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Cortes-Corrales, Sebastián and Gorny, Paul M.

University of Leicester, Universtiy of East Anglia

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Sebastián Cortes-Corrales[‡] and Paul M. Gorny[§]

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Abstract

We investigate the behaviour of agents in bilateral contests within arbitrary network structures when valuations and efficiencies are heterogenous. These parameters are interpreted as measures of strength. We provide conditions for when unique, pure strategy equilibria exist. When a player starts attacking one player more strongly, others join in on fighting the victim. Different efficiencies in fighting make players fight those of similar strength. Centrality of a player (having more enemies) makes a player weaker and her opponents are more likely to attack with more effort.

Keywords: Contest, networks, optimal allocation, games on networks

JEL: C72; D74; D85

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[‡]University of Leicester, scc37@le.ac.uk

[§]University of East Anglia, P.Gorny@uea.ac.uk

1 Introduction

Competition takes the most vigorous form when the parties involved do not use resources for production or consumption, but rather to disable, destroy or appropriate resources from others (Hirshleifer, 1995; Sandler, 2000). The resources employed for these goals, in the form of soldiers, military equipment and time spent are sunk, irrespective of the final outcome. This form of competition can broadly be defined as conflict. It is this wasteful nature and the strategic considerations that spurred the interest of economists and game theorists. The theoretical contributions in this domain are typically built on models with a single conflict with two or more parties. Advancements in transportation and information technology allow states and other international, and potentially militant interest groups to engage in multiple conflicts around the globe. That has added more complexity in how such agents are related to each other. In most of the existing models it is impossible to distinguish a fight *for* a single prize from a fight *against* specific enemies. The motivation of agents is not to fight someone specific within the 'aggregate others', although in reality individuals, political groups and nation states typically have a sense of who each of their opponents are.

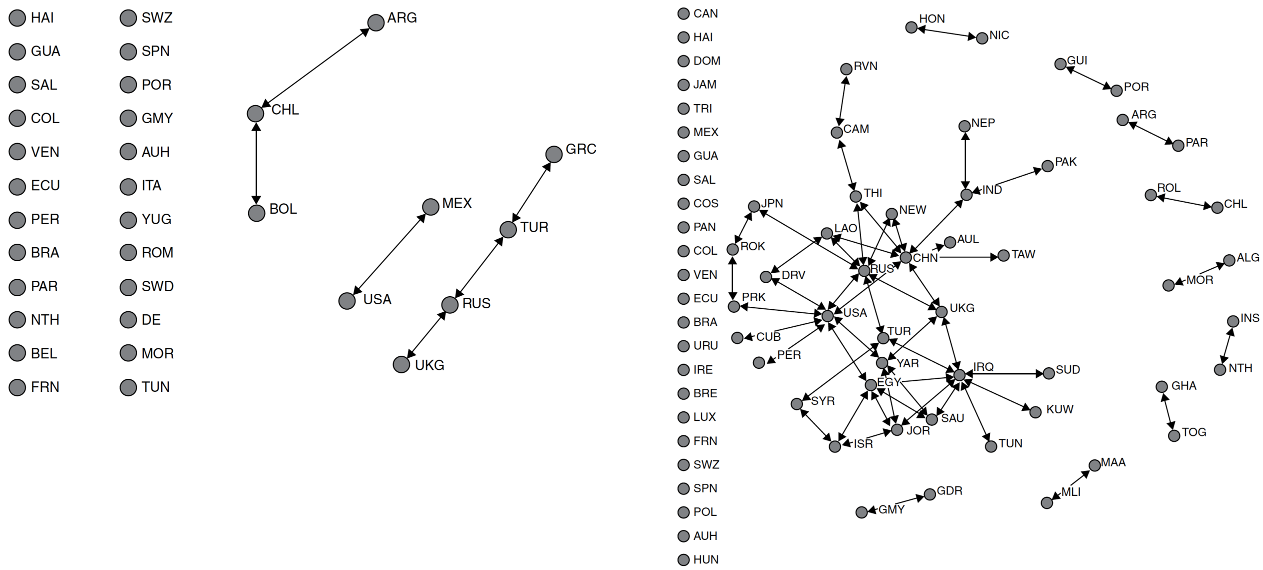
The aim of this paper is to develop a framework with multiple interconnected opponents, to understand how the differences between rivals (i.e position in the conflict structure, efficiency in the conflict technology and prizes within and across the bilateral conflicts) shape the optimal strategies in conflict games.

Two considerations give rise to the type of model we suggest. On the one hand, conflict is characterised by sunk effort investments aiming at increasing the probability of winning a prize. This prize can be land, power or natural resources in the case of war or market share and influence in the case of marketing and lobbying, respectively. This trade-off is frequently modelled by a contest (e.g. Konrad, 2009; Vojnovic, 2016).

On the other hand, conflicts often have a structure of multiple, simultaneous battlefields between the different parties involved. Just as much as strength, the degree of centrality mattered when Germany engaged in battles on multiple fronts in WWI and WWII. These considerations give rise to a network of conflictual links between agents, where each link represents a bilateral contest.

To illustrate this, take a look at the set of Militarised Interstate Disputes between states[1] in 1878 – the year when the Congress of Berlin ended the Russo-Turkish War – and the type of relations in

[1] A Militarised Interstate Dispute is a set of interactions involving the threat, display, or use of force between or among states (Gochman and Maoz, 1984).



(1878) (1962)

Figure 1: Militarised interstate disputes

Source: *Networks of Nations: The Evolution, Structure, and Impact of International Networks, 1816-2001*, (Maoz, 2010)

1962 – the height of the Cold War–, depicted in Figure 1.
In both periods of time the overall conflict structure within states is represented by networks of bilateral conflicts. In 1878, the structure of disputes was characterised mainly by a network with line components in which a state had at most two different conflicts at the same time. In 1962 the picture is different, whilst there is a non-negligible number of isolated bilateral conflicts, there is a less trivial cluster of nodes centred around military powerful and/or resource rich states like the United States, Russia, China, or Iraq among others. It conceptually is hard to judge whether it was the strength of Russia that made other countries engage in conflict with the US, or whether they did so in order to oppose the threat that a potential US hegemony meant to them. In the literature on International Relations, the former is broadly comprised by the term *Bandwagoning*, while the latter is frequently referred as *Balancing* (Waltz, 1979). The paper at hand sheds light on this question in a stylised setting.

While the number of Militarised Interstate Disputes have declined over the second half of the 20th century, the was a sharp increase in internal and internationalised internal conflict. These types of conflict are often referred to as civil wars and include recent examples like the Syrian war, the civil war in Ukraine and the Colombian conflict.[2] As figure 2 suggests, the number of parties in military conflicts has increased on average over time, with the sharpest increase happening

[2]See figure 11 in the appendix for that claim and a more detailed definition of the conflict types mentioned here.

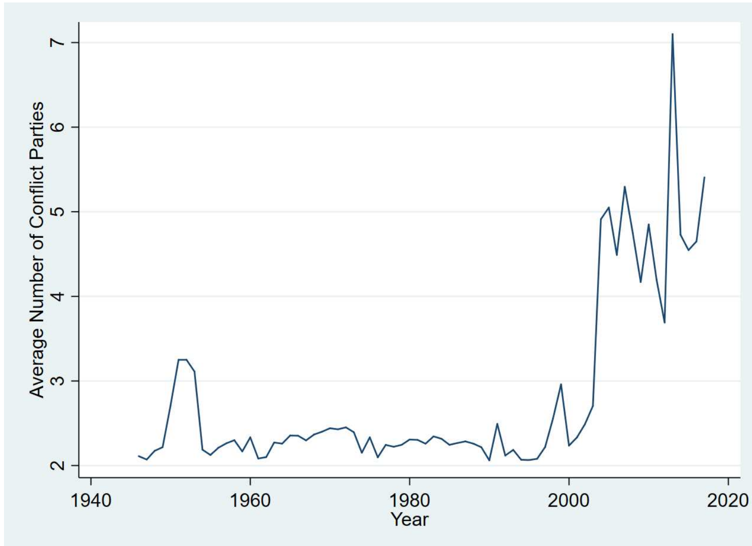


Figure 2: Violent Conflicts Involve More Parties Over Time

after 9/11. Since every additional agent can have conflicts with multiple agents in the network, the increase in conflictual links is likely to be even more pronounced, giving rise to the study at hand.

We propose a setting in which there is an exogenous structure of bilateral conflicts across n opponents as in Franke and Öztürk (2015) (from heron FÖ). Each link between a pair of players represents a bilateral conflict. While FÖ considers cases of symmetric characteristics, we allow for heterogeneity between opponents in terms of their efficiency of conflict investment and the valuations of winning, within and between different conflicts. We model conflict using a *contest success function* based on the axiomatisation proposed by Skaperdas (1996). Finally, we induce a trade-off between different conflicts through a *budget-constrained cost function* – following the formulation proposed by Kovenock and Roberson (2012a), which can accommodate two canonical cases already studied in the literature: *i)* pure cost convex case and *ii)* budget constrained case. The latter yields to the fact that military budgets can hardly ever be altered in the short run when conflicts were not anticipated.

We show that a unique and interior Nash Equilibrium exists in the pure cost case, independently of the model parameters. In contrast, in the budget constrained case the uniqueness and interiority of the equilibrium rely on the spread of valuations a player holds for the different conflicts she is involved in and the characteristics of the impact function. In line with FÖ, we show that a general algebraic solution for the equilibrium strategies does not exist for most types of networks.

We thus use implicit methods to investigate some local effects of the asymmetries between the players' characteristics. We find that asymmetries in the prizes leads to Bandwagoning behaviour, mainly driven by the interaction of local network externalities induced by the conflict structure.[3] If there are players of different strength, as indicated by their efficiency to transform resources into winning probabilities, players tend to fight opponents more strongly that are similar with respect to their strength. Finally, we investigate the changes in the strategies due to heterogeneity in the number of conflicts of a player's opponents. We find that more central players tend to be fought more fiercely.

Related Literature: Modelling conflict on networks is a relatively recent stream of research in economics (Dziubiński et al., 2016), starting with the model proposed by Franke and Öztürk (2015). The authors define a model where players are embedded in a network of bilateral conflicts and each player chooses the amount of resources that they want to invest to each conflict. The conflict is modelled using a lottery success function. The trade-off between different conflicts is induced through a convex cost function of the total amount of resources employed. They relate the total conflict investment with the player's number of conflicts. The article focusses on aggregate behaviour and thus abstracts from individual characteristics by assuming symmetry with respect to all model parameters.

Beside the seminal paper by Franke and Öztürk (2015) and the subsequent studies looking at this type of environment (e.g. König et al., 2017; Dziubiński et al., 2017; Matros and Rietzke, 2018)[4] there are different fields that are related to the paper at hand. The key distinction between our paper and the afore-mentioned (among other differences) is that the players choose an effort level for each opponent they share a link with, rather than choosing a single effort that they employ against all players. [5] Settings with link specific actions, not necessarily conflict or contest though, are quite recent. To the best of our knowledge in addition to Franke and Öztürk (2015) the only models on games on networks that introduce multidimensional strategies are Goyal et al. (2008) or Bourlès et al. (2017). In analysing heterogeneity of players and their response with respect to specific opponents, this characteristic is crucial.

[3]Klose and Kovenock (2015) refer to these as identity-dependent externalities. The first formalisation in contest to our knowledge can be found in Linster (1993).

[4]Huremovic (2016) studies as well conflicts on networks but with a different perspective. He is interested in the endogenous network formation of a network of conflict.

[5]Even though Dziubiński et al. (2017) studies an environment of conflicts on network with a multidimensional strategy space, this feature comes from the fact that is a dynamic game rather than considering a link specific action space.

Apart from that, our paper is related to the literature on multi-battle contests based on the canonical Colonel Blotto game. The Colonel Blotto game has been studied extensively since its first formulation by Borel (1921), where two players need to allocate simultaneously a finite number of resources over k different battlefields, the outcome over each battlefield is modelled as an *all-pay-auction*. This specification is well-researched with characterisations of heterogeneity between players, complementarity of prizes and other modifications to the standard formulation (Borel and Ville, 1938; Gross and Wagner, 1950; Laslier, 2002; Roberson, 2006; Hart, 2008; Hortala-Vallve and Llorente-Saguer, 2012; Weinstein, 2012; Kovenock and Roberson, 2012a; Kovenock et al., 2015; Macdonell and Mastronardi, 2015; Thomas, 2018).

Using a lottery to determine the outcome in each battlefield following the Tullock (1980) *contest success function*, where the probability of winning a specific battlefield is a non-decreasing function in the own resource allocation and decreasing in the enemy's allocation, is another approach to the two player Colonel Blotto problem. Based on this *contest success function*, Friedman (1958) is able to characterise the equilibrium in pure strategies of the two-player game with symmetric and asymmetric budgets and battlefield valuations. The main result of that study is that the optimal allocation is proportional to the valuation of the prizes.[6] This result is a special case of our model. This strand of the literature relies on models with only two players. Our paper is a contribution to the theory of contests in which we extend the current set of models by considering a multi-player environment with asymmetric efficiency of resources toward the conflict outcome and conflict prizes. This variation allow us to find new insights about the effects of the interactions of local network externalities across different conflicts.

We also add to a debate in the literature of international relations. In a war and many other conflictual settings, there is no (strong) institution that allows parties to come to a peaceful agreement. This state can be referred to as Hobbesian anarchy, due to Hobbes (1998). It is the law of the Jungle that should determine the winner(s) in such a state. Differences in the parties' strengths should thus be crucial to any analysis of conflict. Early on, Waltz (1979) and Walt (1987) coined the terms *Balancing* and *Bandwagoning*. *Balancing* is a behaviour where weaker parties ally to balance the power of a strong common opponent. *Bandwagoning* refers to the case where weak parties rally behind the strategic goals of the hegemon. There has been an ongoing discussion about which of these is more likely to occur in situations of armed conflict.[7] Our model allows to introduce a hegemon into the model, using different measures of strength, in order to shed light on the

[6] Robson (2005) generalises it by allowing the contest success function to include an effectiveness advantage and idiosyncratic noise.

[7] See for example Schweller (1994), Waltz (1997) and Lieber and Alexander (2005).

behaviour of the remaining players.

The rest of the paper is structured as follows. In the next section, we describe the setup of the model and discuss some its properties. In Section 3, we prove the uniqueness of an interior Nash Equilibrium, discuss some of its properties and show the non-existence of a general explicit algebraic solution to the model. In section 3.1, we study the specific family of k-regular network structures that enables us to have sharper predictions regarding the equilibrium behaviour. In this section we also present some comparative statics. In section 3.2, we present results concerning the asymmetries coming from the network structure. Section 4 concludes.

2 The Model

Let $\mathcal{I} = \{1, \ldots, n\}$ be a finite set of players with $n \geq 3$. All battlefields are contained in $B \subseteq \mathcal{I}^2$ where $\mathcal{I}^2$ is the set of unordered pairs of $\mathcal{I}$ with typical element (ij). The underlying conflict network $\mathcal{G}$ is represented by the connected graph associated with the pair of sets $(\mathcal{I}, B)$.[8] We say that any pair of players i and j is involved in a bilateral conflict on battlefield (ij) if and only if $(ij) \in B$. The network $\mathcal{G}$ is *undirected* ($\forall \{i,j\} : (ij) \in B \Leftrightarrow (ji) \in B$) and *irreflexive* ($\forall i \in \mathcal{I} : (ii) \notin B$). Let $N_i = \{j \in \mathcal{I} | (ij) \in B\}$ denote the set of i's rivals. The total number of rivals of i is given by $d_i = |N_i|$. The set $\tilde{N}_i = \{1, ..., d_i\}$ then uses the opponent ordering e_{ij}, which is the row number of each element of the ordered vector N_i from j low to high. The total number of battlefields is $\frac{1}{2}b$ with $b = \sum_i d_i = |B|$. Notice, that every player has at most $(n-1)$ rivals and, consequently, the network $\mathcal{G}$ contains at most $\frac{n(n-1)}{2}$ battlefields. In each bilateral conflict, $(ij) \in B$, players i and j fight for a strictly positive exogenous prize. Player i's valuation of winning the prize against player j is denoted v_{ij}. This framework can accommodate constant-sum bilateral conflicts, when $v_{ij} = v_{ji}$, or non constant-sum bilateral conflicts, when $v_{ij} \neq v_{ji}$.[9]

Each player i can allocate an amount of resources $x_{ij} \in \mathbb{R}_+$ in order to increase her probability of winning battlefield (ij) against player j. Thus, $\mathbf{x_i} = (x_{e_{ij}})_{j \in N_i}$ is a d_i-dimensional vector that contains all effort choices of player i.

The outcome of each bilateral conflict is determined by the total amount of resources allocated to

[8] A graph $\mathcal{G}$ is connected if for every pair of players i and j in N there exists a path between them. We do not consider disconnected network structures. The results that we are presenting hold for any non-trivial component of any disconnected network. A path in a network $\mathcal{G}$ between nodes h and l is a sequence of links $i_1 i_2, i_2 i_3, \ldots, i_{k-1} i_k$ such that every $i_m i_{m+1} \in B$ for each $m \in \{1, \ldots, k-1\}$, with $i_1 = h$ and $i_k = l$. Each node in the sequence $\{i_1, \ldots, i_k\}$ needs to be distinct.

[9] The latter case can also capture the idea of identity externalities as mentioned before.

that specific battlefield. Player i's probability of winning is determined by a *contest success function* (from hereon CSF) $p(a_i x_{ij}, a_j x_{ji})$, where $a_i \in \mathbb{R}_{++}$ captures how efficiently player i can employ her resources to increase this probability. The CSF is increasing and concave in x_{ij} and decreasing and convex in x_{ji}.[10] Further, it does not depend on any x_{lk} with $(lk) \neq (ij)$. The axiomatised class of CSFs by Skaperdas (1996) satisfies these properties. Thus, the probability of i winning the prize in the battlefield against j obtains as

$$\tilde{p}_{ij} = \tilde{p}(a_i x_{ij}, a_j x_{ji}) = \begin{cases} \frac{f(a_i x_{ij})}{f(a_i x_{ij}) + f(a_j x_{ji})} & \text{if } (x_{ij} + x_{ji}) \neq 0 \\ \overline{p}_{ij} & \text{if } (x_{ij} + x_{ji}) = 0 \end{cases} \tag{CSF}$$

where $\overline{p}_{ij} \in (0,1)$ is the tie breaking rule. The impact function $f(.)$ is a twice differentiable, positive and strictly increasing function of its argument with $f(0) = 0$.

The tie breaking rule is defined exogenously in order to define the CSF at $(0,0)$. Since this might cause problems for some of the results we intend to show, we use the fact that

$$\tilde{p}(a_i x_{ij}, a_j x_{ji}) = \lim_{\delta \to 0} \frac{f(a_i x_{ij}) + k\delta}{f(a_i x_{ij}) + f(a_j x_{ji}) + (1+k)\delta}$$

The limit of this function coincides with our previous definition at every point. Even the tie rule can be set flexibly since

$$\lim_{\delta \to 0} \frac{f(a_i x_{ij}) + k\delta}{f(a_i x_{ij}) + f(a_j x_{ji}) + (1+k)\delta} \Bigg|_{x_{ij} = x_{ji} = 0} = \lim_{\delta \to 0} \frac{k\delta}{(1+k)\delta} = \frac{k}{1+k} \in (0,1)$$

for every $k \in (0, \infty)$. This approach is essentially the one suggested in Myerson and Wärneryd (2006) with the slight adjustment to accommodate more flexible tie breaking rules. The function we intend to use as contest success function is thus

$$p(a_i x_{ij}, a_j x_{ji}) = \frac{f(a_i x_{ij}) + k\delta}{f(a_i x_{ij}) + f(a_j x_{ji}) + (1+k)\delta} \tag{1}$$

for some arbitrarily small $\delta > 0$.[11]

For ease of notation, throughout the rest of the paper, let $\omega = (v, a)$ be the combination of b values collected in v and the n population weights collected in a. The space of all such combinations

[10] This translates into the following condition on $f()$: $f''(a_i x_{ij})(f(a_i x_{ij}) + f(a_j x_{ji})) - 2f'(a_i x_{ij}) < 0$. In words, $f()$ can have any degree of concavity but should not be too convex.

[11] Note that this implies $p_{ij} + p_{ji} \neq 1$ for $k \neq 1$. Alternatively one could assume $k = 1$ implying $\overline{p}_{ij} = \frac{1}{2}$, resulting in $p_{ij} + p_{ji} = 1$. All results are robust to with respect to that modelling choice.

is $\Omega \in \mathbb{R}^{b+n}_{++}$. We call the case where all valuations across players and battlefields and all efficiencies across all players are the same a *strictly symmetric* parameterisation and denote it $\overline{\omega}$. Let $\overline{\Omega} = \{(\lambda_1 \mathbf{1}_b, \lambda_2 \mathbf{1}_n) | (\lambda_1, \lambda_2) \in \mathbb{R}^2_{++}\}$ denote the set of all such parameterisations.[12]

Players incur costs for employing resources which are captured by $C(X_i)$ where $X_i = \sum_{j \in N_i} x_{ij}$ denotes total resources spent by a player across all her battlefields. We consider a *budget-constrained cost-based* framework as the one proposed by Kovenock and Roberson (2012b) such that for some $R > 0$, we have

$$C(X_i) = \begin{cases} c(X_i) & \text{if } X_i \leq R \\ \infty & \text{if } X_i > R \end{cases} \tag{2}$$

The properties of $c(X_i)$ determine the magnitude of opportunity costs for player i. If the function is strongly increasing at some level of X_i, she will have to withdraw resources from other battlefields rather than increasing the total amount of resources spent. In general, $c(X_i)$ is continuous, non-decreasing and convex. We use $c'(0) = 0$ for proving our results but they go through for sufficiently small $c'(0) > 0$ as well. This definition of the cost function allows the study of two canonical formulations. In the *budget-constrained case* (BCC) the cost function is $c(X_i) = 0$ for all X_i. In this case the opportunity costs across battlefields are mediated solely through the curvature properties of the CSF. Since the marginal returns on each battlefield are strictly increasing in own allocated resources, each player exerts the total amount $X_i = R$ across all battlefields.

The second case is the *pure cost case* (PCC) in which R is arbitrarily large, such that we can guarantee $X_i < R$ in equilibrium for each player i. The opportunity costs for each battlefield enter the model through the positive cross derivatives of $c(X_i)$ which in this case is assumed to be strictly convex.

For technical tractability we ignore the mixed case in which there is a non-trivial cost function $c(X_i)$ and potentially binding constraints for some players but not all players.

For each battlefield $(ij) \in B$, agent i's expected pay-off is $u_{ij} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $u_{ij} = p_{ij} v_{ij}$. We assume that agents are expected pay-off maximisers with risk-neutral preferences. We con-

[12] $\mathbf{1}_k$ is a $k \times 1$ vector containing only ones.

sider an additively separable utility function given by [13]

$$U_i(\mathbf{x_i}, \mathbf{x_{-i}}, \mathcal{G}) = \sum_{j \in N_i} u_{ij} - C(X_i)$$

The set of players, the network structure, the action spaces and expected payoffs define a simultaneous game. Our objective is to study the Nash equilibrium of this game and how the characteristics of the network structure, distribution of values and assumptions on the cost function influence the equilibrium behaviour.

3 Equilibrium Analysis

Given the above structure, each player faces the following maximisation problem of dimension d_i determined by the network structure $\mathcal{G}$ for given $\mathbf{x_{-i}}$,

$$\max_{\mathbf{x_i} \in \mathbb{R}_+^{d_i}} U_i(\mathbf{x_i}, \mathbf{x_{-i}}, \mathcal{G}) \quad \text{with a given } \mathbf{x_{-i}} \in \mathbb{R}_+^{b-d_i} \tag{3}$$

In the pure cost case, the equilibrium behaviour is described by the balance of marginal benefits and marginal costs for each battlefield $(ij) \in B$ and every player $i \in \mathcal{I}$

$$\frac{a_i f'(a_i x_{ij}) f(a_j x_{ji})}{(f(a_i x_{ij}) + f(a_j x_{ji}))^2} v_{ij} = C'(X_i) \tag{PCC}$$

In the budget-constrained case, when the properties of the cost function induce the opportunity costs through a fixed budget that needs to be spent, the optimal allocation of resources for each player $i \in \mathcal{I}$ for every battlefield $(ij) \in B$ in equilibrium is characterised by

$$\frac{f(a_i x_{ij}) + f(a_j x_{ji})}{\sum_{k \in N_i} [f(a_i x_{ik}) + f(a_k x_{ki})]} = \frac{\sqrt{v_{ij} f'(a_i x_{ij}) f(a_j x_{ji})}}{\sum_{k \in N_i} \sqrt{v_{ik} f'(a_i x_{ik}) f(a_k x_{ki})}} \tag{BCC}$$

Players' marginal benefits are shaped by the resource allocation of their direct rivals but also the rivals of their rivals and so on. This interdependency induced by the cost function determines how an individual reacts to changes in some of the parameters of the environment. Some of the

[13] Robson (2005) show that expected utility maximisers end up maximising the sum of benefits of each conflict if they are risk neutral. For a network setting this result still holds.

behavioural implications thus occur even though a player's preferences are not directly altered. FÖ asserted that a unique equilibrium exists irrespective of the network structure if the players characteristics in terms of their preferences and technologies are homogenous. In the setting of convex costs, this homogeneity is not needed to guarantee uniqueness. Under a strict budget constraint we can guarantee it by staying in an open neighbourhood around symmetric parameters. With the CSF as defined in (1), the game is a continuous game with a finite set of players and compact strategy spaces. Thus, we can apply the theorems due to Debreu (1952), Glicksberg (1952) and Fan (1952) in order to guarantee the existence of a pure-strategy Nash equilibrium. Due to the continuity of the cost function and the (almost) infinite marginal gains close to $(0, 0)$ on each battlefield, equilibria are strictly interior. If that is given, the system of first order conditions (from hereon referred to as F) characterises these equilibria. We show that the determinant of that system is always strictly larger than zero. This provides us with the following result.

Proposition 1 (Pure Cost Case).
For the pure cost case, a unique, interior pure-strategy Nash equilibrium exists. The solution function $x(\omega) : \Omega \mapsto \mathbb{R}^b_{++}$ *has the following properties:*

- *It is of class* C^∞
- *Its derivative is given by* $D_x(\omega) = -\left[D_x F(x(\omega); \omega)\right]^{-1} D_\omega F(x(\omega); \omega)$

The budget-constrained setting is slightly more complicated, as players cannot increase or decrease their total efforts X_i to any other level than R. We need to ensure that every player can equalise marginal returns across battlefields. Consider Figure 3. A player i for which $d_i = 2$ can pin down her decision on both battlefields by choosing only x_{i1} since $x_{i2} = R - x_{i1}$. The figure shows her marginal utilities on both battlefields. Since they are both decreasing in the respective effort, marginal utility on battlefield 2 is increasing in x_{i1}.

With MU_{i1} player i has enough endowment to balance marginal utilities, that is, to reach the intersection of the two functions. If we increase the battlefield value from v_{i1} to v'_{i1}, as is shown by the dotted line MU'_{i1}, it would be optimal to spend all resources on the battlefield with the higher marginal revenue and we obtain a corner solution.
Since the marginal utilities increase in the valuations and efficiencies, this also restricts the relative values $\frac{v_{i1}}{v_{i2}}$ and efficiency parameters (i.e. a_i and a_j). This mutual restraint can be characterised more precisely.

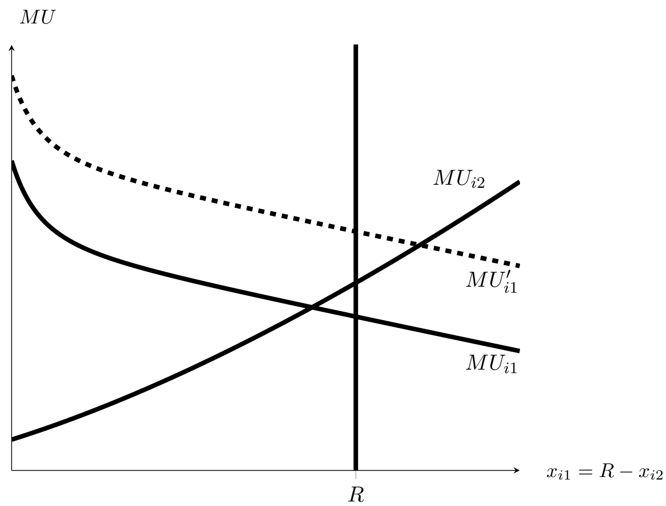


Figure 3: Marginal Utilities for a Player with 2 Battlefields

Take some player $i \in \mathcal{I}$ and denote the highest valuation she holds with $v_{ih} = \max\{v_{ij} | j \in N_i\}$. The highest marginal utility that player i can get in the battlefield against h when $x_{ih} = R$ is

$$\max_{x_{hi} \leq R} \frac{\partial U_i(\mathbf{x}_i, \mathbf{x}_{-i}, \mathcal{G})}{\partial x_{ih}}\Bigg|_{x_{ih}=R} = \begin{cases} \frac{1}{4}\frac{a_i f'(a_i R)}{f(a_i R)} v_{ih} & \text{if} \quad \frac{a_i}{a_h} \leq 1 \\ \frac{a_i f'(a_i R) f(a_h R)}{(f(a_i R) + f(a_h R))^2} v_{ih} & \text{if} \quad \frac{a_i}{a_h} > 1 \end{cases}$$

Since the first order condition of this problem is given by

$$\frac{\partial U_i(\mathbf{x}_i, \mathbf{x}_{-i}, \mathcal{G})}{\partial x_{ih} \partial x_{hi}} = \frac{a_i a_h f'(a_i x_{ih}) f'(a_h x_{hi})(f(a_i x_{ih}) - f(a_h x_{hi}))}{(f(a_i x_{ih}) + f(a_h x_{hi}))^3} \stackrel{!}{=} 0$$

the argument that solves it is $x_{hi} = \frac{a_i}{a_h} x_{ih}$, meaning that the highest marginal utility for a specific battlefield is achieved when winning probabilities are the same for both players.
Equally well, on the battlefield where i holds the lowest valuation $v_{il} = \min\{v_{ij} | j \in N_i\}$, we can find the minimal marginal utility under any profile where $x_{ih} = R$ as

$$\min_{x_{li} \leq R} \frac{\partial U_i(\mathbf{x}_i, \mathbf{x}_{-i}, \mathcal{G})}{\partial x_{il}}\Bigg|_{x_{il}=0} = \frac{a_i f'(0)}{f(a_l R)} v_{il}$$

When this marginal utility (denote it MU_{ij} for some battlefield (ij)) is larger than the one on (ih) when $x_{hi} = \min\{\frac{a_i}{a_h} x_{ih}, R\} = \min\{\frac{a_i}{a_h} R, R\}$, then player i has an incentive to divert resources to other battlefields, since $MU_{ik} \geq MU_{il}$ under this profile for $k \neq h$. Thus, we want to assert that

$$\min_{x_{li} \le R} \frac{\partial U_i(\mathbf{x}_i, \mathbf{x}_{-i}, \mathcal{G})}{\partial x_{il}} \bigg|_{x_{il}=0} > \max_{x_{hi} \le R} \frac{\partial U_i(\mathbf{x}_i, \mathbf{x}_{-i}, \mathcal{G})}{\partial x_{ih}} \bigg|_{x_{ih}=R}$$

To complete this argument of the proof, one has to notice that for $x_{ih} = 0$ and $x_{hi} = R$, we have $MU_{ih} \ge MU_{ik}$ for all $k \ne h$. This implies that each player can achieve $MU_{ij} = MU_{ik}$ for all $\{j, k\} \in N_i$.

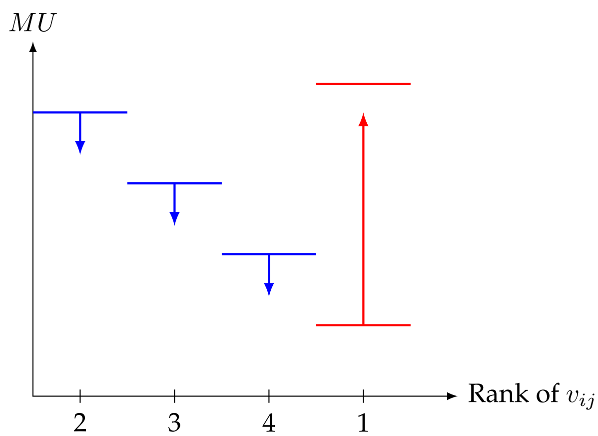


Figure 4: MU_{ih} can be equalised with all MU_{ij}
Note: Valuations are ordered from high to low, i.e. 1 represents

If that holds, we can use the same techniques as in the former proof to obtain the following result.

Proposition 2 (Budget-Constrained Case)**.**
If for all $i \in \mathcal{I}$ we have

$$\frac{v_{il}}{v_{ih}} > \frac{1}{4} \frac{f(a_l R)}{f(a_i R)} \frac{f'(a_i R)}{f'(0)} \tag{4}$$

then for each $\omega \in \Omega$ a strictly interior, unique pure-strategy Nash equilibrium exists.
The solution function $x(\omega) : \Omega \mapsto R^b_{++}$ has the following properties:[14]

- *It is of class C^∞.*
- *Its derivative is given by $D_x(\omega) = -\left[D_x F(x(\omega); \omega)\right]^{-1} D_\omega F(x(\omega); \omega)$.*

[14]This is a slight abuse of notation. Technically, the function is $\tilde{x}(\omega) : \Omega \mapsto R^{b+n}_{++}$. The shadow prices of increasing the total amount of efforts λ is an output of that function for all $i \in \mathcal{I}$. Here, we construct the function from the b components of $\tilde{x}$ that are associated with the b effort levels.

Therefore, the interplay of battlefield values, the initial endowment and properties of the the impact function $f(.)$ determine whether the game has a unique Nash equilibrium. Note that the case $a_i = a_j = 1$ and $v_{ij} = v_{ji} = \overline{v} > 0$ for all $(ij) \in B$ always fulfils the condition.
The analytic condition is particularly interesting as a familiar assumption made in economics could guarantee existence for all kinds of parameters. If we assume that the impact of the first marginal effort increase is infinitely large, that is $\lim_{x \to 0} f'(x) = \infty$, there is no bound on budgets and valuations anymore. While we do not intend to use this assumption for the rest of the paper, it relates to common assumptions made in economics like the *Inada-condition* in production or utility functions. One could state it as the first marginal unit sent to a battlefield having a large impact compared to a further unit sent when there are already many on the field.
Games played on networks give rise to a complex set of dependencies. Any attempt to solve for equilibria should reflect this for the model at hand. The following result tells us that to solve explicitly the system of first order conditions, we need to express the best response functions of some player i as functions of a player's strategies that does not share a link with i.

Lemma 1.
There exists an indirect global dependence where for any pair of agents h and l who are not rivals (i.e. $h \notin N_l$ and $l \notin N_h$), the effort levels as characterised by the system of first order conditions can implicitely be expressed as $\mathbf{x_h}^*(\mathbf{X}^*_{-\{\mathbf{h},\mathbf{l}\}}, \mathbf{x_l}^*)$ and $\mathbf{x_l}^*(\mathbf{X}^*_{-\{\mathbf{l},\mathbf{h}\}}, \mathbf{x_h}^*)$.

In that sense, the fact that we have a connected network creates indirect relations between all players throughout the rivals of rivals along any path of the network. This does not mean that players "respond" to distant players as the word best-response function suggests. It is merely a mathematical fact that we use to show that an algebraic solution for the equilibria can be obtained for hardly any network structure. How much any player's actions affect another player's effort levels in equilibrium depends on how long the shortest possible path between them is. Consider for example a line network with 6 players in a budget constrained setting. In this case the two players in the ends allocate all their resources in their unique battlefield. Let player i be one of the end players in the line. Based on Lemma 1, we can mathematically express any of her best responses as $x^*_{ij} = x_{ij}(x_{jk}(x_{kl}(x_{lp}(x_{pq}))))$ as illustrated by Figure 5.

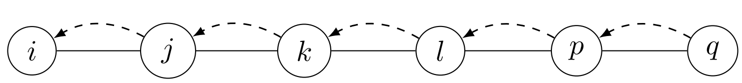


Figure 5: Line Network with 6 Players

Notice, that through the use of the FOCs, each of these nested functions applies a square to a sum. Therefore, to find the equilibrium strategies, we require to find the roots of at least one general polynomial of degree 2^5.

Denoting the length of the longest path in a given network with L, solving the system of first order conditions for all players, generally requires us to find the roots of at least one general polynomial of degree 2^L. This is a mathematical impossibility for any degree greater or equal to 5, according to the Abel-Ruffini Theorem (1779).

Corollary 1.

The equilibrium strategies of the game do not have a generic algebraic solution if the length of the longest path between any two players is greater than or equal to 3*.*

By a generic algebraic solution we mean any formula which would express the roots of the polynomial as functions of the coefficients by means of algebraic operations (i.e $+$,$-$,$\times$ or /) and roots of natural degrees. This is the reason for employing more implicit methods when analysing equilibrium behaviour, or more specifically, changes in equilibrium behaviour that correspond to changes in parameters. Also, we need to restore some degree of symmetry that we impose on the network structure.

3.1 k-Regular Networks

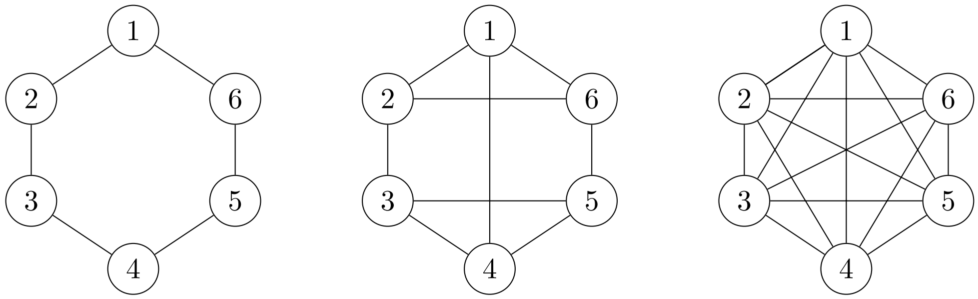
Propositions 1 and 2 provide the general form of the matrix of derivatives for any unique equilibrium. If we focus our attention on a subset of network structures, it is possible to obtain closed forms of these matrices to assess comparative statics more precisely. As these expressions appear more often in the subsequent part of the paper, denote $p_{ij}^1 = p^1(a_i x_{ij}, a_j x_{ji})$ as the derivative of p_{ij} respect to its first argument, $p_{ij}^2 = p^2(a_i x_{ij}, a_j x_{ji})$ the derivative of p_{ij} respect to the second argument. The second and cross-derivatives are equivalently given by $p_{ij}^{11} = p^{11}(a_i x_{ij}, a_j x_{ji}) = -p_{ji}^{22}$ and $p_{ij}^{12} = p^{12}(a_i x_{ij}, a_j x_{ji}) = -p_{ji}^{21}$. The set of graphs we consider is defined as follows.

Definition.

A k-regular network is any graph for which $d_i = k$ for all $i \in \mathcal{I}$ and some $k \in \{2, ..., n-1\}$.

This family of graphs includes the complete network ($k = n-1$) and the ring (or minimal connected) network ($k = 2$) as well as some networks in between these extreme cases as illustrated by Figure 6 for the case of $n = 6$.

Within these networks it is possible to characterise the equilibrium strategies in case of a symmetric parametrisation as in FÖ. We can establish a link between the network structure within that



(a) $k = 2$ (b) $k = 3$ (c) $k = 5$

Figure 6: Examples of k-Regular Networks for $n = 6$

class of graphs and the equilibrium efforts exerted. This means that, even though in a relatively restrictive set of cases, we are able to infer the exact network structure from the strategies played.

Lemma 2.
If $\omega \in \overline{\Omega}$ the following holds:

The graph is k-regular for $k \leq n-1$ if and only if the equilibrium for all $i \in \mathcal{I}$ is $x_{ij} = x^s > 0$ for all $j \in \mathcal{I} \backslash \{i\}$.
This equilibrium obtains as

$$x^s = \frac{1}{k} C'^{-1}(p^1(ax^s, ax^s)av) > 0$$

This result might be useful, as often in real world conflicts effort levels, in the form of soldiers or monetary contributions, are more observable than the underlying network structure. Note that a symmetric equilibrium in a k-regular network does not necessarily imply that $\omega \in \overline{\Omega}$ though. The condition for a strategy profile x^s to constitute an equilibrium in such a network is

$$\frac{p^1(a_i x^s, a_j x^s)}{p^1(a_k x^s, a_l x^s)} = \frac{v_{kl}}{v_{ij}}$$

for all $(ij), (kl) \in B$, which can be satisfied for $\omega \notin \overline{\Omega}$. If we restrict the degrees of freedom further there is some deductions one can make about the parameters as well.

Corollary 2.
If the graph is k-regular for $k < n-1$, and the equilibrium for all $i \in \mathcal{I}$ is uniquely defined by some effort

choice $\frac{R}{k} \geq x^s > 0$ for all $i \in \mathcal{I}$ it follows that

$$a_j = a_i \Leftrightarrow v_{ij} = v_{ji} \quad \forall i, j \in \mathcal{I}$$

Probably the most frequently used CSF is the one suggested by Tullock (1980) for which $f(x) = x$. With the above we can have a first glance at potential comparative statics with respect to parameters.

Example 1 (Relative Battlefield Values)**.**
Let every player in a k-regular network have an endowment equal to one and values such that $v_{ij} = v_{ji}$ for all $(ij) \in B$. Also $V = \sum_{j \in N_i} v_{ij}$ for all $i \in \mathcal{I}$. In a budget-constrained setting, the equilibrium behaviour will be given by

$$\text{For all } i \in \mathcal{I} \text{ and every } (ij) \in B \text{ the optimal allocation strategy is } x^*_{ij} = \frac{v_{ij}}{\sum\limits_{k \in N_i} v_{ik}}$$

When instead of facing a budget constraint, the opportunity cost of allocating resources is induced by a cost function, we observe the same equilibrium strategy if it is of the form $C(X_i) = \frac{1}{4} X_i^V$. The degree of convexity of the cost function thus needs to be proportional to the total value of the contest to each player. Since all involved functions are continuous in their respective arguments, we might infer that increases in own value increase an agent's effort, while those values winning over her decrease it.
Notice, that if we have an environment where players are symmetric and all dimensions (i.e. endowment, battlefield values, number of rivals), our setting collapse to the same equilibrium behaviour already found previously in the literature by Friedman (1958). In there the network structure starts to be redundant and only the relative battlefield values will determine the amount of resources allocated to a given conflict. If all valuations are multiplied by the same constant, strategies do not change. Following up on the notion of externalities, this illustrates why it does not matter how much individual i hates individual j (v_{ij} high). If individual k is hated more ($v_{ik} > v_{ij}$), it appears as if i relatively likes j. This is also in line with observing conflict even in very cohesive groups like tribes and families.

In a symmetric equilibrium within a k-regular network, the first order conditions are the same for every player since $X_i = kx^s$. Thus, we can do a comparative static analysis on the symmetric equilibrium k-regular effort choice with respect to the parameters. Note that this is different from the comparative statics we do later on. There we start at a symmetric parametrisation and then change a particular parameter, rather than the joint value for all player for that parameter.

$$\frac{\partial x^s}{\partial a} < 0, \quad \frac{\partial x^s}{\partial v} > 0, \quad \frac{\Delta x^s}{\Delta k} < 0$$

The effect of increasing everyones' valuations should be correctly anticipated. If we move from one symmetric equilibrium to the next along a linear increase of all efficiencies we do see a reduction in individual efforts. The same holds if the degree k increases for all players. The marginal change in total effort, $X = nkx^s$, is proportional to the first two expressions. Thus, we can say that it increases in valuations and decreases in efficiencies. Notice though that reduced efforts through increases in symmetric efficiencies should not be confused with what could be coined *effective efforts* $\tilde{x}_{ij} = a_i x_{ij}$. For instance in a war between countries, although less soldiers are sent to the battle by each agent, the battle intensity could increase since the weapons got more powerful. We find the same result as FÖ related to the degree of the network (i.e. total effort increasing in k), which should not be too surprising as we are in a strictly symmetric setting.

We stop investigating symmetric changes of parameters here, as this paper is concerned with asymmetries in networks of conflicts. In Proposition 1 and 2 we already gave an abstract characterisation of the derivatives at any given equilibrium. In the the setting of k-regular networks, we can provide general algebraic expressions that allow us to perform comparative statics for which we know the signs and magnitudes.

Proposition 3.

In a k-regular network the partial derivatives around the equilibrium at an arbitrary symmetric parametrisation $\omega \in \overline{\Omega}$ can be obtained analytically as

$$\begin{aligned}
\frac{\partial x_{ij}}{\partial v_{ij}} &= -\frac{z-(k-1)C''(X)}{z-kC''(X)}\frac{\overline{a}p^1}{z} > 0 \\
\frac{\partial x_{il}}{\partial v_{ij}} &= \frac{C''(X)}{z-kC''(X)}\frac{\overline{a}p^1}{z} < 0 \quad \textit{for all } l \neq j \\
\frac{\partial x_{ij}}{\partial a_i} &= -\frac{1+z}{z-kC''(X)}\left(\frac{p^1\overline{v}}{z} + \frac{x^s}{\overline{a}}\right)
\end{aligned}$$

for all $i \in \mathcal{I}$ and $(ij) \in B$ where $z = \overline{a}^2 p^{11}\overline{v}$. All other partial derivatives vanish.

If $X \to R$ with the cost function defined in 2, these become

$$
\begin{aligned}
\frac{\partial x_{ij}}{\partial v_{ij}} &= -\frac{k-1}{k}\frac{p^1}{\overline{a}p^{11}\overline{v}} > 0 \\
\frac{\partial x_{il}}{\partial v_{ij}} &= -\frac{1}{k}\frac{p^1}{\overline{a}p^{11}\overline{v}} < 0 \quad \textit{for } l \neq j \\
\frac{\partial x_{ij}}{\partial a_i} &= 0
\end{aligned}
$$

Let us consider the comparative statics of valuations first. For this it is more illustrative to consider the budget-constrained case. As expected, a player increases her effort on a battlefield as her corresponding value increases. In the constrained setting, she must therefore shift resources from other battlefields. Since these other battlefields are still symmetric with respect to the respective valuations, the reductions in efforts are symmetric. Intuition would thus let us to believe that $\frac{\partial x_{ij}}{\partial v_{ij}} + (k-1)\frac{\partial x_{il}}{\partial v_{ij}} = 0$. That is in fact the case.

If we observe the pure cost case, this trade-off is mediated by the convexity of $C(\cdot)$. If $C''(X) \to 0$, the player just increases her effort on (ij) without reducing it anywhere else. If $C''(X) \to \infty$ we do in fact end up with the constrained solution.[15]

The efficiency parameters cannot make a difference to the effort distribution in the budget-constrained case since the marginal change applies symmetrically to all battlefields and makes us end up in the same symmetric equilibrium for a given k and R. In the pure cost case it is interesting to note that the sign of the effect of an efficiency increase is not clearly defined for all cases. For the class of impact functions that are homogenous of degree r (within the class axiomatised by Skaperdas (1996)) one can show that both signs are indeed possible.

Proposition 4.

Let $f(ax) = (ax)^r$ for $r \in (0,2)$[16] *and $C(X) = \frac{1}{2}X^2$. Then we have*

$$
\begin{aligned}
\frac{\partial x_{ij}}{\partial a_i} &< 0 \quad \textit{for} \quad r = 1 \\
\frac{\partial x_{ij}}{\partial a_i} &> 0 \quad \textit{for} \quad r \to 0 \\
\frac{\partial x_{ij}}{\partial a_i} &> 0 \quad \textit{for} \quad r \to 2
\end{aligned}
$$

[15] This convergence result assumes that $C''(X)$ is continuously differentiable. Since we obtained the results independently from solving the constrained optimisation of 3, this is not necessary and just an additional observation.

[16] See e.g. Pérez-Castrillo and Verdier (1992) applied to bilateral contest for this restriction.

There are two competing effects that determine the sign: (i) the opportunity to reduce costs allows each player to maintain the same probabilities on each battlefield while lowering total costs and (ii) the increased efficiency increases marginal probabilities on each battlefield, thus creating an incentive to increase efforts.

Interestingly, the effect that any change has on the rest of the network seems to be independent of this sign. This is due to the fact that the slope of the best response functions, implicitly characterised by the FOCs, is proportional to the cross-derivatives of the CSF on that battlefield. Since at any equilibrium in which $a_i x_{ij} = a_j x_{ji}$ we have $p_{ij}^{12} = -p_{ji}^{21}$ and $p_{ij}^{12} = p_{ji}^{21}$, which implies $p_{ij}^{12} = 0$. Since one can show that this is a maximum with respect to x_{ji}, player i will reduce her effort if j changes her effort in either direction.

Proposition 5.

Fix some k-regular network G. Let $\omega' = \overline{\omega} + (\mathbf{0}, \mathbf{1}_{ii}\epsilon)$ for some $\overline{\omega} \in \overline{\Omega}$ and $S = (\mathcal{I}', B') \subseteq G$ such that S is a complete network. If there exists some $\epsilon > 0$ such that $\omega' \in V(\overline{\omega})$ and the equilibrium $x(\omega')$ satisfies $x_{ij}(\omega') \lesseqgtr x^s$ for all $\in \mathcal{I}' \backslash \{i\}$, then

$$x_{ji}(\omega') < x^s$$
$$x_{jk}(\omega') > x^s$$

Furthermore, if we define $\Delta x_{lq} = x_{lq} - x^s$, we have

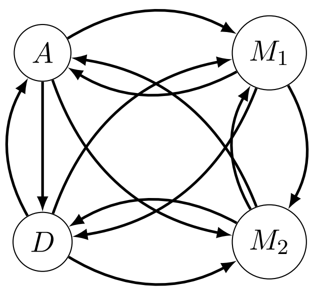
$$\Delta x_{ij} > \Delta x_{ji}, \Delta x_{jk}$$

In words, irrespective of whether she becomes more or less aggressive, the other players in the network reduce their efforts towards her.

Figure 5 illustrates what happens in case one player becomes more aggressive (player A), while another reduces her efforts against her opponents (player D).[17] The magnitude of the arrows between players indicates the relative level of efforts.

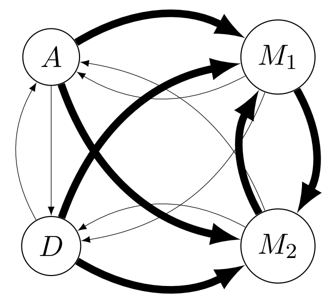
Panel (a) of the figure represents the symmetric case in which all efforts (arrows) are of the same magnitude. Choosing $\mathrm{d}a_A$ and $\mathrm{d}a_D$ such that the indicated changes in A's and D's effort follow, we see in panel (b) that the two remaining players concentrate their efforts against each other.

[17]This is a straightforward extension to the above result. An increase/decrease in efforts for an increase in a implies the opposite behaviour if a decreases.



(a) Symmetric Case

$\xRightarrow{\mathrm{d}a_A, \mathrm{d}a_D}$



(b) One Aggressive, One Defensive Player

Figure 7: Diagrammatic Representation of Proposition 5 for $k = 3$

Note: The size of the arrows refers to the relative amount of resources allocated in each bilateral conflict. Throughout the rest of the paper, numbers will indicate numerical observations, while different sizes of arrows indicate actual results.

Given a specific level of costs that is prohibitive (or a strict budget constraint), efficiencies become a scaling factor of effective budgets. Since after applying the change it must be that either $a_A > a_D$ or $a_A < a_D$, one can interpret the above figure in terms of the fight of rich against the poor. The prediction of the model at hand is that conflict intensity contracts towards the mediocrely endowed individuals and away from the rich and the poor. If efficiency is a measure of strength, the model suggests that the weaker players will rather fight each other in a situation where a single player is stronger than the rest. While Bandwagoning in typically needs the weak to rally behind the strong player who in turns fights them less/ceases to fight them, Balancing (i.e. the weak teaming up to oppose the strong) is not the strategically optimal behaviour.

While any change in efficiencies bears a certain degree of symmetry with it, since ∂a_i is the same for all (ij), a change in valuations induces more asymmetric strategic responses.

Proposition 6.

Fix some k-regular network $\mathcal{G}$. Let $\omega' = \overline{\omega} + (\mathbf{1}_{ij}\epsilon, \mathbf{0})$ for some $\overline{\omega} \in \overline{\Omega}$ and $S = (\mathcal{I}', B') \subseteq \mathcal{G}$ such that S is a complete network.[18] *There exists some $\epsilon > 0$ such that $\omega' \in V(\overline{\omega})$ and the equilibrium $x(\omega')$ for all $k \in \mathcal{I}' \backslash \{i, j\}$ satisfies*

$$\begin{aligned} x_{ij}(\omega') > x^s > x_{ik}(\omega') \\ x_{jk}(\omega') > x^s > x_{ji}(\omega') \\ x_{kj}(\omega') > x^s > x_{ki}(\omega') \end{aligned} \tag{5}$$

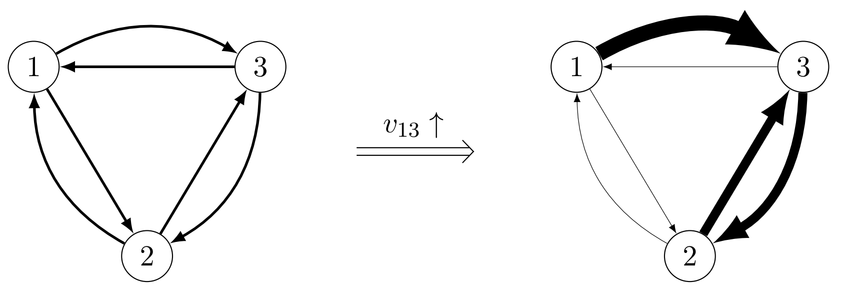
[18] In network terms: S is a subgraph of $\mathcal{G}$ induced by the clique $\mathcal{I}'$.

Furthermore, if we define $\Delta x_{lq} = x_{lq} - x^s$*, we have*

$$\Delta x_{ij} > \Delta x_{jk}, \Delta x_{kj} \tag{6}$$

Figure 8 exemplifies this statement for the case of three players.

The effect of increasing player 1's value on her effort level against player 3 is intuitive. The effects on player j's and all player k's efforts are less obvious. Roughly speaking, the effect of increasing v_{ij} has a first-order effect only on player i's effort levels and a second-order effect on all other players. This is due to how the values feed into the players' payoffs. While each own valuation has a direct effect on her payoff, it can only affect other players' payoffs through the strategic channel. That is, the fact that even 'impartial' players change their equilibrium effort in case of a change in some other player's values is a result of the interdependencies of battlefields.



(a) Symmetric Case (b) Asymmetric Case

Figure 8: Diagrammatic Representation of Proposition 6 for $k = 2$

The behaviour of the preceding proposition, graphically depicted in Figure 8 is an even clearer prediction of Bandwagoning behaviour as opposed to Balancing. Player 1 attacks player 2 more aggressively and thus fights player 2 considerably less. Player two in turn does the same and so they end up fighting player 3 jointly. Interpreting a high valuation as strength is a common interpretation in the contest literature.[19] This result could equally well be described as a form of bullying, where one individual decides to bully a peer and so-called bystanders follow the bully and become hacks. The fact that significant shares of adolescents are observed to behave in that matter is well-established in social psychology (see for example Craig and Pepler (1998), O'connell et al. (1999) and Salmivalli et al. (1996)).

[19] This interpretation as strength or ability comes from the fact that a shift in valuation is isomorphic to a shift in costs, since a contestant with payoff $p(x_i, x_j)v_i b - x_i$ shows the same behaviour as one with payoff $p(x_i, x_j)v_i - \frac{1}{b}x_i$.

3.2 Network Structure and Degree Asymmetry

The preceding comparative statics can be interpreted in terms of a strong player affecting the whole network. So far we have looked at strength by considering the preference and technology parameters. The degree centrality of a player (i.e. the number of links she has to other players) is another measure we could explore with that interpretation. Though we are leaving the realm of k-regular networks here, some of the symmetry features and the associated intuition are important in these discrete comparative statics as well.

Example 2 (Relative Network Structure)**.**
Consider a budget-constrained setting in which every player has an endowment equal to one and faces battlefields with the same value $\bar{v}$ in each conflict. We introduce asymmetries solely a single player's degree. There are 4 players and one of them faces rivals with different degree.

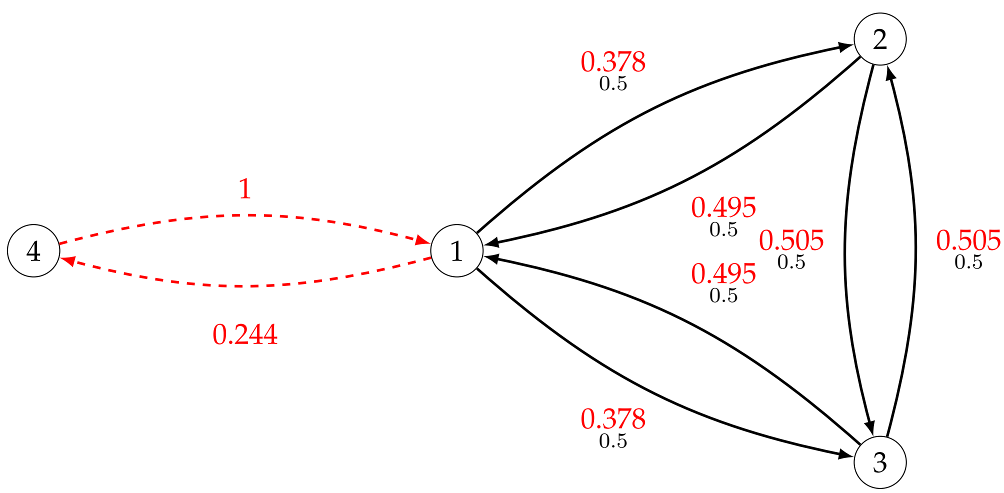


Figure 9: Example asymmetry due to the network structure with budget constraint
Note: Red figures indicate efforts when player 4 is linked to player 1. Black figures give the efforts when only players 1,2 and 3 are connected to each other.

In this network, player 1 chooses the same efforts against players 2 and 3. From her perspective these two players are identical. Player 4 differs from them as she puts all her resources on (14). For player 1 this means that the probability p_{12} is bounded by above by $1/2$. Comparing this to our earlier results, it seems that the number of rivals is weakening player 1 on each battlefield. If the value on (14) is also given by $\bar{v}$ though, her expected payoff in the 4-player game is higher than in the 3-player game ($1.062\bar{v} > \bar{v}$).

How many rivals a player's rival has thus acts as another measure of strength. There is two opposite forces working in asymmetric networks: the own degree and the rivals degree. Given a symmetric parametrisation, every player allocates a strictly positive amount of resources to all her battlefields. That implies that if the amount of battlefields that she is involved in increases, her efforts can be no greater than with fewer battlefield. However, the way that she is going to decrease them depends on the characteristics of her rivals and their rivals. To understand how these forces interact with each other, we consider a simple environment with strict budget constraint equal to one and a network that exhibit high asymmetry in the players' degrees. One such structure is the star network. The periphery players (p) allocate all their resources against the central player (i or j) and the central player allocate the same amount of resources against all her rivals, $\frac{1}{d_i}$ or $\frac{1}{d_j}$.

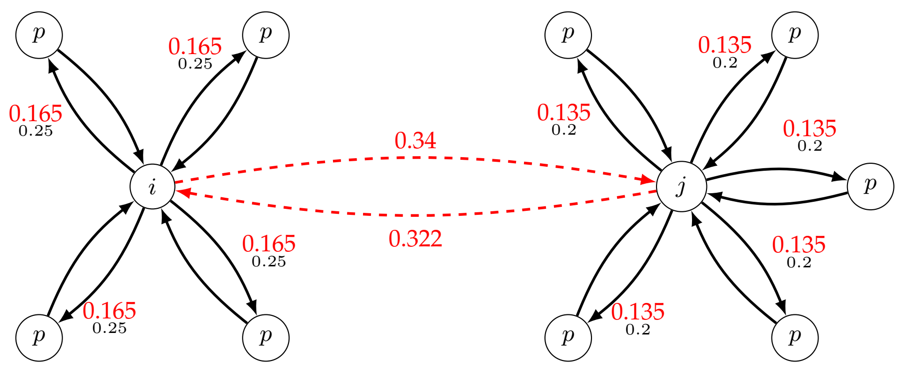


Figure 10: Star networks of order 5 (left) and 6 (right).

The effect of the own degree is straightforward. The central players allocate resources inversely proportional to their degrees. In the star network of order 5 for all $p \in N_i$ we have $x_{ip} = \frac{1}{4}$ and for the star network of order 6 for all $p \in N_j$ we have $x_{jp} = \frac{1}{5}$. In order to create some extra variance in terms of the degree distribution while we keep the setting tractable, we joint the two stars by adding a link between the two central players, as it is shown in Figure 10 by the dashed line. In this case $x_{ij} = 0.34$, $x_{ip} = 0.165$, $x_{ji} = 0.322$ and $x_{jp} = 0.135$. Even though player j has the opportunity to win a prizes on one more battlefield than player i, their payoffs are almost the same ($U_i = 1.080 > 1.081 = U_j$) compared to the case where both star networks are in isolation ($U_i = 0.800 < 0.833 = U_j$). This example shows that the number of battles a player is involved can weaken her. Indeed, the pressure made by the rivals of my rivals is beneficial or *the enemy of my enemy is* (or at least can be) *my friend*.

To understand the magnitude of this effect we can compare the equilibrium allocation with the strategy coming from the heuristic $\frac{1}{d_i}$ for different combinations of d_i and d_j, as it is presented in Table 1. Notice that the difference between having an equal resource allocation and the equilib-

d_i \ d_j	5	10	20	50	100
	$x_{ij} - \frac{1}{d_i}$				
3	0.080	0.073	0.068	0.065	0.064
4	0.116	0.113	0.109	0.107	0.106
5	0.139	0.138	0.135	0.134	0.133
10	0.189	0.191	0.191	0.190	0.190
15	0.207	0.210	0.210	0.210	0.209
20	0.216	0.219	0.220	0.219	0.219
30	0.225	0.229	0.229	0.226	0.229
50	0.233	0.237	0.238	0.238	0.238
100	0.238	0.243	0.244	0.244	0.244

Table 1: Network comparative statics

rium strategies is always positive. This means that the amount of resources allocated to the rival (i.e. j) with higher degree is always higher. Indeed, even though d_i increases for a given d_j we observe a non-linear relation where the marginal increase has a decreasing effect over the difference. This exercise suggests that in settings in which we can isolate the effect of own degree and rivals degree, there is an initial effect related to the opponents' degree leading player i to allocate more resources against her. However, the changes in the magnitude are negligible with respect to the rival's degree changes.

4 Conclusion

We presented a model of conflict networks, focussing on heterogeneity of parameters and changes in individual behaviour. Existence and uniqueness are discussed in a framework that accommodates convex costs and budget constraints. In this framework it is possible to obtain comparative statics with respect to effort efficiency, valuations and – in a more discrete fashion – the degree of centrality of a player. We interpreted these results in terms of Bandwagoning – following a strong player against her opponents – and Balancing – many weak(er) players joining forces to oppose a strong(er) player.

The results seem to advocate Bandwagoning over Balancing, irrespective of the kind of strength we consider. Since part of these considerations also have to do with threats that are dynamic in nature, a full discussion of the two phenomena needs a model with multiple periods of interaction, although the channel we describe here should not cease to exist in any such model.

Endogenous network formation seems to be the natural next step, as strength should be at the heart of the consideration with whom to start a fight. Since for standard methods like backward induction, this requires to pin down payoffs, this is a technical challenge.
Providing the players with a conflict technology only, makes it hard to talk about the potential for peace in this framework. Mutli-graph theory allows for two separate networks, one with conflict and one with cooperative links. The opportunity costs of conflict generated by the opportunity of cooperation can add a new perspective to this line of research. There is a recent, special interest in this type of settings, following the work by Jackson and Nei (2015), Hiller (2017), and König et al. (2017). However, due to the complexity of applying multiple networks simultaneously, these model either focus on endogenous network formation without an explicit allocation stage or on unidimensional action spaces over an exogenous multi-layer network.
Finally, with sufficiently simple networks, it is possible to test how real entities behave under certain parameter constellations. First steps have been made in that direction, but there are many more hypotheses to be tested.

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Appendix

Proofs

For ease of notation, throughout the appendix, let us state the first order conditions and the corresponding Hessian as primitives to the proofs. For each $i \in \mathcal{I}$ we have

$$F_{ij} = \frac{\partial p(a_i x_{ij}, a_j x_{ji})}{\partial x_{ij}} a_i v_{ij} - C'(X_i) = 0 \quad \forall j \in N_i \tag{7}$$

The Hessian H is then a block-symmetric matrix for which each diagonal block associated with some player i's first order condition is given by

$$H_i = [h_i]_{lq} = \begin{cases} \frac{\partial^2 p(a_i x_{ij}, a_j x_{ji})}{(\partial x_{ij})^2} a_i^2 v_{ij} - C''(X_i) & \text{if } l = q \\ -C''(X_i) & \text{else} \end{cases} \tag{8}$$

Each off-diagonal block in row i and column j obtains as

$$O_{ij} = [o_{ij}]_{lq} = \begin{cases} \frac{\partial^2 p(a_i x_{ij}, a_j x_{ji})}{\partial x_{ij} \partial x_{ji}} a_i a_j v_{ij} & \text{if } l = i \wedge q = j \\ 0 & \text{else} \end{cases} \tag{9}$$

Note that $O_{ij}^T = -\frac{v_{ij}}{v_{ji}} O_{ji}$.

Proofs from the main text

Proof of Proposition 1. The proof proceeds in three steps. First, we will show that a pure strategy equilibrium exists for all $\omega \in \Omega$. Then, by means of contradiction, we show that every such equilibrium must be strictly interior and thus be defined by the system of first order conditions F as defined in 7. Finally, we show that the Jacobian of this system has a constant (positive) sign across Ω. By application of the implicit function theorem (IFT) this implies that an open neighbourhood around every $\omega \in \Omega$ exists in which $x(\omega)$ is unique.

Claim 1.

A pure strategy equilibrium exists for all $\omega \in \Omega$.

Proof.

Applying the well-known theorems due to Debreu (1952), Fan (1952) and Glicksberg (1952), the result follows from making the following assertions. The game with the CSF defined in 1 is a continuous game with a finite set of players. The utility functions are strictly concave (and thus quasi-concave) if and only if

$$C''(X_i) > \frac{\prod_{j \in N_i} p_{ij}^{11} v_{ij}}{\sum_{k \in N_i} \prod_{l \neq k} p_{il}^{11} v_{il}}$$

Whenever d_i is odd, the numerator is negative and the denominator is positive and vice versa for d_i even. Since $C''(X_i) \geq 0$ for all $X_i \in \mathbb{R}_+$, this conditon is always fulfilled and the claim follows. □

Lemma 3.
In any equilibrium we have $R_i > x_{ij} > 0$ and $R_j > x_{ji} > 0$ for all $(ij) \in B$.

Proof.

In the pure cost case we asserted R_i to be "sufficiently" high. For some player i with $d_i = 1$ this means R_i is such that

$$C'(R_i) > p_{ij}^{1} v_{ij}$$

Such an $R_i \in \mathbb{R}_{++}$ always exists.
The rest of the proof of the lemma will proceed in two steps.

Claim 2.
Any strategy profile with $x_{ij} = x_{ji} = 0$ for any $(ij) \in B$ can never be an equilibrium.

Proof.

Player i can increase her utility by a (almost)[20] discrete amount while increasing her costs by an infinitesimal increment. This is a profitable deviation. □

Claim 3.
Any strategy profile with $x_{ij} > 0$ and $x_{ji} = 0$ for any $(ij) \in B$ can never be an equilibrium.

[20]We can chose δ arbitrarily small, so the marginal benefit becomes arbitrarily large.

Proof.

Suppose not and consider let player i's strategy profile be given by $X_i = (x_{i1}, ..., x_{ij}, ..., x_{in_i})$. Now consider the alternative profile x'_i which is such that $x'_{ij} = x_{ij} - \epsilon > 0$. The probability of winning on (ij) is still (sufficiently close to) 1 and costs have reduced, thus it constitutes a profitable deviation. A contradiction. □

□

Claim 4.

For every $\omega \in \Omega$ *we have* $\det(H) > 0$.

Proof.

The general formula for $\det(H_i)$ obtains as

$$\det(H_i) = \left(\prod_{j \in N_i} p^{11}_{ij} v_{ij}\right) - C''(X_i)\left(\sum_{j \in N_i} \prod_{l \neq j} p^{11}_{il} v_{il}\right)$$

Note that this also applies to any principal minor of H_i (and their principal minors and so on) simply by summing and taking products over a strict subset of N_i.

Besides the diagonal blocks, H is a sparse matrix with only one (potentially) non-zero element in each O_{ij}. The determinant can thus be expressed as the sum of the determinant of the diagonal matrix and the additional possible permutations with the respective rows.

In order to do so, let the set of all such permutations be denoted S_n with typical element σ. It contains all sets of $(ij) \in B$ that correspond to the additional row permutations. As a last piece of notation we introduce $H_i(\sigma)$ which is the submatrix of H_i obtained by deleting all cofactors (jj) such that $(ij) \in \sigma$. Using the signum function notation for determinants, we get

$$
\begin{aligned}
det(H) &= \prod_{i\in\mathcal{I}} \det(H_i) + \sum_{\sigma\in S_n} \text{sgn}(\sigma) \prod_{(ij)\in\sigma} -\left(p_{ij}^{12}\right)^2 v_{ij}v_{ji} \prod_{i\in\mathcal{I}} \det(H_i(\sigma)) \\
&= \prod_{i\in\mathcal{I}} \det(H_i) + \sum_{\sigma\in S_n} \text{sgn}(\sigma)(-1)^{|\sigma|} \prod_{(ij)\in\sigma} \left(p_{ij}^{12}\right)^2 v_{ij}v_{ji} \prod_{i\in\mathcal{I}} \det(H_i(\sigma)) \\
&= \prod_{i\in\mathcal{I}} \det(H_i) + \sum_{\sigma\in S_n} \prod_{(ij)\in\sigma} \left(p_{ij}^{12}\right)^2 v_{ij}v_{ji} \prod_{i\in\mathcal{I}} \det(H_i(\sigma))
\end{aligned}
$$

where the last equality follows from the fact that the signum is negative when $|\sigma|$ is odd and positive when $|\sigma|$ is even.

One can see that for any $J \subseteq N_i$ (including the empty set)

$$
\begin{aligned}
\frac{\partial \det(H_i(J))}{\partial C''(X_i)} &> 0 \text{ if } d_i - |J| \text{ even} \\
\frac{\partial \det(H_i(J))}{\partial C''(X_i)} &< 0 \text{ if } d_i - |J| \text{ odd}
\end{aligned}
$$

Increasing the degree of convexity of the cost function for all $i \in \mathcal{I}$ and all X_i, this implies[21]

$$
\nabla H_{C''} > 0
$$

Thus, the linear cost case is a lower bound to the determinant. More formally

$$
\det(H)\Big|_{C''(X_i)=0\forall i\in\mathcal{I}} > 0 \Rightarrow \det(H) > 0
$$

Since all off-diagonal elements beside the cross-derivatives in the O_{ij} or O_{ji} matrices are zero, the only permutations that result in a diagonal that has a nonzero product are those where row (ij) is swapped with row (ji). Let $\sigma(B)$ contain the $(ij) \in B$ that are permuted.

The determinant has three types of elements:

- For $\sigma(B) = \emptyset$ we have $\prod_{(ij)\in B} \pi_{ij}^{11} > 0$ since b is always an even number
- For $|\sigma(B)|$ odd we have $(-1)\prod_{(ij)\in\sigma(B)} -(p_{ij}^{12})^2 v_{ij}v_{ji} \prod_{(ij)\notin\sigma(B)} \pi_{ij}^{11} > 0$

[21] In $\prod_{i\in\mathcal{I}} \det(H_i)$ as well as $\prod_{i\in\mathcal{I}} \det(H_i(\sigma))$ for any possible permutation group σ.

- For $|\sigma(B)|$ even we have $(+1)\prod_{(ij)\in\sigma(B)} -(p_{ij}^{12})^2 v_{ij}v_{ji}\prod_{(ij)\notin\sigma(B)} \pi_{ij}^{11} > 0$

Thus, for all $\omega \in \Omega$ we have $\det(H) > 0$. □

We conclude by applying the IFT to each $\omega \in \Omega$ and noting that the system F is a function of class C^∞ over the open set $\mathbb{R}^b_{++} \times \Omega$. □

Proof of Proposition 2. We employ the same steps as in the previous proof with the additional qualification of condition 4.

Claim 5.

A pure strategy equilibrium exists for all $\omega \in \Omega$.

***Proof*.**

Immediate from the previous proof. □

Lemma 4.

If condition 4 is satisfied we have $R_i > x_{ij} > 0$ and $R_j > x_{ji} > 0$ for all $(ij) \in B$.

***Proof*.**

Fix some $(ij) \in B$.

□

Case 1: $d_i, d_j \geq 2$ for some $(ij) \in B$

Claim 6.

In any equilibrium $x_{ij} = x_{ji} = 0$ cannot occur.

***Proof*.**

Suppose not. The following is a profitable deviation: Let $x'_{ik} = x_{ik} - \epsilon$ for some small $\epsilon > 0$. Some strictly positive effort level x_{ik} must exist due to the budget constraint. Due to continuity of the CSF, the winning probability will not be affected by a lot. Employing $x'_{ij} = \epsilon$ increases the probability of winning on (ij) to (almost) one, and thus increases utility as well. A contradiction.

□

Claim 7.

In any equilibrium $x_{ij} > 0$ and $x_{ji} = 0$ can never occur.

Proof.

Suppose not. The probability of winning of player i is one. If player i choses $x'_{ij} = x_{ij} - \epsilon > 0$ with $\gamma \in (0,1)$, the probability is still one. If $x'_{ik} = x_{ik} + \epsilon$, the probability of winning will increase on that battlefield and so will the payoff. A contradiction. $\square$

Case 2: $d_i \geq 2$ and $d_j = 1$ for some $(ij) \in B$

Claim 8.

If condition 4 is fulfilled, then for all $x_{-i} \in \mathbb{R}_+^{b-d_i}$ there exists a vector $x_i = (x_{i1}, ..., x_{ij}, ..., x_{in_i}) \in \mathbb{R}_+^{d_i}$ such that $MU_{ij} = MU_{ik}$ for all $k \in N_i \backslash \{j\}$.

Proof.

The proof is in the main text $\square$

Claim 9.

For every $\omega \in \Omega$ we have $\det(H) > 0$ if 4 is given.

Proof.

Given the structure of each H_i, for each contribution we need at least n permutations. Let $\sigma(B)$ contain the (ij) that are permuted beyond this. Each summand of the determinant is of the following two types:

- $(+1)\left(\prod_{i\in\mathcal{I}}\prod_{\substack{j\in N_i\backslash\{k_i\}\\(ij)\notin\sigma(B)}}\pi_{ij}^{11}\right)\left(\prod_{(lq)\in\sigma(B)} -(p_{lq}^{12})^2 v_{lq}v_{ql}\right)$ if $|\sigma(B)| + n$ is even
- $(-1)\left(\prod_{i\in\mathcal{I}}\prod_{\substack{j\in N_i\backslash\{k_i\}\\(ij)\notin\sigma(B)}}\pi_{ij}^{11}\right)\left(\prod_{(lq)\in\sigma(B)} -(p_{lq}^{12})^2 v_{lq}v_{ql}\right)$ if $|\sigma(B)| + n$ is odd

These expressions are positive for every combination of $|\sigma(B)|$ and n being even or odd, respectively. This implies that $\det(H) > 0$ for all $\omega \in \Omega$ as long as condition 4 is satisfied. $\square$

$\square$

Optimality conditions for the budget-constrained case.

Let us recall the maximisation problem that a given player is facing

$$\max_{\mathbf{x_i}} \sum_{j \in N_i} v_{ij} \frac{f(a_i x_{ij})}{[f(a_i x_{ij}) + f(a_j x_{ji})]} \quad \text{s.t.} \quad \sum_{j \in N_i} x_{ij} \leq R$$

Given the Lagrange associated to this constrained maximisation problem, the first order conditions for the resources allocated to the battlefield (ij) is characterised by the following expression,

$$\frac{\partial \mathcal{L}_i}{\partial x_{ij}} = 0 \Rightarrow \frac{v_{ij}\ a_i\ f'(a_i x_{ij}) f(a_j x_{ji})}{[f(a_i x_{ij}) + f(a_j x_{ji})]^2} = \lambda_i \quad \text{then} \quad f(a_i x_{ij}) = \frac{\sqrt{v_{ij}\ a_i\ f'(a_i x_{ij}) f(a_j x_{ji})}}{\sqrt{\lambda_i}} - f(a_j x_{ji})$$

Summing over the rivals of player i are solving for the square root of $\frac{\lambda_i}{a_i}$,

$$\sqrt{\frac{\lambda_i}{a_i}} = \frac{\sum_{k \in N_i} \sqrt{v_{ik} f'(a_i x_{ij}) f(a_k x_{ki})}}{\sum_{k \in N_i} [f(a_k x_{ik}) + f(a_k x_{ki})]} \quad \text{and} \quad \sqrt{\frac{\lambda_i}{a_i}} = \frac{\sqrt{v_{ij} f'(a_i x_{ij}) f(a_j x_{ji})}}{(f(a_i x_{ij}) + f(a_j x_{ji}))}$$

Using this two expressions we get that

$$\frac{f(a_i x_{ij}) + f(a_j x_{ji})}{\sum_{k \in N_i} [f(a_i x_{ik}) + f(a_k x_{ki})]} = \frac{\sqrt{v_{ij}\ f'(a_i x_{ij}) f(a_j x_{ji})}}{\sum_{k \in N_i} \sqrt{v_{ik}\ f'(a_i x_{ik}) f(a_k x_{ki})}}$$

$\square$

Proof of Lemma 1.

As the network of conflict $\mathcal{G}$ induced by the disjoint pair of sets (N, B) is a connected graph we always can find a path $\mathcal{P}$ between any two nodes. Let us take any two nodes h and l that are not rivals (i.e. $h \notin N_l$ and $l \notin N_h$), we know that there exist a path $\mathcal{P}_{hl} = \{hi_2, i_2 i_3, \ldots, i_{k-1} i_k, i_k l\}$ where $i_j \in \mathcal{I}$ and all $(i_j i_k) \in B$ between them. As usual, the best-response of player h depends on her rivals' actions, thus $\mathbf{x_h^*} = \mathbf{x_h}(\mathbf{x_{r_1}^*}, \ldots, \mathbf{x_{r_j}^*}, \ldots, \mathbf{x_{r_k}^*})$ where all $r_j \in N_h$. In particular, we know that $(hi_2) \in B$ which implies that $i_2 \in N_h$, them $\mathbf{x_h^*} = \mathbf{x_h}(\mathbf{x_{r_1}^*}, \ldots, \mathbf{x_{i_2}^*}, \ldots, \mathbf{x_{r_j}^*}, \ldots, \mathbf{x_{r_k}^*})$. Notice that $\mathbf{x_{i2}^*} = \mathbf{x_{i_2}}(\mathbf{x_{g_1}^*}, \ldots, \mathbf{x_{i_3}^*}, \ldots, \mathbf{x_{g_j}^*}, \ldots, \mathbf{x_{g_k}^*})$ for all $g_i \in N_{i_2}$. Following the sequence of nodes describe path the path $\mathcal{P}_{hl}$, we can rewrite the best-response of player h as a function of her direct rivals and all the nodes in the path such that $\mathbf{x_h^*} = \mathbf{x_h}(\mathbf{x_{r_1}^*}, \ldots, \mathbf{x_{r_j}^*}, \ldots, \mathbf{x_{r_k}^*}, \mathbf{x_{i2}^*}, \mathbf{x_{i3}^*}, \ldots, \mathbf{x_{ik}^*}, \mathbf{x_l^*})$. Thus, even though player are not direct rival between them and they are pay-off irrelevant in the primitives of the game, the best-response of any player will depend on the action of any player that is connected by a path with her. As this the graph is connected this it true for any pair of player. Therefore generically, we have that $\mathbf{x_h^*} = \mathbf{x_h}(\mathbf{X}^*_{-\{h,l\}}, \mathbf{x}_l^*)$ and analogously $\mathbf{x_l^*} = \mathbf{x_l}(\mathbf{X}^*_{-\{h,l\}}, \mathbf{x}_h^*)$ where $\mathbf{X}^*_{-\{h,l\}} = [\mathbf{x_i^*}]_{i\in\mathcal{I}\setminus\{h,l\}}$. □

Proof of Corollary 1.

Recall the first order conditions induced by the optimisation problem of player i in battlefield (ij),

$$\frac{v_{ij}f'(a_ix_{ij})f(a_jx_{ji})}{[f(a_ix_{ij})+f(a_jx_{ji})]^2} = C'(X_i) \Rightarrow f(a_ix_{ij})^2 + 2f(a_ix_{ij})f(a_jx_{ji}) + f(a_jx_{ji})^2 - \frac{v_{ij}f'(a_ix_{ij})f(a_jx_{ji})}{C'(X_i)} = 0$$

Based on the result presented in Lemma 1, we know that player i allocation in the conflict against j will depend also in the j rivals' actions. If $k \in N_j$, player j optimality condition requires that

$$f(a_ix_{ij}) = \frac{[f(a_ix_{ij})+f(a_jx_{ji})]^2}{[f(a_kx_{kj})+f(a_jx_{jk})]^2}\frac{f(a_kx_{kj})f'(a_jx_{jk})}{f'(a_jx_{ji})}\frac{v_{jk}}{v_{ji}}$$

Using this expression in the optimality condition of player i for battlefield (ij), we get

$$\begin{aligned}&\left[\frac{[f(a_ix_{ij})+f(a_jx_{ji})]^2}{[f(a_kx_{kj})+f(a_jx_{jk})]^2}\frac{f(a_kx_{kj})f'(a_jx_{jk})}{f'(a_jx_{ji})}\frac{v_{jk}}{v_{ji}}\right]^2 \\ &\qquad + 2\left[\frac{[f(a_ix_{ij})+f(a_jx_{ji})]^2}{[f(a_kx_{kj})+f(a_jx_{jk})]^2}\frac{f(a_kx_{kj})f'(a_jx_{jk})}{f'(a_jx_{ji})}\frac{v_{jk}}{v_{ji}}\right] f(a_jx_{ji}) + \\ &\qquad\qquad f(a_jx_{ji})^2 - \frac{v_{ij}f'(a_ix_{ij})f(a_jx_{ji})}{C'(X_i)} = 0\end{aligned}$$

We are interested to solve for $f(a_ix_{ij})$ and $f(a_jx_{ji})$, to do that we need to find the roots of this polynomial. Notice that this expression is of the form

$$\begin{aligned}&A\ f(a_ix_{ij})^4 + B\ f(a_ix_{ij})^3 f(a_jx_{ji}) + C\ f(a_ix_{ij})^2(2f(a_jx_{ji}) + 6f(a_jx_{ji})^2) + \\ &\qquad D\ f(a_ix_{ij})(4f(a_jx_{ji})^3 + 4f(a_jx_{ji})^2) + E\ 2f(a_jx_{ji})^3 + F\ f(a_jx_{ji})^4 + C = 0,\end{aligned}$$

which independently of the forms of the scalar A, B, C, D, E and F is an irreducible over $\mathbb{C}$ and therefore irreducible over $\mathbb{R}_+$. We can continue the substitution process along any path between player i and any other player $l \in \mathcal{I}$. Each step further we are going to find a new term in our polynomial which exponent is going to be squared due to the non-linearity of the primitives of the contest success function. Thus, to solve the system of equations induced by the maximisation problem of each individual, we will need to solve at least one polynomial of degree 2^L where L is the largest path between a pair of player i and j. Hence, to solve the system of equations we will need to find the root of at least one polynomial of the form

$$ax^{2^L} + bx^{2^L - 1} + \ldots + cx^{2^2 - 1} + dx^{2^1 - 1} + e = 0$$

Therefore, network structures in which we can find a path of length higher or equal than 3 will require to find the roots of at least one general algebraic equation of degree higher or equal to 8 in the best case scenario.

Theorem 1.

Abel-Ruffini Theorem(1779)

A general algebraic equation of degree ≥ 5 cannot be solved in radicals. This means that there does not exist any formula which would express the roots of such equation as functions of the coefficients by means of the algebraic operations and roots of natural degrees.

By looking at the functional form of the reaction functions and the result of the Abel-Ruffini Theorem, we can say that this type of system will not be an algebraic solution using radicals. Hence, in our setting the equilibrium strategies of the game do not have a generic algebraic solution characterisation if the longest path between any two players is higher than 3. □

***Proof of Lemma 2*.**

$\Rightarrow$:

Let $\omega = \{\mathbf{1}_{2b}v, \mathbf{1}_n a\}$ for some $a, v \in \mathbb{R}_{++}$ and $d_i = k \in \mathbb{N}_+$ for all $i \in \mathcal{I}$.

Consider some FOC of the maximisation problem for some player i and some battlefield (ij)

$$\frac{\partial p(ax_{ij}, ax_{ji})}{\partial x_{ij}} av = C'(X_i)$$

$$\text{assume symmetry: } x_{ij} = x_{ji} = x^s$$

$$\Leftrightarrow \quad p^1(ax^s, ax^s)av = C'(kx^s)$$

$$\Leftrightarrow \quad x^s = \frac{1}{k} C'^{-1}(p^1(ax^s, ax^s)av)$$

It is unique as per proposition 1.

$\Leftarrow$:

Assume $x_{lq} = x^s$ for all $(lq) \in B$ some x^s. Redoing the steps in the first part for any two players i and j, we get

$$\frac{1}{d_i} C'^{-1}(p^1(ax^s, ax^s)av) = \frac{1}{d_j} C'^{-1}(p^1(ax^s, ax^s)av)$$

which implies $d_i = d_j$. □

Proof of Corollary 2.

Take condition **??** for some battlefield $(ij) \in B$ and both players in that battlefield. The condition obtains as

$$\frac{p^1(a_i x^s, a_j x^s)}{p^1(a_j x^s, a_i x^s)} = \frac{v_{ji}}{v_{ij}}$$

Assuming $v_{ij} = v_{ji}$ we have

$$p^1(a_i x^s, a_j x^s) = p^1(a_j x^s, a_i x^s)$$

which can be true if and only if $a_i = a_j$. □

Proof of Proposition 3.

In this case the Jacobian of the system of FOCs contains of the following elements:

$$\begin{aligned}
\frac{\partial F_{ij}}{\partial x_{ij}} &= a_i^2 p_{ij}^{11} v_{ij} - C''(X_i) < 0 \\
\frac{\partial F_{ij}}{\partial x_{ji}} &= a_i a_j p_{ij}^{12} v_{ij} \lesseqgtr 0 \\
\frac{\partial F_{ij}}{\partial x_{ik}} &= -C''(X_i) < 0 \\
\frac{\partial F_{ij}}{\partial x_{ki}} &= 0
\end{aligned}$$

Since at any symmetric equilibrium in a k-regular network $p_{ij}^{12} = 0$ for all $(ij) \in B$, this results in $D_x(F) = \text{diag}(A_1, A_2, ..., A_n)$ with $A_i = B_i + E_i$ and

$$B_i = \begin{pmatrix} f_{i1} & 0 & \cdots & \cdots & 0 \\ 0 & f_{i2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 0 & & & & f_{iN} \end{pmatrix}$$

with $f_{ij} := a_i^2 p_{ij}^{11} v_{ij}$ and $E_i = [e]_{kl} = -C''(X_i)$ for all (kl). Note that at the symmetric equilibrium in a k-regular network we have $f_{ij} = fkl = f = \overline{a}^2 p_{ij}^{11} \overline{v}$.
The inverse of this matrix is given by applying the Sherman-Morrison formula to be

$$A_i^{-1} = \frac{1}{f} I - \frac{\frac{1}{f^2} E}{1 - \frac{1}{f} d_i C''(X)}$$

or, in a more compact way,

$$A^{-1} = G = [g]_{l,q} = \begin{cases} \frac{f-(d_i-1)C''(X)}{f-d_i C''(X)} f^{-1} & \text{if } l = q \\ \frac{C''(X)}{f-d_i C''(X)} f^{-1} & \text{else} \end{cases}$$

The partial effects are then given by

$$\frac{\partial x}{\partial \omega} = - \left[D_x(F) \right]^{-1} D_\omega(F)$$

or more precisely:

$$
\begin{aligned}
\frac{\partial x_{ij}}{\partial v_{ij}} &= -\frac{f-(d_i-1)C''(X)}{f-d_iC''(X)}\frac{p^1}{\overline{a}p^{11}\overline{v}} > 0 \\
\frac{\partial x_{ik}}{\partial v_{ij}} &= \frac{C''(X)}{f-d_iC''(X)}\frac{p^1}{\overline{a}p^{11}\overline{v}} < 0 && \text{for } k \neq j \\
\frac{\partial x_{ij}}{\partial a_i} &= -\frac{1+f}{f-d_iC''(X)}\left(\frac{p^1}{\overline{a}^2p^{11}} + \frac{\overline{x}}{\overline{a}}\right)
\end{aligned}
$$

The effects for the budget-constrained case obtain for $C'' \to \infty$:[22]

$$
\begin{aligned}
\frac{\partial x_{ij}}{\partial v_{ij}} &= -\frac{d_i-1}{d_i}\frac{p^1}{\overline{a}p^{11}\overline{v}} > 0 \\
\frac{\partial x_{ik}}{\partial v_{ij}} &= \frac{1}{d_i}\frac{p^1}{\overline{a}p^{11}\overline{v}} < 0 && \text{for } k \neq j \\
\frac{\partial x_{ij}}{\partial a_i} &= 0 \\
\frac{\partial x_{ij}}{\partial R} &= \frac{1}{d_i}
\end{aligned}
$$

□

<u>Proof of Proposition 4</u>.

Under the assumed analytical forms, the expression for $\frac{\partial x_{ij}}{\partial a_i}$ obtains as

$$
\frac{1}{\overline{a}^3r\overline{v}+\overline{a}k}\left[(x^s)^2 - \frac{1}{2}\overline{a}^{r-1}(x^s)^{1+r} - \overline{a}^2r\overline{v}x^s + \frac{1}{2}\overline{a}^{1+r}\overline{v}(x^s)^r\right] \tag{P1}
$$

while the equilibrium effort for each player and battlefield is given by

$$
x^s = \left(\overline{a}^{1+r}\frac{r}{2k}\overline{v}\right)^{\frac{1}{2-r}}
$$

<u>Case 1: $r = 1$</u>

Given the claim in the proof we need to assert that the term in square parenthesis in the first

[22] Again, we also obtained these solving the constrained problem independently.

expression is indeed positive. Under $r = 1$ it becomes

$$(x^s)^2 - \overline{a}^2\overline{v}x^s < 0$$

Since we know that $x^s > 0$ this is equivalent to

$$x^s = \frac{1}{2k}\overline{a}^2\overline{v} < \overline{a}^2\overline{v}$$

or $k > \frac{1}{2}$, which is true for every possible (non-trivial) k-regular network.

<u>Case 2: $r \to 0$</u>

In this case $x^s \to 0$ and P1 becomes[23]

$$\frac{1}{2}\frac{\overline{v}}{k}\sqrt{\frac{\overline{av}}{2k}} > 0$$

<u>Case 3: $r \to 2$</u>

Under this limit the expression in P1 changes to

$$(x^s)^2 - \left(\frac{2}{\overline{a}} + \overline{a}^2\overline{v}\right)x^s + 4\overline{av}$$

The equilibrium will be $x^s \to 0$ if $\overline{a}^3\overline{v} < k$ and $x^s \to \infty$ if $\overline{a}^3\overline{v} > k$. In both cases the last expression is tending towards something strictly positive. □

<u>Proof of Proposition 5</u>.

From the derivatives obtained in the earlier result, we know already that $x_{ij}(\omega') > x^s$ as well as $x_{ik}(\omega') < x^s$ for all $k \neq j$. Note that the change induced in a nested function f is less than that induced in g whenever

$$\begin{aligned} |Df(g(x))g'(x)| &< |g'(x)| \\ |Df(g(x))||g'(x)| &< |g'(x)| \\ |Df(g(x))| &< 1 \end{aligned}$$

[23] It is the last summand that contains an expression that is proportional to $\lim_{r\to 0^+} r^r = 1$. We are thus not using the controversial mathematical convention that $0^0 = 1$, but obtain the result from an actually defined limit.

The changes we consider are either $\frac{\partial x_{ij}}{\partial x_{ji}}$, which is zero at the symmetric parametrisation and close to zero near it, and

$$\left|\frac{\partial x_{ij}}{\partial x_{ik}}\right| = \left|-\frac{-C''(X_i)}{p^{11} - C''(X_i)}\right| = \left|\frac{C''(X_i)}{C''(X_i) - p^{11}}\right| < 1$$

This implies that any effect of a sufficiently small change in parameters from $\overline{\omega}$ diminishes over with increasing length of a path.

We denote each best response function as a nested function of the strategies that constitute the shortest path through the graph to a nonzero derivative. In a slight abuse of notation, let us denote player i's best response function on battlefield (ij) as $x_{ij}(x_{ji}(a_i))$

$$\begin{aligned}
x_{ji}(x_{ij}(a_i)) =& x_{ji}(x_{ij}(\overline{a})) + \frac{\partial x_{ji}}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial a_i}(\overline{a})(a_i - \overline{a}) \\
&+ \frac{1}{2}\left(\frac{\partial^2 x_{ji}}{(\partial x_{ij})^2}\left(\frac{\partial x_{ij}}{\partial a_i}\right)^2 + \frac{\partial x_{ji}}{\partial x_{ij}}\frac{\partial^2 x_{ij}}{(\partial a_i)^2}\right)(\overline{a})(a_i - \overline{a})^2 \\
|_{\frac{\partial x_{ji}}{\partial x_{ij}}(\overline{a})=0} =& x^s - \frac{1}{2}\frac{p^{122}\overline{a}^3 v(z - C''(X))}{z - C''(X)}\left(\frac{\partial x_{ij}}{\partial a_i}\right)^2 (a_i - \overline{a})^2 \\
<& x^s
\end{aligned}$$

Note that this is irrespective of the sign of $\frac{\partial x_{ij}}{\partial a_i}$.

Similarly, it follows that

$$x_{jk}(x_{ji}(x_{ij}(a_i))) = x^s + \frac{\partial x_{jk}}{\partial x_{ji}}\frac{\partial^2 x_{ji}}{(\partial x_{ij})^2}\left(\frac{\partial x_{ij}}{\partial a_i}\right)^2 (a_i - \overline{a})^2 > x^s$$

□

Proof for Proposition 6.

Just as in the above proof the effects of the other players in S can be obtained via a Taylor approximation as

$$x_{ji}(x_{ij}(v_{ij})) \tilde{=} x^s + \frac{1}{2}\frac{\partial^2 x_{ji}}{(\partial x_{ij})^2}\left(\frac{\partial x_{ij}}{\partial v_{ij}}\right)^2 (v_{ij} - \overline{v})^2 < x^s$$

$$x_{jk}(x_{ji}(x_{ij}(v_{ij}))) \tilde{=} x^s + \frac{1}{2}\frac{\partial x_{jk}}{\partial x_{ji}}\frac{\partial^2 x_{ji}}{(\partial x_{ij})^2}\left(\frac{\partial x_{ij}}{\partial v_{ij}}\right)^2 (v_{ij} - \overline{v})^2 > x^s$$

$$x_{kj}(x_{ki}(x_{ik}(v_{ij}))) \tilde{=} x^s + \frac{1}{2}\frac{\partial x_{kj}}{\partial x_{ki}}\frac{\partial^2 x_{ki}}{(\partial x_{ik})^2}\left(\frac{\partial x_{ik}}{\partial v_{ij}}\right)^2 (v_{ij} - \overline{v})^2 > x^s$$

$$x_{ki}(x_{ik}(v_{ij})) \tilde{=} x^s + \frac{1}{2}\frac{\partial^2 x_{ki}}{(\partial x_{ik})^2}\left(\frac{\partial x_{ik}}{\partial v_{ij}}\right)^2 (v_{ij} - \overline{v})^2 < x^s$$

Since the $|\frac{\partial x_{ij}}{v_{ij}}|, |\frac{\partial x_{ik}}{v_{ij}}| > \frac{\partial x_{lq}}{v_{ij}} = 0$ for any (lq) such that $l \neq i$, there exists some $\epsilon = v_{ij} - \overline{v}$ such that for any $A \neq 0$ any $|A\epsilon^2|$ is strictly between the absolute value of these partial derivatives and 0. $\square$

Data and Graphs

The data we used for figure 2 stems from the Uppsala Conflict Data Program (UCDP). The dataset is *[a] dyad-year version of the UCDP/PRIO Armed Conflict Dataset. A dyad consists of two opposing actors in an armed conflict where at least one party is the government of a state.* (UCDP, 2018). For a more detailed description of the dataset also see Harbom et al. (2018) or Pettersson and Eck (2018).

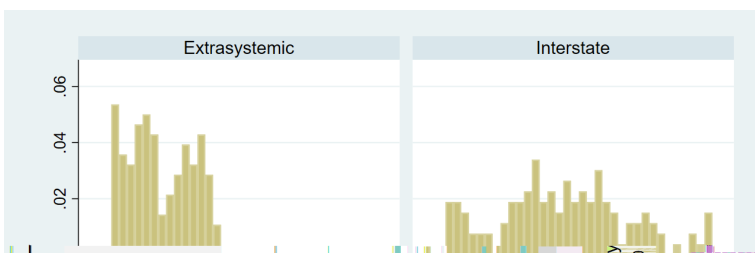

Figure 11: Frequency of Conflict Types over Time

Extrasystemic armed conflict is defined as a conflict between a state and a non-state group outside its own territory. Interstate armed conflicts are between two or more states. Internal armed conflicts are between the government of a state and one or more internal opposition groups, without intervention from other states. For Internationalised Internal conflict, intervention from other states on one or both sides is added to the definition.