractal signature of the process, and it is used in Table 2.

Table 2, on the other hand, shows the results of scaling the S&P-500 over different periods. The first column is the Hausdorff dimension D_0. That value was obtained from an algorithm due to Trusoft International. D_2 is the correlation dimension which is also a *probabilistic dimension* it is recalled, it ac-

---

[3] $D_0 + H = 1 + D_T$, where D_T is the topological dimension.

Table 1 Dimensions of Unimodal Maps from Various Methods

Studies	D_0	D_1	D_2
Schroeder (2009)[1]	0.537	0.515	0.497
Grassberger (1981)	0.538	-----	------
Gluzman and Yukalov (1998)	0.5380451	0.5170975	0.49783645

(1) Schroeder uses the Mandelbrot Method; Grassberger uses analytical and numeral methods; Gluzman and Yukalov employ their Renormalization Group Analysis.

Table 2 Information Dimensions of the S&P-500 Index for Different Periods

Period	The H Exponent	D_0	D_1[1]
1961-72	0.52	0.48	$0.33 < D_1 > 0.48$
1972-80	0. 22	0. 78	$0.33 < D_1 > 0.78$
1980-83	0.28	0.77	$0.33 < D_1 > 0.77$
1983-87	0.56	0.44	$0.33 < D_1 > 0.44$
1988-92	0.53	0.47	$0.33 < D_1 > 0.47$
1992-97	0.46	0.54	$0.33 < D_1 > 0.54$
1998-02	0.61	0.39	$0.33 < D_1 > 0.39$
2003-07	0.11	0.89	$0.33 < D_1 > 0.89$
2007-08	0.28	0.72	$0.33 < D_1 > 0.72$
2009-11	0.14	0.86	$0.33 < D_1 > 0.86$

(1) The embedding dimension of 0.33 around a point on the attractor is from Peters (1991)

counts for the frequency with which orbits visit different parts of the attractor. It is the equivalent of the procedure recommended by Grassberger and Procaccia (1983) to compute the correlation dimension[4].

Assuming that D_1 remains between D_0 and D_2 throughout, it is clear that variations in H result in huge gaps in the value of D_1. Anyway, with the help of these two tables, we can now compute the difference between FA and SA.

As it can be observed below, the FA exhibits aperiodic orbits leaving the observer in the dark as to which speck of dust the orbit will finally converge on in a set of Lebesgue measure zero. That uncertainty lasts until convergence on the attractor. In the meantime, however, all forecasts of equilibrium must necessarily be probabilistic. We also recall that below Ω_0, orbits are periodic, hence no new information would be expected as the Shannon entropy is zero. At Ω_0, there is no SDIC but orbits become aperiodic. The onus then is on the observer to produce market information.

In SA, in contrast, we have a countable set of periodic orbits and an uncountable set of aperiodic orbits. The Shannon entropy of periodic trajectories is zero, but we are unable to say that SDIC is caused by aperiodic trajectories since FA has aperiodic trajectories but no SDIC. Therefore, it is reasonable to argue that information in SA is produced by both the effort of the observer and SDIC.

---

[4] The correlation integral of a time series is a normalized count of the number of close pairs of points of the series lying within a certain fixed distance from another.

Consequently, we then contain that in random determinism or under Shannon entropy, the search for the balanced state between demand and supply (after losing a significant portion of information) is the incitation or the force driving the production of information and innovations in market economies.

In fact, at this point we can even answer the question asked by the Queen of England, namely: "How could economists have missed the crash of 2008"? The answer is: Your Majesty, it is a question of the wrong probability distribution.

It should be recalled that the selected process is a mixed fractional Brownian motion (see Mandelbrot and van Ness, 1968; Thale, 2009, among others), indexed by the Hurst exponent. Scaling it from 1961 to 2011 shows that the S&P-500 index undergoes periodic change in frequency as well. As already indicated, this is attributed to a characteristic of the Hurst exponent. To take an example, consider the

Differences Between FA and SA

Characteristics of FA:

i) Type: Strange or Multifractal Cantor dust
ii) Motion: aperiodic
iii) SDIC: no
iv) Information Entropy: o.602 ban /unit of time
v) Dissipation Rate: 0.515 percent
vi) Dissipation Time: ≈ after 5 periods, with 1.5 percent remaining.

Characteristics of SA

vii) Type: Strange
viii) Motion: Countable set of periodic orbits and uncountable set of aperiodic orbits
ix) SDIC: yes
x) Information Entropy: based on the time period 1983-87: 0.602 ban /unit of time
xi) Dissipation Rate: Assume 0. 36 percent
xii) Dissipation Time: ≈ after 8 periods, with 1.7 percent remaining.

time period 2003 to 2011; one observes a surge in entropy, characterizing an epoch of high hesitation, doubt and low confidence level in the American economy. But from 1983-97, on the other hand, system's frequency was relatively low; consequently, the loss of information was also less. Previously Dominique (2018) has found that during period of low system's frequency, the economy grew and the inherent inequality in income distribution did improved. The opposite occurred during high frequency periods.

3- Final Remarks

We began by considering processes in a family of models in which strange attractors are produced upon changes in their parameter sets. We next demonstrates that at critical values of the parameter set there is a surge in Shannon entropy in FA. Below that critical value, orbits of this type of dynamical systems are periodic. At the critical value, the attractor consisting of Cantor set of dust concentrated on some small parts of the attractor, entropy surges, orbits become aperiodic, but nevertheless are attracted to some speck of Cantor dust. In SA, one observes about the outcome, but in addition, orbits are periodic, aperiodic, and SDIC exists. The attempt to reduce the uncertainty, i. e., in order to reach a balanced position, is the spur of production of market information in FA, whereas in SA there are two spurs namely aperiodicity and SDIC.

It is worth repeating at this juncture what we have said above. In the history of science, for example, concepts and narratives arise from the dominant paradigm, but the dominant paradigm may eventually be supplanted by a new paradigm. For instance, a concept such as space-time has undoubtedly shined light on mechanics. Similarly, as we are now in the age of information, that concept undoubtedly stands to shine light on space-time itself. In a way, we have argued in this note that the same concept will eventually answer questions that are now beyond the purview of neo-classical economic analysis.

The findings of Table 2 also indicate that during high frequency mode, the whole singularity spectrum of the process undergoes an enlargement, following the increase in entropy. For example, during the period 1983-87, the S&P-500 index was in a low frequency mode; consequently the loss of information at Ω_0 was between 33% and 44%. During the period 2003-11 on the other hand, the loss of information was substantial as the index was in high frequency mode. The loss of information is the Achilles heel of forecasters. In that connection, it is interesting to note that in the last meeting of the International Monetary Fund (IMF) and the World Bank in Bali Indonesia, the IMF had to downgrade its 2018 forecast of the global growth rate from 3.9 % to 3.7 %, and no doubt that forecast will be revised again and again in 2019. This is not surprising for it will always be so. In fact, from Table 2, forecasters should realize that in order to forecast the equilibrium of the economy, they would have to have first a probability distribution. But the value of D_1 would also tell them how difficult such a task would be.

These findings fit rather well the stylized facts. Relations between agents are known to be cyclic, and the model replicates cyclic motions and fluctuations in frequencies. It shows that the difficulty of making accurate forecasts is due to the surge in Shannon entropy when the system is at a critical point. Incidentally, this offers a simple answer to the critiques of the notion of equilibrium as well as those who fault economists for not having predicted the 2008 financial crisis. It explains the ups and downs in the inequality of income distribution as well as the variations in the confidence level of investors. It shows that chaos (or SDIC) is not the only source of information production, and it vindicates Paul Romer's explanation of the role of information in economic growth. Finally, these results are only a few examples of how the concept of information can shed light on neo-classical economics.

References

Alvarez-Ramirez, J. and Alvarez B. et al. (2008). Time Varying Hurst Exponent for the US Stock Market. *Physica A*, 1959-69.

Bayraktar, E., Poor, V. H. et al. (2004). Estimating the Fractal Dimension of the S&P-500 Index Using Wavelet Analysis. Dep. Of Electrical Engineering, Princeton Univ.: Princeton, N. J.

Berliner, M. (1992). Statistics, Probability and Chaos. *Stat. Science*, 7, (1), 69-90.

Catsigeras, E. and Enrich, H. (1997). Persistence of the Feigenbaum Attractor in one-parameter Families. *Commm Math. Phys.Preprint* 97/11.

Dominique, C-R. (2018). Could Noise Spectra of Strange Attractors Better Explain Wealth and Income Inequality? Evidence from the S&P-500 Index. *Modern Economy*, 9, 1-12.

Eckmann, J. P. and Ruelle, D. (1986). Ergodic Theory of Chaos and Strange Attractors. *Rev. of Modern Physics*, 67, (3), 617-56.

Feigenbaum, M. J. (1978). Quantitative Universality for a Class of Non-linear Transformations. *Journal of Stat. Physics*, 19, 25-52.

Grassberger, P. (1981).On the Hausdorff Dimension of Fractal Attractors. J. *of Stat. Physics*, 26, 173-79.

Grassberger, P. and Procaccia, I. (1983). Characterization of Strange Attractors. *Phy. Rev. Letters*, 50, 346-49.

Gluzman, S. and Yukalov, V. I. (1998). Renormalization Group Analysis of October Market Crashes. *Modern Phys. Letters B*, 12, 75-84.

Greene, M. T. and Fielitz B. D. (1977). Long Term Dependence in Common Stock Returns. *Jour. of Financial Economics,* 4, 339-49.

Hurst, E. *et al.* (1951). Long-Term Storage: An Engineering Study. *Transactions of the American Society of Civil Engineers*, 110, 770-90.

Kaplan, L. M. and Jay, K. (1993). Fractal Estimation from Noisy Data via Discrete Fractional Gaussian Noise and the Haar Basis. *IEEE Transactions*, 41, 3554-62.

Kolmogorov, A. (1958). A New Metric Invariant of transitive Dynamic Systems and Automorphisms of Lebesgue Space. *Dokl. Acad. Nauk, SSSR*, 119, 861-64.

Kutnetsov, S. P. an Osbaldestin, A. H. (2002). Generalized Dimensions of Feigenbaum's Attractor from Renormalization-Group Functional Equation. http://citeseerx.ist.psu.edu/viewdoc/download? Doi-10.1.1.390. 8884 &rep-rep1&type, pdf; retrieved on Oct 3rd, 2018.

Mandelbrot, B. (1974). Intermittent Turbulence and Self-Similar Cascades: Divergence of High Moments and Dimension of Carriers. *J. of Fluid Mechanics*, 62, 331-5

……………. (2003). Multifractal Power Law Distributions: Negative and Critical Dimensions and Other Anomalies. *J. of Stat. Physics*, 110, 739-79.

Mandelbrot B. and van Ness, J. W. (1968). Fractional and Brownian Motion, Fractional Noises, and Applications. *SIAM Rev*., 10, 422-37.

Medio, A. (1992). *Chaotic Dynamics: Theory and Applications to Economics*. Cambridge Univ. Press: Cambridge.

Monte, B. M. (1994). Chaos and the Stock Market. Masters Thesis Presented at the California State Univ., San Bernardino, CA, June.

Olofsen, E. (1991). The Identification of Strange Attractors Using Experimental Time Series. Masters Thesis, Dept. of Physiology, Leiden Univ. Medical Center.

Peters, E. (1991). A Chaotic Attractor for the S&P-500. *Financial Analyst Journal*, 47, 55-62.

Renyi, A. (1970). *Probability Theory*. North-Holland: Amsterdam.

………… …(1995). On a New Axiomatic Theory of Probability. *Acta Math*., 6, 285-335.

Robledo, A. and Moyano, L. G. (2009) Dynamics Towards the Feigenbaum Attractor. *Brazilian Jour. of Phys*.,39, 2A, August.

Schroeder, M. (2009). *Fractals, Chaos, Power Laws*. Dover Pub. Inc.: New York.

Shannon, C. and Weaver, W. (1949). *The Mathematical Theory of Communication*. Univ. of Illinois: Urbana.

Shaw, R. (1981). Strange Attractors, Chaotic Behavior, and Information Flow. Zfn.mpdl.de/ Reihe A136 Z-NA-1981-.36a-0080.pdf; retrieved on October 19th, 2018.

Takens, F. (1981).Detecting Strange Attractors in Turbulence. In Rand, D. A. and Young, L. S. (eds) *Dynamical Systems in Turbulence*, 366-81. New York: Springer-Verlag.

Thale, C. (2009). Further Remarks on the Mixed Fractional Brownian motion. *Applied Math. Sci*., 3, 1-17.