Time-varying congestion tolling and urban spatial structure

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Abstract

This study develops a model in which heterogeneous commuters choose their residential locations and departure times from home in a monocentric city with a bottleneck located at the entrance to the central business district (CBD). We systematically analyze the model by utilizing the properties of complementarity problems. This analysis shows that, although expanding the capacity of the bottleneck generates a Pareto improvement when commuters do not relocate, it can lead to an unbalanced distribution of benefits among commuters: commuters residing closer to the CBD gain and commuters residing farther from the CBD lose. Furthermore, we reveal that an optimal time-varying congestion toll alters the urban spatial structure. We then demonstrate through examples that (a) if rich commuters are flexible, congestion tolling makes cities denser and more compact; (b) if rich commuters are highly inflexible, tolling causes cities to become less dense and to spatially expand; and (c) in both cases, imposing a toll helps rich commuters but hurts poor commuters.

JEL classification: D62; R21; R41; R48

Keywords: time-varying congestion toll; bottleneck congestion; urban spatial structure; heterogeneity

1 Introduction

Traditional residential location models (Alonso, 1964; Mills, 1967; Muth, 1969) have succeeded in predicting the empirically observed patterns of residential location based on the trade-off between land rent and commuting costs and have been used for evaluating the efficacy of urban policies. These traditional models, however, mostly describe traffic congestion by using *static* congestion models, in which congestion at a location depends only on the total traffic demand (i.e., the total number of commuters passing a location), regardless of the time-of-use pattern. This indicates that these models do not capture peak-period traffic congestion that takes the form of queuing at a bottleneck. Consequently, we cannot use traditional models to examine the effects of transportation demand management (TDM) measures intended to alleviate it (e.g., time-varying congestion tolling).

The bottleneck model can adequately capture the *dynamic* nature of traffic congestion (Vickrey, 1969; Hendrickson and Kocur, 1981; Arnott et al., 1990b, 1993). This model provides a simple

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framework for describing peak-period congestion that allows us to study the effects of various TDM measures, thereby inspiring numerous extensions and modifications. The standard bottleneck model, however, cannot be easily applied in the context of the residential location model, as stated by Ross and Yinger (2000). Therefore, these models essentially ignore a spatial dimension.

Several studies have incorporated a spatial dimension to the standard bottleneck model by embedding the dynamic bottleneck congestion into a simple monocentric model to study the effects of imposing an optimal time-varying congestion toll to eliminate the queue. Arnott (1998) provides an integrated treatment of equilibrium location and trip timing for the case in which commuters are identical and the closed city comprises two islands with the central business district (CBD) located on one of the islands. He shows that imposing a toll without redistributing its revenues affects neither commuting costs nor commuters' residential locations. Gubins and Verhoef (2014) treat the conventional continuous location monocentric model, assume that a bottleneck exists at the entrance to the CBD, and consider the case with identical commuters and a closed city. Their model introduces an incentive for commuters to spend time at home and assumes that a commuter's house size affects the marginal utility of spending time at home.[1] They then demonstrate that congestion tolling eliminates waiting time in a queue and allows commuters to spend more time at home. It causes them to have larger houses, leading to spatial expansion of the city. Fosgerau et al. (2018) develop a different type of model by incorporating location choices of homogeneous commuters into the dynamic congestion model of Fosgerau and de Palma (2012) which introduces commuting costs (scheduling preferences) that are not separable in trip duration and arrival time. That is, they assume that commuters' scheduling preferences depend on their travel time.[2] This study considers the same spatial structure as in Gubins and Verhoef (2014), but assumes that the city is open. They show that an optimal toll changes commuters' scheduling preferences, which induces lower density in the center and higher density farther out.

The results of these studies fundamentally differ from the standard results given by traditional location models with static congestion, which predict that cities become denser with tolling (e.g., Kanemoto, 1980; Wheaton, 1998; Anas et al., 1998). This difference arises from the following reasons: in the model with static congestion, imposing a toll makes commuting more expensive; in the model with dynamic congestion, tolling does not change the cost of traversing the bottleneck if commuters are homogeneous. These results also show that an optimal time-varying congestion toll itself plays no essential role in changing commuters' location incentives. Indeed, we can see that the assumption on commuters' scheduling preferences that the standard bottleneck model disregards is the key to altering urban spatial structure.

Takayama and Kuwahara (2017) recently reveal that an optimal time-varying toll can cause cities to become less dense and to spatially expand. To this end, they extend Arnott (1998) by treating the continuous location monocentric model and by incorporating heterogeneity in commuters' value of time (i.e., willingness to pay for reducing travel time) and flexibility. Their analysis shows that a toll changes commuters' commuting costs, thereby altering their spatial distribution. Furthermore, they demonstrate that congestion tolling helps commuters with a high value of time but hurts commuters with a low value of time. These results, however, essentially depend on the assumption of quasi-linear utility (i.e., the income elasticity of the demand for land

[1] Vickrey (1973), Tseng and Verhoef (2008), Fosgerau and Lindsey (2013), and Fosgerau and Small (2017) also introduce the utility from spending at home.

[2] The assumptions of our model imply that commuters' scheduling preferences depend on their type of job (e.g., shift worker, academic) but not on their travel time.

is zero), which is inconsistent with empirical evidence (Wheaton, 1977; Glaeser et al., 2008).

This study develops a model of trip timing and residential location choices of heterogeneous commuters that resolves the limitations of the previous literature discussed above. We consider a monocentric city with a bottleneck located at the entrance to the CBD as in Gubins and Verhoef (2014) and Fosgerau et al. (2018) and employ a utility function that allows the income elasticity of the demand for land to be positive. We then systematically analyze our model using the properties of complementarity problems that define the equilibrium and show that commuters sort themselves both temporally and spatially on the basis of their income[3] and flexibility. We further reveal that, although the bottleneck capacity expansion generates a Pareto improvement when commuters do not relocate, it can lead to an unbalanced distribution of benefits among commuters: commuters residing closer to the CBD gain and commuters residing farther from the CBD lose. This occurs because alleviating peak-period congestion causes the city to spatially expand and thus increases commuting distance of commuters residing farther from the CBD. This result contrasts with that obtained in Takayama and Kuwahara (2017), which observe that the capacity expansion helps all commuters.

In addition, this study investigates the effects of an optimal time-varying congestion toll on urban spatial structure. We show that imposing the toll changes commuting costs, thereby altering commuters' lot sizes and spatial distribution. To concretely demonstrate the effects of tolling, we analyze the model for cases in which rich commuters are flexible and highly inflexible. This analysis reveals that (a) tolling makes cities denser and more compact when rich commuters are flexible; (b) it causes cities to become less dense and to spatially expand when rich commuters are highly inflexible; and (c) in both cases, imposing a toll helps rich commuters but hurts poor commuters. These findings contrast not only with the standard results of traditional location models, but also with those of Arnott (1998), Gubins and Verhoef (2014), and Fosgerau et al. (2018).

This study proceeds as follows. Section 2 presents a model in which heterogeneous commuters choose their departure times from home and residential locations in a monocentric city. Sections 3 and 4 characterize equilibria with and without tolling, respectively, by utilizing the properties of complementarity problems. Section 5 clarifies the effects of an optimal time-varying congestion toll. Section 6 concludes the study.

2 The model

2.1 Assumptions

We consider a long narrow city with a spaceless CBD, in which all job opportunities are located. The CBD is located at the edge of the city and a residential location is indexed by distance x from the CBD (see Figure 1). In the city, land is uniformly distributed with unit density along a road. As is common in the literature, the land is owned by absentee landlords. The road has a single bottleneck with capacity s at the entrance to the CBD (i.e., $x = 0$). If arrival rates at the bottleneck exceed its capacity, a queue develops. To model queuing congestion, we employ first-in-first-out (FIFO) and a point queue, in which vehicles have no physical length as in standard bottleneck models (Vickrey, 1969; Arnott et al., 1993). Free-flow travel time per unit distance is assumed to be constant at $\tau > 0$ (i.e., free-flow speed is $1/\tau$).

[3] We assume that commuters' value of time is positively correlated to their income.

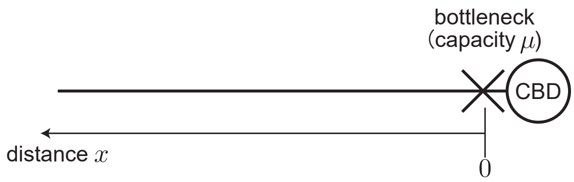


Figure 1: Urban spatial structure

There are G groups of commuters, who differ in their income, value of (travel) time, and schedule delay cost for arriving at work earlier or later than desired. The number of commuters of group $i \in \mathcal{G} \equiv \{1, 2, \cdots, G\}$, whom we call "commuters i," is fixed and denoted by N_i. They have a common desired arrival time t^* at work. The commuting cost of commuter i who resides at x and arrives at work at time t is the sum of travel time cost $\alpha_i\{q(t) + \tau x\}$ and schedule delay cost $d_i(t - t^*)$:

$$c_i(x, t) = \alpha_i\{q(t) + \tau x\} + d_i(t - t^*), \tag{1a}$$

$$d_i(t - t^*) = \begin{cases} \beta_i(t^* - t) & \text{if} \quad t \leq t^*, \\ \gamma_i(t - t^*) & \text{if} \quad t \geq t^*, \end{cases} \tag{1b}$$

where $\alpha_i > 0$ is the value of time of commuters i, $q(t)$ denotes the queuing time of commuters arriving at work at time t, and τx represents the free-flow travel time of commuters residing at x. $\beta_i > 0$ and $\gamma_i > 0$ are the marginal early and late delay costs, respectively.

This study imposes the following assumptions about the value of time and the marginal schedule delay costs, which is common to the literature employing a bottleneck model with commuter heterogeneity (e.g., Vickrey, 1973; Arnott et al., 1992, 1994; van den Berg and Verhoef, 2011b; Hall, 2018).

Assumption 1

(i) $\alpha_i > \beta_i$ *for all* $i \in \mathcal{G}$.

(ii) $\gamma_i/\beta_i = \eta > 1$ *for all* $i \in \mathcal{G}$.

Assumption 1 (i) requires that the value of time α_i is higher than the marginal early delay cost β_i for all commuters $i \in \mathcal{G}$. This assumption implies that commuters prefer to wait at the office rather than wait in traffic. If this condition is violated, there is no equilibrium that satisfies the FIFO property (i.e., vehicles must leave the bottleneck in the same order as their arrival at the bottleneck). Assumption 1 (ii) means that commuters with a high early delay cost also have a high late delay cost. Under this assumption, β_i (or γ_i) provides a measure of the *inflexibility* of commuters i.

It is well known that the primary source of heterogeneity in the value of time is variation in their income.[4] Thus, we suppose that commuters with a high (low) value of time are assumed to be *rich (poor)*.

Assumption 2 *If* $\alpha_i \geq \alpha_j$*, then* $y_i \geq y_j$.

[4]Other sources of heterogeneity in the value of time include trip purpose (work or recreation), time of day, physical or psychological amenities available during travel, and the total duration of the trip (Small and Verhoef, 2007).

Each commuter consumes a numéraire good and land. The preferences of commuter i who resides at x and arrives at work at time t are represented by the Cobb-Douglas utility function

$$u(z_i(x,t), a_i(x,t)) = \{z_i(x,t)\}^{1-\mu}\{a_i(x,t)\}^{\mu}, \tag{2}$$

where $\mu \in (0,1)$, $z_i(x,t)$ denotes consumption of the numéraire good, and $a_i(x,t)$ is the lot size. The budget constraint is given by

$$y_i = z_i(x,t) + \{r(x) + r_{\mathrm{A}}\}\, a_i(x,t) + c_i(x,t), \tag{3}$$

where $r_A > 0$ is the exogenous agricultural rent and $r(x) + r_{\mathrm{A}}$ denotes land rent at x.

The first-order conditions of the utility maximization problem give

$$z_i(x,t) = (1-\mu)I_i(x,t), \qquad a_i(x,t) = \frac{\mu I_i(x,t)}{r(x) + r_{\mathrm{A}}}, \qquad I_i(x,t) \equiv y_i - c_i(x,t), \tag{4}$$

where $I_i(x,t)$ denotes the income net of commuting cost earned by commuters i who reside at x and arrive at work at t. Substituting this into the utility function, we obtain the indirect utility function

$$v(I_i(x,t), r(x) + r_{\mathrm{A}}) = (1-\mu)^{1-\mu}\mu^{\mu} I_i(x,t)\{r(x) + r_{\mathrm{A}}\}^{-\mu}. \tag{5a}$$

2.2 Equilibrium conditions

Similar to models in Gubins and Verhoef (2014) and Takayama and Kuwahara (2017), we assume commuters make short-run decisions about day-specific trip timing and long-run decisions about residential location. In the short-run, commuters i minimize commuting cost $c_i(x,t)$ by selecting their arrival time t at work taking their residential location x as given. In the long-run, each commuter i chooses a residential location x so as to maximize his/her utility. We therefore present the short- and long-run equilibrium conditions.

2.2.1 Short-run equilibrium conditions

In the short-run, commuters determine only their day-specific arrival time t at work, which implies that the number $N_i(x)$ of commuters i residing at x (spatial distribution of commuters) is assumed to be a given. It follows from (1) that the commuting costs $c_i(x,t)$ of commuters i consists of a cost $\alpha_i \tau x$ of free-flow travel time depending only on residential location x and a bottleneck cost $c_i^b(t)$ owing to queuing congestion and a schedule delay depending only on arrival time t at work:

$$c_i(x,t) = c_i^b(t) + \alpha_i \tau x, \tag{6a}$$
$$c_i^b(t) \equiv \alpha_i q(t) + d_i(t - t^*). \tag{6b}$$

This implies that each commuter i chooses arrival time t so as to minimize his/her bottleneck cost $c_i^b(t)$. Therefore, short-run equilibrium conditions coincide with those in the standard bottleneck model, which are given by the following three conditions:

$$\begin{cases} c_i^b(t) = c_i^{b*} & \text{if} \quad n_i(t) > 0 \\ c_i^b(t) \geq c_i^{b*} & \text{if} \quad n_i(t) = 0 \end{cases} \quad \forall i \in \mathcal{G}, \tag{7a}$$

$$\begin{cases} \sum_{k\in\mathcal{G}} n_k(t) = s & \text{if} \quad q(t) > 0 \\ \sum_{k\in\mathcal{G}} n_k(t) \leq s & \text{if} \quad q(t) = 0 \end{cases} \quad \forall t \in \mathbb{R}_+, \tag{7b}$$

$$\int n_i(t)\mathrm{d}t = N_i \qquad \forall i \in \mathcal{G}, \tag{7c}$$

where $n_i(t)$ denotes the number of commuters i who arrive at work at time t (i.e., arrival rate of commuters i at the CBD) and c_i^{b*} is the short-run equilibrium bottleneck cost of commuters i.

Condition (7a) represents the no-arbitrage condition for the choice of arrival time t. This condition means that, at the short-run equilibrium, no commuter can reduce the bottleneck cost by altering arrival time unilaterally. Condition (7b) is the capacity constraint of the bottleneck, which requires that the total departure rate $\sum_{k\in\mathcal{G}} n_k(t)$ at the bottleneck equals capacity s if there is a queue; otherwise, the total departure rate is (weakly) lower than s. Condition (7c) is flow conservation for commuting demand.

These conditions give $n_i(t)$, $q(t)$, and c_i^{b*} at the short-run equilibrium as functions of $(N_i)_{i\in\mathcal{G}}$.[5] The short-run equilibrium commuting cost $c_i^*(x)$ and the income net of commuting cost $I_i(x)$ of commuters i residing at x are given by

$$c_i^*(x) = c_i^{b*} + \alpha_i \tau x, \tag{8a}$$

$$I_i(x) \equiv y_i - c_i^*(x). \tag{8b}$$

2.2.2 Long-run equilibrium conditions

In the long-run, each commuter i chooses a residential location x so as to maximize indirect utility (5). Thus, long-run equilibrium conditions are expressed as the following complementarity problems:

$$\begin{cases} v(I_i(x), r(x) + r_{\mathrm{A}}) = v_i^* & \text{if} \quad N_i(x) > 0 \\ v(I_i(x), r(x) + r_{\mathrm{A}}) \leq v_i^* & \text{if} \quad N_i(x) = 0 \end{cases} \quad \forall x \in \mathbb{R}_+,\ \forall i \in \mathcal{G}, \tag{9a}$$

$$\begin{cases} \sum_{k\in\mathcal{G}} a(I_i(x), r(x) + r_{\mathrm{A}})N_k(x) = 1 & \text{if} \quad r(x) > 0 \\ \sum_{k\in\mathcal{G}} a(I_i(x), r(x) + r_{\mathrm{A}})N_k(x) \leq 1 & \text{if} \quad r(x) = 0 \end{cases} \quad \forall x \in \mathbb{R}_+ \tag{9b}$$

$$\int_0^\infty N_i(x)\,\mathrm{d}x = N_i \qquad \forall i \in \mathcal{G}, \tag{9c}$$

where v_i^* is the long-run equilibrium utility level of commuters i and $a(I_i(x), r(x) + r_{\mathrm{A}})$ denotes the lot size of commuters i at location x, which is given by

$$a(I_i(x), r(x) + r_{\mathrm{A}}) = \frac{\mu I_i(x)}{r(x) + r_{\mathrm{A}}}. \tag{10}$$

Condition (9a) is the equilibrium condition for commuters' choice of residential location. This

[5] Note that the short-run equilibrium conditions depend on $(N_i)_{i\in\mathcal{G}}$ but not on $N_i(x)$.

condition implies that, at the long-run equilibrium, no commuter has incentive to change residential location unilaterally. Condition (9b) is the land market clearing condition. This condition requires that, if total land demand $\sum_{k\in\mathcal{G}} a(I_k(x), r(x) + r_{\mathrm{A}})N_k(x)$ for housing at x equals supply 1, land rent $r(x) + r_{\mathrm{A}}$ is (weakly) larger than agricultural rent r_{A}. Condition (9c) expresses the population constraint.

As is discussed in Takayama and Kuwahara (2017), *traditional bid-rent approach (Alonso, 1964; Kanemoto, 1980; Fujita, 1989; Duranton and Puga, 2015) is equivalent to our approach using complementarity problems* (for the proof, see Appendix A.1). Specifically, long-run equilibrium conditions (9) coincide with those of the bid-rent approach. Therefore, even if we use the traditional bid-rent approach, we obtain the same results as those presented in this study.

3 Equilibrium

3.1 Short-run equilibrium

The short-run equilibrium conditions (7) coincide with those in the standard bottleneck model, as discussed above. Therefore, we can invoke the results of studies utilizing the bottleneck model to characterize the short-run equilibrium (Arnott et al., 1994; Lindsey, 2004; Iryo and Yoshii, 2007; Liu et al., 2015). In particular, the following properties of the short-run equilibrium are useful for investigating the properties of our model.

Lemma 1 (Lindsey, 2004; Iryo and Yoshii, 2007) *Suppose Assumption 1 (i). Then, the short-run equilibrium has the following properties:*

(a) The short-run equilibrium bottleneck cost c_i^{b*} is uniquely determined.

(b) The short-run equilibrium number $(n_i^*(t))_{i\in\mathcal{G}}$ of commuters arriving at time t coincides with the solution of the following linear programming problem:

$$\min_{(n_i(t))_{i\in\mathcal{G}}} \sum_{i\in\mathcal{G}} \int \frac{d_i(t-t^*)}{\alpha_i} n_i(t)\mathrm{d}t \tag{11a}$$

$$s.t. \quad \sum_{i\in\mathcal{G}} n_i(t) \leq s \quad \forall t \in \mathbb{R}, \tag{11b}$$

$$\int n_i(t)\ \mathrm{d}t = N_i \quad \forall i \in \mathcal{G}, \tag{11c}$$

$$n_i(t) \geq 0 \quad \forall i \in \mathcal{G},\ \forall t \in \mathbb{R}. \tag{11d}$$

Let us define time-based cost as the cost converted into equivalent travel time. Since that cost for commuters i is given by dividing the cost by α_i, we say that $\frac{d_i(t-t^*)}{\alpha_i}$ represents the time-based schedule delay cost of commuters i. Therefore, Lemma 1 (b) shows that, at the short-run equilibrium, the total *time-based* schedule delay cost is minimized, but the total schedule delay cost is not necessarily minimized.[6]

We let $\mathrm{supp}\,(n_i^*) = \{t \in \mathbb{R}_+ \mid n_i^*(t) > 0\}$ be the support of the short-run equilibrium number

[6]As will be shown in Section 4.1, under an optimal time-varying congestion toll, the total schedule delay cost (the social cost of commuting) is minimized at the short-run equilibrium.

$n_i^*(t)$ of commuters i who arrive at work at t. From Lemma 1 (b), we have

$$\text{supp}\left(\textstyle\sum_{i\in\mathcal{G}} n_i^*\right) = [t^{\mathrm{E}}, t^{\mathrm{L}}], \tag{12}$$

where t^{E} and t^{L} denote the earliest and latest arrival times of commuters, which satisfy

$$t^{\mathrm{L}} = t^{\mathrm{E}} + \frac{\sum_{i\in\mathcal{G}} N_i}{s}. \tag{13}$$

This indicates that, at the short-run equilibrium, a rush hour in which queuing congestion occurs must be a single time interval.

By using short-run equilibrium condition (7a), we obtain

$$c_i(t_i) + c_j(t_j) \leq c_i(t_j) + c_j(t_i) \quad \forall t_i \in \text{supp}\,(n_i^*),\ t_j \in \text{supp}\,(n_j^*). \tag{14}$$

Substituting (6b) into this, we have

$$\begin{cases} \left(\frac{\beta_i}{\alpha_i} - \frac{\beta_j}{\alpha_j}\right)(t_i - t_j) \geq 0 & \text{if} \quad \max\{t_i, t_j\} \leq t^* \\ \left(\frac{\gamma_i}{\alpha_i} - \frac{\gamma_j}{\alpha_j}\right)(t_i - t_j) \leq 0 & \text{if} \quad \min\{t_i, t_j\} \geq t^* \end{cases} \quad \forall i, j \in \mathcal{G}. \tag{15}$$

This leads to the following proposition as given in Arnott et al. (1994) and Liu et al. (2015):

Proposition 1 *Suppose Assumption 1. Then, at the short-run equilibrium, commuters with a high marginal time-based schedule delay cost (β_i/α_i) arrive closer to their preferred arrival time t^*.*

This proposition indicates that the short-run equilibrium has the following properties: if marginal schedule delay cost of commuters i is lower than that of commuters j (i.e., $\beta_i/\alpha_i < \beta_j/\alpha_j$), early-arriving commuters i arrive at the CBD earlier than early-arriving commuters j and late-arriving commuters i arrive at the CBD later than late-arriving commuters j. This occurs because commuters with a lower time-based schedule delay cost avoid queuing time rather than a schedule delay.

By using Proposition 1, we can explicitly obtain the short-run equilibrium bottleneck cost. For the moment, we assume, without loss of generality, that commuters with small i have a (weakly) higher marginal time-based schedule delay cost:

Assumption 3 $\frac{\beta_{i-1}}{\alpha_{i-1}} \geq \frac{\beta_i}{\alpha_i}$ *for any $i \in \mathcal{G}\backslash\{1\}$.*

Under this assumption, commuters with smaller i arrive (weakly) closer to their preferred arrival time t^*. Therefore, the short-run bottleneck cost c_i^{b*} of commuters i is derived by following the procedure employed in literature employing a bottleneck model with commuter heterogeneity (see, e.g., van den Berg and Verhoef, 2011a):

$$c_i^{b*} = \frac{\eta}{1+\eta}\left\{\beta_i \frac{\sum_{k=1}^{i} N_k}{s} + \alpha_i \sum_{k=i+1}^{G} \frac{\beta_k}{\alpha_k}\frac{N_k}{s}\right\} \qquad \forall i \in \mathcal{G}. \tag{16}$$

This indicates that commuters with high value of travel time or high schedule delay cost incur higher bottleneck costs at the short-run equilibrium.

We see from the results of this subsection that the indirect utility (5) is uniquely determined. Therefore, in the following subsection, we characterize the long-run equilibrium by using the properties of the complementarity problems (9).

3.2 Long-run equilibrium

We examine the properties of urban spatial structure at the long-run equilibrium. From (9b) and (10), we have

$$r(x) + r_A = R(I(x)) = \begin{cases} \mu I(x) & \text{if} \quad \mu I(x) \geq r_{\mathrm{A}}, \\ r_{\mathrm{A}} & \text{if} \quad \mu I(x) \leq r_{\mathrm{A}}, \end{cases} \tag{17a}$$

$$I(x) \equiv \sum_{i \in \mathcal{G}} I_i(x) N_i(x), \tag{17b}$$

where $I(x)$ denotes the total income net of commuting cost in location x. Substituting this into (5), the indirect utility is expressed as

$$v_i(x) = (1-\mu)^{1-\mu} I_i(x) \{R(I(x))\}^{-\mu} \tag{18}$$

Therefore, the long-run equilibrium conditions in (9) are rewritten as

$$\begin{cases} v_i(x) = v_i^* & \text{if} \quad N_i(x) > 0 \\ v_i(x) \leq v_i^* & \text{if} \quad N_i(x) = 0 \end{cases} \qquad \forall x \in \mathbb{R}_+, \ \forall i \in \mathcal{G}, \tag{19a}$$

$$\int_0^{X^{\mathrm{B}}} N_i(x) \, \mathrm{d}x = N_i \qquad \forall i \in \mathcal{G}. \tag{19b}$$

The equilibrium conditions (9) or (19) are equivalent to the Karush-Kuhn-Tucker (KKT) conditions of the following optimization problems:

Lemma 2

(a) The spatial distribution $(N_i(x))_{i \in \mathcal{G}}$ of commuters is a long-run equilibrium if and only if it satisfies the KKT conditions of the following optimization problem:

$$\max_{(N_i(x))_{i \in \mathcal{G}}} \ P((N_i(x))_{i \in \mathcal{G}}) = P_1((N_i(x))_{i \in \mathcal{G}}) + P_2((N_i(x))_{i \in \mathcal{G}}) \tag{20a}$$

$$\text{s.t.} \quad \int_0^{\infty} N_i(x) \mathrm{d}x = N_i \quad \forall i \in \mathcal{G}, \tag{20b}$$

$$N_i(x) \geq 0 \quad \forall i \in \mathcal{G}, \forall x \in \mathbb{R}_+, \tag{20c}$$

where $P_1((N_i(x))_{i \in \mathcal{G}})$ and $P_2((N_i(x))_{i \in \mathcal{G}})$ are expressed as

$$P_1((N_i(x))_{i \in \mathcal{G}}) = \int_0^{\infty} \sum_{i \in \mathcal{G}} v(I_i(x), R(I(x))) \, N_i(x) \, \mathrm{d}x, \tag{20d}$$

$$P_2((N_i(x))_{i \in \mathcal{G}}) = (1-\mu)^{-\mu} \mu^{\mu} \int_0^{\infty} \left\{ R(I(x))^{1-\mu} - r_{\mathrm{A}}^{1-\mu} \right\} \mathrm{d}x. \tag{20e}$$

(b) The set of utility level $(v_i^*)_{i \in \mathcal{G}}$ and land rent $r(x) + r_{\mathrm{A}}$ is a long-run equilibrium if and only

if it satisfies the KKT conditions of the following optimization problem:

$$\min_{r(x),(v_i^*)_{i\in\mathcal{G}}} \quad D((v_i^*)_{i\in\mathcal{G}}, r(x)) = D_1((v_i^*)_{i\in\mathcal{G}}) + D_2(r(x)) \tag{21a}$$

$$\text{s.t.} \quad v_i^* \geq v(I_i(x), r(x) + r_\mathrm{A}) \quad \forall i \in \mathcal{G}, \forall x \in \mathbb{R}_+, \tag{21b}$$

$$r(x) \geq 0 \quad \forall x \in \mathbb{R}_+, \tag{21c}$$

where $D_1((v_i^*)_{i\in\mathcal{G}})$ and $D_2(r(x))$ are expressed as

$$D_1((v_i^*)_{i\in\mathcal{G}}) = \sum_{i\in\mathcal{G}} N_i v_i^* \tag{21d}$$

$$D_2(r(x)) = (1-\mu)^{-\mu}\mu^\mu \int_0^\infty \left\{ [r(x) + r_\mathrm{A}]^{1-\mu} - r_\mathrm{A}^{1-\mu} \right\} \mathrm{d}x \tag{21e}$$

Proof The KKT conditions of problem (20) correspond to the long-run equilibrium conditions (19). The KKT conditions of problem (21) correspond to the conditions (9a). Thus, we have Lemma 2.

Note that $P_2((N_i(x))_{i\in\mathcal{G}})$ can be rewritten as

$$P_2((N_i(x))_{i\in\mathcal{G}}) = \int_0^\infty \int_{r_\mathrm{A}}^{R(I(x))} \frac{\partial v(I_i(x), r)}{\partial I_i(x)} \,\mathrm{d}r \,\mathrm{d}x, \tag{22}$$

which represents the revenue from land expressed in utility term. Therefore, we can say that $P((N_i(x))_{i\in\mathcal{G}})$ *denotes the social welfare* and that *Lemma 2 (a) shows the efficiency of land market.*

Note also that $D_1((v_i^*)_{i\in\mathcal{G}})$ and $D_2(r(x))$ can be rewritten as

$$D_1((v_i^*)_{i\in\mathcal{G}}) = \int_0^\infty \sum_{i\in\mathcal{G}} \frac{\partial v(I_i(x), r(x) + r_\mathrm{A})}{\partial I_i(x)} e(r(x) + r_\mathrm{A}, v_i^*) \, N_i(x) \,\mathrm{d}x, \tag{23a}$$

$$D_2(r(x)) = \int_0^\infty \int_{r_\mathrm{A}}^{r(x)+r_\mathrm{A}} \frac{\partial v(I_i(x), r)}{\partial I_i(x)} \,\mathrm{d}r \,\mathrm{d}x, \tag{23b}$$

$$e(r(x) + r_\mathrm{A}, v_i^*) = \frac{v_i^*}{(1-\mu)^{1-\mu}\mu^\mu \{r(x) + r_\mathrm{A}\}^{-\mu}}, \tag{23c}$$

where $e(r, v)$ is the minimum expenditure to attain utility v given the land rent r (i.e., expenditure function). This shows that $D_1((v_i^*)_{i\in\mathcal{G}})$ and $D_2(r(x))$ denote the total expenditure of commuters and the revenue from land, respectively, evaluated in utility terms. Hence, $D((v_i^*)_{i\in\mathcal{G}}, r(x))$ *represents the loss function* (Harris and Wildasin, 1985), which is interpreted as the amount of total disposable income in the economy that can achieve utility level $(v_i^*)_{i\in\mathcal{G}}$ under the land rent $r(x) + r_\mathrm{A}$. Thus, *Lemma 2 (b) shows that the loss is minimized at the long-run equilibrium.*[7]

Since the long-run equilibrium conditions are represented by (19), the model of commuters' location choice can be viewed as *a multiple population game* in which the set of population is $\mathcal{G}$, the set of players of population i is $[0, N_i]$, the common action set is $\mathbb{R}_+$, and the payoff is $(v_i(x))_{i\in\mathcal{G}}$. Furthermore, $P((N_i(x))_{i\in\mathcal{G}})$ is *a potential function* of the game since $\frac{\partial P((N_i(x))_{i\in\mathcal{G}})}{\partial N_i(x)} = v_i(x)$ for all $i \in \mathcal{G}$ and $x \in \mathbb{R}_+$. Therefore, Lemma 2 (a) suggests that *a long-run equilibrium of our model can be considered a Nash equilibrium of the potential game with a continuous player set,* which is

[7] For a discussion of the duality between welfare-maximization and loss-minimization problems, see Harris and Wildasin (1985).

studied in Cheung and Lahkar (2018).

The objective function $P((N_i(x))_{i\in\mathcal{G}})$ of the optimization problem (20) is concave, but it is not strictly concave. This implies that the equilibrium spatial distribution of commuters $(N_i^*(x))_{i\in\mathcal{G}}$ is not necessarily unique. However, by using Lemma 2 (b), we can show the uniqueness of $r(x)$ and $(v_i^*)_{i\in\mathcal{G}}$.

Lemma 3 *The long-run equilibrium land rent $r(x) + r_{\rm A}$ and utility level $(v_i^*)_{i\in\mathcal{G}}$ are uniquely determined.*

Proof See Appendix B.

By using the equilibrium condition (19a), we can see that there is no vacant location between any two populated locations, as shown in Lemma 4.

Lemma 4 *The long-run equilibrium number $\sum_{i\in\mathcal{G}} N_i^*(x)$ of commuters residing at x has the following properties:*

(a) the support of $\sum_{i\in\mathcal{G}} N_i^*(x)$ is given by

$$\mathrm{supp}\,(\textstyle\sum_{i\in\mathcal{G}} N_i^*) = [0, X^{\rm B}], \tag{24}$$

where $X^{\rm B}$ denotes the residential location for commuters farthest from the CBD (i.e., city boundary).

(b) the land rent $r(x) + r_{\rm A}$ satisfies

$$r(x) + r_{\rm A} = \mu I(x) > r_{\rm A} \quad \forall x \in \mathrm{supp}\,(\textstyle\sum_{i\in\mathcal{G}} N_i^*)\backslash\{X^{\rm B}\}, \tag{25a}$$

$$r(X^{\rm B}) + r_{\rm A} = \mu I(X^{\rm B}) = r_{\rm A}. \tag{25b}$$

Proof See Appendix C.

It follows immediately from Lemma 4 that the indirect utility $v_i(x)$ of commuters i is given by

$$v_i(x) = (1-\mu)^{1-\mu} I_i(x)\{I(x_i)\}^{-\mu} \qquad \forall x \in \mathrm{supp}\,(\textstyle\sum_{i\in\mathcal{G}} N_i^*),\ \forall i \in \mathcal{G}. \tag{26}$$

Furthermore, the long-run equilibrium condition (9a) yields

$$v_i(x_i)\cdot v_j(x_j) \geq v_i(x_j)\cdot v_j(x_i) \qquad \forall x_i \in \mathrm{supp}\,(N_i^*),\ \forall x_j \in \mathrm{supp}\,(N_j^*),\ \forall i,j \in \mathcal{G}, \tag{27}$$

where $N_i^*(x)$ denotes the long-run equilibrium number of commuters i residing at x. Substituting (26) into this, we have

$$\left\{\frac{y_i - c_i^{b*}}{\alpha_i} - \frac{y_j - c_j^{b*}}{\alpha_j}\right\}(x_i - x_j) \geq 0 \qquad \forall x_i \in \mathrm{supp}\,(N_i^*),\ \forall x_j \in \mathrm{supp}\,(N_j^*),\ \forall i,j \in \mathcal{G}. \tag{28}$$

This condition implies that if $\frac{I_i(x)}{\alpha_i} > \frac{I_j(x)}{\alpha_j}$, then $x_i \geq x_j$ at the long-run equilibrium,[8] which yields the following proposition.

[8] Let $\Psi_i(x, v_i^*)$ denote bid-rent function of commuters i. Then, as shown in Appendix A.2, $\Psi_i(x, v_i^*)$ is steeper than $\Psi_j(x, v_j^*)$ if and only if the condition $I_i(x)/\alpha_i > I_j(x)/\alpha_j$ holds. Therefore, we can say that Proposition 2 is consistent with the standard results obtained in the literature studying the traditional location model (e.g., Kanemoto, 1980; Fujita, 1989; Duranton and Puga, 2015).

Proposition 2 *Commuters with a high time-based income net of commuting cost ($I_i(x)/\alpha_i$) reside farther from the CBD at the long-run equilibrium.*

This proposition states that *commuters sort themselves spatially depending not only on their income and value of time, but also on their flexibility.* This is because commuters with a high income net of commuting cost consume a larger amount of land and commuters with a high value of time want to reduce their free-flow travel time cost.

Proposition 2 also indicates that if $\frac{y_i - c_i^{b*}}{\alpha_i} \neq \frac{y_j - c_j^{b*}}{\alpha_j}$ for all $i, j \in \mathcal{G}$, $(N_i^*(x))_{i\in\mathcal{G}}$ is uniquely determined. If there exist $i, j \in \mathcal{G}$ such that $\frac{y_i - c_i^{b*}}{\alpha_i} = \frac{y_j - c_j^{b*}}{\alpha_j}$, $(N_i^*(x))_{i\in\mathcal{G}}$ is not unique because the locations of commuters i and j are interchangeable without affecting their utilities.

By using Proposition 2, we examine properties of the long-run equilibrium. For this, we assume, without loss of generality, that commuters with small i have lower time-based income net of commuting cost:

Assumption 4 $\frac{I_{i-1}(x)}{\alpha_{i-1}} \leq \frac{I_i(x)}{\alpha_i}$ *for any* $i \in \mathcal{G}\backslash\{1\}$.

For the moment, we suppose that all commuters $i-1$ reside closer than every commuter i for examining the properties of $r(x)$ and $(v_i^*)_{i\in\mathcal{G}}$ at the long-run equilibrium, each of which is uniquely determined. Let X_i denote the location for commuters i residing nearest the CBD. It follows from Proposition 2 that commuters i reside in $[X_i, X_{i+1}]$ (i.e., $\mathrm{supp}\,(N_i^*) = [X_i, X_{i+1}]$). Therefore, we have $v_i(x) = v_i(X_i)$ for all $x \in \mathrm{supp}\,(N_i^*)$. This, together with the population constraint (19b), yields the following lemma

Lemma 5 *Suppose Assumption 4 and* $\mathrm{supp}\,(N_i^*) = [X_i, X_{i+1}]$ *for any* $i \in \mathcal{G}$. *Then, the long-run equilibrium land rent at location* X_i *is given by*

$$r(X_i) + r_{\mathrm{A}} = r_i \equiv \sum_{k=i}^{G} \alpha_k \tau N_k + r_{\mathrm{A}}. \tag{29}$$

Proof See Appendix D.

Substituting this into (62), we obtain X_i as follows:

$$X_1 = 0, \qquad X_{i+1} = \sum_{j=1}^{i} \left[\{r_{j+1}\}^{-\mu} - \{r_j\}^{-\mu}\right] \{r_{i+1}\}^{\mu} \frac{y_j - c_j^{b*}}{\alpha_j \tau} \quad \forall i \in \mathcal{G}, \tag{30}$$

From these results, we have the following lemma:

Lemma 6 *Suppose Assumption 4. Then, at the long-run equilibrium,*

(a) the city boundary X^{B} *is given by*

$$X^{\mathrm{B}} = \sum_{i\in\mathcal{G}} \left[\{r_{i+1}\}^{-\mu} - \{r_i\}^{-\mu}\right] \{r_{\mathrm{A}}\}^{\mu} \frac{y_i - c_i^{b*}}{\alpha_i \tau} \tag{31}$$

where r_i *is represented as* (29).

(b) the long-run equilibrium utility level $(v_i^*)_{i\in\mathcal{G}}$, *land rent* $r(x) + r_{\mathrm{A}}$, *and lot size* $a_i(x)$ *are given*

by

$$v_i^* = (1-\mu)^{1-\mu}\mu^\mu\alpha_i \left[\{r_{i+1}\}^{-\mu}\frac{y_i - c_i^{b*}}{\alpha_i} - \sum_{j=1}^{i}\left[\{r_{j+1}\}^{-\mu} - \{r_j\}^{-\mu}\right]\frac{y_j - c_j^{b*}}{\alpha_j}\right] \qquad \forall i \in \mathcal{G}, \tag{32a}$$

$$r(x) + r_{\mathrm{A}} = (1-\mu)^{\frac{1-\mu}{\mu}}\mu\left\{\frac{I_i(x)}{v_i^*}\right\}^{\frac{1}{\mu}} \qquad \forall x \in \mathrm{supp}\,(N_i^*), \tag{32b}$$

$$a_i(x) = (1-\mu)^{-\frac{1-\mu}{\mu}}\{I_i(x)\}^{-\frac{1-\mu}{\mu}}\{v_i^*\}^{\frac{1}{\mu}} \qquad \forall x \in \mathrm{supp}\,(N_i^*). \tag{32c}$$

We see from Lemma 6 (a) that the city boundary X^{B} increases with an increase in the time-based income net of bottleneck cost ($\frac{y_i - c_i^{b*}}{\alpha_i}$). This shows that *the spatial size of the city is affected not only by commuters' income and value of time, but also by their flexibility.*

From Lemma 6 (b), we have

$$\frac{\mathrm{d}\{r(x) + r_{\mathrm{A}}\}}{\mathrm{d}x} = -\frac{\alpha_i \tau}{a_i(x)} < 0 \qquad \forall x \in \mathrm{supp}\,(N_i^*), \tag{33}$$

which is known as the Alonso-Muth condition. This states that, at the long-run equilibrium, the marginal commuting cost $\alpha_i\tau$ equals the marginal land cost saving $-\frac{\mathrm{d}\{r(x)+r_{\mathrm{A}}\}}{\mathrm{d}x}a_i(x)$. Thus, the land rent $r(x) + r_{\mathrm{A}}$ decreases with distance x from the CBD.

Lemma 6 (b) also allows us to examine the long-run effect of the bottleneck capacity expansion. It follows from (16) that the short-run equilibrium bottleneck cost c_i^{b*} decreases with the bottleneck capacity s. That is, *in the short-run, the capacity expansion generates a Pareto improvement.* However, we can see by differentiating the equilibrium utility level $(v_i^*)_{i\in\mathcal{G}}$ with respect to the capacity that there can exist $i \in \mathcal{G}$ such that $\frac{\mathrm{d}v_i^*}{\mathrm{d}s} < 0$. More specifically, since we have

$$\frac{\mathrm{d}v_i^*}{\mathrm{d}s} = (1-\mu)^{1-\mu}\mu^\mu\alpha_i\left[-\{r_{i+1}\}^{-\mu}\frac{1}{\alpha_i}\frac{\mathrm{d}c_i^{b*}}{\mathrm{d}s} + \sum_{j=1}^{i}\left[\{r_{j+1}\}^{-\mu} - \{r_j\}^{-\mu}\right]\frac{1}{\alpha_j}\frac{\mathrm{d}c_j^{b*}}{\mathrm{d}s}\right], \tag{34a}$$

$$\frac{\mathrm{d}v_1^*}{\mathrm{d}s} = -(1-\mu)^{1-\mu}\mu^\mu\{r_1\}^{-\mu}\frac{\mathrm{d}c_1^{b*}}{\mathrm{d}s} > 0, \tag{34b}$$

$$\frac{1}{\alpha_{i-1}}\frac{\mathrm{d}v_{i-1}^*}{\mathrm{d}s} > \frac{1}{\alpha_i}\frac{\mathrm{d}v_i^*}{\mathrm{d}s} \qquad \forall i \in \mathcal{G}\backslash\{1\}, \tag{34c}$$

the capacity expansion cannot lead to a Pareto improvement in the long-run if there exists $i \in \mathcal{G}$ such that

$$\frac{\{r_{i+1}\}^{-\mu}}{\alpha_i}\frac{\mathrm{d}c_i^{b*}}{\mathrm{d}s} > \sum_{j=1}^{i}\frac{\{r_{j+1}\}^{-\mu} - \{r_j\}^{-\mu}}{\alpha_j}\frac{\mathrm{d}c_j^{b*}}{\mathrm{d}s}. \tag{35}$$

That is, if (35) holds for some i, *commuters residing closer to the CBD gain, but those residing farther from the CBD lose from the capacity expansion.* This is due to the fact that the expansion increases the city boundary X^{B}, thereby increasing commuting distance of commuters residing farther from the CBD.

The results obtained thus far are summarized as follows.

Proposition 3 *The bottleneck capacity expansion generates a Pareto improvement in the short-*

run, but it can lead to an unbalanced distribution of benefits in the long-run: commuters residing closer to the CBD gain and those residing farther from the CBD lose.

4 Optimal time-varying congestion toll

Studies utilizing the standard bottleneck model show that queuing time is a pure deadweight loss. Hence, in our model, there is no queue at the social optimum, and the social optimum is achieved by imposing an optimal time-varying congestion toll (e.g., Arnott, 1998; Gubins and Verhoef, 2014; Takayama and Kuwahara, 2017). This section examines the effect of introducing an optimal congestion toll $p(t)$ by analyzing equilibrium under this pricing policy.

4.1 Short-run equilibrium

An optimal time-varying congestion toll $p(t)$ eliminates queuing congestion. Thus, the commuting cost $c_i^o(x,t)$ of commuters i is given by

$$c_i^o(x,t) = c_i^{bo}(t) + \alpha_i \tau x, \tag{36a}$$

$$c_i^{bo}(t) \equiv p(t) + d_i(t - t^*). \tag{36b}$$

Superscript o describes variable under the optimal congestion toll.

Since we consider heterogeneous commuters, the congestion toll $p(t)$ does not equal the queuing time cost $\alpha_i q(t)$ at the no-toll equilibrium, and it is set so that travel demand $n^o(t)$ at the bottleneck equals supply (i.e., capacity) s. Therefore, the short-run equilibrium conditions are expressed as

$$\begin{cases} c_i^{bo}(t) = c_i^{bo*} & \text{if} \quad n_i^o(t) > 0 \\ c_i^{bo}(t) \geq c_i^{bo*} & \text{if} \quad n_i^o(t) = 0 \end{cases} \qquad \forall i \in \mathcal{G},\ \forall t \in \mathbb{R}, \tag{37a}$$

$$\begin{cases} \sum_{i \in \mathcal{G}} n_i^o(t) = s & \text{if} \quad p(t) > 0 \\ \sum_{i \in \mathcal{G}} n_i^o(t) \leq s & \text{if} \quad p(t) = 0 \end{cases} \qquad \forall t \in \mathbb{R}, \tag{37b}$$

$$\int n_i^o(t)\ \mathrm{d}t = N_i \qquad \forall i \in \mathcal{G}. \tag{37c}$$

Condition (37a) is the no-arbitrage condition for commuters' arrival time choices. Condition (37b) denotes the bottleneck capacity constraints, which assure that queuing congestion is eliminated at the equilibrium. Condition (37c) provides the flow conservation for commuting demand. These conditions give $n_i^o(t), p(t), c_i^{bo*}$ at the short-run equilibrium.

As in the case without the congestion toll, by invoking the results of studies employing the bottleneck model, we have the following lemma.

Lemma 7 (Lindsey, 2004; Iryo and Yoshii, 2007) *Suppose Assumption 1 (i). Then, the short-run equilibrium under the congestion toll has the following properties:*

(a) The bottleneck cost c_i^{bo*} is uniquely determined.

(b) The short-run equilibrium number $(n_i^{o*}(t))_{i \in \mathcal{G}}$ of commuters arriving at time t coincides with

the solution of the following linear programming problem:

$$\min_{(n_i^o(t))_{i\in\mathcal{G}}} \sum_{i\in\mathcal{G}} \int d_i(t-t^*)\, n_i^o(t)\ \mathrm{d}t \tag{38a}$$

$$s.t. \quad \sum_{i\in\mathcal{G}} n_i^o(t) \leq s \quad \forall t \in \mathbb{R}, \tag{38b}$$

$$\int n_i^o(t)\ \mathrm{d}t = N_i \quad \forall i \in \mathcal{G}, \tag{38c}$$

$$n_i^o(t) \geq 0 \quad \forall i \in \mathcal{G},\ \forall t \in \mathbb{R}. \tag{38d}$$

Lemma 7 (b) suggests that total schedule delay cost is minimized at the short-run equilibrium under the congestion toll. Note that total schedule delay cost equals total commuting cost minus total toll revenue. Hence, Lemma 7 (b) indicates that, *in the short-run, the optimal congestion toll minimizes the social cost of commuting.*

From the short-run equilibrium condition (37a), we have

$$c_i^{bo}(t_i) + c_j^{bo}(t_j) \leq c_i^{bo}(t_j) + c_j^{bo}(t_i) \qquad \forall t_i \in \mathrm{supp}\,(n_i^{o*}),\ \forall t_j \in \mathrm{supp}\,(n_j^{o*}),\ \forall i,j \in \mathcal{G}. \tag{39}$$

Substituting (36b) into this, we have

$$\begin{cases} (\beta_i - \beta_j)\,(t_i - t_j) \geq 0 & \text{if} \quad \max\{t_i, t_j\} \leq t^*, \\ (\gamma_i - \gamma_j)\,(t_i - t_j) \leq 0 & \text{if} \quad \min\{t_i, t_j\} \geq t^*. \end{cases} \tag{40}$$

Therefore, we obtain the following proposition.

Proposition 4 *Suppose Assumption 1. Then, at the short-run equilibrium, commuters with a high marginal schedule delay cost (β_i) arrive closer to their preferred arrival time t^*.*

Propositions 1 and 4 show that *the equilibrium bottleneck cost under the congestion toll c_i^{bo*} generally differs from the no-toll equilibrium bottleneck cost c_i^{b*} when we consider commuter heterogeneity in the value of time.* To see this concretely, we assume, without loss of generality, that commuters with small i have a (weakly) higher marginal schedule delay cost:

Assumption 5 $\beta_{i-1} \geq \beta_i$ *for any* $i \in \mathcal{G}\backslash\{1\}$.

Then, we can obtain the short-run equilibrium bottleneck cost c_i^{bo*} and commuting cost $c_i^{o*}(x)$ under the toll in the same manner as in (16).

$$c_i^{bo*} = \frac{\eta}{1+\eta}\left\{\beta_i \frac{\sum_{k=1}^{i} N_k}{s} + \sum_{k=i+1}^{G} \beta_k \frac{N_k}{s}\right\} \qquad \forall i \in \mathcal{G}, \tag{41a}$$

$$c_i^{o*}(x) = c_i^{bo*} + \alpha_i \tau x. \tag{41b}$$

This shows that inflexible commuters have higher bottleneck costs at the equilibrium under the toll, which is fundamentally different from the properties of the no-toll equilibrium bottleneck cost c_i^{b*}.

4.2 Long-run equilibrium

We characterize the urban spatial structure at the long-run equilibrium under the toll by using the short-run equilibrium bottleneck cost c_i^{bo*}. In the long-run, the difference between cases with and without tolling appears only in the income net of commuting cost. Specifically, under the congestion toll, the income net of commuting cost is expressed as

$$I_i^o(x) \equiv y_i - c_i^{o*}(x), \qquad I^o(x) \equiv \sum_{i\in\mathcal{G}} I_i^o(x) N_i(x). \tag{42}$$

The long-run equilibrium conditions are thus represented as (9) with the use of (42).

Let us introduce the following assumption, as in the case without tolling.

Assumption 6 $\frac{I_{i-1}^o(x)}{\alpha_{i-1}} \leq \frac{I_i^o(x)}{\alpha_i}$ *for any* $i \in \mathcal{G}\backslash\{1\}$.

Then, following the same procedure as in Section 3.2 reveals that the urban spatial structure at the long-run equilibrium under the congestion toll has the same properties as the case without tolling.

Lemma 8 *Under the congestion toll, the long-run equilibrium has the following properties.*

(a) The spatial distribution $(N_i^o(x))_{i\in\mathcal{G}}$ *of commuters is a long-run equilibrium if and only if it satisfies the KKT conditions of the following optimization problem:*

$$\max_{(N_i(x))_{i\in\mathcal{G}}} P^o((N_i(x))_{i\in\mathcal{G}}) = P_1^o((N_i(x))_{i\in\mathcal{G}}) + P_2^o((N_i(x))_{i\in\mathcal{G}}) \tag{43a}$$

$$\text{s.t.} \quad \int_0^\infty N_i(x)\mathrm{d}x = N_i \quad \forall i \in \mathcal{G}, \tag{43b}$$

$$N_i(x) \geq 0 \quad \forall i \in \mathcal{G}, \forall x \in \mathbb{R}_+, \tag{43c}$$

where $P_1^o((N_i(x))_{i\in\mathcal{G}})$ *and* $P_2^o((N_i(x))_{i\in\mathcal{G}})$ *are expressed as*

$$P_1^o((N_i(x))_{i\in\mathcal{G}}) = \int_0^\infty \sum_{i\in\mathcal{G}} v(I_i^o(x), R(I^o(x)))\, N_i(x)\, \mathrm{d}x, \tag{43d}$$

$$P_2^o((N_i(x))_{i\in\mathcal{G}}) = (1-\mu)^{-\mu}\mu^\mu \int_0^\infty \left\{R(I^o(x))^{1-\mu} - r_{\mathrm{A}}^{1-\mu}\right\} \mathrm{d}x. \tag{43e}$$

(b) Let $\mathrm{supp}\,(N_i^{o*})$ *be the support of the long-run equilibrium number* $N_i^{o*}(x)$ *of commuters residing at* x*. Then, for any* $x_i \in \mathrm{supp}\,(N_i^{o*})$ *and* $x_j \in \mathrm{supp}\,(N_j^{o*})$*,*

$$\left\{\frac{y_i - c_i^{bo*}}{\alpha_i} - \frac{y_j - c_j^{bo*}}{\alpha_j}\right\}(x_i - x_j) \geq 0. \tag{44}$$

(c) Suppose Assumption 6. Then, the city boundary X^{B} *and equilibrium utility level* $(v_i^{o*})_{i\in\mathcal{G}}$ *are uniquely determined and are given by*

$$X^{o\mathrm{B}} = \sum_{i\in\mathcal{G}} \left[\{r_{i+1}\}^{-\mu} - \{r_i\}^{-\mu}\right]\{r_{\mathrm{A}}\}^\mu \frac{y_i - c_i^{bo*}}{\alpha_i \tau}, \tag{45a}$$

$$v_i^{o*} = (1-\mu)^{1-\mu}\mu^{\mu}\alpha_i \left[\{r_{i+1}\}^{-\mu}\frac{y_i - c_i^{bo*}}{\alpha_i} - \sum_{j=1}^{i}\left[\{r_{j+1}\}^{-\mu} - \{r_j\}^{-\mu}\right]\frac{y_j - c_j^{bo*}}{\alpha_j}\right] \quad \forall i \in \mathcal{G}, \tag{45b}$$

where r_i is represented as (29).

Lemma 8 (a), together with Lemma 7 (b), demonstrates that the equilibrium with tolling corresponds to the social optimum given that the social cost of commuting is minimized in the short-run and the social welfare is maximized in the long-run.

Lemma 8 (b) shows that as in the case without tolling, commuters with a high income net of commuting cost reside farther from the CBD. This and Proposition 2 clearly indicate that an optimal congestion toll generally alters the urban spatial structure.

From Lemma 8 (c), we can say that the capacity expansion causes the city to physically expand outward. Furthermore, although the expansion generates a Pareto improvement in the short-run, it does not necessarily lead to a Pareto improvement in the long-run like the case without tolling.

This lemma yields the following proposition.

Proposition 5

(a) Commuters with a high time-based income net of commuting cost ($I_i^o(x)/\alpha_i$) reside farther from the CBD at the long-run equilibrium under an optimal time-varying congestion toll.

(b) Imposing an optimal time-varying congestion toll alters the urban spatial structure if commuters are heterogeneous in their value of time.

(c) The bottleneck capacity expansion generates a Pareto improvement in the short-run, but it can lead to an unbalanced distribution of benefits in the long-run: commuters residing closer to the CBD gain and those residing farther from the CBD lose.

5 Comparison between equilibria with and without tolling

5.1 Short- and long-run equilibria

In the previous sections, we have investigated the properties of equilibria with and without tolling and have shown that the urban spatial structure changes with the imposition of an optimal congestion toll. This section considers a simple setting to demonstrate the effects of the congestion toll on the urban spatial structure. Specifically, we suppose Assumptions 1–6.

Under this setting, commuters with small i are inflexible and have a high marginal time-based schedule delay cost. Therefore, they are willing to pay in travel time or money to reduce schedule delay, thereby arriving closer to their preferred arrival time t^* at the short-run equilibrium. The difference between short-run equilibrium bottleneck costs with and without tolling is thus given by

$$c_i^{bo*} - c_i^{b*} = \frac{\eta}{1+\eta}\sum_{k=i+1}^{G}(\alpha_k - \alpha_i)\frac{\beta_k}{\alpha_k}\frac{N_k}{s} \qquad \forall i \in \mathcal{G}. \tag{46}$$

This clearly shows that the sign of $c_i^{bo*} - c_i^{b*}$ depends on the difference in commuters' value of time.

Commuters with small i have a low time-based income net of commuting costs both before and after imposing the toll. This implies that they reside closer to the CBD at the long-run equilibrium. Therefore, we have

$$X^{o\mathrm{B}} - X^{\mathrm{B}} = \sum_{i \in \mathcal{G}} \left[\{r_{i+1}\}^{-\mu} - \{r_i\}^{-\mu} \right] \{r_{\mathrm{A}}\}^{\mu} \frac{c_i^{b*} - c_i^{bo*}}{\alpha_i \tau}, \tag{47a}$$

$$v_i^{o*} - v_i^{*} = (1-\mu)^{1-\mu} \mu^{\mu} \left[\{r_{i+1}\}^{-\mu} \{c_i^{b*} - c_i^{bo*}\} - \sum_{j=1}^{i} \left[\{r_{j+1}\}^{-\mu} - \{r_j\}^{-\mu} \right] \frac{\alpha_i}{\alpha_j} \{c_j^{b*} - c_j^{bo*}\} \right] \quad \forall i \in \mathcal{G}. \tag{47b}$$

(47a) indicates that the spatial size of the city can expand or shrink by imposing the toll due to changes in the short-run bottleneck cost. (47b) shows that the difference between the bottleneck costs with and without tolling affects the commuters' benefits from the imposition of the toll.

The difference of the equilibrium utility level (47b) also shows that even if tolling generates a Pareto improvement in the short-run (i.e., $c_i^{bo*} \leq c_i^{b*}$ for all $i \in \mathcal{G}$), it does not necessarily leads to a Pareto improvement in the long-run (i.e., $v_i^{o*} \geq v_i^{*}$ for all $i \in \mathcal{G}$). This can occur in the following mechanism: improvements in the bottleneck cost increase the income net of commuting cost and the lot size of commuters residing near the CBD; this causes the city to expand outward; the spatial expansion of the city increases the commuting distance of commuters residing farther from the CBD, which decreases their income net of commuting cost.

To see the effects of tolling more concretely, we introduce an additional assumption on the value of time in the following subsection. Specifically, we analyze the following two cases:

Case A: rich commuters are flexible

Case B: rich commuters are highly inflexible

5.2 Simple examples

5.2.1 Case A: rich commuters are flexible

We first introduce the following assumption in addition to Assumptions 1–6.

Assumption 7 $\alpha_{i-1} < \alpha_i$ *for all* $i \in \mathcal{G} \backslash \{1\}$.

Note that Assumptions 2–7 are not too restrictive. Indeed, if the income y_i of commuters i is proportional to their value of time α_i (i.e., $y_i = \phi \alpha_i$), Assumptions 5 and 7 are sufficient conditions for these assumptions to hold.

In Case A, rich commuters are flexible and have a lower marginal time-based schedule delay cost. This implies that rich commuters tend to avoid queuing time and paying the toll rather than schedule delay. Thus, they arrive farther from their preferred arrival time t^* at the short-run equilibria with and without tolling. The short-run equilibrium bottleneck costs with and without tolling satisfy

$$c_{i-1}^{bo*} - c_{i-1}^{b*} > c_i^{bo*} - c_i^{b*} \qquad \forall i \in \mathcal{G} \backslash \{1\}, \tag{48a}$$

$$c_G^{bo*} - c_G^{b*} = 0. \tag{48b}$$

We see from (48) that *optimal congestion tolling increases short-run equilibrium bottleneck costs of all commuters other than richest ones.* This reflects the fact that poor commuters pay a higher toll and that the richest commuters are those who face no queuing cost at the equilibrium without tolling and face no toll at the equilibrium with tolling.

The toll decreases the income net of commuting cost, which leads to a decrease in lot size and the spatial size of city. This can be seen by substituting (48) into (47a). This means that *the city becomes denser with tolling,* which is same as the standard results of traditional location models considering static congestion (Kanemoto, 1980; Wheaton, 1998; Anas et al., 1998).

We see from (47b) and (48) that the difference between the equilibrium utility levels with and without tolling satisfy

$$v_{i-1}^{o*} - v_{i-1}^{*} < v_i^{o*} - v_i^{*} \qquad \forall i \in \mathcal{G}\backslash\{1\}, \tag{49a}$$

$$v_1^{o*} - v_1^{*} < 0, \tag{49b}$$

$$v_G^{o*} - v_G^{*} > 0. \tag{49c}$$

This shows that *rich commuters gain and poor commuters lose from tolling in Case A.* This occurs because the spatial shrinkage of the city reduces the commuting distance, which helps commuters residing farther from the CBD.

These results establish the following proposition.

Proposition 6 *Suppose Assumptions 1–7. Then,*

(a) rich commuters arrive farther from their preferred arrival time and reside farther from the CBD at the equilibria with and without tolling;

(b) an optimal congestion tolling weakly increases the bottleneck costs of all commuters, which causes the city to become denser and more compact;

(c) rich commuters gain and poor commuters lose from imposing the toll.

5.2.2 Case B: rich commuters are highly inflexible

In Case B, we assume that rich commuters are inflexible and have a higher marginal time-based schedule delay cost. That is, we suppose Assumptions 1–6 and 8.

Assumption 8 $\alpha_{i-1} > \alpha_i$ *for all* $i \in \mathcal{G}\backslash\{1\}$.

Note that Assumptions 2–6 and 8 are also not too restrictive. Indeed, if the income y_i of commuters i is given by $\phi\alpha_i + \psi$ with $\phi > 0$ and $\psi > c_1^{bo*}$, the sufficient conditions for these assumptions to hold are given by Assumptions 3 and 8.

In Case B, rich commuters are willing to pay in travel time or money to reduce schedule delay, thereby arriving closer to their preferred arrival time t^* at the short-run equilibria with and without tolling. Thus, the short-run equilibrium bottleneck costs with and without tolling satisfy

$$c_{i-1}^{bo*} - c_{i-1}^{b*} < c_i^{bo*} - c_i^{b*} \qquad \forall i \in \mathcal{G}\backslash\{1\}, \tag{50a}$$

$$c_G^{bo*} - c_G^{b*} = 0. \tag{50b}$$

The conditions in (50) shows that, *in the short-run, a Pareto improvement is achieved by imposing an optimal congestion toll.* This happens because rich commuters experience larger queuing time at the no-toll equilibrium and imposing the toll eliminates all queuing.

The conditions in (50) also indicate that the toll increases their income net of commuting cost. This leads to increases in their lot size, thereby increasing the city boundary (i.e., $X^{o\mathrm{B}} > X^{\mathrm{B}}$). This can be confirmed by substituting (50) into (47a). This means that *the city becomes less dense with tolling,* which contrasts with the standard results of traditional location models that consider static congestion.

By using (50), we obtain the following conditions on the equilibrium utility level, which reveal that *rich commuters gain and poor commuters lose from tolling in Case B.*

$$v_{i-1}^{o*} - v_{i-1}^{*} > v_i^{o*} - v_i^{*} \qquad \forall i \in \mathcal{G}\backslash\{1\}, \tag{51a}$$

$$v_1^{o*} - v_1^{*} > 0, \tag{51b}$$

$$v_G^{o*} - v_G^{*} < 0. \tag{51c}$$

This is due to the fact that the spatial expansion of the city increases the commuting distance, thereby increasing commuting cost of poor commuters who reside farther from the CBD.

We summarize the results as the following proposition.

Proposition 7 *Suppose Assumptions 1–6 and 8. Then,*

(a) rich commuters arrive closer to their preferred arrival time and reside closer to the CBD at the equilibria with and without tolling.

(b) an optimal congestion tolling generates a Pareto improvement in the short-run, but it causes the city to become less dense and to spatially expand outward in the long-run.

(c) rich commuters gain and poor commuters lose from imposing the toll.

5.3 Numerical examples

We numerically analyze the model to show the effects of an optimal congestion toll and the bottleneck capacity expansion. In this analysis, we use the following parameter values:

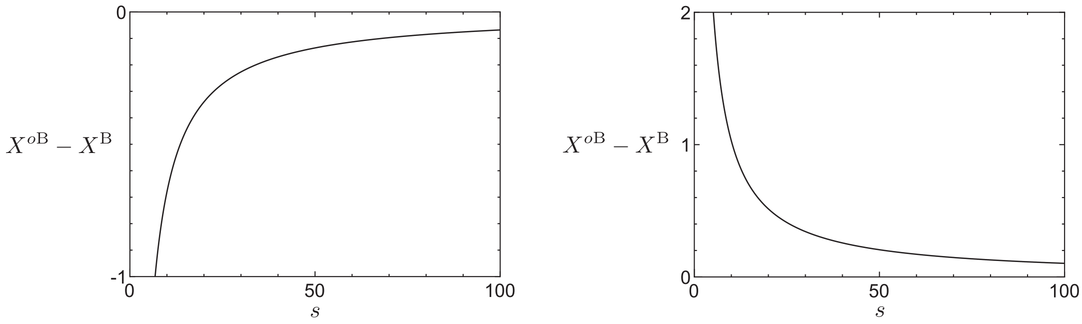
$$G = 4, \quad \mu = 0.2, \quad \tau = 2 \text{ (min/km)}, \quad r_{\mathrm{A}} = 500, \quad (N_i)_{i\in\mathcal{G}} = (200, 1000, 1000, 200). \tag{52a}$$

The values of y_i, α_i, β_i, η are set to be consistent with Assumptions 2–8 and the empirical result (Small, 1982):

$$\text{Case A:} \quad \begin{cases} (y_i)_{i\in\mathcal{G}} = (90, 120, 150, 240), & (\alpha_i)_{i\in\mathcal{G}} = (0.3, 0.4, 0.5, 0.8), \\ (\beta_i)_{i\in\mathcal{G}} = (0.28, 0.25, 0.15, 0.05), & \eta = 4.0. \end{cases} \tag{52b}$$

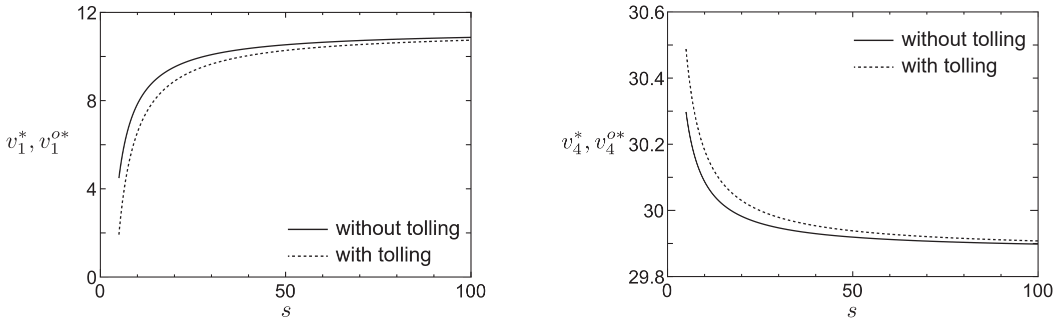
$$\text{Case B:} \quad \begin{cases} (y_i)_{i\in\mathcal{G}} = (195, 190, 185, 180), & (\alpha_i)_{i\in\mathcal{G}} = (0.45, 0.4, 0.35, 0.3), \\ (\beta_i)_{i\in\mathcal{G}} = (0.4, 0.35, 0.3, 0.05), & \eta = 4.0. \end{cases} \tag{52c}$$

We conduct comparative statics with respect to bottleneck capacity s. As we can see from Figure 2, imposing the toll results in a denser urban spatial structure in Case A, whereas it leads to spatial expansion of the city in Case B. Figures 3 and 4 indicate that, in both cases, congestion



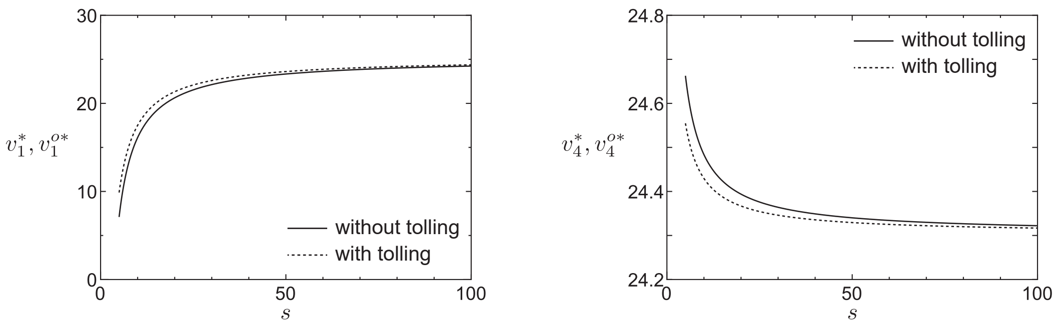
(a) Case A (b) Case B

Figure 2: Difference between city boundaries with and without tolling



(a) commuters 1 (the poorest commuters) (b) commuters 4 (the richest commuters)

Figure 3: Utility of commuters in Case A



(a) commuters 1 (the richest commuters) (b) commuters 4 (the poorest commuters)

Figure 4: Utility of commuters in Case B

tolling leads to an unbalanced distribution of benefits among commuters: rich commuters gain and poor commuters lose. These results are consistent with those presented in Section 5.2.

Figures 3 (b) and 4 (b) also show that $\frac{\mathrm{d}v_4^*}{\mathrm{d}s} < 0$ and that $\frac{\mathrm{d}v_4^{o*}}{\mathrm{d}s} < 0$. This implies that a Pareto improvement is not achieved by expanding the bottleneck capacity in both cases. More specifically, commuters residing farthest from the CBD lose from a capacity improvement. This is also consistent with Propositions 3 and 5.

6 Conclusion

This study develops a model in which heterogeneous commuters choose their departure times from home and residential locations in a monocentric city. By using the properties of the complementarity problems, we systematically examine equilibrium urban spatial structure. Our analysis shows that commuters temporally and spatially sort according to their income, value of time, and flexibility. Furthermore, although the bottleneck capacity expansion generates a Pareto improvement when commuters do not relocate, it can lead to an unbalanced distribution of benefits among commuters: commuters residing closer to the CBD gain and commuters residing farther from the CBD lose. We further reveal that the imposition of an optimal congestion toll causes the city to spatially shrink or expand—this can help rich commuters but hurt poor commuters. Since these findings are fundamentally different from results obtained in Arnott (1998), Gubins and Verhoef (2014), and Fosgerau et al. (2018) that consider homogeneous commuters, it is important to consider commuters' heterogeneity when examining the effectiveness of transportation policies intended to alleviate peak-period congestion.

In this paper, we consider a city with a single bottleneck. Therefore, we need to examine the robustness of our result by analyzing a model with multiple bottlenecks.[9] In addition, it would be valuable for future research to investigate effects of policies other than optimal congestion tolling, such as step tolls (Arnott et al., 1990a; Laih, 1994, 2004; Lindsey et al., 2012) and TDM measures (Mun and Yonekawa, 2006; Takayama, 2015) for alleviating traffic congestion.

A Equivalence between the bid-rent and complementarity approaches

A.1 Equilibrium conditions

We show that long-run equilibrium conditions (9) coincide with those of the bid-rent approach. The condition (9a) can be rewritten as

$$\begin{cases} r(x) + r_{\mathrm{A}} = \Psi_i(x, v_i^*) & \text{if} \quad N_i(x) > 0 \\ r(x) + r_{\mathrm{A}} \geq \Psi_i(x, v_i^*) & \text{if} \quad N_i(x) = 0 \end{cases} \quad \forall x \in \mathbb{R}_+, \quad \forall i \in \mathcal{G}. \tag{53}$$

$\Psi_i(x, v_i^*)$ is given by

$$\Psi_i(x, v_i^*) = \left\{ \frac{(1-\mu)^{1-\mu}\mu^{\mu} I_i(x)}{v_i^*} \right\}^{\frac{1}{\mu}}. \tag{54}$$

Furthermore, since $\max_{a_i}[I_i(x) - \{v_i^*\}^{1/(1-\mu)} a_i^{-\mu/(1-\mu)}]/a_i = \Psi_i(x, v_i^*)$,[10] $\Psi_i(x, v_i^*)$ can be interpreted as the bid-rent function of commuters i.[11] This shows that conditions in (9b), (9c), and (53) are the equilibrium conditions of the bid-rent approach (see, e.g., Fujita, 1989, Definition 4.2).

[9] Kuwahara (1990) and Akamatsu et al. (2015) have shown properties of equilibrium of a bottleneck model with multiple bottlenecks.

[10] $\{v^*\}^{1/(1-\mu)} a_i^{-\mu/(1-\mu)}$ represents the amount of numéraire good that is necessary to achieve utility level v^* when the lot size of the house is a_i.

[11] As shown in, e.g., Fujita (1989), this maximization problem defines the bid-rent function.

A.2 Relative steepness of bid-rent curves

As is shown in Fujita (1985), we can say that $\Psi_i(x, v_j^*)$ is steeper than $\Psi_j(x, v_i^*)$ if and only if the following condition holds:

$$\frac{\partial \Psi_i(x, v_i^*)}{\partial x} < \frac{\partial \Psi_j(x, v_j^*)}{\partial x} \quad \text{whenever} \quad \Psi_i(x, v_i^*) = \Psi_j(x, v_j^*). \tag{55}$$

Differentiating the bid-rent function $\Psi_i(x, v_i^*)$ with respect to location x, we have

$$\frac{\partial \Psi_i(x, v_i^*)}{\partial x} = -\frac{\Psi_i(x, v_i^*)^{1/\mu}}{\mu} \frac{\alpha_i \tau}{I_i(x)}. \tag{56}$$

Therefore, the condition (55) can be rewritten as

$$\frac{I_i(x)}{\alpha_i} > \frac{I_j(x)}{\alpha_j}. \tag{57}$$

B Proof of Lemma 3

The optimization problem (21) is equivalent to

$$\min_{r(x)} \sum_{i \in \mathcal{G}} N_i \max_x v(I_i(x), r(x) + r_{\mathrm{A}}) + D_2(r(x)) \tag{58a}$$

$$\text{s.t.} \quad r(x) \geq 0 \quad \forall x \in \mathbb{R}_+. \tag{58b}$$

Since the objective function of this problem is strictly convex, $r(x)$ is uniquely determined. Furthermore, the uniqueness of $r(x)$ implies that the indirect utility $v(I_i(x), r(x) + r_{\mathrm{A}})$ is uniquely determined. Therefore, $(v_i^*)_{i \in \mathcal{G}}$ is also uniquely determined.

C Proof of Lemma 4

For any $x^a, x^b (> x^a) \in \operatorname{supp}(\sum_{i \in \mathcal{G}} N_i^*)$, there is no $x^c \in (x^a, x^b)$ such that $\sum_{i \in \mathcal{G}} N_i^*(x^c) = 0$ since the indirect utility is given by (18). Thus, we obtain Lemma 4 (a).

Differentiating the indirect utility with respect to location x, we have

$$\frac{\mathrm{d}v_i(x)}{\mathrm{d}x} = \begin{cases} v_i(x) \left\{ -\frac{\alpha_i \tau}{I_i(x)} - \frac{\mu}{I(x)} \frac{\mathrm{d}I(x)}{\mathrm{d}x} \right\} & \text{if} \quad \mu I(x) \geq r_{\mathrm{A}}, \\ v_i(x) \left\{ -\frac{\alpha_i \tau}{I_i(x)} \right\} < 0 & \text{if} \quad \mu I(x) \leq r_{\mathrm{A}}. \end{cases} \tag{59a}$$

Therefore, at the long-run equilibrium, the total income net of commuting costs satisfies

$$\begin{cases} \frac{\mathrm{d}I(x)}{\mathrm{d}x} < 0 \\ \mu I(x) \geq r_{\mathrm{A}} \end{cases} \qquad \forall x \in \operatorname{supp}(\textstyle\sum_{i \in \mathcal{G}} N_i^*). \tag{60}$$

Furthermore, it follows from the long-run equilibrium condition (9a) that $I(X^{\mathrm{B}})$ also satisfies

$$\mu I(X^{\mathrm{B}}) = r_{\mathrm{A}}. \tag{61}$$

Thus, we have Lemma 4 (b).

D Proof of Lemma 5

At the long-run equilibrium, the indirect utility (5) satisfies $v_i(x) = v_i(X_{i+1})$ for all $x \in \mathrm{supp}\,(N_i^*)$, and this condition gives

$$\frac{r(x) + r_{\mathrm{A}}}{r(X_{i+1}) + r_{\mathrm{A}}} = \left\{ \frac{I_i(x)}{I_i(X_{i+1})} \right\}^{\frac{1}{\mu}}. \tag{62}$$

Furthermore, from (9b) and Proposition 2, we have $a_i(x) = \frac{1}{N_i^*(x)}$. Substituting these into (4), we obtain $N_i^*(x)$ as follows:

$$N_i^*(x) = \frac{1}{\mu}\{I_i(x)\}^{\frac{1-\mu}{\mu}}\{I_i(X_{i+1})\}^{-\frac{1}{\mu}}\{r(X_{i+1}) + r_{\mathrm{A}}\}. \tag{63}$$

Therefore, the population constraint (19b) can be rewritten as

$$\begin{aligned} N_i &= -\frac{1}{\alpha_i \tau}\{r(X_{i+1}) + r_{\mathrm{A}}\}\left[1 - \left\{\frac{I_i(X_i)}{I_i(X_{i+1})}\right\}^{\frac{1}{\mu}}\right] \\ &= -\frac{r(X_{i+1}) - r(X_i)}{\alpha_i \tau}. \end{aligned} \tag{64}$$

Since $r(X_{I+1}) = 0$, we have Lemma 5.

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