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Market sentiment and heterogeneous fundamentalists in an evolutive financial market model

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Abstract

We study a financial market populated by heterogeneous fundamentalists, whose decisions are driven by "animal spirits". Each agent may have optimistic or pessimistic beliefs about the fundamental value, which are selected from time to time on the basis of an evolutionary mechanism. The evolutionary selection depends on a weighted evaluation of the general market sentiment perceived by the agents and on a profitability measure of the existent strategies. As the relevance given to the sentiment index increases, a herding phenomenon in agents behavior may take place and the animal spirits can drive the market toward polarized economic regimes, which coexist and are characterized by persistent high or low levels of optimism and pessimism. This conduct is detectable from agents polarized shares and beliefs, which in turn influence the price level. Such polarized economic regimes can consist in stable steady states or can be characterized by endogenous complex dynamics, generating persistent alternating waves of optimism and pessimism, as well as return distributions displaying fat tails and excess volatility.

Keywords: heterogeneous fundamentalists; animal spirits; behavioral finance; sentiment index; complex dynamics.

JEL classification: D84, G41, C62, B52

1 Introduction

Representing agents as boundedly rational actors has become a quite common modeling assumption in several economic contexts. Such an assumption relies on the evidence that the complexity of the economic environment restricts the actual agents capability to have a complete knowledge about it, so that agents take decisions that are unavoidably cursed by uncertainty (see also [35]). Moreover, the psychological investigation about humans, and hence about economic agents, shows that most of the decisions are taken on the basis of simple heuristics (see e.g. [15], [18] and [37] among others). Individuals, being affected by psychological and emotional factors, rely more on impressions and common feelings than on a precise knowledge and evaluation of the environment they live in.

Nowadays, the approach based on boundedly rational agents is widely applied in the modeling of financial markets (see e.g. [3], [8], [19] and [24]), which are intrinsically characterized by a high degree of complexity and in which the behavior of the agents can not be neglected in order to understand and replicate the dynamics exhibited by the real-world economic variables. The literature that stems from these ideas is burgeoning and widespread (see, just to cite a few, [4], [5], [7], [27], [28], [30], [31]).

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Concerning the research strand that is closer to the present contribution, we mention: the paper by De Grauwe and Rovira Kaltwasser [9], in which the emergence of waves of optimism/pessimism is explained in terms of an evolutionary selection between optimistic/pessimistic exogenous beliefs about the fundamental value; the work by Naimzada and Pireddu [29], that studies the evolution of agents beliefs with endogenously varying levels of optimism/pessimism; the paper by Cavalli et al. [6], in which endogenously changing optimistic/pessimistic beliefs can not be disentangled from their evolutionary selection in view of understanding the emergence of waves of optimism/pessimism. In all the above mentioned works, the main mechanisms (imitation and evolutionary selection) ground on precise evaluations of economic indicators (like profits or forecasting errors), which are assumed to be correctly estimated by agents and on which agents base their choices.

However, the previous bunch of works deserves two remarks. The first one concerns the modeling side. Although being in line with a general idea of bounded rationality, such a literature systematically disregards the Keynes' insights "that a large proportion of our positive activities depend on spontaneous optimism rather than on a mathematical expectation, whether moral or hedonistic or economic", and that "there is the instability due to the characteristic of human nature", namely that economic agents can act as "animal spirits" (see [22]). In fact, in the aforementioned contributions, the role of animal spirits behaviors is relegated to a qualitative outcome (in terms of waves of optimism/pessimism), because agents divert from a rational expectation assumption and have biased beliefs. However, agents still take decisions "rationally", measuring precise economic quantities. In this sense, animal spirits are not among the endogenous drivers of the economic decisions. The second remark concerns the lack of a relevant result: when the agents decisions are driven by animal spirits and rely on a general perceived opinion, optimism and pessimism should be able to self-sustain themselves, polarizing the majority of agents toward one or the other feeling and leading them to "herd" in persistent groups where almost all members behave as optimists or as pessimists. Thus a natural question arises, which constitutes the motivation of the present research: "What happens when decisions (in the present case, strategies in a financial market) are driven by animal spirits?".

We try to tackle both previous concerns, without losing the undebatably positive qualities of the results arising in the existing literature, in terms of relevant endogenous dynamics and qualitatively good properties in the behavior of the economic variables. Therefore, the present contribution aims at providing a rigorous formal modeling of animal spirits as one of the drivers of the agents and, subsequently, of the market behavior. A further motivation of this approach is the content of a recent survey by Franke and Westerhoff (see [14]), in which a similar viewpoint is expressed. Using their words, the above mentioned literature provides a "weak form" of animal spirits, in the sense that the "model is able to generate waves of, say, an optimistic and pessimistic attitude, or waves of applying a forecast rule 1 as opposed to a forecast rule 2." (p. 3). Conversely a "strong form" of animal spirits modeling approaches "exists if agents also rush toward an attitude, strategy, or so on, simply because it is being applied at the time by the majority of agents." (p. 3). In the present work, agents rush toward optimism and pessimism depending on what they observe or feel about the behavior of the majority of the other agents, which is what we call the "general sentiment". Hence, differently from the work in [9] that adopts the "weak" idea of animal spirits, the present model provides a "strong" form of animal spirits modeling in order to retrieve the Keynesian seminal idea.

More precisely, we consider a financial market model populated by optimistic and pessimistic fundamentalists[1] who respectively overestimate and underestimate the true fundamental value, as in

[1]It may be argued that professional market participants trade on the basis of either technical or fundamental analysis. Nonetheless, there exist several papers in which agents are endowed with biased beliefs about the fundamental (or target) value of a certain economic variable, such as the price, the inflation, output gap or exchange rate target (see, just to cite a few, Brock and Hommes (1998), Rovira Kaltwasser (2010), Anufriev et al. (2013), Agliari et al. (2017) and Hommes and Lustenhouwer (2017)). Moreover, due to the findings of our work, there is also a deeper reason to consider only fundamentalist agents in the model. Namely, in this way, even if a major role is played by the negative feedback of investors on prices (as the behavior of fundamentalists refers to buying and selling decisions that are made with a view to the underlying value of the asset, and tend to push prices back toward fundamentals), we find the emergence of

[6]. Agents are allowed to switch between their pessimistic or optimistic attitude and, accordingly, we propose an evolutionary mechanism that governs their choices. Such a mechanism ends up in a combination of the average mood perceived by the agents about the status of the market (i.e., the *sentiment index*) and of a measure of profitability of the existent strategies. The sentiment index depends on how many optimists and pessimists populate the market and on how much their beliefs about the fundamental value are optimistically/pessimistically biased. In this way, the psychological and emotional components become a constitutive part of the decision process.

The results emerging from our analysis are interesting under several perspectives. Moving from a modeling approach relying on a weak form of animal spirits to a different one which relies on a strong form, if the final outcomes were the same in both approaches, there would not be much difference between the two forms of animal spirits. But this is not the case. We shall show the emergence of herding phenomena which occurs "because individual agents believe that the majority will probably be better informed and smarter than they themselves" (see [14]), and thus agents can pay more attention to the general sentiment than to the comparison of realizable profits. Herding drives the occurrence of long lasting waves of optimism and pessimism, which are a consequence of the (strong form of) animal spirits behavior of the agents. Furthermore, the emergence of coexisting economic regimes (consisting of stable steady states or of more complex attractors), that did not emerge in the above mentioned literature and that are characterized by persistently polarized levels of optimism and pessimism, mirrors the eventuality of such herding behaviors. When agents "endeavor to conform with the behavior of the majority or the average" [23], it is more likely that animal spirits generate outcomes which can be seen as a result of a herding phenomenon around a polarized situation. Finally, the present setting does not rule out the possibility of endogenous dynamics with neat and persistent periods of optimism alternating with those of pessimism, and the resulting dynamics provide a clearer representation of the stylized facts occurring in real-world financial markets.

The remainder of the paper is organized as follows. In Section 2 we outline the model, which is analytically investigated in Section 3. Simulative results are collected in Section 4. In Section 5 we present the findings deriving from the extension of the benchmark model when considering an endogenous mechanism in the formation of agents beliefs and the presence of another class of agents, namely technical traders. Finally, Section 6 concludes. All the proofs of our analytical results can be found in the Appendix.

2 The model

The evolutive financial model we study takes inspiration from [9] and grounds on a market in which, at each discrete time period t, a normalized population of size 1 is composed by boundedly rational fundamentalists.

Agents are divided into optimists and pessimists[2], whose shares are respectively described by $\omega_t \in (0, 1)$ and $1 - \omega_t$. Fundamentalists buy/sell stocks in undervalued/overvalued markets. In particular, pessimistic and optimistic agents, who systematically underestimate and overestimate the true fundamental value F, have biased beliefs respectively equal to $X = F - \Delta/2$ and $Y = F + \Delta/2 = X + \Delta$, where $\Delta > 0$ is the belief bias and assesses the degree of heterogeneity among the agents.[3]

herding phenomena, which are instead usually triggered by a positive feedback, typical of the technical traders behavior. Therefore, including technical traders in our model could conceal the effect of the sentiment index on herding. For a classical work in the herding literature we refer the interested reader to Kirman (1993), which proposes a model of stochastic recruitment to explain both herding and epidemics phenomena observed in financial markets.

[2] A clear explanation of the empirical basis for the assumption that fundamentalists are either optimistic or pessimistic can be found in the introduction of [9]. Indeed, authors recall, e.g., that in the past decade two sets of beliefs emerged about the fundamental value of the US dollar, according to which the large account deficits of the US observed since the second half of the 1990s were unsustainable (see [32] and [33]) or perfectly sustainable (see [17]).

[3] As in [9], we deal with a symmetric framework in order to focus on the most significant form of heterogeneity, i.e. the maximum possible degree of polarization, represented by Δ, between different attitudes of agents toward the reference

The fraction of each kind of agents varies according to an evolutionary selection of the best performing optimistic/pessimistic heuristic. The stock price P_t is adjusted by a market maker by means of a nonlinear mechanism.

As in [9], the demand functions of pessimistic and optimistic agents are respectively given by $d_{t,pes} = \alpha(X - P_t)$ and $d_{t,opt} = \alpha(Y - P_t)$, where P_t is the asset price at time t and $\alpha > 0$ is a demand reactivity parameter, which we can assume to be the same for the two kinds of agents, being them both fundamentalists. If at time t the share of pessimists (resp. optimists) is $\omega_t \in [0,1]$ (resp. $1-\omega_t$), the total excess demand is $D_t = \omega_t\alpha(X - P_t) + (1-\omega_t)\alpha(Y - P_t)$, which, recalling that $Y = X + \Delta$, can be rewritten as $D_t = \alpha(F - P_t + \Delta(1/2 - \omega_t))$. We assume that the price variation is described by the nonlinear, bounded mechanism

$$P_{t+1} - P_t = f(\gamma D_t) = f(\gamma\alpha(F - P_t + \Delta(1/2 - \omega_t))), \tag{1}$$

where $\gamma > 0$ represents the price adjustment reactivity and $f : \mathbb{R} \to (-a_2, a_1)$, with $-a_2 < 0 < a_1$, is a twice differentiable sigmoidal function, i.e., an increasing function, satisfying $f(0) = 0$, $f'(0) = 1$, $f''(z) > 0$ on $(-\infty, 0)$ and $f''(z) < 0$ on $(0, +\infty)$. Moreover, as it is evident from (1), without loss of generality we can set $\alpha = 1$, encompassing both the demand and the price adjustment reactivities in the parameter γ.

The nonlinear mechanism introduces a cautious price adjustment, as from t to $t+1$ prices can only increase or decrease by a bounded quantity, respectively given by a_1 or a_2. Namely, the mechanism in (1) encompasses a conservative behavior for the market maker, induced by a central authority that, trying to limit overreaction phenomena with the consequent occurrence of an excessive stock volatility, imposes limits to price variations (see [12], [16] and [26]). As a consequence, the market maker prudently adjusts prices in the presence of extreme excess demand, while when excess demand is small the price adjustment is nearly proportional to it. In particular, the price variation limiter mechanism can be modeled by a sigmoidal adjustment rule that determines a bounded price variation in every time period, thanks to the presence of two asymptotes that limit the price changes. We recall that, in the literature on behavioral financial markets, nonlinear price adjustment mechanisms have been already considered, among others, by [36, 38]. More precisely, thanks to the previous assumptions on f, from (1) we have that the stock price respectively increases or decreases when the excess demand is positive or negative, and that the variation rate increases as the excess demand vanishes, since the derivative of the right-hand side of (1) with respect to D_t is given by $\gamma f'(\gamma D_t)$, and attains its maximum value γ when $D_t = 0$.

The last part of the model to be described concerns the evolution of the shares of optimists/pessimists, based on an evolutionary competition between the two behavioral rules. Differently from [6, 9], where the fitness measure in the switching mechanism relied only on the comparison between the profits that could be realized by the two groups of agents, in the present contribution the evolutionary selection of beliefs depends also on the general feeling perceived by the agents about the market status. Such a feeling is described by the sentiment index

$$I_t = \omega_t X + (1-\omega_t)Y - F = X + (1-\omega_t)\Delta - F, \tag{2}$$

which measures the difference between the average belief about the fundamental value, represented by $\omega_t X + (1-\omega_t)Y$, and the true fundamental value F. The average belief about fundamental value depends on both the beliefs and the shares, whose effects can not be completely disentangled. The sign of I_t gives information about the general degree of optimism or pessimism in the market, as I_t is positive (negative), portraying the underlying optimistic (pessimistic) perceived market mood, when

value F. This also allows us to keep the model analytically tractable and to provide a neater interpretation of the results.

$\omega_t X + (1-\omega_t)Y$ is larger (smaller) than F. In particular, since $X = F - \Delta/2$, we find that

$$I_t = \Delta(1/2 - \omega_t). \tag{3}$$

Hence, I_t is positive (negative) when the share of pessimists is below (above) 1/2.
Then, the population fraction composed by pessimists evolves depending on a convex combination of the general market sentiment and of the profits realized by the two kinds of agents, according to the following updating rule

$$\omega_{t+1} = \frac{e^{\beta(\sigma(-I_t)+(1-\sigma)\pi_{X,t+1})}}{e^{\beta(\sigma(-I_t)+(1-\sigma)\pi_{X,t+1})} + e^{\beta(\sigma I_t+(1-\sigma)\pi_{Y,t+1})}} = \frac{1}{1+e^{\beta(2\sigma I_t+(1-\sigma)(\pi_{Y,t+1}-\pi_{X,t+1}))}}, \tag{4}$$

where β is a positive parameter representing the intensity of choice of the switching mechanism and $\sigma \in [0,1]$ is the *sentiment weight*.
Equation (4) describes the probability that agents will be pessimistic in their belief. Such a probability depends on two factors: the general market sentiment at time t and a measure of profitability of the existent strategies.[4] The latter is given by the most recent profit

$$\pi_{j,t+1} = (P_{t+1} - P_t)(j - P_t), \quad j \in \{X, Y\}, \tag{5}$$

that would have been realized adopting a pessimistic belief (corresponding to $j = X$) or an optimistic one (corresponding to $j = Y$).[5] When $\sigma = 0$, the evolutionary mechanism in (4) is exactly the same as in [9], while, as σ increases, the relevance assigned by the agents to the general perceived mood increases, raising the impact of the crowd psychology and of the herd instinct. If $\sigma = 1$, the profitability measure is no more influencing the switching mechanism, which only depends on the sentiment index (2), and the share of pessimists will increase (decrease) when I_t is negative (positive), i.e., when the average belief about fundamental value is smaller (larger) than F. We note that in [6] the average belief about the fundamental value (and its generalization over a window of $n > 1$ preceding periods) was introduced and used to study the possible emergence of waves of optimism and pessimism. However, in [6], the index I_t only helped in describing the dynamics without affecting them, being not taken into account by the agents in their decisions. Finally, we stress that β in (4) measures how much relevance agents assign to the forecasting rules. If β is small, agents are quite indifferent to the signals coming from the two heuristics (in terms of general sentiment and/or profitability), and they tend to equally distribute themselves between optimism and pessimism. Conversely, if more relevance is given to the perceived market mood and/or to the eventual profitability, a larger share of agents will switch to the most performing attitude toward the reference value.

Our model is obtained collecting the price adjustment mechanism (1) and the evolutionary mechanism (4), being described by the two-dimensional map $G = (G_1, G_2) : (0, +\infty) \times (0,1) \to \mathbb{R}^2$, $(P_t, \omega_t) \mapsto (G_1(P_t, \omega_t), G_2(P_t, \omega_t))$, defined as:

$$\begin{cases} P_{t+1} = G_1(P_t, \omega_t) = P_t + f\left(\gamma(F - P_t + \Delta(1/2 - \omega_t))\right), \\ \omega_{t+1} = G_2(P_t, \omega_t) = \dfrac{1}{1 + e^{\beta(\sigma\Delta(1-2\omega_t)+(1-\sigma)\Delta f(\gamma(F-P_t+\Delta(1/2-\omega_t))))}}, \end{cases} \tag{6}$$

where, in the last equation, we replaced I_t with its expression provided in (3) and we employed the identity $\pi_{Y,t+1} - \pi_{X,t+1} = (Y - X)(P_{t+1} - P_t) = \Delta f(\gamma D_t)$.[6]

[4] We stress that the opposite signs preceding I_t in the argument of the exponential functions in (4) are a consequence of the different attitude of optimists and pessimists toward positive or negative values of the sentiment index.

[5] We would like to point out that more refined fitness measures could encompass portfolio balances and wealth, as the literature on market microstructure deals with (see e.g. [11] and [21]). But this is beyond the scope of this paper and we leave the investigation of such measures for future research.

[6] In the present work we only consider the case of exogenous biased beliefs, to better focus on the role of the market

We stress that for $\sigma = 0$ the model in (6) reduces to that studied in [6] when $\mu = 0$ or in [9] for a linear price adjustment mechanism. Therefore, in what follows we investigate what happens when $\sigma > 0$, focusing in particular on the extreme case $\sigma = 1$ (see Section 4). It is worth noticing that the present framework significantly diverts from [9] under several aspects. As explained above, the key economic element to be here considered is that the choice between optimism and pessimism is not only driven by a rational computation of the profitability of the existent strategies, but, as σ increases, the psychological and emotional aspects increasingly assume a central role, being the unique impulse when $\sigma = 1$. This reinforces the behavior of agents as "animal spirits", which is partially encompassed in [6, 9] in the optimistically /pessimistically biased beliefs, but which, as $\sigma \to 1$, becomes the main motivational driver of the agents choice about the forecasting rule in the present model. However, the significance of the considered framework is not limited to the economic interpretation of the model, but, as it will become evident from the analytical results in Section 3 and the numerical simulations in Section 4, the possible dynamical outcomes arising when the sentiment index drives the stock market significantly differ from those found in the existing literature, so that the proposed approach provides a "strong form" ([14]) of animal spirits modeling, differently from the "weak form" of [6, 9].

3 Analytical results on steady states and local stability

In this section we determine the possible steady states of the model outlined before and provide analytical conditions for the local stability of the fundamental steady state. Additionally, we also present some results on how the additional steady states vary when the relevant parameters change. We start by investigating the existence of the steady states of (6).

Proposition 1. *System* (6) *has*

a) a unique steady state $S^* = (P^*, \omega^*) = (F, 1/2)$ *if* $\sigma \in [0, 1]$ *and*

$$\sigma \leq \frac{2}{\beta\Delta}; \tag{7}$$

b) three steady states $S^*, S^o = (P^o, \omega^o)$ *and* $S^p = (P^p, \omega^p)$ *if* $\frac{2}{\beta\Delta} < \sigma \leq 1$. *In particular,* S^o *and* S^p *are symmetric w.r.t.* S^*, *with* $P^p < P^* < P^o$ *and* $\omega^o < \omega^* < \omega^p$.

The previous result bears relevance for our analysis. In fact, differently from what is found in the existing literature (see e.g. [6, 9, 29]), where the unique steady state is given by the fundamental steady state S^*, when animal spirits affect economic decisions the system can be driven toward a steady state characterized by either greater (P^o) or smaller (P^p) prices than the fundamental value. In these steady states, the population consists of a larger share of optimists ($\omega^o < 1/2$) or pessimists ($\omega^p > 1/2$), respectively. We then have two more steady economic regimes that can be identified as "pessimistic" (S^p) and "optimistic" (S^o), coexisting with S^*. This eventuality occurs if agents give a sufficiently large relevance to the perceived market mood, as the additional steady regimes can emerge only for suitably large sentiment weight values. If agents rely just on a "rational" comparison of the performance of pessimism and optimism in terms of profits (as in [6]), the equilibrium configuration can solely consist in an even distribution of pessimists/optimists, with the stock price corresponding to the true fundamental value. However, as the sentiment weight approaches 1, the switching mechanism is more and more influenced by the sentiment index, whose size is not only determined by the population share of pessimists, but also by the distance Δ between optimistic and pessimistic beliefs. More precisely, if the relevance given by the agents to the perceived mood is small (i.e., β is low), agents will more likely

sentiment on the results. The generalization to the case in which the biases on the beliefs depend on the relative ability to guess the actual realized price is considered in Section 5, from which it is possible to see that the results we shall present in Sections 3 and 4 are robust with respect to the endogenization of the beliefs.

choose indifferently one of the two heuristics, so that deviations from a uniform distribution have a little consequence and shares will settle back to a uniform distribution. Conversely, if the relevance is large (i.e., β is high), even a small excess of pessimistic agents ($\omega_t > 1/2$) triggers a diffusion of pessimism, that leads the majority of agents to become pessimists ($\omega^p > 1/2$). We will come back to this aspect after Proposition 4 with the help of the stability analysis and of some simulated time series. Moreover, as the factors characterizing the agents behaviors become more extreme (i.e., as the intensity of choice and/or the polarization of the beliefs increase), the effect of animal spirits is bolstered and a progressively reduced sentiment weight is enough to trigger the emergence of the polarized steady states. In this respect, we stress that both the relevance given to the evolutionary selection of heuristics ($\beta > 0$) and the heterogeneity degree of beliefs ($\Delta > 0$) are essential, as otherwise only the intermediate steady state is possible (as (7) is fulfilled).

However, the agents features, encompassed in parameters β and Δ, and the sentiment weight σ do not only foster the emergence of the steady states S^o and S^p, but also significantly affect their position, as well as the values of the sentiment index I^o and I^p at S^o and S^p, as shown in the next result.

Proposition 2. *Let $\sigma > 2/(\beta\Delta)$, $\sigma \in [0, 1]$. Then, on increasing σ, β and Δ we have that ω^o decreases, while P^o and I^o increase, and that ω^p increases, while P^p and I^p decrease.*

This result reinforces that in Proposition 1. Firstly, the more the psychological and emotional components are determinant for the choice of optimistic/pessimistic heuristics, the more the polarized steady states divert from the intermediate one. Indeed, increasing the relevance of the perceived market mood leads to final outcomes that are more strongly characterized (both in terms of prices and shares) by optimism and pessimism. Such a feature is reflected in the resulting sentiment index at the polarized steady states, which is not "neutral" ($I = 0$) as at S^*, but consistently portrays the pessimistic ($I^p < 0$) and optimistic ($I^o > 0$) mood perceived in the market. Psychological factors can then strengthen the role of the market sentiment in determining the prices and the shares.

The next level of investigations concerns how the stability of S^* is affected by the sentiment weight and, more generally, we analyze the effects of the sentiment index on the resulting dynamics.[7] Before presenting such results, we compare the roles of β, Δ and γ on the stability of S^*, when the evolutionary mechanism is only driven either by the profitability measure[8] ($\sigma = 0$) or by the perceived sentiment ($\sigma = 1$). To this end we recall that, if we consider homogeneous beliefs ($\Delta = 0$) and we consequently neglect the evolutionary switching mechanism ($\beta = 0$), it is easy to see that dynamics only consist in the price adjustment mechanism and that S^* is stable when $\gamma < 2$.

Proposition 3. *When $\sigma = 0$, the steady state S^* is locally asymptotically stable provided that $2(\gamma - 2)/\gamma < \Delta^2\beta < 4/\gamma$. When $\sigma = 1$, the steady state S^* is locally asymptotically stable provided that $\gamma < 2$ and $\beta\Delta < 2$.*

When the price mechanism does not introduce instabilities ($\gamma < 2$), increasing the intensity of choice and the heterogeneity degree has a destabilizing effect for both extremal choices of fitness measures (i.e., $\sigma = 0$ and $\sigma = 1$). Conversely, when the price mechanism introduces instability ($\gamma > 2$), suitable intermediate values of β and Δ can stabilize S^* (as discussed in [9]) when agents choose their strategy on the basis of the profitability measure. Such stabilization does not occur when the fitness measure is given by the sentiment index, as the steady state S^* is unstable regardless of Δ and β.

[7]In what follows, we say that an unconditionally stable/unstable scenario is realized when the steady state is locally asymptotically stable/unstable independently of the parameter values; a stabilizing/destabilizing scenario occurs when the steady state is locally asymptotically stable only above/below a certain threshold and unstable otherwise; a mixed scenario arises when the steady state is locally asymptotically stable only for intermediate parameter values, between two stability thresholds, and unstable otherwise.

[8]Indeed, this case corresponds to that studied in [9, Proposition 1].

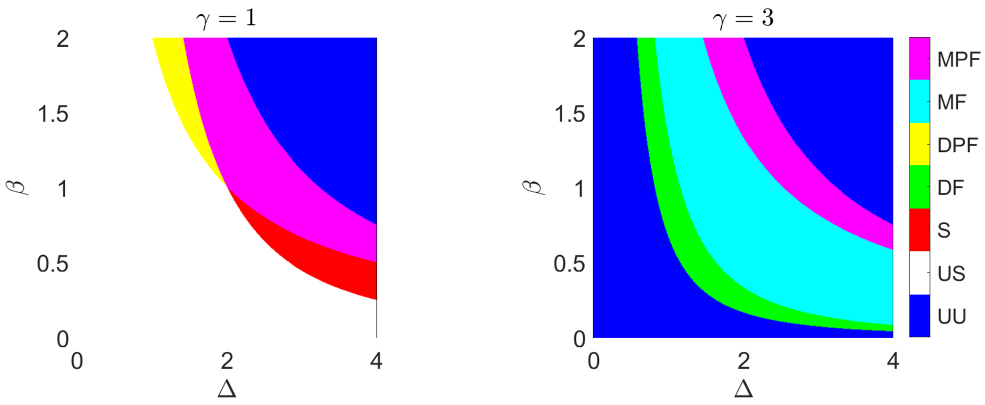

Figure 1: For each couple of β and Δ, different colors correspond to different stability scenarios on increasing σ. Red color denotes values (Δ, β) for which a stabilizing scenario (S) occurs (i.e., according to Proposition 4, S^* is stable for $\sigma \in (\sigma_{ns}, 1]$), green color denotes values for which a destabilizing scenario through a flip bifurcation (DF) occurs (i.e. S^* is stable for $\sigma \in [0, \sigma_{fl})$), yellow color denotes values for which a destabilizing scenario through a pitchfork bifurcation (DPF) occurs (i.e. S^* is stable for $\sigma \in [0, \sigma_{pf})$), cyan color denotes values for which a mixed scenario in which instability occurs through a flip bifurcation (MF) occurs (i.e. S^* is stable for $\sigma \in (\sigma_{ns}, \sigma_{fl})$), magenta color denotes values for which a mixed scenario in which instability occurs through a pitchfork bifurcation (MPF) occurs (i.e. S^* is stable for $\sigma \in (\sigma_{ns}, \sigma_{pf})$). White and blue colors respectively denote values (Δ, β) for which S^* is either stable (US) or unstable (UU) independently of σ.

Proposition 3 shows that the same degree of beliefs heterogeneity has a different effect when the fitness criterion is represented by the profitability measure or by the sentiment index. In the latter case, the role of the parameters affecting the selection mechanism (β) and the agents heterogeneity (Δ) is essentially the same, as the stability of S^* depends on $\Delta\beta$: therefore, increasing either β or Δ by the same amount has an identical effect. Conversely, beliefs heterogeneity has a more intense effect when the strategy choice depends only on profits, as in that case the stability of S^* is affected by $\beta\Delta^2$. Such dissimilarity can be easily understood in terms of the effect of beliefs polarization on the fitness measure: the degree of heterogeneity affects the sentiment index through beliefs only "once" while it affects the profits twice, both directly, through the excess demand, and through prices, resulting in a "squared" influence. This may occur since agents assign more emphasis to the possibility of realizing a profit evaluating their own strategy rather than considering the general mood perceived by the market, which can manifest in a sluggish manner due to some form of slowness in the news diffusion about the market status.

As a consequence, a large (resp. small) belief polarization has a greater (resp. reduced) effect on the stability of the fundamental steady state when the sentiment has no relevance, compared to the case when it drives the choice of strategies. It is then predictable that, ceteris paribus, as σ increases, the effect of Δ on the stability of S^* can increase or decrease. This will help to understand what happens as the relevance of the sentiment index increases, an issue which is studied in the next proposition. In particular, in order to better focus on the role of σ on varying β and Δ, we make explicit the stability conditions considering two distinct frameworks in relation to the price dynamics. Namely, we study the stability of S^* for System (6) considering two regimes, represented by $\gamma = 1$ and $\gamma = 3$, for which the price mechanism respectively is/is not a source of instability.

Proposition 4. *On varying $\sigma \in [0, 1]$, we have that the sentiment weight can have a destabilizing, stabilizing, mixed or neutral effect on the stability of S^*. In particular, condition (7) is necessary for*

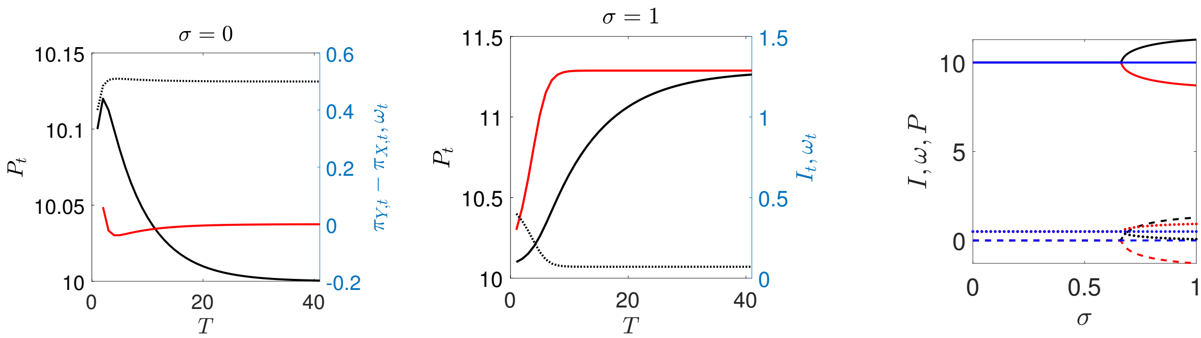

Figure 2: The left and middle panels represent time series when fitness measure coincides with the profitability measure (left panel) with the sentiment index (middle panel). Prices (solid black line, referred to the left axis), shares of pessimists (dotted black line, referred to the right axis) and fitness measure (red line, referred to the right axis) are represented. The right panel represents the evolution of the optimistic (black), pessimistic (red), intermediate (blue) steady states on varying σ. Solid line: P; dotted line: ω; dash-dotted line: I. Parameter values used in all the simulations are $F = 10$, $\beta = 1$, $\gamma = 0.1$ and $\Delta = 3$.

the stability of S^*. If $\gamma = 1$, then S^* is stable for

$$\begin{array}{lll}
\sigma \in [0,1] & \textit{if} & \beta < \min\left\{\frac{4}{\Delta^2}, \frac{2}{\Delta}\right\} \\
\sigma \in (\sigma_{ns}, \sigma_{pf}) = \left(\frac{\Delta^2\beta-4}{\Delta^2\beta}, \frac{2}{\beta\Delta}\right) & \textit{if} & \max\left\{\frac{4}{\Delta^2}, \frac{2}{\Delta}\right\} < \beta < \frac{2\Delta+4}{\Delta^2} \\
\sigma \in [0, \sigma_{pf}) = \left[0, \frac{2}{\beta\Delta}\right) & \textit{if} & \frac{2}{\Delta} < \beta < \frac{4}{\Delta^2} \ \textit{(possible only if } \Delta < 2) \\
\sigma \in (\sigma_{ns}, 1] = \left(\frac{\Delta^2\beta-4}{\Delta^2\beta}, 1\right] & \textit{if} & \frac{4}{\Delta^2} < \beta < \frac{2}{\Delta} \ \textit{(possible only if } \Delta > 2)
\end{array}$$

and it is unstable for any σ if $\beta > \frac{2\Delta+4}{\Delta^2}$.
Conversely, if $\gamma = 3$, we have that S^* is stable for

$$\begin{array}{lll}
\sigma \in [0, \sigma_{fl}) = \left[0, \frac{3\Delta^2\beta-2}{\Delta\beta(3\Delta+1)}\right) & \textit{if} & \frac{2}{3\Delta^2} < \beta < \frac{4}{3\Delta^2} \\
\sigma \in (\sigma_{ns}, \sigma_{fl}) = \left(\frac{3\Delta^2\beta-4}{\Delta\beta(3\Delta+4)}, \frac{3\Delta^2\beta-2}{\Delta\beta(3\Delta+1)}\right) & \textit{if} & \frac{4}{3\Delta^2} < \beta < \frac{6\Delta+4}{3\Delta^2} \\
\sigma \in (\sigma_{ns}, \sigma_{pf}) = \left(\frac{3\Delta^2\beta-4}{\Delta\beta(3\Delta+4)}, \frac{2}{\Delta\beta}\right) & \textit{if} & \frac{6\Delta+4}{3\Delta^2} < \beta < \frac{2\Delta+4}{\Delta^2}
\end{array}$$

and it is unstable for any σ if $\beta > \frac{2\Delta+4}{\Delta^2}$ or if $\beta < 2/(3\Delta^2)$.
Finally, crossing the threshold values σ_{fl}, σ_{ns} and σ_{pf} a flip, Neimark-Sacker and pitchfork bifurcation respectively occurs.

We start by analyzing the case $\gamma = 1$, making reference to the left panel in Figure 1. In this case, the occurrence of instability only depends on the switching mechanism, and, according to Proposition 3, in the two extremal frameworks corresponding to $\sigma = 0$ and $\sigma = 1$ we have that S^* is stable for $\Delta^2\beta < 4$ and $\Delta\beta < 2$, respectively.

If β and Δ are suitably small, the increasing relevance given to the perceived sentiment has no effect on stability. Either if the context is affected by a very reduced heterogeneity so that the evolutionary pressure is not enough to induce agents to change strategy or if the beliefs are polarized but the agents do not rely very much on them, we have that S^* is stable regardless of the adopted fitness measure (white region).

If the beliefs polarization is sufficiently large ($\Delta > 2$) and we increase the intensity of choice, since in this case the effect of heterogeneity plays a major role on the profitability measure than on the general sentiment, we have a situation in which S^* is unstable for $\sigma = 0$ and stable for $\sigma = 1$ (see the comments after Proposition 3) and thus the sentiment index has a stabilizing effect on S^* (red region).

Conversely, if the beliefs polarization is small ($\Delta < 2$) and we increase the intensity of choice, since in this case the effect of heterogeneity on the profitability measure is weaker than before, we have a situation in which S^* is stable for $\sigma = 0$ and unstable for $\sigma = 1$. The consequence is that, as the relevance of the sentiment index increases, S^* can lose stability (yellow region). However, such instability does not mean that dynamics become erratic, but that agents rather start herding toward the same strategy, and this drives the evolution of prices toward either the optimistically or the pessimistically biased new steady state. That mechanism can be better understood looking at the left and middle panels in Figure 2, where the parameter setting as well as the initial conditions are essentially the same as in Figure 1.[9] If the fitness measure of the evolutionary process is the profitability measure ($\sigma = 0$, left panel), since initial conditions depict a slightly optimistic state in which $P_0 = 10.1 > F = 10$ and $\omega_0 = 0.4$, there is a positive excess demand ($D_0 = 0.2$) that induces a slight increase in prices at $t = 1$ ($P_1 = 10.12$). However, profits of optimistic agents are only a little larger than those of pessimists, so that the updated share of pessimists is actually larger ($\omega_1 = 0.485$) than the initial one. That is where the typically negative feedback about the market prices of fundamentalists comes in: if prices are too large, pessimists have a more pronounced propensity to sell than optimistic agents have to buy, so that the price increase slows down and a larger share of agents believe that a change is about to occur. This leads prices to decrease at $t = 2$, so that the profits of pessimists are larger than those of optimists and most agents switch to the pessimistic strategy ($\omega_2 > 0.5$). The next trajectories lead prices to monotonically decrease toward the fundamental value, as well as agents choices to evenly distribute between the two strategies. Conversely, if the fitness measure is the sentiment index ($\sigma = 1$, central panel), since at $t = 1$ the price increases, the overall sentiment increases, too. Thus, if the evolutionary tendency to adopt the most fitting strategy is suitably strong and strategies are heterogeneous enough, this leads to a decrease in the share of pessimists. In such a case, the switching mechanism and the heterogeneity strengthen the optimistic mood, which gets a positive feedback by the agents. Indeed, if pessimism is initially quite pronounced, we then observe the opposite effect, and what actually happens on the steady state values as σ increases is reported in the right panel of Figure 2, with the emergence of increasingly polarized prices, shares and general sentiment.

Going back to the role of σ, if we further suitably increase β, for any degree of beliefs heterogeneity $\Delta > 2$ we have that the fundamental steady state is unstable in both the two extremal frameworks identified by $\sigma = 0$ and $\sigma = 1$. However, for intermediate values of σ the negative feedback of the profitability measure is offset by the positive feedback of the sentiment index, and their effects cancel out, giving rise to stable dynamics (magenta region). Finally, for extreme values of β and Δ both mechanisms are very strong and stability can be not recovered (blue region).

Concerning the case of $\gamma = 3$ (right panel in Figure 1), the main difference with respect to the framework with $\gamma = 1$ is that the price mechanism can introduce or bolster instabilities and that, as discussed in [9], an intermediate joint effect of evolutionary pressure and heterogeneity has a stabilizing effect. Hence, when β and Δ are small, the price mechanism introduces instability for $\sigma = 0$ that can not be recovered by increasing the role of the sentiment, due to the reduced relevance and polarization of the beliefs (bottom left blue region). If $\beta\Delta^2$ assumes an intermediate value, the price dynamics are stabilized by the joint effect of evolutionary pressure and heterogeneity when the fitness measure is the profitability measure. However, as the relevance of the sentiment increases, the actual effect of heterogeneity is reduced, and S^* becomes unstable due to the price mechanism when σ becomes suitably large (green region). Since β and Δ are not sufficiently relevant, no herding effect can take place. If β and Δ further increase, the joint effect of evolutionary pressure and heterogeneity now introduces new instability, which can be recovered by appeasing their influence as the sentiment index increases. In this case, we may have again that, if σ is sufficiently large, increasing polarization of the

[9] We stress that, with respect to the situation reported in Figure 1, in the simulations reported in Figure 2 we have reduced the value of γ, which now is 0.1. This has been done to avoid an uninteresting oscillating convergence toward the steady state and to provide a neater interpretation, which is still valid for any value of $\gamma < 2$.

economic variables takes place and dynamics still converge toward a steady state[10] (magenta region). Finally, when endogenous dynamics induce price fluctuations, agents erratically switch between the optimistic and the pessimistic behaviors.

4 Numerical analysis

In the present section we complement with numerical simulations the analytical results on the steady states obtained in Section 3, in order to deepen the understanding of the economic relevance of the arising dynamics in model (6) and to investigate the qualitative properties of the time series when exogenous non-deterministic effects are taken into account.

4.1 Deterministic simulations

We consider two sets of simulations, characterized by different values of the heterogeneity degree parameter, in order to provide a portrait of the possible dynamic scenarios that may occur in our model economy and to highlight the influence of the behavioral parameters on the financial market. To run the simulations, we specify the sigmoid function as

$$f(z) = \begin{cases} a_1 \tanh(z/a_1) & \text{if } z \geq 0, \\ a_2 \tanh(z/a_2) & \text{if } z < 0, \end{cases} \tag{8}$$

setting $F = 10$, $a_1 = 2$, $a_2 = 1$, while we let β, σ, γ and Δ vary from time to time. In the reported two-dimensional bifurcation diagrams we use different colors to identify the degree of complexity of the attractor corresponding to the parameters coupling: white color refers to parameter pairs for which convergence is toward a steady state (either S^*, S^o or S^p), while other colors refer to attractors consisting of more than one point (in particular cyan color identifies cycles of high periodicity, quasi-periodic or complex attractors). The initial datum for the left simulations in Figures 3-4 is $(X_0, P_0, \omega_0) = (X^* + 0.01, P^* + 0.01, \omega^* + 0.01)$. Moreover, the stability threshold curves of the steady state S^* are drawn in black. In particular, the solid line refers to the bifurcation curve $\sigma = \sigma_{pf}$, whose crossing can give rise to a pitchfork bifurcation, while the dashed and dash-dotted lines refer to the bifurcation curves $\sigma = \sigma_{ns}$ and $\sigma = \sigma_{fl}$, whose crossing can give rise to a Neimark-Sacker or to a flip bifurcation, respectively, where the threshold values σ_{pf}, σ_{ns} and σ_{fl} have been introduced in Proposition 4. In the two- and one-dimensional bifurcation diagrams in Figures 3-4 the values of X and ω are related to the price variable, recalling that, when prices are large/small due to the pitchfork bifurcation, shares and beliefs move accordingly. The initial datum in the central plots of Figures 3-4, as well as in the right plots of Figures 3 and 4, is $(X_0, P_0, \omega_0) = (X^* + 0.01, P^* + 0.01, \omega^* + 0.01)$ in black bifurcation diagrams and $(X_0, P_0, \omega_0) = (X^* - 0.01, P^* - 0.01, \omega^* - 0.01)$ in red ones.

The first group of simulations (see Figures 3 and 4) deals with a case in which beliefs are strongly polarized, that is, agents heterogeneity about the fundamental value is high ($\Delta = 3$). We start by considering a moderate ($\gamma = 1$) reactivity to price variation. Looking at the vertical sections of the two-dimensional bifurcation diagram reported in the left plot of Figure 3, we can see examples of unconditionally stable (e.g. for $\beta = 0.1$), stabilizing ($\beta = 0.6$) and mixed ($\beta = 1$) scenarios on varying σ. We stress that in the white region below the black solid line, convergence is toward S^*, while in that above the black solid line, convergence is toward S^o. As shown in the one-dimensional bifurcation diagram in the middle plot of Figure 3, if we keep the value of the intensity of choice fixed at $\beta = 0.6$ and we let σ vary, we observe a stabilizing effect played by the sentiment index on price dynamics. When the sentiment index has no relevance ($\sigma = 0$), the occurring quasi-periodic dynamics are a consequence of the switching mechanism and of a suitably large intensity of choice (in fact, from the

[10] We stress that, when the stability of S^* is not recovered, we are in the unconditionally unstable case again, but now dynamics can converge to the polarized steady states at least for intermediate values of σ.

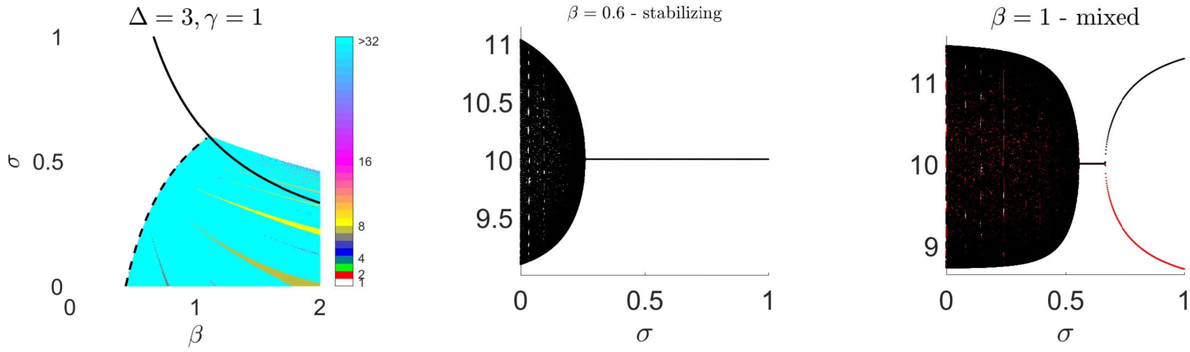

Figure 3: Two-dimensional bifurcation diagram (left plot). The solid and dashed curves represent the bifurcation curves $\sigma = \sigma_{pf}$ and $\sigma = \sigma_{ns}$, respectively, with σ_{pf} and σ_{ns} introduced in Proposition 4. Central and right panels depict two possible scenarios about P on varying σ for different values of β.

left panel in Figure 3 we note that, if we decrease β, we have stable dynamics), and they are transferred to the price mechanism, which would be otherwise stable.[11] In this setting, the evolutionary selection only depends on the profitability measure, which in turn is affected by excess demand and agents heterogeneity. As σ increases, the strength of the interplay between prices and shares decreases, as the switching mechanism is more affected by the sentiment index and less by the profits, which are in this setting the source of instability. The result is that endogenous oscillations firstly decrease and disappear, so that agents evenly distribute among beliefs and the stock price converges toward the fundamental value. On the other hand, if we slightly increase the intensity of choice up to $\beta = 1$ (see the right plot of Figure 3), we again have the initial stabilizing effect of animal spirits, which then gives rise to a polarization in the shares due to an increased relevance assigned to the utility of being either pessimistic or optimistic, as remarked in the comments after Proposition 1. Depending on an initial deviation in shares toward optimism or pessimism, we have that most of the agents herd around pessimism (red bifurcation diagram) or optimism (black bifurcation diagram). Again, dynamics become convergent toward a stable steady state (either S^* or S^o/S^p) as σ increases.
If we raise γ, we obtain the two-dimensional bifurcation diagram reported in the left plot of Figure 4. In this case, the unconditionally unstable scenario for small values of β is due to the price mechanism. Considering $\beta = 1$ as in the previous simulation, we observe in the central plot of Figure 4 the occurrence of a mixed scenario, similar to the one described in Figure 3, but now the polarized prices undergo a period-doubling cascade of bifurcations that leads to chaotic dynamics. We stress that we still observe a herding phenomenon as σ increases, which now, according to the initial conditions, gives rise to price dynamics that endogenously fluctuate around large or small values. To conclude, in the right plot of Figure 4 we report the basin of attraction obtained when $\sigma = 1$ and $\beta = 16$. We stress that we have checked through simulations that the shape of the basin for $\sigma = 1$ is robust with respect to the considered parameters setting. It is evident that a sufficiently high degree of optimism or pessimism (which is in turn determined by both beliefs and shares values) uniquely determines the convergence toward an attractor that reflects the same polarized optimism or pessimism. The final state to which the economic variables converge is then affected by the sentiment perceived by the agents in a very neat way, self-sustaining and reinforcing the emergence of more extreme levels of optimism/pessimism. This last aspect, together with the static and local stability analysis of Section 3, will allow to understand the behavior of economic observables when a non-deterministic effect is introduced.

[11] Indeed, the derivative of the price at the steady states is given by $1 - \gamma$.

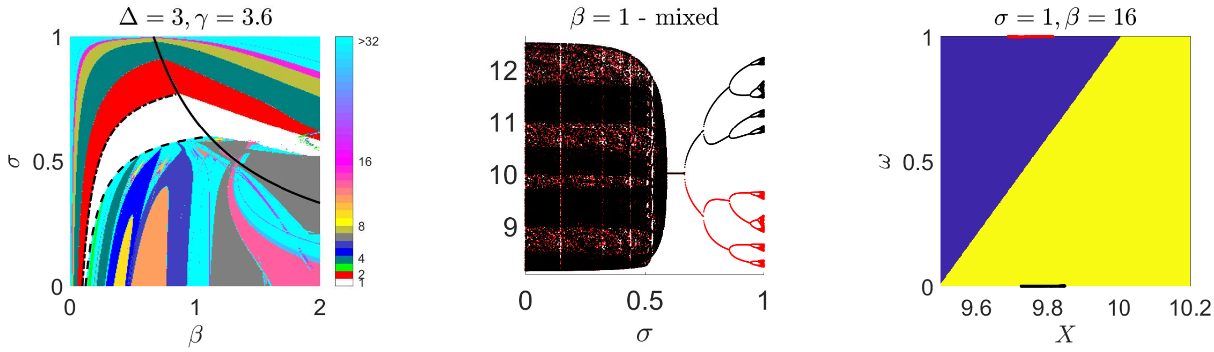

Figure 4: Two-dimensional bifurcation diagram (left panel). The solid, dashed and dash-dotted curves refer to the bifurcation curves $\sigma = \sigma_{pf}$, $\sigma = \sigma_{ns}$ and $\sigma = \sigma_{fl}$, whose crossing can give rise to a pitchfork, to a Neimark-Sacker or to a flip bifurcation, respectively, where the threshold values σ_{pf}, σ_{ns} and σ_{fl} have been introduced in Proposition 4. The one-dimensional bifurcation diagram on the central panel refers to the parameter setting used in the left two-dimensional bifurcation diagram. The right panel depicts the basin of attraction of the optimistic (yellow region) and pessimistic (blue region) attractors, from which the polarization of beliefs is clearly visible.

4.2 Stochastic simulations

The analysis performed in the previous subsection confirmed that animal spirits can drive the market toward economic regimes characterized by either optimism or pessimism.

In order to study the emergence of bubbles and crashes for stock prices, excess volatility and deviation from normality of the distributions of returns (defined by $R_{t+1} = 100((P_{t+1} - P_t)/P_t))$, we follow the approach in [13], introducing structural stochastic terms into the dynamical system in (6). In particular, we assume that the true fundamental value follows the random walk

$$F_{t+1} = F_t + \varepsilon_{F,t} F_t \,, \tag{9}$$

where $\{\varepsilon_{F,t}\}$ are normally distributed random variables with standard deviation $s_1 > 0$ and zero mean. Recalling that in the deterministic version of the model it holds that $X = F - \Delta/2$ and $Y = F + \Delta/2$, we now consider $X_t = F_t - \Delta/2$ and $Y_t = F_t + \Delta/2$, so that the condition $Y_t - X_t = \Delta$ is still fulfilled. We stress that also in [9] the bias about the fundamental remains unchanged, while the fundamental may vary. Moreover, as in [13], we introduce a random perturbation of beliefs, proportional to the price, i.e.,

$$X_{t+1} = F_t - \Delta/2 + \varepsilon_{X,t} P_t, \quad Y_{t+1} = F_t + \Delta/2 + \varepsilon_{Y,t} P_t, \tag{10}$$

where $\{\varepsilon_{X,t}\}$ and $\{\varepsilon_{Y,t}\}$ are normally distributed random variables with standard deviation $s_2 > 0$ and zero mean, which describe a temporary perturbation of the agents heterogeneity level.[12]

The shock on F is an improvement of that considered in [9], as in (9) the random component structurally depends on the size of the fundamental.[13] The resulting stochastic model is then obtained by replacing F, $X = F - \Delta/2$ and $Y = F + \Delta/2$ with F_t, $X_t = F_t - \Delta/2$ and $Y_t = F_t + \Delta/2$, respectively, in the construction of the model in Section 2. This allows to get the stochastic version of System (6), with F replaced by F_t, to which equations (9) and (10) have to be added. In particular, we notice that the shock $\varepsilon_{F,t}$ enters the expression of X_{t+1} and Y_{t+1} through F_t, together with $\varepsilon_{X,t}$ and $\varepsilon_{Y,t}$.

[12] Being both groups of agents fundamentalists, it is more economically reasonable to consider the same standard deviation for both beliefs perturbations. In any case, we have checked that the results we present are robust with respect to the introduction of a suitable asymmetry in the standard deviations of $\{\varepsilon_{X,t}\}$ and $\{\varepsilon_{Y,t}\}$.

[13] As noted in [13], an approach like that adopted in [9] is able to replicate only a first family of stylized facts, like bubbles, crashes and persistent price deviation from fundamental. However, in order to mimic fat tails, volatility clustering and long memory effects, a structural element has to be introduced in the description of random components.

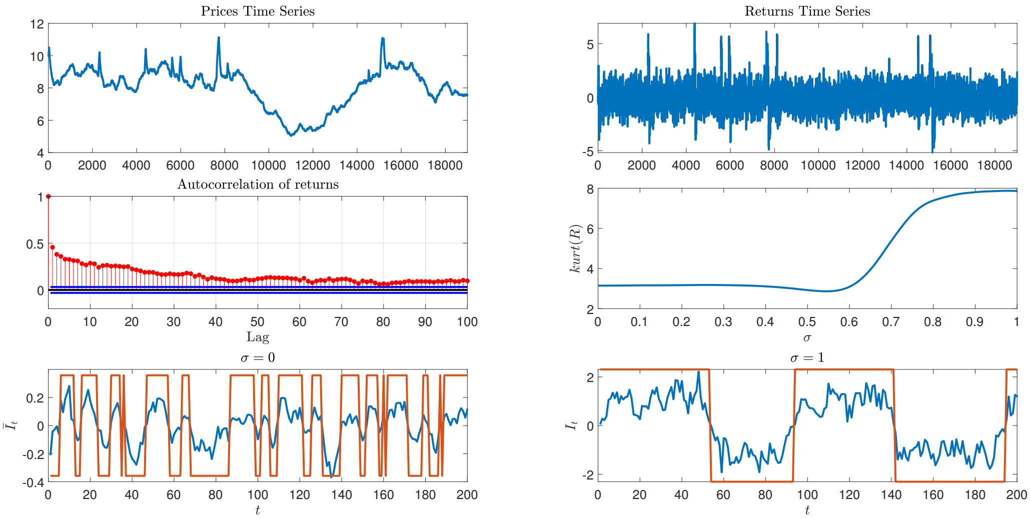

Figure 5: The plots in the first row show the time series of prices and returns; in the second row we show the autocorrelation of returns when $\sigma = 1$ and the kurtosis of returns distribution as σ increases; the third row highlights the emergence of optimistic and pessimistic waves in the market sentiment index for different values of the sentiment weight ($\sigma = 0$ in the left panel and $\sigma = 1$ in the right panel, respectively).

We report some possible outcomes of our stochastic model in Figure 5, where we consider the parameter setting used for the simulation reported in the left panel of Figure 3, with the exception of γ that is set equal[14] to 0.02.
In the presence of exogenous shocks on the fundamental value, periods of high volatility in the price course may alternate with periods in which prices do not depart too much from the fundamental value. Such a behavior can arise when the parameter setting is located near the pitchfork bifurcation boundary and exogenous noise can occasionally spark long-lasting endogenous fluctuations around the new occurring steady states.
More precisely, the top left panel in Figure 5 displays a typical plot for the price time series obtained for $s_1 = 0.003$ and $s_2 = 0.035$, which highlights the very erratic price course with alternating bubbles and crashes. The corresponding time series of returns is reported in the top right panel of Figure 5, which still reflects the alternating periods with high and low volatility, and exhibits volatility clustering, highlighted by the strongly positive, slowly decreasing autocorrelation coefficients of absolute returns (central left plot of Figure 5). Moreover, deviation from normality in the returns distribution only occurs as the herding phenomenon takes place, i.e., as the sentiment index plays an increasing relevant role in determining agents choices, as shown in the central right plot of Figure 5. The presence of fat tails implies that, when the sentiment index drives the market, large returns often occur, corresponding to strong movements in prices, and thus to more volatility in the financial market, in agreement with the well-known stylized facts empirically observed. The bottom panels of Figure 5 compare the time series of the sentiment index I_t when $\sigma = 0$ and $\sigma = 1$ (respectively in the bottom left and right plots). When $\sigma = 0$, it is possible to observe the emergence of periods of prevailing optimism or pessimism only if we consider a moving average $\bar{I}_t$ of I_t on a suitable number of periods (in the reported simulation $\bar{I}_t$ is computed considering, at each time period t, the last 5 values assumed by the sentiment index). This means that there is indeed alternation of periods characterized by the prevalence of a certain sentiment, but such phenomenon is quite weak and can be perceived only considering an average behavior over a suitable amount of periods. Conversely, when agents choose strategies on the basis

[14] This is agreement with [13], in which, when structural volatility is considered for the first model taken into account, parameter are changed so that "the price converges monotonically ... though only (very) slowly so". This also enforces the random walk nature of the asset prices. We stress that results are qualitatively robust with respect to parameters modifications in suitable ranges.

of the sentiment index, there exist waves of optimism and pessimism that are much more long-lasting than when the influence from behavioral aspects is neglected in agents choices. Namely, in such latter case optimism and pessimism quickly alternate due to a continuous and recurrent evaluation of market beliefs based only on the market performance. The rationale for the occurrence of the waves of optimism and pessimism can be explained as follows: suppose that agents have the choice of using biased beliefs about the fundamental value of the asset, and that they seek to opt for the one that provides them a higher profitability. When the price volatility is low, the biases would not diverge too much from the fundamental and agents act more or less independently (some of them being optimists and the others pessimists). Accordingly, the market maker price adjustment will not be too strong and the price volatility remains low. In other words, the negative feedback induced by the traders compounds the one of the market maker, and prices may converge, with alternating periods of optimism and pessimism. On the other hand, when the price dynamics is more turbulent, agents may prefer to observe other agents choices more closely and possibly imitate them. The resulting herding behavior implies that agents choices become increasingly aligned (i.e. they behave less independently), eventually on values that differ from the fundamental steady state. This may be the case in which the optimistic and pessimistic steady states emerge. In such an eventuality, agents orders are less balanced around the fundamental value and the market maker is no longer able to mediate among them. Therefore, the market maker's price adjustments over/under react to those misalignments and the volatility remains high.

The emergence of long-lasting alternating periods of optimism and pessimism can be understood in the light of the stability analysis reported in Section 3. As we have shown, the most economically relevant phenomenon occurring when the market is driven by the general mood is that polarized states emerge, in terms of both possible steady states, attractors and basins of attractions. In the simulation reported in Figure 5, for $\sigma = 1$ the fundamental steady state is unstable and deterministic trajectories can converge toward the polarized steady states. Since the stock price is affected by shocks, thanks to the "polarized" structure of the basins of attractions, the trajectories persist in the basin of the same attractor (e.g. of the optimistic one) for several periods, until a random deviation moves them into the basin of the other attractor (e.g. of the pessimistic one), in which trajectories wander until a similar phenomenon drives them into the basin of the former attractor. In this process, prices are close to a random walk, with optimistic and pessimistic agents frequently switching between the two strategies.

5 Generalization of the model

In this section we generalize the benchmark model in two directions. At first, we consider the possibility for agents beliefs to endogenously evolve over time, that is, agents still underestimate or overestimate the true fundamental value F and form their beliefs X_t and Y_t about F on the basis of an imitative rule. Secondly, since professional market participants trade on the basis of different techniques, we also extend the model by including an additional type of traders, acting on the basis of technical analysis.

5.1 Endogenous beliefs formation

We study the robustness of the results obtained in the case of exogenously fixed beliefs about the fundamental by considering an endogenous mechanisms for the beliefs formation. In what follows we introduce the rules describing the updating of beliefs, the price variation and the evolutionary selection of heuristics. As we shall see, the evolutionary selection rule of the best performing optimistic/pessimistic heuristic and the nonlinear price adjustment mechanism remain essentially unchanged with respect to the benchmark model.

We assume that agents know the true fundamental value F but that, due to the ambiguity in the financial market generated by the uncertainty about the future stock price, agents do not rely on F. Considering it just as a reference value, they take into account the relative ability shown by optimists and pessimists in guessing the realized stock price P_t in the previous period and, still remaining optimists or pessimists, update their beliefs about the fundamental according to the proportional imitation rules

$$\begin{aligned} X_{t+1} &= (F-\Delta)\frac{e^{\mu(Y_t-P_t)^2}}{e^{\mu(X_t-P_t)^2}+e^{\mu(Y_t-P_t)^2}} + F\left(1-\frac{e^{\mu(Y_t-P_t)^2}}{e^{\mu(X_t-P_t)^2}+e^{\mu(Y_t-P_t)^2}}\right) \\ &= F - \frac{\Delta}{e^{\mu((X_t-P_t)^2-(Y_t-P_t)^2)}+1}, \end{aligned} \tag{11a}$$

$$\begin{aligned} Y_{t+1} &= F\left(1-\frac{e^{\mu(X_t-P_t)^2}}{e^{\mu(X_t-P_t)^2}+e^{\mu(Y_t-P_t)^2}}\right) + (F+\Delta)\frac{e^{\mu(X_t-P_t)^2}}{e^{\mu(X_t-P_t)^2}+e^{\mu(Y_t-P_t)^2}} \\ &= F + \frac{\Delta}{e^{\mu((Y_t-P_t)^2-(X_t-P_t)^2)}+1}, \end{aligned} \tag{11b}$$

where Δ and μ are two positive parameters describing the degree of heterogeneity among the agents and the strength of the imitative process, respectively.

More precisely, by (11a) we have that beliefs X_{t+1} of pessimistic agents consist in a convex combination of the lowest possible pessimistic belief $F-\Delta$ and of the true fundamental value F, in which weights depend on the relative ability shown at time t by each kind of agent in guessing the realized price P_t. Such performance is estimated by comparing the errors of pessimists $(X_t-P_t)^2$ and of optimists $(Y_t-P_t)^2$. Similarly, by (11b), the beliefs Y_{t+1} of optimistic agents are updated as a convex combination of the true fundamental value F and the greatest possible optimistic belief $F+\Delta$, with weights still representing the effectiveness of each heuristic in providing the price approximation.[15] Indeed, we have $F-\Delta < X_t < F < Y_t < F+\Delta$, which portrays the systematic underestimation/overestimation of the fundamental by pessimists/optimists. Moreover, it is straightforward to check that (11) implies $Y_{t+1} = X_{t+1}+\Delta$, so that the beliefs of both groups can be actually described by using just one equation, which for us will be that concerning the beliefs of pessimistic agents, i.e., (11a). As concerns parameter μ it is worth noticing that when there is no imitation (i.e. $\mu = 0$) then $X_{t+1} = F-\Delta/2$ and $Y_{t+1} = F+\Delta/2$, so that both X_{t+1} and Y_{t+1} lie at the midpoint of the interval in which they may respectively vary and we are led back to the benchmark model, while as $\mu \to +\infty$ we have that, when $(X_t-P_t)^2 < (Y_t-P_t)^2$, so that pessimists proved to be more able in guessing the fundamental value, then both X_{t+1} and Y_{t+1} tend to assume the lowest possible value, i.e., $X_{t+1} \to F-\Delta$ and $Y_{t+1} \to F$, while, when $(X_t-P_t)^2 > (Y_t-P_t)^2$, then both X_{t+1} and Y_{t+1} tend to assume the highest possible value, i.e., $X_{t+1} \to F$ and $Y_{t+1} \to F+\Delta$.

The market maker sets the stock price for the next period on the basis of the total excess demand, which is now given by $D_t = \omega_t\alpha(X_t-P_t) + (1-\omega_t)\alpha(Y_t-P_t)$, where $\alpha > 0$ is a demand reactivity parameter, and which, using $Y_t = X_t+\Delta$, can be rewritten as $D_t = \alpha(X_t-P_t+(1-\omega_t)\Delta)$. The price variation is then described by

$$P_{t+1}-P_t = f(\gamma D_t) = f(\gamma\alpha(X_t-P_t+(1-\omega_t)\Delta)), \tag{12}$$

with γ and f as described in Section 2. Without loss of generality, we can still assume that $\alpha = 1$, encompassing both the demand and the price adjustment reactivities in the parameter γ.

When beliefs are endogenous, the sentiment index becomes

$$I_t = \omega_t X_t + (1-\omega_t)Y_t - F = X_t + (1-\omega_t)\Delta - F, \tag{13}$$

[15] We remark that also for the case of endogenous beliefs we deal with a symmetric framework in which now the lowest possible pessimistic belief $F-\Delta$ and the greatest possible optimistic belief $F+\Delta$ lie at the same distance Δ from the fundamental value F, in order to focus on the most significant form of heterogeneity, i.e., the polarization, represented by Δ, between different attitudes of agents toward the reference value F.

which measures the difference between the average belief about the fundamental value, represented by $\omega_t X_t + (1-\omega_t)Y_t$, and the true fundamental value F. Hence, the population fraction composed by pessimists evolves according to the following updating rule

$$\omega_{t+1} = \frac{e^{\beta(\sigma(-I_t)+(1-\sigma)\pi_{X,t+1})}}{e^{\beta(\sigma(-I_t)+(1-\sigma)\pi_{X,t+1})} + e^{\beta(\sigma I_t+(1-\sigma)\pi_{Y,t+1})}} = \frac{1}{1+e^{\beta(2\sigma I_t+(1-\sigma)(\pi_{Y,t+1}-\pi_{X,t+1}))}}, \tag{14}$$

with $\pi_{j,t+1} = (P_{t+1} - P_t)(j_t - P_t)$, $j \in \{X, Y\}$.

The final model is obtained collecting the imitative updating rule of the pessimistic beliefs (11a), the price adjustment mechanism (12) and the evolutionary selection rule (14), and it is described by the three-dimensional map $G = (G_1, G_2, G_3) : (F-\Delta, F) \times (0, +\infty) \times (0, 1) \to \mathbb{R}^3$, $(X_t, P_t, \omega_t) \mapsto (G_1(X_t, P_t, \omega_t), G_2(X_t, P_t, \omega_t), G_3(X_t, P_t, \omega_t))$, defined as:

$$\begin{cases} X_{t+1} = G_1(X_t, P_t, \omega_t) = F - \dfrac{\Delta}{e^{\mu((X_t-P_t)^2-(X_t+\Delta-P_t)^2)}+1}, \\ P_{t+1} = G_2(X_t, P_t, \omega_t) = P_t + f\left(\gamma(X_t - P_t + (1-\omega_t)\Delta)\right), \\ \omega_{t+1} = G_3(X_t, P_t, \omega_t) = \dfrac{1}{1+e^{\beta(2\sigma(X_t+(1-\omega_t)\Delta-F)+(1-\sigma)\Delta f(\gamma(X_t-P_t+(1-\omega_t)\Delta)))}}, \end{cases} \tag{15}$$

where we used the identity $Y_t = X_t + \Delta$ and, in the last equation, we replaced I_t with its expression provided in (13) and we employed the condition $\pi_{Y,t+1} - \pi_{X,t+1} = (Y_t - X_t)(P_{t+1} - P_t) = \Delta f(\gamma D_t)$.

We stress that for $\sigma = 0$ the model in (15) reduces to that studied in [6].

We start by investigating the existence of the steady states of (15).

Proposition 5. *System* (15) *has*

a) a unique steady state $S^* = (X^*, P^*, \omega^*) = (F - \Delta/2, F, 1/2)$ *if* $\sigma \in [0, 1]$ *and*

$$\sigma \leq \frac{4}{\beta\Delta(\Delta^2\mu+2)}; \tag{16}$$

b) three steady states $S^*, S^o = (X^o, P^o, \omega^o)$ *and* $S^p = (X^p, P^p, \omega^p)$ *if* $\frac{4}{\beta\Delta(\Delta^2\mu+2)} < \sigma \leq 1$. *In particular,* S^o *and* S^p *are symmetric w.r.t.* S^*, *having* $X^p < X^* < X^o, P^p < P^* < P^o$ *and* $\omega^o < \omega^* < \omega^p$.

The conclusions of Proposition 1 are thus robust with respect to the introduction of an endogenous mechanism for the selection of beliefs. In addition to the comments about the steady pessimistic (S^p) and optimistic (S^o) economic regimes, coexisting with S^*, in this model with endogenous beliefs at the steady states agents have higher (X^o) or reduced (X^p) beliefs with respect to the average $F - \Delta/2$. In agreement with Proposition 1, this again occurs if agents give a sufficiently large relevance to the perceived market mood. However, the occurrence of polarized steady states is now also influenced by the strength of the imitative process, and, as the strength of the imitation process increases, the effect of animal spirits is bolstered and a progressively reduced sentiment weight is enough to trigger the emergence of the polarized steady states. Finally, we stress that (16) reduces to (7) when $\mu = 0$.
In the next result we study the effect of the behavioral parameters on the optimistic and pessimistic steady states when $\mu > 0$.

Proposition 6. *Let* $\sigma > 4/(\beta\Delta(\Delta^2\mu+2))$, $\sigma \in [0, 1]$. *Then, on increasing* σ, β, Δ *and* μ, *we have that* ω^o *decreases, while* P^o *and* I^o *increase, and that* ω^p *increases, while* P^p *and* I^p *decrease. Moreover, on increasing* σ, β, Δ *and* μ, *the distance from* X^o *and* X^p *to* $X^* = F - \Delta/2$ *increases.*

The above result reinforces that in Proposition 5, underlining how the psychological and emotional components are determinant for the choice of optimistic/pessimistic heuristics. Psychological factors

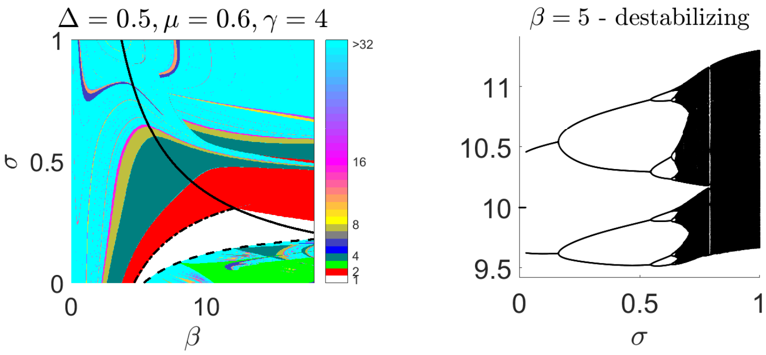

Figure 6: Two-dimensional bifurcation diagram (left plot). The solid, dashed and dash-dotted curves represent the bifurcation curves (iv'), (i') and (ii') in (30) in the Appendix, respectively. The right plot represents the evolution of the stock price on varying σ.

as well as the imitative behavior may then strengthen the role of market sentiment in determining the beliefs[16], the shares and the prices.

The next level of investigation concerns how the stability of S^* is affected by the sentiment weight and, more generally, the effects of the sentiment index on the resulting dynamics.[17]

In Proposition 7 we derive the local stability conditions for System (15) at S^*.

Proposition 7. *On varying $\sigma \in [0,1]$, we have that the set on which the steady state S^* is locally asymptotically stable can be*[18]

a) connected, being an interval, in which case the sentiment weight can have a destabilizing, stabilizing, mixed or neutral effect;

b) unconnected.

In particular, condition (16) is necessary for the stability of S^*.

As in the case with $\mu = 0$, when the condition (16) is violated, a pitchfork bifurcation occurs, in which the stable steady state S^* loses stability and is replaced by the pair of steady states S^o and S^p. We can also note that the sentiment weight may have a quite ambiguous effect on the stability of S^*, being both stabilizing and destabilizing. It is interesting to remark that, if S^* is unstable when the fitness measure of the evolutionary mechanism is only based either on the profit evaluation ($\sigma = 0$) or on the sentiment index ($\sigma = 1$), it may happen that for a suitable combination of them ($0 < \sigma < 1$) S^* will become stable. This is a consequence of the mixed role of β on stability, also emerging and analyzed in [6, 9]. Such a framework occurs in the mixed scenarios we shall observe in Figure 6. Finally, we stress that, even if the unconnected scenario is possible, its actual relevance is quite marginal, since we numerically checked that it occurs on a very small set of parameter configurations.

We now present some numerical simulations to show the dynamics emerging from the general model in which agents endogenously select their beliefs. We recall that the sigmoid function is the same as described by (8) while the other parameters are set as $F = 10$, $a_1 = 2$, $a_2 = 1$, while we let μ, β, σ, γ and Δ vary from time to time.
In Figure 6 we consider a low degree of beliefs polarization ($\Delta = 0.5$), as well as sufficiently high levels of imitation ($\mu = 0.6$) and of reactivity to price variations ($\gamma = 4$). In this case we have either unconditionally unstable ($\beta = 1$), destabilizing ($\beta = 5$) or mixed ($\beta = 6$) scenarios.

[16] We stress that, in order to ensure that $X^o > 0$, in Proposition 6 we need to consider $\Delta < F$.
[17] We talk about an unconnected scenario if the set on which the steady state is stable is unconnected.
[18] We checked via numerical simulations that all the scenarios listed in Proposition 7 can actually occur.

Comparing the two-dimensional bifurcation diagram in Figure 6 with those in Figures 3 and 4, we note that the white region, in which the steady states are stable, is reduced in size, due to the increase in γ and μ.
In the left panel of Figure 6 we highlight a destabilizing role for σ, meaning that an increasing relevance assigned to the market sentiment translates into periodic and chaotic dynamics as well. In particular, we can observe that, for $\beta = 5$, when the sentiment weight is sufficiently low, the price dynamics are relatively regular, being characterized by convergence toward P^* (when $\sigma \approx 0$) or by periodic oscillations (as σ slightly raises). When the sentiment weight grows, an increasing importance is assigned to the difference between the beliefs and the fundamental value than compared to the difference between the beliefs and the current price. As a consequence, the relative share of agents adopting the optimistic or the pessimistic rule assumes extreme values and, accordingly, the price reaches furthest levels. Hence, it is the distribution of the shares, which is more strongly affected by the market sentiment, that determines the amplification in the price variation. Accordingly, the role of the profits is less intense, while the behavioral component driven by the animal spirits turns out to be determinant in the price evolution. Such phenomenon, in turn, is able to amplify price variations with resulting complicated dynamics.

5.2 Introducing technical traders

Instead of considering endogenous beliefs on the fundamental, we shall now study which are the effects of enriching the set of agents types in order to account also for the presence of technical traders. It is well known that agents can choose their strategies not only on the basis of an evaluation of the fundamental value, but also on the basis of the market trend. The behavior of technical traders is usually characterized by a positive feedback about price movements, buying and selling when price respectively increases and decreases. Because of such a positive feedback, the presence of technical traders can give rise to spontaneous herding phenomena, even when agents switching choices just depend on profits. Also for this reason they have not been taken into account in the original model. We shall now show that the results obtained for model (6) are robust with respect to the introduction of technical traders. In order to do so, while keeping the model simple and close to (6), we assume that the chartists share $\omega_c \in [0, 1)$ is exogenously given, whereas the share of optimistic/pessimistic agents depends as in (6) on a fitness measure corresponding to a weighted average of a sentiment index and of profits. Namely, we still assume that beliefs of fundamentalists are optimistically (resp. pessimistically) biased, so that they are[19] $P^e_{o,t+1} = F + \Delta/2$ (resp. $P^e_{p,t+1} = F - \Delta/2$), while beliefs of chartists are $P^e_{c,t+1} = P_{t-1} + \delta(P_{t-1} - P_{t-2})$, with $\delta > 0$ representing their reactivity. The demand of each kind of agent is $D_{i,t} = \alpha_i(P^e_{i,t+1} - P_t)$, with $i \in \{o, p, c\}$, where $\alpha_i > 0$ are demand parameters and $\alpha_p = \alpha_o = \alpha_F$. The excess demand is then

$$ED_t = \sum_{i\in\{o,p,c\}} \alpha_i\omega_{i,t}(P^e_{i,t+1} - P_t) = \alpha_F\omega_{p,t}(X - P_t) + \alpha_F(1 - \omega_{p,t} - \omega_c)(Y - P_t) + \alpha_c\omega_c(P_{t-1} + \delta(P_{t-1} - P_{t-2}) - P_t).$$

The sentiment I_t perceived at time t by the fundamentalists about the market is given by the difference between the beliefs of each agent and the fundamental, weighted by the share of each group, and now takes into account the choices of technical traders, i.e.

$$\begin{aligned} I_t = \textstyle\sum_{i\in\{o,p,c\}} \omega_{i,t}(P^e_{i,t+1} - F) &= \omega_{p,t}(X - F) + (1 - \omega_{p,t} - \omega_c)(Y - F) + \omega_c(P_{t-1} + \delta(P_{t-1} - P_{t-2}) - F) \\ &= \omega_{p,t}X + (1 - \omega_{p,t} - \omega_c)Y + \omega_c(P_{t-1} + \delta(P_{t-1} - P_{t-2})) - F. \end{aligned}$$

[19] We stress that the beliefs of the fundamentalists concern the fundamental value, not the price. However, in this section, in order to keep the notation more homogeneous with the case of chartists, we denote the beliefs of the fundamentalists by P^e, as well.

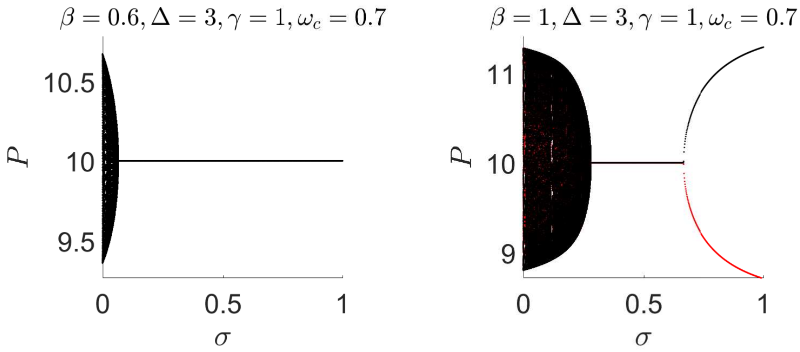

Figure 7: Bifurcation diagrams on varying σ for different values of the intensity of choice β. The other parameter values are: $\alpha_F = 3.5$, $\alpha_c = 0.4$, $\delta = 0.6$. With a stable price mechanism, even a large share of technical traders does not amplify the dynamics, which may settle down to the fundamental steady state S^* (left panel) or to the optimistic/pessimistic steady states S^o, S^p (right panel).

The model is then given by

$$\begin{cases} P_{t+1} = P_t + f(\gamma ED_t) \\ \omega_{p,t+1} = (1-\omega_c)\frac{e^{-\sigma\beta I_t + (1-\sigma)\beta\pi_{o,t+1}}}{e^{-\sigma\beta I_t + (1-\sigma)\beta\pi_{p,t+1}} + e^{\sigma\beta I_t + (1-\sigma)\beta\pi_{o,t+1}}} \end{cases}$$

We stress that, due to the presence of technical traders, the joint share of optimists and pessimists satisfies $\omega_{p,t} + \omega_{o,t} = 1 - \omega_c$, and thus $\omega_{p,t}$, $\omega_{o,t} \in (0, 1 - \omega_c)$. It is possible to analytically prove that the condition for the emergence of two polarized steady states is not affected by the fraction of chartists and it is again $\sigma \in (2/(\beta\Delta), 1]$.

Scenarios occurring when $\omega_c = 0$ are essentially robust with respect to the introduction of chartists. If we consider a stable price mechanism (i.e. if $\gamma < 2$), the bifurcation diagrams with respect to γ are basically unaffected by the introduction of an even large share of chartists, as shown in Figure 7. Conversely, when the price mechanism is unstable, the introduction of chartists can spread instabilities in the market, as shown in Figure 8, even if the polarization of attractors still occurs. This may be due to the fact that a turbulent market is considered as more attractive by the chartists with higher possibilities of realizing gains. In turn, if chartists act aggressively, they may lead the price to divert even from the optimistic or pessimistic steady states, with consequent possible oscillations around them, where prices may trace out bubble or crash paths from time to time.

6 Concluding remarks

In the economic literature there exist several contributions (e.g. [6, 9, 29]) showing that, diverting from the perfect rationality assumption on the agents behavior, the waves of optimism and pessimism observed in financial markets can be explained in terms of endogenous fluctuations originated by the evolutionary selection of simple heterogeneous heuristics and/or by imitation mechanisms in forming beliefs about the fundamental. However, changes in the psychological and emotional perception of the market are not only consequences of the agents choices, being also part of the process on which decisions are taken. When the mechanism which regulates the evolutionary selection of forecasting rules is based on a combination of the average mood perceived by the agents about the status of the market and a precise evaluation of the profits, new economic regimes arise, different from those occurring when agents decisions are not driven by "animal spirits". Such regimes are characterized by persistently polarized levels of optimism and pessimism, highlighted by high/small beliefs and prices, as well as by a large share of optimists or pessimists. An excess of optimism and pessimism may endogenously generate outcomes which can be seen as the result of a self-sustaining herding

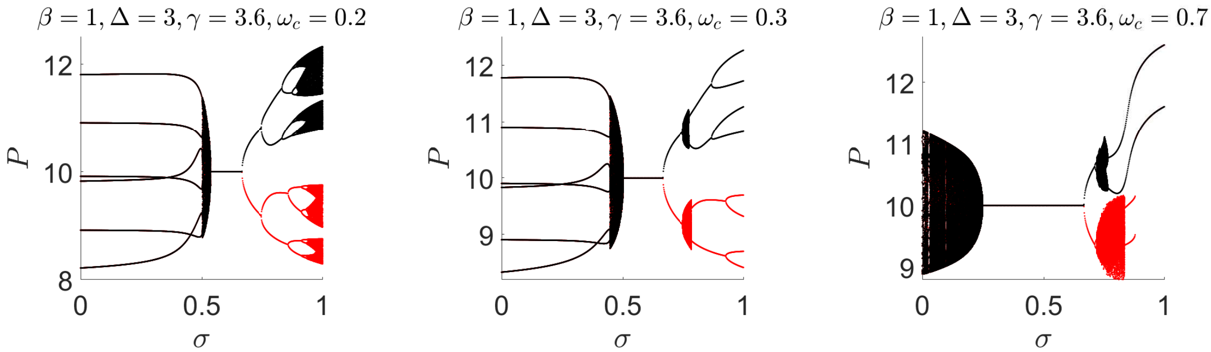

Figure 8: Bifurcation diagrams on varying σ for different values of the share of technical traders ω_c. The other parameter values are: $\alpha_F = 1.25$, $\alpha_c = 0.6$, $\delta = 0.4$. When the price mechanism is unstable, the introduction of technical traders may amplify the resulting dynamics in a way that, after the occurrence of the pitchfork bifurcation, further complexity issues take place, with the occurrence of periodic dynamics due to a period-doubling bifurcation (left panel) or endogenous oscillations around the polarized regimes as the effect of a secondary Neimark-Sacker bifurcation (central and right panels).

phenomenon. On the other hand, endogenous waves of optimism and pessimism are not ruled out by animal spirits, especially when decision mechanisms are based on both sentiment and evaluation of profits. Moreover, as the role of the sentiment index becomes predominant, those waves are reinforced by possible endogenous dynamics around self-fulfilling economic regimes and, when non-deterministic effects are taken into account, give rise to alternating long-lasting periods of polarized economic regimes. Our future researches will aim at deepening the study of the role of animal spirits as the drivers of economic decisions, extending the pursued approach to other macroeconomic frameworks, also involving the real market side.

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A Proof of the analytical results

Proof of Proposition 1. A straightforward check shows that $(P^*, \omega^*) = (F, 1/2)$ is always a steady state. However, in general, it is not the only one. Indeed, setting $P_{t+1} = P_t = P$ and $\omega_{t+1} = \omega_t = \omega$ in (6), we obtain $F - P + \Delta(1/2 - \omega) = 0$, from which, recalling (3), it follows that

$$P = F + \Delta(1/2 - \omega) = F + I \tag{17}$$

and

$$\omega = \frac{1}{1 + e^{\beta\Delta\sigma(1-2\omega)}}. \tag{18}$$

Equation (18) is indeed solved by $\omega^* = 1/2$, so that by (17) we obtain $P^* = F$.

Let us introduce function $h : (0, 1) \to \mathbb{R}$ whose expression is given by the right-hand side of (18). We have that $h(1/2) = 1/2$, $\lim_{\omega\to-\infty} h(\omega) = 0$ and $\lim_{\omega\to+\infty} h(\omega) = 1$, and a straightforward check shows that $h'(\omega) > 0$ and that $h'(1/2) = \beta\Delta\sigma/2$. Moreover, h is convex (resp. concave) for $\omega < 1/2$ (resp. $\omega > 1/2$) with $h''(1/2) = 0$.

From the previous considerations, recalling that the left-hand side of (18) is ω, we can conclude that if $h'(1/2) \le 1$ there is exactly one solution to (18), that is, $\omega^* = 1/2$, while if $h'(1/2) > 1$ there are three distinct solutions to (18). Indeed, it is easy to show that h is symmetric w.r.t. $\omega = 1/2$, i.e., that $h(1/2 + \varepsilon) - h(1/2) = h(1/2) - h(1/2 - \varepsilon)$ for every $\varepsilon > 0$, which, recalling that $h(1/2) = 1/2$, reduces to $h(1/2 + \varepsilon) = 1 - h(1/2 - \varepsilon)$.
Noting that $h'(1/2) \le 1$ corresponds to (7) allows concluding that, when (7) is violated, (18) is solved by some $\omega^o < 1/2 < \omega^p$, which satisfy $\omega^o = 1 - \omega^p$, and for which, by (17), there is a correspondence to $P^p < F < P^o$, respectively. Moreover, replacing in (17) ω with $1-\omega$ we obtain $F - I$, which means that P^o and P^p are symmetric with respect to F. □

Proof of Proposition 2. We consider (18), in which we put in evidence the dependence of h on the parameter we study. We start noting that h is greater (resp. smaller) than $1/2$ for $\omega > 1/2$ (resp. $\omega < 1/2$) and we recall that it is continuous on $(0, 1)$. Since the roles of σ and Δ in the expression of h is exactly the same as that of β we can just deal with this last parameter. We have that if $\beta_1 < \beta_2$ then $h_{\beta_2}(\omega) > h_{\beta_1}(\omega)$ (resp. $h_{\beta_2}(\omega) < h_{\beta_1}(\omega)$) on $\omega > 1/2$ (resp. $\omega < 1/2$). A simple geometrical consideration allows concluding that ω^o (resp. ω^p) is strictly decreasing (resp. increasing) with respect to β.

Finally, from the result about ω^o, by (17) we have that $P^o = (1/2 - \omega^o)\Delta$ increases as β, σ or Δ increase and I^o increases, too. □

Proof of Proposition 3. The local asymptotic stability of S^* is guaranteed if

$$\begin{aligned}
&1 + \det(J^*) + \mathrm{tr}(\mathrm{J}^*) > 0 \Leftrightarrow -\Delta\beta\sigma\left(\gamma + \Delta\gamma - 2\right) + \gamma(\Delta^2\beta - 2) + 4 > 0\\
&1 - \det(J^*) > 0 \Leftrightarrow \Delta\beta\sigma\left(\gamma(2 + \Delta) - 2\right) - \Delta^2\beta\gamma + 4 > 0\\
&1 + \det(J^*) - \mathrm{tr}(\mathrm{J}^*) > 0 \Leftrightarrow 2 - \Delta\beta\sigma > 0
\end{aligned} \tag{19}$$

where

$$J^* = \begin{pmatrix} 1-\gamma & -\Delta\gamma \\ \dfrac{\Delta\beta\gamma\left(1-\sigma\right)}{4} & \dfrac{\beta\Delta\left(2\sigma + \Delta\gamma\left(1-\sigma\right)\right)}{4} \end{pmatrix}$$

is the Jacobian matrix of System (15) evaluated at S^*. We recall that when stability is lost due to a violation of the first (resp. second) condition of (19), steady state S^* incurs a flip (resp. Neimark-Sacker) bifurcation. Conversely, also recalling Theorem 5, the third condition of (19) is the same as (15), so when it is violated a pitchfork bifurcation occurs.

If $\sigma = 0$, conditions (19) become

$$\begin{cases} -2\gamma + \Delta^2\beta\gamma + 4 > 0 \\ -\Delta^2\beta\gamma + 4 > 0 \end{cases}$$

which easily provides $2(\gamma - 2) < \Delta^2\beta\gamma < 4$, while if $\sigma = 1$ we have

$$\begin{cases} (\Delta\beta + 2)(\gamma - 2) < 0 \\ \Delta\beta - 2 < 0 \\ \Delta\beta - 2 - \Delta\beta\gamma < 0 \end{cases}$$

which, after noting that the third condition is implied by the second one, leads to the assertion. □

Proof of Proposition 4. Let us set $\gamma = 1$. The conditions in (19) become

$$\begin{aligned} &\Delta\beta\sigma(1-\Delta) + \Delta^2\beta + 2 > 0 \\ &\Delta^2\sigma\beta - \Delta^2\beta + 4 > 0 \\ &2 - \Delta\beta\sigma > 0 \end{aligned} \tag{20}$$

As it is easy to check, the first condition in (20) is satisfied for any $\sigma \in [0,1]$, both when $\Delta \le 1$ and when $\Delta > 1$. If fact, if $\Delta \le 1$ the left-hand side is clearly strictly positive, while if $\Delta > 1$ we have $\Delta\beta\sigma(1-\Delta) + \Delta^2\beta + 2 > -\Delta^2\beta\sigma + \Delta^2\beta + 2 = \Delta^2\beta(1-\sigma) + 2$, which is indeed positive for any $\sigma \in [0,1]$.
The third condition of (20) is always satisfied if $\beta < 2/\Delta$ while if $\beta > 2/\Delta$ it holds true provided that

$$\sigma < \frac{2}{\Delta\beta}.$$

The second condition of (20) is fulfilled for any $\sigma \in [0,1]$ if $\beta < 4/\Delta^2$, otherwise it is fulfilled for $\sigma \in \left(\frac{\Delta^2\beta - 4}{\Delta^2\beta}, 1\right]$ if $\beta > 4/\Delta^2$.

Stability is then unconditional if

$$\beta < \min\left\{\frac{4}{\Delta^2}, \frac{2}{\Delta}\right\}$$

while a pitchfork bifurcation occurs for $\sigma = 2/\Delta\beta$ if

$$\frac{2}{\Delta} < \beta < \frac{4}{\Delta^2}.$$

To have a mixed scenario, we need

$$\begin{cases} \Delta^2\beta > 4 \\ \Delta\beta > 2 \end{cases}$$

and

$$\frac{\Delta^2\beta - 4}{\Delta^2\beta} < \frac{2}{\Delta\beta}$$

which requires $\Delta^2\beta < 2\Delta + 4$, i.e

$$\max\left\{\frac{4}{\Delta^2}, \frac{2}{\Delta}\right\} < \beta < \frac{2\Delta + 4}{\Delta^2},$$

otherwise S^* is unconditionally unstable, i.e. for

$$\beta > \frac{2\Delta + 4}{\Delta^2}\,.$$

Finally, we have a stabilizing scenario if

$$\frac{4}{\Delta^2} < \beta < \frac{2}{\Delta}\,.$$

If we set $\gamma = 3$, stability is guaranteed for

$$\begin{aligned} &-\Delta\beta\sigma(3\Delta+1) + 3\Delta^2\beta - 2 > 0 \\ &\Delta\beta\sigma(3\Delta+4) - 3\Delta^2\beta + 4 > 0 \\ &2 - \Delta\beta\sigma > 0 \end{aligned} \tag{21}$$

The first condition of (21) is satisfied by

$$\sigma < \frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta+1)} < 1$$

which requires $\Delta^2\beta > 2/3$ (since the rightmost condition is always fulfilled), otherwise it is never fulfilled for $\sigma \in [0, 1]$.
The second condition of (21) is always satisfied if $\Delta^2\beta < 4/3$, while if $\Delta^2\beta > 4/3$ it can be rewritten as

$$\sigma > \frac{3\Delta^2\beta - 4}{\Delta\beta(3\Delta+4)} < 1$$

where the rightmost inequality holds true independently of Δ and β.
The third condition in (21) is the same as that in (20).
If $\beta < 2/(3\Delta^2)$ then the steady state is unconditionally unstable.
If $2/(3\Delta^2) < \beta < 4/(3\Delta^2)$, then the steady state is stable on

$$\sigma \in \left[0, \min\left\{\frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta+1)}, \frac{2}{\Delta\beta}\right\}\right)\,. \tag{22}$$

However

$$\frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta+1)} < \frac{2}{\Delta\beta}$$

is equivalent to

$$\beta < \frac{6\Delta + 4}{3\Delta^2}$$

so it is granted if $\beta < 4/(3\Delta^2)$. Then (22) reduces to

$$\sigma \in \left[0, \frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta+1)}\right)\,.$$

If $\beta > 4/(3\Delta^2)$, then S^* is stable on

$$\sigma \in \left(\frac{3\Delta^2\beta - 4}{\Delta\beta(3\Delta+4)}, \min\left\{\frac{3\Delta^2\beta - 2}{\Delta\beta(3\Delta+1)}, \frac{2}{\Delta\beta}\right\}\right)\,.$$

We start noting that

$$\frac{3\Delta^2\beta - 4}{\Delta\beta(3\Delta+4)} < \frac{2}{\Delta\beta}$$

if

$$\beta < \frac{2\Delta+4}{\Delta^2}.$$

From the previous considerations we have that if

$$\frac{4}{3\Delta^2} < \beta < \frac{6\Delta+4}{3\Delta^2} \left(< \frac{2\Delta+4}{\Delta^2} \right)$$

then S^* is stable on

$$\sigma \in \left(\frac{3\Delta^2\beta-4}{\Delta\beta(3\Delta+4)}, \frac{3\Delta^2\beta-2}{\Delta\beta(3\Delta+1)} \right)$$

and on decreasing σ we have a Neimark-Sacker bifurcation at $\frac{3\Delta^2\beta-4}{\Delta\beta(3\Delta+4)}$, while on increasing it we have a flip bifurcation at $\frac{3\Delta^2\beta-2}{\Delta\beta(3\Delta+1)}$.

If

$$\frac{6\Delta+4}{3\Delta^2} < \beta < \frac{2\Delta+4}{\Delta^2}$$

S^* is stable on

$$\sigma \in \left(\frac{3\Delta^2\beta-4}{\Delta\beta(3\Delta+4)}, \frac{2}{\Delta\beta} \right)$$

and on decreasing σ we have a Neimark-Sacker bifurcation at $\frac{3\Delta^2\beta-4}{\Delta\beta(3\Delta+4)}$, while on increasing it we have a pitchfork bifurcation at $2/(\Delta\beta)$.

Finally, if $\beta > \frac{2\Delta+4}{\Delta^2}$ the steady state is unconditionally unstable. $\square$

Proof of Proposition 5. A straightforward check shows that $(X^*, P^*, \omega^*) = (F-\Delta/2, F, 1/2)$ is always a steady state. However, in general, it is not the only one. Indeed, setting $X_{t+1} = X_t = X$, $P_{t+1} = P_t = P$ and $\omega_{t+1} = \omega_t = \omega$, from the second equation in (15) we obtain $X - P + (1-\omega)\Delta = 0$, from which, recalling (13), it follows that

$$P = X + (1-\omega)\Delta = I + F \tag{23}$$

and, from the third equation in (15), we have

$$\omega = \frac{1}{1 + e^{2\beta\sigma(X+(1-\omega)\Delta-F)}}. \tag{24}$$

By (23) we find $X - P = -\Delta(1-\omega)$ and $X + \Delta - P = \Delta\omega$, which inserted into the first equation in (15) provide

$$X = F - \frac{\Delta}{e^{\mu(\Delta^2(1-\omega)^2-\Delta^2\omega^2)}+1} = F - \frac{\Delta}{e^{\mu\Delta^2(1-2\omega)}+1}. \tag{25}$$

Inserting (25) into (24), we obtain the implicit equation

$$\omega = \frac{1}{1 + e^{2\beta\sigma\Delta\left(-\frac{1}{e^{\mu\Delta^2(1-2\omega)}+1}+(1-\omega)\right)}}, \tag{26}$$

which is indeed solved by $\omega^* = 1/2$, so that by (25) we find $X^* = F - \Delta/2$, and then by (23) we obtain $P^* = F$. After some algebraic manipulations, we can rewrite (26) as

$$-\frac{1}{e^{\mu\Delta^2(1-2\omega)}+1} + (1-\omega) = \frac{1}{2\beta\sigma\Delta} \ln\left(\frac{1-\omega}{\omega}\right). \tag{27}$$

Let us introduce functions $h : (0,1) \to \mathbb{R}$ and $g : (0,1) \to \mathbb{R}$, whose expressions are given by the left-

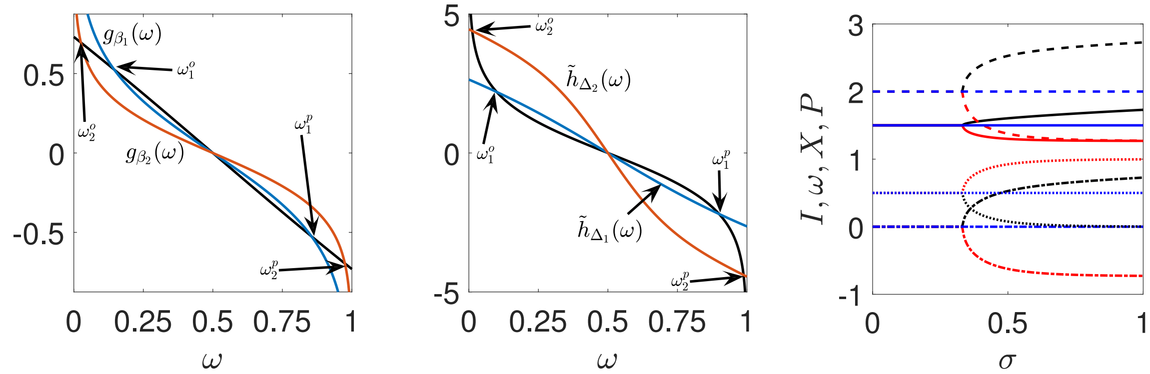

Figure 9: Graph of $g(\omega)$ on varying β (left panel) and of $\tilde{h}(\omega)$ on varying Δ (central panel), respectively. In the left panel, for $\beta_1 < \beta_2$, function $g_{\beta_2}(\omega)$ (orange) lies above (below) $g_{\beta_1}(\omega)$ (cyan) on $\omega < 1/2$ (on $\omega > 1/2$). Function h is represented in black. In the central panel, for $\Delta_1 < \Delta_2$, function $\tilde{h}_{\Delta_1}(\omega)$ (cyan) lies above (below) $\tilde{h}_{\Delta_2}(\omega)$ (orange) on $\omega < 1/2$ (on $\omega > 1/2$). Function $\tilde{g}$ is represented in black. In both plots the steady state ω^o (resp. ω^p), located at the intersection between the functions, is decreasing (resp. increasing) with respect to β and Δ (e.g. $\omega_2^o < \omega_1^o$ and $\omega_1^p < \omega_2^p$.) The right panel represents the evolution of the optimistic (black), pessimistic (red), intermediate (blue) steady states on varying σ. Solid line: X; dotted line: ω; dashed line: P; dash-dotted line: I. The parameter values are: $F = 2$, $mu = 1$, $\beta = 4$, $\Delta = 1$.

and the right-hand sides of (27), respectively. We have that $h(1/2) = g(1/2) = 0$ and a straightforward check shows that $h'(\omega) < 0$, $g'(\omega) < 0$, that $h'(1/2) = -(\Delta^2\mu)/2 - 1$, $g'(1/2) = -2/(\beta\Delta\sigma)$, and that $h''(1/2) = g''(1/2) = 0$ and h is concave (resp. convex) for $\omega < 1/2$ (resp. $\omega > 1/2$), while g is convex (resp. concave) for $\omega < 1/2$ (resp. $\omega > 1/2$).

Since g has two vertical asymptotes in $\omega = 0$ and $\omega = 1$, while h is always defined, we can conclude that if $h'(1/2) \leq g'(1/2)$ there is exactly one solution to (26), that is, $\omega^* = 1/2$, while if $h'(1/2) > g'(1/2)$ there are three distinct solutions. Indeed, it is easy to show that h and g are symmetric w.r.t. $\omega = 1/2$, i.e., that $h(1/2+\varepsilon) - h(1/2) = h(1/2) - h(1/2-\varepsilon)$ and $g(1/2+\varepsilon) - g(1/2) = g(1/2) - g(1/2-\varepsilon)$, for every $\varepsilon \in (0, 1/2)$, which, recalling that $h(1/2) = 0 = g(1/2)$, reduce respectively to $h(1/2+\varepsilon) = -h(1/2-\varepsilon)$ and $g(1/2+\varepsilon) = -g(1/2-\varepsilon)$.
Noting that $h'(1/2) > g'(1/2)$ corresponds to the opposite inequality to (16) allows concluding that, when (16) is violated, (26) is solved by some $\omega^o < 1/2 < \omega^p$, which satisfy $\omega^o = 1 - \omega^p$, to which, by (25) and (23), correspond $X^p < F - \Delta/2 < X^o$ and $P^p < F < P^o$, respectively. To complete the proof, we notice that (25) is left unchanged when replacing ω and X with $1-\omega$ and $2F - \Delta - X$, respectively, which guarantees that X^o and X^p are symmetric with respect to $F - \Delta/2$. Moreover, replacing in (23) ω and X with $1-\omega$ and $2F - \Delta - X$, respectively, we obtain $2F - P$, which means that P^o and P^p are symmetric with respect to F. □

Proof of Proposition 6. We consider (27), in which we put in evidence the dependence of h and g on the parameter we study. We start noting that both h and g are positive (resp. negative) for $\omega < 1/2$ (resp. $\omega > 1/2$) and we recall that they are continuous on $(0, 1)$. Let us deal with β. We have that if $\beta_1 < \beta_2$ then $g_{\beta_2}(\omega) < g_{\beta_1}(\omega)$ (resp. $g_{\beta_2}(\omega) > g_{\beta_1}(\omega)$) on $\omega < 1/2$ (resp. $\omega > 1/2$), while $h_{\beta_1} \equiv h_{\beta_2}$. A simple geometrical consideration (see Figure 9, left panel) allows concluding that ω^o (resp. ω^p) is strictly decreasing (resp. increasing) with respect to β.

Since the role of σ in the expression of g is exactly the same as that of β, we can also conclude that ω^o (resp. ω^p) is strictly decreasing (resp. increasing) with respect to σ. As a consequence, if either β or σ increases, from (25), we have that X^o increases (since we have that $\omega^o < 1/2$ decreases). Finally, combining the results about ω^o and X^o, by (23) we have that $P^o = X^o + (1-\omega^o)\Delta$ increases as β or σ increase and I^o increases, too. The opposite conclusions hold for X^p, P^p and I^p if either β or σ increases.

Let us now consider μ. In this case, we have that if $\mu_1 < \mu_2$ then $h_{\mu_1}(\omega) < h_{\mu_2}(\omega)$ (resp. $h_{\mu_2}(\omega) < h_{\mu_1}(\omega)$) on $\omega < 1/2$ (resp. $\omega > 1/2$), while $g_{\mu_1} \equiv g_{\mu_2}$, so that a simple geometrical consideration allows concluding that ω^o (resp. ω^p) is strictly decreasing (resp. increasing) with respect to μ. As a consequence, if μ increases, from (25) we have that X^o increases (since we have that $\mu(1-2\omega^o)$ increases). Finally, combining the results about ω^o and X^o, by (23) we have that $P^o = X^o + (1-\omega^o)\Delta$ increases as μ increases and I^o increases, too. The opposite conclusions hold for X^p, P^p and I^p if μ increases.

To investigate the role of Δ, we rewrite (27) as

$$\tilde{h}_\Delta(\omega) = 2\beta\sigma\Delta\left((1-\omega) - \frac{1}{e^{\mu\Delta^2(1-2\omega)}+1}\right) = \ln\left(\frac{1-\omega}{\omega}\right) = \tilde{g}(\omega). \tag{28}$$

Since $-\frac{1}{e^{\mu\Delta^2(1-2\omega)}+1}$ is increasing with respect to Δ if $\omega < 1/2$, also $\tilde{h}_\Delta(\omega)$ is increasing with respect to Δ if $\omega < 1/2$. When $\omega > 1/2$, a direct computation shows that $\tilde{h}'_\Delta(\omega) < 0$. Since $\tilde{g}$ does not depend on Δ, also in this case (see Figure 9, central panel) we can conclude that ω^o (resp. ω^p) is strictly decreasing (resp. increasing) with respect to Δ. Recalling that, from (23), we have $I = P - F$ and using it in (24), we can conclude that the monotonicity behaviors of P^o, I^o and P^p, I^p are respectively the opposite with respect to those of ω^o and ω^p.

Now we focus on the behavior of X. To prove that X^p and X^o increasingly deflect from $F - \Delta/2$ as Δ increases, we note that from (25) we have

$$X^i - \left(F - \frac{\Delta}{2}\right) = \frac{\Delta}{2}\cdot\frac{e^{\mu\Delta^2(1-2\omega^i)}-1}{e^{\mu\Delta^2(1-2\omega^i)}+1} = \frac{\Delta}{2}\cdot r(\Delta,\omega^i),\ i\in\{o,p\},$$

where we set $r(\Delta,\omega) = (e^{\mu\Delta^2(1-2\omega)}-1)/(e^{\mu\Delta^2(1-2\omega)}+1)$. We indeed have $r(\Delta,\omega^o) > 0$ and $r(\Delta,\omega^p) < 0$ and

$$\frac{dr(\Delta,\omega^i)}{d\Delta} = \frac{\partial r(\Delta,\omega^i)}{\partial\Delta} + \frac{\partial r(\Delta,\omega^i)}{\partial\omega^i}\cdot\frac{\partial\omega^i}{\partial\Delta},$$

where

$$\frac{\partial r(\Delta,\omega^i)}{\partial\Delta} = \frac{4\Delta\mu e^{\Delta^2\mu(1-2\omega^i)}(1-2\omega^i)}{(e^{\Delta^2\mu(1-2\omega^i)}+1)^2}$$

and $\partial_{\omega^i} r(\Delta,\omega^i) < 0$. Since $\omega^o < 1/2$, we have $\partial_\Delta r(\Delta,\omega^o) > 0$ and $\partial_\Delta\omega^o < 0$ and hence $dr(\Delta,\omega^o)/d\Delta > 0$, while, recalling that $\omega^p > 1/2$, from $\partial_\Delta r(\Delta,\omega^p) < 0$ and $\partial_\Delta\omega^p > 0$ we have $dr(\Delta,\omega^p)/d\Delta < 0$. The monotonicity of $X^i - \left(F - \frac{\Delta}{2}\right)$ then follows from the previous considerations about the sign of $dr(\Delta,\omega^i)/d\Delta$, from

$$\frac{d\left(X^i - \left(F - \frac{\Delta}{2}\right)\right)}{d\Delta} = \frac{1}{2}\cdot r(\Delta,\omega^i) + \frac{\Delta}{2}\frac{dr(\Delta,\omega^i)}{d\Delta}$$

and from the sign of $r(\Delta,\omega^i)$. □

Proof of Proposition 7. To determine the conditions which ensure the local asymptotic stability of S^*, we evaluate the Jacobian matrix for G at S^*, obtaining

$$J^* = \begin{bmatrix} \frac{-\mu\Delta^2}{2} & \frac{\mu\Delta^2}{2} & 0 \\ \gamma & 1-\gamma & -\gamma\Delta \\ \frac{-\beta(2\sigma+(1-\sigma)\gamma\Delta)}{4} & \frac{\beta\gamma\Delta(1-\sigma)}{4} & \frac{\beta\left(2\sigma\Delta+(1-\sigma)\gamma\Delta^2\right)}{4} \end{bmatrix} \tag{29}$$

The characteristic polynomial of J^* is $p(x) = x^3 + C_1 x^2 + C_2 x + C_3$, where

$$C_1 = \frac{\mu\Delta^2}{2} + \gamma - 1 - \frac{\beta\Delta}{4}\left(2\sigma + (1-\sigma)\gamma\Delta\right), \; C_3 = \frac{\beta\Delta^3\mu}{8}(2\sigma + (1-\sigma)\tilde{\gamma}\Delta - 2\sigma\tilde{\gamma}),$$
$$C_2 = \frac{\Delta}{2}\left(\frac{\beta}{2}\left(2\sigma + (1-\sigma)\gamma\Delta - 2\sigma\gamma - \sigma\Delta^2\mu - \frac{\mu\gamma\Delta^3(1-\sigma)}{2}\right) - \mu\Delta\right).$$

Stability is guaranteed by the Farebrother conditions in [10], which provide

$$\begin{aligned} &(i')\; 1 - C_1 + C_2 - C_3 = \sigma C_{11} + C_{10} > 0; && (ii')\; 1 - C_2 + C_1 C_3 - (C_3)^2 = -\sigma^2 C_{22} + \sigma C_{21} + C_{20} > 0; \\ &(iii')\; 3 - C_2 = \sigma C_{31} + C_{30} > 0; && (iv')\; 1 + C_1 + C_2 + C_3 = 4 - \sigma\beta\Delta(\Delta^2\mu + 2) > 0, \end{aligned} \tag{30}$$

where coefficients C_{ij} in $(i') - (iii')$ depend on γ, β, Δ and μ, and (iv') is the same as condition (16) with strict inequality.

In particular the neutral scenarios are realized either when $(i') - (iv')$ are simultaneously satisfied for each $\sigma \in [0, 1]$ (unconditionally stable scenario) or when at least one condition among $(i') - (iv')$ is not fulfilled for any $\sigma \in [0, 1]$ (unconditionally unstable scenario). In what follows, we only focus on the case in which all inequalities do not reduce to an empty set, as otherwise we are in the unconditionally unstable scenario.

The discriminant between the occurrence of a connected (namely, an interval) or unconnected stability region is given by the solution of condition (ii'). Since such inequality is represented by a second degree polynomial, it can be solved on

$$(ii'a)\; \sigma_{2,l} < \sigma < \sigma_{2,r}, \qquad (ii'b)\; \sigma < \sigma_{2,l} \vee \sigma > \sigma_{2,r}.$$

Since inequalities $(i'), (iii')$ and (iv') are linear with respect to σ and thus the corresponding stability conditions can just give rise to intervals, the unconnected stability region may arise only in case $(ii'b)$, as otherwise the final intersection is connected.

In particular, inequalities $(i'), (iii')$ and (iv') are satisfied when $\sigma \lesseqgtr \sigma_1, \sigma \lesseqgtr \sigma_3$ and $\sigma < \sigma_4$, where $\sigma_i, i \in \{1, 3, 4\}$, are the solutions of the equations corresponding to $(i'), (iii')$ and (iv'), respectively. The possible resulting stability conditions are then

$$(I)\; \sigma < \min\{\sigma_1, \sigma_3, \sigma_4\}, \qquad (II)\; \sigma_a < \sigma < \sigma_b,$$

where σ_a and σ_b are, respectively, either σ_3 and $\min\{\sigma_1, \sigma_4\}$, or σ_1 and $\min\{\sigma_3, \sigma_4\}$, or $\max\{\sigma_1, \sigma_3\}$ and σ_4. This means that combining case $(ii'a)$ with case I) we have a destabilizing scenario if $\sigma_{2,l} < 0 < \min\{\sigma_1, \sigma_3, \sigma_4, \sigma_{2,r}\} < 1$, a mixed one if $0 < \sigma_{2,l} < \min\{\sigma_1, \sigma_3, \sigma_4, \sigma_{2,r}\} < 1$, and a stabilizing one if $0 < \sigma_{2,l} < 1 < \min\{\sigma_1, \sigma_3, \sigma_4, \sigma_{2,r}\}$, while the remaining possible configurations can only provide neutral scenarios. Similarly, combining case $(ii'a)$ with case II) we have a destabilizing scenario if $\max\{\sigma_{2,l}, \sigma_a\} < 0 < \min\{\sigma_b, \sigma_{2,r}\} < 1$, a mixed one if $0 < \max\{\sigma_{2,l}, \sigma_a\} < \min\{\sigma_b, \sigma_{2,r}\} < 1$, and a stabilizing one if $0 < \max\{\sigma_{2,l}, \sigma_a\} < 1 < \min\{\sigma_b, \sigma_{2,r}\}$, while the remaining possible configurations can only provide neutral scenarios.

On the other hand, combining $(ii'b)$ with I), when $0 < \sigma_{2,l} < \sigma_{2,r} < \min\{\sigma_1, \sigma_3, \sigma_4, 1\}$, we have a mixed scenario in which S^* is stable on $[0, \sigma_{2,l}) \cup (\sigma_{2,r}, \min\{\sigma_1, \sigma_3, \sigma_4, 1\})$ and unstable otherwise.[20] □

[20] We thoroughly checked via numerical simulations that other unconnected stability sets, that may in principle arise from $ii'b$) and either case I) or II), are actually not possible, as they result in empty intersections.