Oil Price Dynamics and Business Cycles in Nigeria: A Bayesian Time Varying Analysis

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Abstract

The study investigated the dynamic relationship between oil prices and Real GDP growth in Nigeria over time by making use of time varying Bayesian VAR with Stochastic volatility. I distinguished between supply and demand shocks by means of a sign restriction. First, the study found that, the response of the Nigerian economy has been varying overtime. Also, GDP growth responds positively to both supply and demand shocks. However, the magnitude of the response to demand shock is larger compared to that of supply shocks. This suggests that overtime an increase in oil prices due to shocks in demand matter most for the Nigerian economy.

Keywords: Crude oil Price, Real GDP growth, Time Varying Coefficients, Stochastic Volatility

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1 Introduction

One of the reasons ascribed to the resource curse (the observation that countries rich in natural resources perform poorly compared to their resource poor counterparts in terms of growth is the volatility of commodity prices that results in macroeconomic instability (Van der Ploeg and Poelhekke 2010). The reliance on the natural resource sector makes them vulnerable to swings in commodity prices. Economic diversification has been touted as one of ways resource rich countries can shield themselves from the resource curse (Collier and Venables 2007; Gelb and Grasmann 2010; Gelb 2009). Diversification can make natural resource-rich developing countries less vulnerable to fiscal shocks, reduce their dependence on a single or a few resources and boost their productivity and employment (Gelb and Grasmann 2010).

Nigeria is not only the largest oil producer in Sub Saharan Africa but also Africa’s largest economy. The economy has been reliant on the oil and gas sector since the discovery and subsequent production of crude oil. Petroleum exports account for 91% and 80% of total exports in 2015 and 2016 respectively (OPEC 2017). Also, Oil revenues account for 55.4% and 47.4% of total revenues in 2015 and 2016 respectively. However, the slump in crude oil prices in 2016 took a serious toll on the Nigerian economy leading it into recession when growth entered a negative territory, -0.67%, -1.49%, -2.34% and -1.73% in the first, second, third and fourth quarters of 2016 respectively.

Despite the fact several oil exporting countries in the developing world have experienced substantial fluctuations due to oil price shocks, a great body of literature in this area have focused on analyzing these impacts from the context of the developed world especially the US economy (Jiménez-Rodríguez* and Sánchez 2005; Berument et al. 2010). The results from these studies in general cannot be extended to the context of developing countries. It is therefore imperative to investigate this issue in the context of developing countries. In
view of this, this study therefore seeks to answer the following questions; What has been the effect of fluctuations in oil prices on economic growth in Nigeria? Is there time variation in the transmission of oil price shocks to economic growth in Nigeria?

To answer these questions, I first differentiate between oil demand and supply shocks to ascertain which of these matter for the Nigerian economy. The recent literature has demonstrated that the underlying reason for a change in oil prices is necessary to determine the economic consequences (Kilian, 2009). For instance, a decline in oil prices due to slow down in global economic activity (a demand shock) are expected to lead to a decline in economic activity in an oil exporting country. These shocks are identified by means of sign restrictions.

Secondly, there are several reasons to believe that the transmission of shocks to an economy has changed over time; The Nigerian economy has undergone several reforms aimed at stabilizing the economy. It is therefore expected that the response / the vulnerability of the economy to shocks in oil prices will change over time if indeed these reforms have been effective. In addition, some studies (see Baumeister and Peersman, 2013) have suggested that the volatilities of oil market variables have changed over time. This suggests that the transmission of the shocks to the economy will differ over time. Therefore I use a made use of a time varying parameter with stochastic volatility (TVPSV) to investigate the impact of the oil shocks on the Nigerian economy over time.

2 A brief Introduction to the Nigerian Economy

The Nigerian economy is largely dependent on the petroleum sector. The sector contributes most to GDP, government revenue and the export sector (see Table 1). The economy went into recession in 2016 due to the slump in crude oil prices when growth entered a negative territory. The economy has began to pick up and is projected to grow by 2.1% and 1.9% in
2018 and 2019 respectively largely due to the petroleum sector.

Table 1: Selected Economic Indicators (2016-2019)

<table>
<thead>
<tr>
<th>Indicator (Annual percentage change, unless otherwise specified)</th>
<th>2016</th>
<th>2017</th>
<th>2018</th>
<th>2019</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real GDP (2010 market prices)</td>
<td>-1.6</td>
<td>0.8</td>
<td>2.1</td>
<td>1.9</td>
</tr>
<tr>
<td>Oil and Gas GDP</td>
<td>-14.4</td>
<td>7.6</td>
<td>10.8</td>
<td>5.7</td>
</tr>
<tr>
<td>Non oil GDP</td>
<td>-0.3</td>
<td>0.2</td>
<td>1.3</td>
<td>1.5</td>
</tr>
<tr>
<td>CPI (annual average)</td>
<td>15.7</td>
<td>16.5</td>
<td>14</td>
<td>14.8</td>
</tr>
<tr>
<td>Exports</td>
<td>-21.6</td>
<td>25.7</td>
<td>30.1</td>
<td>0</td>
</tr>
<tr>
<td>Imports</td>
<td>-34.7</td>
<td>2.5</td>
<td>34.3</td>
<td>2.9</td>
</tr>
<tr>
<td>Terms of Trade</td>
<td>-6.3</td>
<td>9.5</td>
<td>10.7</td>
<td>-3.9</td>
</tr>
<tr>
<td>Total Revenues and grants (percent of GDP)</td>
<td>5.6</td>
<td>5.7</td>
<td>7.4</td>
<td>7.1</td>
</tr>
<tr>
<td>of which: oil and gas revenue</td>
<td>2.1</td>
<td>2.5</td>
<td>3.7</td>
<td>3.3</td>
</tr>
<tr>
<td>Total Expenditure (percent of GDP)</td>
<td>9.5</td>
<td>11.2</td>
<td>11.9</td>
<td>11.6</td>
</tr>
<tr>
<td>Overall Balance</td>
<td>-3.9</td>
<td>-5.5</td>
<td>-4.5</td>
<td>-4.4</td>
</tr>
<tr>
<td>Non-oil Primary Balance (percent of non oil GDP)</td>
<td>-5.1</td>
<td>-7</td>
<td>-7.6</td>
<td>-6.7</td>
</tr>
<tr>
<td>Non oil Revenue (percent of non oil GDP)</td>
<td>3.6</td>
<td>3.4</td>
<td>3.9</td>
<td>4</td>
</tr>
<tr>
<td>Public Gross Debt (percent of GDP)</td>
<td>19.6</td>
<td>22.3</td>
<td>25.3</td>
<td>26</td>
</tr>
</tbody>
</table>


Notes: (2018 & 2019 figures are projected)

3 Empirical Evidence on the effects of Oil prices on Economic activity

The importance of the role of oil prices in influencing economic activity depends on the nature of the relationship assumed. Earlier studies assume a linear relationship between oil prices and economic activity. In an influential study, Hamilton (1983) extended the model of Sims (1980) to study how several macroeconomic variables and oil prices are related since
He found a negative correlation between oil prices and output between 1948-1972 i.e oil price increases experienced during this period have led to a reduction in output.

However, the slump in oil prices experienced in the mid-1980s were found to have smaller positive effects on economic activity than predicted by linear models making the linear relationship to start losing their significance. The focus therefore moved from linear relationship to non-linear relationship between oil price changes and economic activity. Mork (1989) extended the work of Hamilton (1983) to allow oil prices to have an asymmetric effect on the US economy. By specifying oil price increases and decreases as separate variables, he found that only increases in oil prices have a significant effect on the US economy. This result have been confirmed by Mork et al. (1994) for other countries other than the US. Specifically, they found asymmetry in the cases of Norway and all G-7 countries in exception of Italy. They also found the correlations with oil-price increases to be negative and significant for most countries, but positive for Norway, whose oil-producing sector is large relative to the economy as a whole.

With time the interest in non-linear models increased leading to the development of 2 main additional non-linear transformation of oil prices. The first transformation of oil price known as the scaled oil price model was done by Lee et al. (1995). This involves standardizing oil prices by their time varying variability in a GARCH framework. This transformation allows to differentiate between changes in oil prices which are sharp and frequent and changes in oil prices are small but sporadic.

Motivated by the fact that increases in oil prices since 1985 are a correction to earlier increases in oil prices Hamilton (1996) introduced what is popularly known as the Net Oil Price Increase(NOPI) model. This consists of transforming the oil price variable by comparing the price of oil today with the maximum value of oil price for the past four quarters. If the price of oil today is higher, then NOPI is the difference between the maximum and the
current price of oil, otherwise it is zero. This transformation reflects asymmetry in the sense that it ignores oil price decreases and only takes into consideration oil price increases.

In a recent study, Kilian and Vigfusson (2017) showed that the quantitative importance of oil price shocks in explaining recessions in the US depends on whether a linear or non-linear relationship is assumed. Using a conditional approach, they found that conditional response to oil price shocks in the net oil price increase model is best characterized as time-varying, but symmetric, rather than asymmetric. They also showed that in the case of the US, on average the linear model fits the data better after net oil price increases than the nonlinear net oil price increase model.

Very few studies have studied the effect of oil prices on growth in the context of oil exporting countries in developing countries. Omojolaibi (2014) investigated the transmission mechanism of oil prices in Nigeria and concludes that crude oil price has very important impact on the Nigerian economy through monetary policy channel. This result is buttressed by Eagle (2017) who also found that the volatility of oil prices affect the Nigerian Economy adversely. However they found that the main transmission channel is through the exchange rate. All these two studies have assumed a linear relationship and assume the effect have been constant over time. This research departs from these studies by taking a time varying approach.

4 Empirical Methodology

Following Primiceri, 2005, I extend the traditional constant parameter VAR(p) model to estimate a time varying VAR model with stochastic volatility:

\[ y_t = B_{0,t} + B_{1,t}y_{t-1} + B_{2,t}y_{t-2} + \ldots + B_{p,t}y_{t-p} + v_t \quad v_t \sim N(0, \Omega_t) \quad t = 1, 2, \ldots, T \]  

(1)
where \( y_t \) is an \( n \times 1 \) variables of interest in this study, \( B_{0,t} \) is an \( n \times 1 \) vector of time varying intercepts, \( B_{i,t} \) \( i = 1, \ldots, p \), are \( n \times n \) matrices of time varying coefficients, \( v_t \) is a Gaussian white noise with zero mean and time-varying \( n \times n \) covariance matrix \( \Omega_t \).

Let \( X_t = [1, y_{t-1}', y_{t-2}', \ldots, y_{t-p}'] \) and \( B_t = [B_{0,t}, B_{1,t}, B_{1,t}, \ldots, B_{p,t}]' \)

We can then rewrite (1) as;

\[
y_t = (I_n \otimes X_t) \beta_t + v_t \tag{2}
\]

where \( I_n \) is an identity matrix with dimension \( n \), \( \otimes \) denotes the Kronecker product, \( \beta_t \) is \( \text{vec}(B_t) \). Since the time varying error covariance matrix in (1) contains time varying variance and covariance terms, estimating of the model therefore requires identification of each element within the matrix. To this end most studies (see Primiceri (2005)) propose the following structure;

\[
\Omega_t = A_t^{-1} D_t A_t^{-1} \tag{3}
\]

Where \( A_t \) is a lower triangular matrix with ones in the main diagonal and elements \( a_{ij,t} \) and \( D_t \) is a diagonal matrix with the stochastic volatilities, \( h_{i,t} \) as diagonal elements. For example for a 4 variable VAR;

\[
A_t = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & a_{23,t} & 1 & 0 \\
0 & a_{24,t} & a_{34,t} & 1
\end{pmatrix}, \quad D_t = \begin{pmatrix}
h_{1,t} & 0 & 0 & 0 \\
0 & h_{2,t} & 0 & 0 \\
0 & 0 & h_{3,t} & 0 \\
0 & 0 & 0 & h_{4,t}
\end{pmatrix}
\]

This model therefore has 2 sets of time varying coefficients \( \beta_t \) and \( a_{ij,t} \)\(^1\) in addition to time varying covariance. By allowing for time variation in both the coefficients and the

\(^1\)see Appendix A
variance covariance matrix, the model allows for the data to determine whether any time variation exists in both the size and frequency of exogenous shocks as well the contemporaneous responses and lagged propagation of the variables to those shocks (Primiceri, 2005)

The driving processes of the system are specified as follows:

\[ \beta_t = \beta_{t-1} + u_t \quad u_t \sim N(0, \Sigma) \quad (4) \]
\[ a_{ij,t} = a_{ij,t-1} + V_t \quad V_t \sim N(0, \nu) \quad (5) \]
\[ \ln h_{i,t} = \ln h_{i,t-1} + \xi_{i,t} \quad \xi_{i,t} \sim N(0, \omega_i) \quad (6) \]

This VAR model is estimated using Bayesian method by means of the Carter and Kohn algorithm to draw both \( \beta_t \) and \( a_{ij,t} \) and independence Metropolitan & Hasting(MH) algorithm to draw the stochastic volatility. An optimal lag length of two(2) was selected based on the Schwarzs Bayesian information criterion (SBIC).

### 4.1 Data

Real Gross Domestic Product(GDP) was obtained from the statistical bulletins of the Central Bank of Nigeria(CBN). Oil price defined as the US refiners acquisition cost of crude oil in real terms was obtained from the Energy Information Administration(EIA) of the US. The refiners acquisition cost for imports refers to the price of oil paid by U.S. refiners for crude oil purchased from abroad, which is a commonly used proxy for the global price of crude oil. Oil supply defined as global oil production data was sourced from EIA Monthly bulletin. For global economic activity, I use the updated version of economic activity index constructed by Kilian (2009). This index have been widely used in most oil market VAR models because it includes emerging economies such as China & India and does not require exchange rate

---

2 I model the volatility as \( y_{it} = \epsilon_{it}\sqrt{h_{it}} \) with the state \( h_{it} \) evolving according to (6). One can see that the observation equation is non-linear in the state \( h_{it} \) therefore the Carter and Kohn algorithm which is based on the Kalman filter does not apply. Jacquier et al. (2004) suggests using the Independent MH algorithm.

3 Data available at [http://www-personal.umich.edu/~kilian/reupdate/new.txt](http://www-personal.umich.edu/~kilian/reupdate/new.txt)
weighting. The data used in the study span from 1982Q2 to 2017Q4 but the first 10 years (40 quarters) of data was used as a training sample to generate the priors for estimation.

### 4.2 Identification

The model up to this point can only allow us estimate a VAR in a reduced form. A structural interpretation of the reduced form model is thus required to be able to provide a meaningful economic interpretation. In view of this, my interest lies in recovering the structural shocks denoted as $\epsilon_t$ through the following relationship;

$$ v_t = A_t \epsilon_t $$

or in explicit form;

$$
\begin{pmatrix}
v_{OilSupply,t} \\
v_{OilDemand,t} \\
v_{OilPrice,t} \\
v_{GDP,t}
\end{pmatrix} =
\begin{pmatrix}
\phi_{11,t} & \phi_{12,t} & \phi_{13,t} & \phi_{14,t} \\
\phi_{21,t} & \phi_{22,t} & \phi_{23,t} & \phi_{24,t} \\
\phi_{31,t} & \phi_{32,t} & \phi_{33,t} & \phi_{34,t} \\
\phi_{41,t} & \phi_{42,t} & \phi_{43,t} & \phi_{44,t}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{OilSupply,t} \\
\epsilon_{OilDemand,t} \\
\epsilon_{OilPrice,t} \\
\epsilon_{GDP,t}
\end{pmatrix}
$$

Where $\phi_{ij,t}$ denotes contemporaneous time varying impact of the jth innovation on the ith variable at time t. $A_t$ is normally referred to as the impact matrix because the jth column represents the contemporaneous impact of the jth innovation on all other variables. Various methods have been proposed in the literature for the identification of a structural model from the reduced form. Key among them are the recursive form identification strategy and identifying by sign restriction. According to Kilian (2011) the recursive form identification strategy is subject to order effects. In view of this, the identification strategy used in this paper is by sign restrictions which has become popular in the recent literature (see for example Kilian and Murphy (2012); Baumeister and Peersman (2013)). In order to differentiate between supply and demand shocks, I follow the restriction commonly used in the oil economics literature. Supply shocks can be due to an exogenous disruption in world oil
production supply shocks that may be caused by geopolitical turmoil for example or the decision by OPEC to strategically reduce oil production. I therefore define supply shock to be one that causes negative responses of global crude oil production and world economic activities but increases the real oil price. Demand shocks are due to a rise in the economic activity globally which tend to lead to an increase in the price of commodities. Therefore I define a demand shock as one that causes positive responses to global economic activity, the price of oil and world oil production. I impose no restriction on GDP growth and allowed that to be determined by the data since that is the variable I am mainly interested in.

4.3 Priors and Starting Values

Due to the large number of parameters to be estimated, the easiest and convenient way is by Bayesian Inference. This requires priors and starting values. I set the priors for Σ using the Inverse Wishart. \( P(\Sigma) \sim IW(\Sigma_0, T_0) \). This is normally set using a training sample, \( T_0 \). Following the literature, I use a training sample of 10 years i.e 40 quarters to estimate a constant coefficient VAR; \( \beta_0 = (X'_{0,t}X_{0,t})^{-1}(X_{0,t}Y_{0,t}) \) \( P_0 = \Phi_0 \otimes (X'_{0,t}X_{0,t})^{-1} \), where \( \Phi_0 = \frac{(Y_{0,t}-X_{0,t}\beta_0)'(Y_{0,t}-X_{0,t}\beta_0)}{(T_0-n)} \). The scale matrix, \( \Sigma_0 \) is then set equal to \( P_0 \times T_0 \times \kappa \), where \( \kappa \) is the scaling factor. Most empirical research set this to \( 3.510 \times 10^{-4} \), a small number to reflect the fact that the training sample is typically short and the resulting estimates of \( P_0 \) maybe imprecise. Initial values for the state, \( \beta_t \) and its covariance are set to \( vec(\beta_0) \) and \( P_0 \) respectively.

The priors for \( \iota \) is set to inverse gamma(IG). \( P(\iota_i) \sim IG(\iota_{i0}, T_0) \) for \( i=1,2,3 \). Following Mumtaz and Zanetti (2013), I set the prior scale matrices, \( \iota_{10} = 0.001 \), \( \iota_{20} = \begin{pmatrix} 0.001 & 0 \\ 0 & 0.001 \end{pmatrix} \) and \( \iota_{30} = \begin{pmatrix} 0.001 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.001 \end{pmatrix} \). I set the initial values of the state vector, \( a_{ij,0} \) to be the non zero elements of \( Z0 = (\Phi_0^{1/2})^{-1} \).
and its variance to be \( \text{abs}(a_{ij,0}) \times 10 \).

The priors for \( \ln h_0 \sim N(\bar{\rho}_i, \bar{\sigma}) \), with \( \bar{\rho}_i \) equal to the log of the ith diagonal element of \( \Phi_0 \). The starting values for \( h_{it} \) is set as \( \hat{v}_{it} \). The priors for \( \omega_i \) is set as \( P(\omega_i) \sim IG(\omega_0, v_0) \).

### 4.4 Gibbs Sampling Algorithm

1. Draw \( \beta_t \), conditional on \( A_t, H_t \) and \( Q \) using the Carter and Kohn algorithm

2. Draw \( a_{ij,t} \) (the elements of \( A_t \)) conditional on \( \beta_t \), using the Carter and Kohn algorithm

3. Compute \( V_{1t}, V_{2t}, V_{3t}, V_{4t}, V_{5t} \) and \( V_{6t} \) conditional on \( a_{12,t}, a_{13,t}, a_{23,t}, a_{14,t}, a_{24,t} \) and \( a_{34,t} \)

4. Compute \( A_tv_t = \epsilon_t \) and draw \( h_{it} \) using the Independence MH algorithm conditional on \( \omega_i \)

5. Draw \( \omega_i \) conditional on \( h_{it} \).

6. Repeat these steps \( K \) times. The last \( M \) draws provide an approximation to the marginal posterior distributions of the model parameters.

### 4.4.1 Convergence

Theoretically, the Gibbs Sampler converges exponentially to the posterior distribution as the number of draws approaches infinity [Geman and Geman (1984)]. However, the major hurdle in empirical work is that it is impossible to make an infinite amount of draws. This brings into question the convergence of the Gibbs Sampling algorithm. Following [Blake et al. (2012)], I compute the recursive means from the retained draws of the 3 parameters. According to the authors, convergence is achieved if the recursive means display random fluctuations around

\footnote{In this study, I made 150,000 draws and discarded the first 149,000.}
their steady values and not display any trend. The recursive means are shown in figure 7 in Appendix A. We can see that the parameters show a very random variation around their mean indicating convergence for the 3 parameters.

5 Empirical Results

5.1 Time Varying Parameters

The empirical analysis of the results starts with providing an evidence of variation of estimated parameters over time. I plot the posterior means of the VAR coefficients figure 1. It is clear from the figure that there is time variation in these coefficients.

![Figure 1: Plot of posterior means of time varying VAR coefficients](image)

I plot the stochastic volatilities of each of the variable in Figure 2. We can see evidence of time varying volatility of each variable. The volatility in global oil supply has been declining since the 2000s. This is in contrast to volatilities of global economic activity and oil prices which have increased over the same period. Oil price tends to fluctuate a lot compared to the other variables. This is very typical of the oil prices which have been found
to fluctuate considerably. Finally, the volatility in GDP growth in Nigeria has fluctuated over time reaching a peak in 2010/2011 and has been on the decline since. The implication of these variation in oil prices is that a standard VAR with constant innovations might not be adequate to capture these dynamics.

Figure 2: Plot of posterior median (blue line), 16th percentile (red line) and 84th percentile (gold line) of estimated stochastic volatilities of each variable

5.2 Response of GDP to demand shock

Figure 3 shows the response of the Nigerian economy to a 1% increase in oil price due to a demand shock. I report the responses for different horizons over time. I define 4 quarters horizons as the short-term, 8 and 12 quarters horizon as the medium term and 24 quarters horizons as the long term. The first thing to note from figure 3 is that the response of the Nigerian economy to a demand shock vary over time. The responses are large in 2010, 2014 and 2015-2016 compared to 1999 and 2017. Secondly, there is a negative relationship between the magnitude of responses and the impulse horizons. The responses are bigger in the short term compared to the medium term and eventually dies out in the long term. Over all the response of the Nigerian economy to a positive demand shock is largely positive over time.
Figure 3: Time Varying median Impulse response of real GDP growth to a 1% increase oil prices due to a demand shock over different horizons

To further show that the responses have been different over time, I randomly select 4 periods to show their responses. The periods are 1999Q1, 2008Q3, 2014Q4 and 2016Q1. It is evident from figure 4 that the responses during this periods are varied.
5.3 Response of GDP to Supply shock

Next, I present the response of GDP growth to a positive oil supply shock. We can see time variation is also evident in the response of GDP growth. In the short term, the response of GDP growth are largely positive (after 4 quarters). However in the medium term, the initial responses is negative (after 8 quarters) tends to diminish and eventually dies out in the long term.
Figure 5: Time Varying median Impulse response of real GDP growth to a 1% increase oil prices due to a demand shock over different horizons

The difference in the responses over time is evident in figure 6 even though the responses are of lesser magnitude as compared to demand shocks.

Figure 6: Time Varying Impulse response of real GDP growth to a 1% increase oil prices due to a demand shock in different periods
6 Conclusion

The study investigated the dynamic relationship between oil prices and key macroeconomic variables in Nigeria over time by making use of time varying VAR with Stochastic volatility. This enabled me to differentiate between inter-temporal variations in the size and frequency of exogenous shocks from the endogenous propagation of those shocks. The study found that both the VAR coefficients and the variance have varied over time. The study found that demand shocks are the most important for the Nigerian economy.
References


Appendix A

To see the second case it is worth noting the following relationship;

\[ A_t v_t = \epsilon_t \]  \hspace{1cm} (A.1)

where \( \epsilon_t \sim N(0, D_t) \)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
a_{12,t} & 1 & 0 & 0 \\
a_{13,t} & a_{23,t} & 1 & 0 \\
a_{14,t} & a_{24,t} & a_{34,t} & 1
\end{pmatrix}
\begin{pmatrix}
v_{1t} \\
v_{2t} \\
v_{3t} \\
v_{4t}
\end{pmatrix}
=
\begin{pmatrix}
\epsilon_{1t} \\
\epsilon_{2t} \\
\epsilon_{3t} \\
\epsilon_{4t}
\end{pmatrix}
\]

\[ v_{1t} = \epsilon_{1t} \]

\[ a_{12,t}v_{1t} + v_{2t} = \epsilon_{2t} \]

\[ a_{13,t}v_{1t} + a_{23,t}v_{2t} + v_{3t} = \epsilon_{3t} \]

\[ a_{14,t}v_{1t} + a_{24,t}v_{2t} + a_{34,t}v_{3t} + v_{4t} = \epsilon_{4t} \]

Rearranging we have

\[ v_{1t} = \epsilon_{1t} \]

\[ v_{2t} = -a_{12,t}v_{1t} + \epsilon_{2t} \]

\[ v_{3t} = -a_{13,t}v_{1t} - a_{23,t}v_{2t} + \epsilon_{3t} \]

\[ v_{4t} = -a_{14,t}v_{1t} - a_{24,t}v_{2t} - a_{34,t}v_{3t} + \epsilon_{4t} \]

\[ \text{var}(\epsilon_{2t}) = (h_{2t}) \quad \text{var}(\epsilon_{3t}) = (h_{3t}) \quad \text{var}(\epsilon_{4t}) = (h_{4t}) \]

\[
\begin{pmatrix}
a_{12,t} \\
a_{13,t} \\
a_{23,t}
\end{pmatrix}
= \begin{pmatrix}
a_{12,t-1} + V_{1,t} \\
a_{13,t-1} + V_{2,t} \\
a_{23,t-1} + V_{3,t}
\end{pmatrix}
\]
\[
\begin{pmatrix}
a_{14,t} \\
a_{24,t} \\
a_{34,t}
\end{pmatrix} =
\begin{pmatrix}
a_{14,t-1} \\
a_{24,t-1} \\
a_{34,t-1}
\end{pmatrix} +
\begin{pmatrix}
V_{4t} \\
V_{5t} \\
V_{6t}
\end{pmatrix}
\]

Therefore \(a_{ij,t}\) are the time varying coefficients on regressions involving the VAR residuals.

Figure 7: Recursive mean for the 3 key parameters