# Inequalities and zones. New mathematical results for behavioral and social sciences

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**Inequalities and zones.**
**New mathematical results for behavioral and social sciences**

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A theorem, mathematical method and model are introduced in the present article. Inequalities, allowed and forbidden zones, their relations, consequences, and applications are considered for the expectations of random variables. The method and model are based on the inequalities and zones of the theorem. The article is motivated by the need for theoretical support for the practical analysis performed for the purposes of behavioral economics.

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Content

# 1. Introduction

Random variables whose values lie within finite intervals are analyzed in the present article. Inequalities for their variances and ranges are used in a theorem to establish the existence and estimate the parameters of inequalities for their expectations. The established inequalities can be considered as the conditions for some allowed and forbidden zones for the expectations near the boundaries of the intervals.

This theorem is used as the basis for an applied mathematical method (approach) and models, and other applications that can support and improve the analysis (see, e.g., [25]) of some well-known generic problems (see, e.g., [31]) of behavioral economics and for the purposes of psychology, decision theory, and the social sciences.

## 1.1. Main contributions. Organization of the article

The four main applied mathematical contributions of the present article are:

1) Theoretical support for the analysis of the problems of behavioral economics.

2) An applied mathematical method.

3) A special practical mathematical model.

4) The successful uniform application of the model in more than one domain.

Section 1 of the article presents its motivations and sources. Section 2 presents relations between the inequalities and the allowed and forbidden zones. Section 3 presents the theorem. Section 4 presents examples and properties of the forbidden zones. Section 5 presents the mathematical method. Section 6 presents the mathematical models. Section 7 presents applications of the theorem and method. Section 8 presents a discussion and conclusions. The Appendix presents lemmas for the theorem.

## 1.2. Moments, functions, utility, noise. Review of the literature

Inequalities for moments, means and functions are considered in a wealth of works, see, e.g., [12], [16], [33], [34]. The works [8], [19], [36], [37], [43], [44] and works devoted to the Hermite–Hadamard inequalities, see, e.g., [1], [6], [10], [14], [18], [20], [24], [35], [39], [48], [53] consider situations that are, in the purely mathematical aspects, the most similar to that analyzed here. Additionally, a discrete part of the proof in the Appendix of the present article can be considered as another variant of the proof of [8] used in [43], [44]. The continuous and mixed parts of the Appendix can be considered as its developments.

Mathematical aspects of utility and prospect theories are considered in a number of works, see, e.g., [2], [9], [17], [47], [50], [51]. Such aspects are considered in the present article as well. In particular, [2], [47] constitute one of the two starting points for the considerations of the next subchapter.

A noise and its influence are the subject of a wealth of works.

Channel capacity and noise are considered in a lot of works, see, e.g., [15], [42], [45], [52]. The above allowed zone is in a sense similar to the channel capacity.

Some qualitative influences of noise are analyzed as well. For example, stabilization and synchronization by noise are considered in a number of works, see, e.g., [4], [5], [7], [13], [22], [30]. A noise as a possible cause of some periodic behavior is considered in, e.g., [23], [40]. So the cited and also, in a sense, this article show that a noise can exert not only a quantitative but also some qualitative influence.

## 1.3. Practical need for such considerations

### 1.3.1. Problems of uncertain and sure games

A man as an individual actor is a key subject of economics and some other sciences. There are a number of problems concerned with the mathematical description of the behavior of an individual. Examples of these problems are the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, risk premium, fourfold pattern of risk preferences, etc.

The essence of these examples of the problems consists in biases of preferences and choices of people (subjects) for the uncertain and sure games in comparison with the predictions of the theory of probability. These problems are basic and well-known. They are the most important in behavioral economics in utility and prospect theories and also in psychology, decision theory, and the social sciences.

The above basic problems are pointed out in a wealth of works.

For example, we see in [31] page 222:

"A long series of modern challenges to utility theory, starting with the paradoxes of Allais (1953) ..., have demonstrated inconsistency in preferences"

For example, we see in [32], page 265:

"PROBLEM 1: Choose between

A: 2,500 with probability .33, // 2,400 with probability .66, // 0 with probability .01;

B: 2,400 with certainty.

$N$ = 72 [18] [82]" (My note: that is 72 trials: 18% of "A" and 82% of "B")

For example, we see in [46], page 974:

"... a choice between two lotteries *R'* (for "riskier") and *S'* (for "safer"). *R'* gave a 0.2 chance of winning £10.00 and a 0.75 chance of winning £7.00 (with the residual 0.05 chance of winning nothing); *S'* gave £7.00 for sure."

So, *R'* = £10.00*0.2+£7.00*0.75 = £7.25 and *S'* = £7:00*1 = £7:00. Here *R'* = £7:25 > S0 = £7:00, but the results were 13 choices for *R'* and 27 choices for *S'*.

### 1.3.2. Problems of varied domains

An additional and, probably, harder problem is, moreover, the radically different behavior of people (subjects) in different domains.

Thaler wrote in 2016 in [49], pages 1581–1582 (the italics are my own):

"Kahneman and Tversky's research documented numerous choices that *violate any sensible definition of rational*. ... subjects *were risk averse in the domain of gains* but *risk seeking in the domain of losses*." (My note: at high probabilities)

For example, the data in [32], page 268 Table 1 can be represented as:
Problem 3: (4,000 at 0.80) > (3,000 at 1.00) leads to choices [20%] < [80%].
Problem 3': (-4,000 at 0.80) < (-3,000 at 1.00) leads to choices [92%] > [8%].

The present article is motivated in large measure by the need for mathematical support for the already performed analysis of the influence of the scatter and noisiness of data. This idea, that is mathematically supported here, has explained, at least partially or qualitatively (see, e.g., [25]), the above problems.

### 1.4. Two ways. Variance, expectation, and forbidden zones

Many efforts have been made to explain the above basic problems of behavioral economics and other sciences.

One of the possible ways to explain them has been widely discussed, e.g., in [11], [28], [41]. It consists in paying proper attention to imprecision, noise, incompleteness, and other reasons that can cause dispersion, scattering and spread of data.

Another possible way is to consider the vicinities of the boundaries of the probability scale, e.g., at $p \to 1$. So [2], [47] emphasized a fundamental question: whether Prelec's weighting function $W(p)$ (see [38]) is equal to 1 at $p = 1$.

In any case, one may suppose that a synthesis of these two ways can be of some interest. This idea of a synthesis turned out to be useful indeed. It has successfully explained, at least partially, the underweighting of high and the overweighting of low probabilities, risk aversion, and some other problems (see, e.g., [25]). There are also works providing experimental support of this synthesis (see, e.g., [26], [46]).

Here it is proven that the inequalities for the variances and ranges of random variables lead to inequalities (or forbidden zones) for their expectations. The role of noise, as a possible cause of these forbidden zones, and their possible influence on the results of measurements, are considered in a preliminary way as well.

Keeping in mind the above inequalities for functions [16], [19], [33], [37], functions of the expectations of random variables can be also investigated.

## 2. Inequalities and zones

Let us give some necessary short considerations to relations between inequalities and allowed and forbidden zones. Let us consider some main quantity $u \in R$ and some supplementary quantities $v, v_{Min}, v_{Max} \in R,\ v_{Min} < v_{Max}$.

### 2.1. One inequality. Semi-infinite zones

Let us consider examples of a single inequality.

Consider a strict inequality. For example,

$$u > v, \quad \text{i.e.,} \quad u \in (v, +\infty)$$

means that $u$ can belong to the open semi-infinite interval $(v, +\infty)$. In other words, the interval $(v, +\infty)$ can be evidently considered as the open semi-infinite allowed zone for $u$ and the semi-open (or semi-closed) semi-infinite interval $(-\infty, v]$ can be evidently considered as the semi-open semi-infinite forbidden zone for $u$.

Consider a non-strict inequality. For example,

$$u \leq v, \quad \text{i.e.,} \quad u \in (-\infty, v]$$

means that $u$ can belong to the semi-open semi-infinite interval $(-\infty, v)$, or the interval $(-\infty, v)$ can be considered as the semi-open semi-infinite allowed zone for $u$ and the open semi-infinite interval $(v, +\infty)$ as the open semi-infinite forbidden zone for $u$.

So one inequality can be equivalent to semi-infinite allowed and forbidden zones.

## 2.2. Two inequalities. Semi-infinite, finite and point zones

### 2.2.1. Two points. Semi-infinite and finite zones

Now consider conjunctions of two conditions for two points. For example,

$$(u \geq v_{Min}) \cap (u \leq v_{Max}), \quad \text{i.e.,} \quad v_{Min} \leq u \leq v_{Max}$$

means that u belongs to the closed finite interval $[v_{Min}, v_{Max}]$. In other words, the interval $[v_{Min}, v_{Max}]$ can be evidently considered as the closed finite allowed zone for $u$ and the two open semi-infinite intervals $(-\infty, v_{Min})$ and $(v_{Max}, +\infty)$ can be evidently considered as the two open semi-infinite forbidden zones for $u$.

Now consider disjunctions of two conditions for two points. For example,

$$(u > v_{Max}) \cup (u < v_{Min}), \quad \text{i.e.,} \quad u \notin [v_{Min}, v_{Max}]$$

means that $u$ belongs to at least one of the two open semi-infinite intervals $(-\infty, v_{Min})$ and $(v_{Max}, +\infty)$. In other words, $(-\infty, v_{Min})$ and $(v_{Max}, +\infty)$ can be evidently considered as the two open semi-infinite allowed zones for $u$ and $[v_{Min}, v_{Max}]$ can be evidently considered as the closed finite forbidden zone for $u$.

So two inequalities for two points can be equivalent to semi-infinite and finite allowed and forbidden zones.

### 2.2.2. One point. Semi-infinite and point zones

Let us consider examples of two inequalities associated with one point.

Consider conjunctions of two conditions. For example,

$$(u \geq v) \cap (u \leq v), \quad \text{i.e.,} \quad u = v$$

means that $u$ is equal to $v$, or $v$ can be considered as an allowed point for $u$ and the two open semi-infinite intervals $(-\infty, v)$ and $(v, +\infty)$ can be considered as the two open semi-infinite forbidden zones for $u$. This corresponds to the equality $u = v$.

Consider disjunctions of two conditions. For example,

$$(u > v) \cup (u < v), \quad \text{i.e.,} \quad u \neq v$$

means that $u$ is not equal to $v$ or $v$ can be considered as the forbidden point for $u$ and the two open semi-infinite intervals $(-\infty, v)$ and $(v, +\infty)$ can be considered as the open semi-infinite allowed zones for $u$. This corresponds to $u \neq v$.

So two inequalities for one point can be equivalent to semi-infinite and point allowed and forbidden zones.

## 2.3. Hermite–Hadamard inequalities

For example, the Hermite–Hadamard inequalities (see, e.g., [1], [6], [10], [14], [18], [24], [35], [39], [48], [53] evidently correspond to this case of two inequalities.

There are indeed either an allowed point or an allowed zone for their middle term of the mean, when the outermost terms of these inequalities are (the case of a point) or are not (the case of a zone) equal to each other. This mean is also similar to the expectation of a random variable, especially for the continuous case.

## 2.4. Implicit or hidden inequalities of the theorem. Bounds. Denotations

Implicit or hidden inequalities can occur along with explicit ones. Let us consider some two examples from next sections, particularly from the theorem.

1. Suppose the values of a random variable $X$ lie within an interval $[a, b]$. Hence $E(X) \geq a$. If there is an inequality for $E(X)$, then the inequality imposed by the boundary $a$ can be added to it and they can produce some finite or point zone.

2. Suppose there are two inequalities for $E(X)$, the first inequality in the presence and the second one in the absence of some condition for the variance. Then these two inequalities can produce some finite or point zone as well.

We see that combining implicit inequalities can produce some new zones.

So inequalities can determine allowed and forbidden zones and vice versa. Inequalities are often treated as bounds (see, e.g., [8], [12], [33], [44]). Since the equalities and their inversions are shown here to be representable by combinations of inequalities, I will refer to all of them also as inequalities. Though the term "inequalities" and its applications are more rigorous, general and usually shorter than, e.g., "forbidden zones," the term "zones" is sometimes more intuitive and convenient in practical applications and qualitative considerations.

## 3. Theorem

### 3.1. Preliminaries

Let us consider a set $\{X_i\}$, $i = 1, \dots, n$, of random variables $X_i$ whose values lie within an interval $[a; b]$. For the sake of simplicity, $X_i$, $\mu_i$, $\sigma_i^2$ and similar symbols will often be written without the subscript "$i$."

If there is at least one discrete value of $X$, then let us denote the discrete value(s) of $X$ by $\{x_k\}$, $k = 1, \dots, K$, where $K \geq 1$, and the probability mass function by $p_X(x_k)$. If there are none, then let us ignore all the expressions involving discrete value(s).

If there are continuous values of $X$, then let us denote them by $x$ and the probability density function by $f_X(x)$. If there are none, then let us ignore all the expressions involving continuous values.

Under the normalizing condition

$$\sum_{k=1}^{K} p_X(x_k) + \int_{-\infty}^{+\infty} f_X(x)dx = \sum_{x_k \in [a,b]} p_X(x_k) + \int_a^b f_X(x)dx = 1\,, \tag{1}$$

let us consider the expectation $\mu$ and variance $\sigma^2$ of $X$ and their relations.

In connection with the terms "bound" and "forbidden zone," the abbreviation "$r_\mu$" (arising from the first letter "$r$" of the term "restriction") will be used here, due to its consonance with the usage in previous works. Non-trivial forbidden zones of non-zero width will sometimes be referred to as non-zero forbidden zones.

### 3.2. Conditions for maximality of the variance

A proof is given in [8] that for the variance $\sigma^2$ of a discrete random variable with range $[a, b]$ and expectation $\mu$, the following inequality holds:

$$\sigma^2 \leq (\mu - a)(b - \mu)\,. \tag{2}$$

The Appendix gives an alternate proof that the same inequality holds also for the variance of any real-valued random variable $X_i$ as in the above subsection 3.1.

### 3.3. The theorem. Two formulations

Theorem. (Formulation in terms of inequalities)

Consider a set $\{X_i\}$, $i = 1, \dots, n$, of random variables $X_i$ whose values lie within an interval $[a, b]$. If $0 < (b-a) < \infty$ and there exists a non-zero minimal variance $\sigma^2_{\mathrm{Min}}$ such that for all $i$,

$$\sigma^2_i \geq \sigma^2_{Min} > 0 \tag{3}$$

where $\sigma_i^2$ denotes the variance of $X_i$, then there exist non-zero bounds (restrictions) $r_\mu$ such that the following inequalities are true for the expectations $\mu_i$ of the $X_i$

$$a < (a + r_\mu) \leq \mu_i \leq (b - r_\mu) < b, \tag{4}$$

Proof.

Inequalities (2) and (3) lead to

$$0 < \sigma^2_{Min} \leq \sigma_i^2 \leq (\mu_i - a)(b - \mu_i).$$

At the boundary $a$, this leads to $\sigma^2_{\mathrm{Min}} \leq \sigma^2_{\mathrm{i}} \leq (\mu_i - a)(b-a)$ and

$$\mu_i \geq a + \frac{\sigma^2_i}{b-a} \geq a + \frac{\sigma^2_{Min}}{b-a}.$$

At the boundary $b$, the considerations are similar and give

$$\mu_i \leq b - \frac{\sigma^2_i}{b-a} \leq b - \frac{\sigma^2_{Min}}{b-a}.$$

Defining the bounds (restrictions) $r_\mu$ on the expectation $\mu_i$ as

$$r_\mu \equiv \frac{\sigma^2_{Min}}{b-a} \leq \frac{\sigma^2_i}{b-a}, \tag{5}$$

we obtain the generalized inequalities

$$a + r_\mu \leq \mu_i \leq b - r_\mu .$$

Since $0 < (b-a) < \infty$ and $\sigma^2_{\mathrm{Min}} > 0$, these bounds $r_\mu$ are non-zero and this leads to the inequalities

$$a < a + r_\mu \quad \text{and} \quad b - r_\mu < b .$$

Therefore these generalized inequalities can be transformed into (4) which proves the theorem.

We see that the particular bounds for the expectation of some particular random variable $X_i$ are determined by its variance $\sigma_i^2$. If its variance is non-zero, then these bounds are non-zero also. If the minimal variance $\sigma^2_{\mathrm{Min}}$ in for the set of random variables $\{X_i\}$ is non-zero, then the common bounds for the set of all $X_i$ are non-zero as well. These bounds cannot be less than $r_\mu$, but, if the strict inequality $\sigma_i^2 > \sigma^2_{\mathrm{Min}}$ holds, then they are greater than $r_\mu$.

The bounds (restrictions) $r_\mu$ can be considered as some forbidden zones of the width $r_\mu$ for the expectations of the random variables $X_i$ near the boundaries of the interval $[a, b]$. Consequently the allowed zone for the expectations of $X_i$ is located in the center of the interval.

An alternate version of the theorem (but with the same proof) can be represented by means of the "Forbidden zone formulation":

Theorem. (Alternate version. Forbidden zone formulation)

Consider a set $\{X_i\}$, $i = 1, \dots ,$ of random variables $X_i$ whose values lie within an interval $[a, b]$. If $0 < (b\text{-}a) < \infty$ and there exists a forbidden zone (or lower bound) of some non-zero width (3) for the variances of $X_i$, then forbidden zones (4) of non-zero width exist for the expectation of each $X_i$ near the boundaries of the interval.

This formulation is clearer for practical and qualitative applications, although it sometimes needs explanation.

The importance of the theorem for applied problems, especially in behavioral sciences, will be revealed in next sections.

## 4. Consequences of the theorem. Practical examples

### 4.1. General consequences. Mathematical support. Noise

The theorem provides the mathematical support for the analysis (see, e.g., [25]) of some experiments in behavioral economics. It proves the possibility of the existence of forbidden zones for the expectations of the discrete random variables that take a limited number of values used in the above analysis.

Due to the theorem, the forbidden zones for the expectation of one of the random variables $X_i$ from the above set $\{X_i\}$, e.g., of $X_c$, are determined by its variance, $\sigma_c^2$. If $\sigma_c^2 > 0$, then the width of these particular zones for $X_c$ is non-zero, i.e., these zones are non-trivial. If there exists a non-zero minimal variance for the set $\{X_i\}$ (as a whole), then there are non-trivial forbidden zones for the expectation of any $X_i$.

The list of possible causes of this non-zero minimal variance includes noise, imprecision, errors, incompleteness, various types of uncertainty, etc. Such causes are considered in a lot of works, e.g., [11], [28], [41]. Noise can be one of usual sources of the non-zero minimal variance (3). There are many types and subtypes of noise. The hypothetical task of determining an exact general relation between the level of noise and the non-zero minimal variance 3 of the random variables can be rather complicated.

If, nevertheless, noise leads to some non-zero minimal variance (3) of some considered set of random variables, then, due to the theorem, such noise leads evidently to the above non-zero forbidden zones. If the noise leads to some increasing of the value of the minimal variance, then the width of these zones increases as well.

So the theorem can provide a new mathematical tool for the description of the influence of at least some types of noise near the boundaries of intervals.

## 4.2. Practical examples of the occurrence of forbidden zones

**1. Ships and waves.**

Consider a calm or mirror-like sea and a small rigid boat or any other small rigid floating body at rest in the sea. Suppose that this boat or body rests in the mirror-like sea right against (or is constantly touching) a rigid moorage wall. As long as the sea is calm, the expectations of their sides can touch the wall.

Suppose there is a heavy sea. Consider a small rigid boat or any other small rigid floating body which oscillates on the waves in the heavy sea. Suppose that this boat or body oscillates on the waves near this rigid moorage wall.

When the boat is oscillated by sea waves, then its side oscillates also (both up–down and left–right) and it can touch the wall only in the (nearest) extremity of the oscillations. Therefore, the expectation of the side cannot touch the wall. Therefore, the expectation of the side is biased away from the wall.

So, one can say that, in the presence of waves, a forbidden zone exists between the expectation of the side and the wall.

This forbidden zone biases the expectation away from the wall. The width of the forbidden zone is roughly one-half of the amplitude of the oscillations.

**2. Washing machine, drill.**

Consider a washing machine that can vibrate when it works. Suppose it is near a rigid wall. Suppose an edgeless side of a drill (or any other rigid body that can vibrate) is located near a rigid surface or wall.

If the washing machine (or drill) is at rest, then the expectation of its edgeless side can be located right against (be constantly touching) the wall.

If the washing machine (or drill) vibrates, then the expectation of its edgeless side is biased and kept away from the rigid wall due to its vibrations.

The same is evidently true for any rigid body near any rigid surface or wall.

## 4.3. Vibration suppression. Sure games

Vibrations or oscillations can be suppressed with some effort. Such effort can be, e.g., physical in the case of physical vibrations. A vibrating rigid body can be pressed by some means. The corresponding forbidden zone can be also suppressed either partially or even totally, depending on the parameters of the suppression.

This suppression can correspond to the case of sure games (and outcomes) in behavioral economics, decision theory, the social sciences, and psychology.

The term “sure” presumes usually that some efforts are applied to guarantee this sure game in comparison with the uncertain ones. Due to these guaranteeing efforts, the width of the forbidden zones and hence the bias for sure games can be less than the width and biases for the uncertain games. In the limiting case, when the efforts are sufficiently hard, there are no forbidden zones. Note, that in the limiting case for the middle integral term for the mean (that can be considered as analogous to the expectation) in the Hermite–Hadamard inequalities (see, e.g., [1], [6], [10], [14], [18], [20], [24], [35], [39], [48], [53]), point (or trivial) allowed zones can be obtained.

So, sure games are guaranteed by some efforts. Due to these efforts, the forbidden zones and biases for the sure games can be suppressed and reduced.

## 5. Applied mathematical method of biases of expectations (AMMBE)

### 5.1. Preliminary considerations. Two main presuppositions

**Preliminary principle**. First of all, the above hard and complex problems evidently cannot be solved by a single researcher and all the more by a single theorem and single article. Any essential and elaborated contribution to the modern behavioral and social sciences needs elaborated works of a sufficient number of research teams.

The preliminary principle should be therefore "*stage by stage and step by step*." Consequently, the applied mathematical method (or approach), that will be proposed in this article, should be only a preliminary stage for subsequent verifications, changes, modifications and refinements by various research teams. So for such a preliminary stage, some good step can be even the above theorem with its consequences and a collection of some suppositions and mathematical relations.

**Basic working premise of the method**. The practical examples of the previous section evidently illustrate possible forbidden zones of the theorem. Similar examples are widespread in real life. Due to this prevalence, the subjects (people) can keep in mind the feasibility of such forbidden zones and the biases of the expectations caused by the zones. This can influence the behavior and choices of the subjects.

Two main presuppositions can be proposed due to this premise:

**1. Biases of expectations**.

The subjects make their choices (at least to a considerable degree) as if there were some biases of the expectations of the games.

This first main presupposition can be supported by the reason that such biases may be proposed and tested even from a purely formal point of view. Due to it, the method (approach) can be called the Applied Mathematical Method of Biases of Expectations, or AMMBE, or MMBE, or MBE. MMBE in total is to explain not only the objective situations but also and mainly the subjective behavior and choices of subjects.

**2. Explanation by the theorem**.

These biases (real biases or subjective reactions and choices of the subjects) can be explained (at least to a considerable degree) with the help of the forbidden zones of the theorem.

## 5.2. Notation

Denote the expectations of the uncertain and sure games by

$$\mu_{Uncert} \equiv \mu_{Uncertain} \quad \text{and} \quad \mu_{Sure}.$$

Denote the presupposed biases of the expectations for the uncertain and sure games that are required to obtain the data corresponding to these choices by

$$\Delta_{Uncert} \equiv \Delta_{Uncertain} \equiv \Delta_{Choice.Uncertain} \quad \text{and} \quad \Delta_{Sure} \equiv \Delta_{Choice.Sure}.$$

One can introduce some accessory modes (of the games) indicated by subscripts, e.g., #1 and #2. One of these modes can correspond to, e.g., the uncertain games (this may be either mode #1 or mode #2) and the other to the sure ones (#2 or #1). The corresponding expectations are $\mu_1$ and $\mu_2$ and the biases are $\Delta_1$ and $\Delta_2$.

Due to these accessory modes, we can use two convenient notations:

a) the real difference between the expectations of the compared modes

$$d_\mu \equiv \mu_2 - \mu_1, \quad \text{or} \quad d_\mu \equiv \mu_1 - \mu_2,$$

b) the difference between the presupposed biases of the compared modes

$$d_{Choice} \equiv \Delta_2 - \Delta_1, \quad \text{or} \quad d_{Choice} \equiv \Delta_1 - \Delta_2,$$

that is required to obtain the data corresponding to the revealed choices.

The simplicity of the mathematical calculations and transformations allows omitting most of the intermediate mathematical manipulations in what follows.

## 5.3. General inequalities

Let us consider some essential features of the examined situations and, using the above notation, develop some inequalities.

**1. Condition for MMBE.**

Due to the first presupposition, MMBE can be useful only if there is some presupposed non-zero difference $d_{\mathrm{Choice}}$ between the biases for the choices

$$\exists\, d_{Choice} : |\, d_{Choice} \,| > 0 \quad \text{or} \quad \exists\, d_{Choice} : \operatorname{sgn} d_{Choise} \neq 0. \qquad (6)$$

**2. Forbidden zones as, at least, one of the origins of biases.**

The presupposed $d_{\mathrm{Choice}}$ may be introduced and considered purely formally. The question is not only whether $d_{\mathrm{Choice}}$ can explain the problems. Due to the second presupposition, $d_{\mathrm{Choice}}$ itself should be explained by the theorem, at least partially.

First of all, the theorem should be applicable. Therefore inequalities (3)

$$\sigma^2 \geq \sigma^2{}_{Min} > 0$$

of the non-zero minimal variance are required to be true.

Further, let us denote the biases caused by the forbidden zones of the theorem by $\Delta_{\text{Theorem}}$ and the difference that can be explained by the theorem as $d_{\text{Theorem}}$. The sign of the difference for the choice should coincide with that for the theorem

$$\operatorname{sgn} d_{Choice} = \operatorname{sgn} d_{Theorem}\,.$$

Then the conditions for the explanation can be represented as $d_{\text{Theorem}} \approx d_{\text{Choice}}$, in the case when the forbidden zones of the theorem are the main source of the biases. If these forbidden zones are one of the essential source of the biases, then the conditions for the explanation can be represented as $d_{\text{Theorem}} = \text{O}(d_{\text{Choice}})$.

So the relations of the explanation by the theorem can be represented by

$$\sigma^2 \geq \sigma^2{}_{Min} > 0 \text{ and either } d_{Theorem} \approx d_{Choise} \text{ or at least } d_{Theorem} = O(d_{Choise})\,. \quad (7)$$

**3. Biases for sure games.**

The above considerations about noise suppression and sure games emphasize the condition that the sure games are guaranteed by some guaranteeing efforts. Due to these efforts, the biases for the sure games can be suppressed and reduced. They can be moreover equal to zero.

In accordance with these deductions, I assume that the presupposed bias of the measurement data for the sure games is equal to zero or, more generally, is strictly less than the presupposed bias for the uncertain games.

So, the inequality relating the sure and uncertain games is

$$|\Delta_{Uncert}| > |\Delta_{Sure}| \quad \text{or} \quad \operatorname{sgn} d_{Choise} = \operatorname{sgn} \Delta_{Uncert}\,. \qquad (8)$$

## 5.4. First stage of the approach. Qualitative problems, models and explanations

Due to the above preliminary principle, the first stage of the approach (method) can be constituted by qualitative models. This means that the models of the method can both deal with qualitative problems and give qualitative explanations.

The preliminary statements of the first stage can be formulated as follows:

**Qualitative analysis**. Only a qualitative analysis will be performed.

**Qualitative problems**. Only qualitative problems will be considered.

**Qualitative explanation**. Only qualitative explanations of the existing problems will be given. No predictions will be made in during this first stage.

**Choices of subjects**. The models will explain mainly the subjective behavior and choices of subjects.

## 6. Qualitative mathematical models

### 6.1. Need for qualitative models

First of all, is there a real need for qualitative models?

Suppose you are considering a confused situation where you know the exact magnitude of some effect, which can be either positive or negative, but you cannot predict its sign. Evidently the goal is, first of all, to understand and explain the origins of the effect and predict its sign and only then to calculate its exact magnitude.

The literature analysis states that this problem of the determination of the signs was posed not later than in 1979 (see, e.g., [32] page 268 ”The reflection effect”), but is still unsolved (see, e.g., [49] pages 1581–1582 “violate any sensible definition of rational. ... subjects were risk averse in the domain of gains but risk seeking in the domain of losses”). So the theory takes into account the observed signs of the biases but does not explain them, and there is a need for such an explanation.

### 6.2. Restrictions on the models. The main question

**First**. Evidently, if $\sigma_{\mathrm{Min}} \to 0$ then, due to (5), $r_\mu/\sigma_{\mathrm{Min}} \to 0$ as well.

**Second**. The preliminary estimate [27] shows that the real relative biases are sometimes comparable to the upper bound of the relative bias guaranteed by the theorem.

Due to these two reasons, and also from general and formal points of view, one may suppose: “In general cases, along with the non-zero minimal variance, other sources of the biases cannot be excluded so far.” Hence, a general model can be considered at present as only a preliminary one. So, the main question is to determine whether the forbidden zones can lead to sufficiently high values for the biases (both for low and high minimal variances). So, the main question of future research is to analyze the possible widths of the forbidden zones for various types of distributions.

## 6.3. Basics of a general qualitative model

There can only be three combinations for the real expectations: the expectation for the uncertain game (or outcome) can be greater than, less than, or equal to that for the sure one. So, the signs of their differences can be positive, negative, or zero.

The inalienable feature of the analyzed qualitative problems is the necessary change of sign. That is: the signs of the presupposed differences (for the choices of subjects), that are required to obtain the observed data, should be not equal to the signs of the real differences between the expectations for the uncertain and sure games.

In other words, when this real difference is, e.g., positive (that is, $sgn\ d_\mu > 0$), then, to obtain the observed data, this presupposed difference should be negative, that is, $sgn\ d_{\mathrm{Choice}} < 0$ (note that, due to (6), $d_{\mathrm{Choice}} \neq 0$). When the real difference equals zero, then the presupposed difference should be undoubtedly positive or negative.

This feature of these qualitative problems can be represented by

$$\operatorname{sgn} d_{Choice} \neq \operatorname{sgn} d_\mu . \tag{9}$$

These types of qualitative problems are chosen as the examples that are usual in experiments (see, e.g., [32], [46], [49]. They can make manifest qualitative representations of the problems and can be a background for further research.

To change the real difference of the expectations for the uncertain and sure games to another qualitative situation, the presupposed bias of choices should be evidently not less than this difference, that is

$$| d_{Choice} | \geq | d_\mu | . \tag{10}$$

So, inequalities 9 and 10 constitute the addition to the basis of the method.

Note. Inequality 10 implies, in particular, that for the certainty equivalents

$$| d_{Choice} | = | d_\mu | \quad \text{and, due to (6) and (9),} \quad d_{Choice} = -d_\mu ,$$

and for the other problems

$$| d_{Choice} | > | d_\mu | .$$

The trial examples of [27] of applications of the general model show that it can qualitatively explain the practical examples cited here. Nevertheless, the general formal preliminary qualitative mathematical model still needs proofs.

### 6.4. Special qualitative model (SQM)

Let us consider the qualitative problems under the special condition

$$\operatorname{sgn} d_{\mu} = 0 \quad \text{or} \quad d_{\mu} = 0 \quad \text{or} \quad \mu_{Uncertain} = \mu_{Sure}. \qquad (11)$$

This special condition asserts that the expectations for the uncertain games are exactly equal to the expectations of the corresponding sure games.

Such a special situation enables avoiding the constraints of preliminary estimate [27] of the secure upper bound for the bias and making the special model less formal. The biases can be selected to be much less than the secure upper bound and the suppositions will be simpler. This Special Practical Qualitative Mathematical Model (SPQMM or SQM) can be considered as a first step of the first stage of the approach (method) and of an explanation of the above problems.

The relations of the SQM can be summarized as follows:

Inequality (6) of the non-zero difference between the biases in the choices

$$\exists\, d_{Choice} : |\, d_{Choice} \,| > 0 \quad \text{or} \quad \exists\, d_{Choice} : \operatorname{sgn} d_{Choise} \neq 0.$$

Relations (7) of the theorem and choices

$$\sigma^2{}_{Min} > 0 \text{ and either } d_{Theorem} \approx d_{Choise} \text{ or at least } d_{Theorem} = O(d_{Choise}).$$

Inequality (8) for the choices for the sure and uncertain games

$$|\, \Delta_{Uncert} \,| > |\, \Delta_{Sure} \,| \quad \text{or} \quad \operatorname{sgn} d_{Uncert} = \operatorname{sgn} \Delta_{\Pr ob}.$$

Condition (11) of the special qualitative problems

$$\operatorname{sgn} d_{\mu} = 0 \quad \text{or} \quad d_{\mu} = 0 \quad \text{or} \quad \mu_{Uncertain} = \mu_{Sure}.$$

## 7. Applications of the theorem and method. Novelty and importance

### 7.1. Practical applications in behavioral economics

The idea of the considered forbidden zones was applied in the analysis [25] of some well-known problems (see, e.g., [31] of utility and prospect theories. The analysis was performed for the purposes of behavioral economics, psychology, decision theory, and the social sciences. In [25] some examples of typical paradoxes were studied. These paradoxes can concern problems such as the underweighting of high and the overweighting of low probabilities, risk aversion, etc.

The dispersion and noisiness of the initial data can lead to the above non-zero minimal variance (3) and forbidden zones for the expectations of these data and to corresponding non-zero biases. This idea explained, at least partially, the analyzed examples of paradoxes. The theorem proves the possibility of the existence of such non-zero zones and, consequently, non-zero biases under the condition (3).

Experimental and analytical works (see, e.g., [26] and [46]) devoted to the experimental methods of behavioral economics support this idea as well.

### 7.2. Practical numerical example. First domain. Gains

The special practical qualitative mathematical model enables to use small and convenient biases. In particular, integer numbers are convenient for consideration. The minimal non-zero integer is one. Suppose that the parameters of the SQM for the gains are: the presupposed bias for the choices for the uncertain games is equal to \$2, and for the sure game it is equal to \$1 or to zero.

The typical examples (see, e.g., [32] and [46]) can be simplified to the special qualitative situations similar to that of the preceding section and [25]:

Imagine that you face the following pair of concurrent games (a sure game and an uncertain game) with their corresponding sets of one sure and two uncertain outcomes.

Choose between:

A) A sure gain of \$99.

B) A 99% chance to gain \$100 and a 1% chance to gain or lose nothing.

#### 7.2.1. Ideal case

In the ideal case, without taking into account the dispersion of the data, the expectations for the sure game and for the uncertain game are both equal:

$$\$99 \times 100\% = \$99,$$
$$\$100 \times 99\% = \$99.$$

The expectations for the ideal case are exactly equal to each other

$$\$99 = \$99.$$

So, in the ideal case, the uncertain game and the sure game are equally preferable.

### 7.2.2. Forbidden zones and biases

In the real case, one should take into account some dispersion of the data, the minimal non-zero variance (3) caused by this dispersion, and the forbidden zones (4) caused by this variance. These forbidden zones can lead to biases of the expectations, at least for the uncertain games (and outcomes). Let us consider the case of a non-zero variance of the data, the corresponding forbidden zones, and the resulting biases.

The biases are $\Delta_{\text{Uncertain}} = \$2$ and $\Delta_{\text{Sure}} = \$1$ and we have

$$\$99 \times 100\% - \Delta_{Sure} = \$99 - \$1 = \$98,$$
$$\$100 \times 99\% - \Delta_{Uncert} = \$99 - \$2 = \$97.$$

The expectation for the uncertain game is biased more than that for the sure one and

$$\$98 > \$97.$$

Let us consider the case when $\Delta_{\text{Sure}} = \$0$. We have

$$\$99 \times 100\% - \Delta_{Sure} = \$99 - \$0 = \$99,$$
$$\$100 \times 99\% - \Delta_{Uncert} = \$99 - \$2 = \$97.$$

The expectation for the uncertain game is biased but that for the sure one is not, and

$$\$99 > \$97.$$

In both these cases, the expectation of the uncertain game is biased more than that of the sure one. This bias decreases the preferability of the uncertain game. Therefore the uncertain gain (game) is less preferable than the sure one.

We see the clear and evident difference between the expectations and its correspondence with the salient and unequivocal preferences and choices.

So the theorem and SQM provide mathematical support for the above analysis in the domain of gains. So, the forbidden zones and their natural difference for uncertain and sure games can explain the experimental fact that the subjects are risk averse in the domain of gains. They explain, at least qualitatively or partially, the analyzed example of [49] and many other similar results.

### 7.3. Practical numerical example. Second domain. Losses

The case of gains has been explained many times and in a lot of ways. The uniform explanation for both gains and losses, without additional suppositions, as, e.g., in [32], had nevertheless not been recognized by the author of the present article (see a slightly similar work [21]). The theorem, method, and models (in particular SQM) turn out to be useful for such a uniform explanation.

Let us consider the case of losses under the same suppositions as gains

Imagine that you face the following pair of concurrent games (a sure game and an uncertain game). Choose between:

A) A sure loss of $99.

B) A 99% chance to lose $100 and a 1% chance to lose or gain nothing.

#### 7.3.1. Ideal case

In the ideal case, the expectations for these choices are

$$-\$99\times 100\% = -\$99,$$

$$-\$100\times 99\% = -\$99.$$

Here, the expectations are exactly equal to each other:

$$-\$99 = -\$99.$$

Therefore both choices (games) should be equally preferable.

#### 7.3.2. Forbidden zones and biases

Let us consider the forbidden zones and biases under the same suppositions as for the gains. That is for the same, uniform parameters of the models.

The forbidden zone biases the expectation from the boundary of the interval to its middle. The biases are therefore subtracted from the absolute values for both cases, gains and losses. That is, due to the opposite signs of the values for gains and losses, the bias is subtracted for the gains and added for the losses.

Note. This is not a supposition but a rigorous conclusion. Hence the applications of the general model and SQM are naturally uniform for more than one domain.

The parameters of SQM for the losses are: the bias for the uncertain games equals $2 and for the sure game $1 or zero.

Let us consider the case when the bias for the sure game equals $1

$$-\$99\times 100\% + \Delta_{Sure} = -\$99 + \$1 = -\$98,$$

$$-\$100\times 99\% + \Delta_{Uncert} = -\$99 + \$2 = -\$97.$$

The expectation of the uncertain game is biased more than that for the sure one and

$$-\$98 < -\$97 .$$

Let us consider the case when the width of the forbidden zones for the expectations of data in the sure game is equal to zero. We have

$$-\$99 \times 100\% + \Delta_{Sure} = -\$99 + \$0 = -\$99 ,$$

$$-\$100 \times 99\% + \Delta_{Uncert} = -\$99 + \$2 = -\$97 .$$

The expectation for the uncertain game is biased but that for the sure one is not and

$$-\$99 < -\$97 .$$

In both these cases, the expectation for the uncertain game is biased more than that for the sure one, as was also the case for the gains, but here the bias increases the preferability of the uncertain game and the uncertain loss is (due to the obvious difference between the expectations) more preferable than the sure one. We see the clear difference between the expectations and its correspondence with the salient choices.

So the SQM provides support for an analysis in the domain of losses as well.

### 7.4. Novelty and importance

There are a lot of real examples of the forbidden zones for the expectations. The idea of such zones helps in the above analysis of the generic problems. The theorem provides the mathematical description of the forbidden zones and the mathematical support for this analysis. MMBE is an application of the theorem to it. The qualitative mathematical models are the first stage of MMBE and SQM is its first step.

The literature analysis and comments of journals' editors and reviewers on similar articles and on the previous versions of the present article allow stating reliably that the idea, theorem and its support of the above analysis, the method and models have not been described before and are new.

But such forbidden zones are evident and often well-known. Why were not they mathematically described before? The long lack of such a description can be probably explained by reason that the phenomena similar to the above examples are evident, can be as a rule easily estimated as approximately one-half of the amplitude of the vibrations, and did not need a more detailed investigation. However, in the considered problems and paradoxes of behavioral economics, such phenomena are hidden by other details of the experiments (see, e.g., [26]) and hence are not as evident. In addition, the well-known law of diminishing marginal utility proposes other ways to analyze it.

The special practical qualitative mathematical model (SQM) can be naturally, uniformly and successfully applied in more than one domain. This has been shown in particular for gains and losses at high probabilities and enables solving the problem of explanation and even prediction of the signs of the biases for these domains.

As was considered in subsection 6.1, the determination or, at least, explanation of the sign of an analyzed effect, in particular of the bias of subjects' choices, is the first, indispensable and, hence, important goal of this research.

## 7.5. Possible general applications

### 7.5.1. Possible general applications. Noise

Let us make some preliminary considerations for possible applications of the theorem to a mathematical description of noise.

If some type of noise leads to some non-zero minimal variance 3 for the considered set of random variables, then this non-zero minimal variance (and, consequently, this type of noise) leads to the above non-trivial forbidden zones 4 for the expectations of these variables. If some type of noise leads to an increase in the value of this minimal variance, then the width of these forbidden zones increases also.

The proposed theorem and method are a step towards developing new general mathematical tools to describe the influence of noise near the boundaries of finite intervals. In particular, if the noise leads to a non-zero minimal variance $\sigma^2_{\mathrm{Min}}$ : $\sigma^2_i > \sigma^2_{\mathrm{Min}} > 0$ for the set $\{\sigma^2_i\}$ of variances of the random variables $X_i$, then the theorem predicts there will be forbidden zones whose width $r_{\mathrm{Noise}}$ is not less than

$$r_{Noise} \geq \frac{\sigma^2{}_{Min}}{b-a}.$$

So, the proven theorem can be a preliminary step towards a general mathematical description of the possible influence of noise near the boundaries of finite intervals.

Some general questions concerning this item can arise. For example, general determinations of level, strength, power, etc. of noise are needed. They should lead to the general determinations of the non-negligible noise. There are many types of noise. Another thing that is needed is the specification of common widespread types of noise of a measurement those lead to a non-zero minimal variance of the measurement data in the usual circumstances and environments.

Due to the general character of the above questions and due to the demand for widespread experimental support, there is a need for a wide variety of research teams to give reliable answers to these questions.

### 7.5.2. Possible general applications. Biases of measurement data

Let us make some preliminary considerations on potential applications of the theorem to a general mathematical description of the possible biases of data.

The forbidden zones (4) can lead evidently to some biases in measurements.

Suppose a set (like the above $\{X_i\}$) of series of measurements whose data all lie within a common finite interval. The set of the data series forms the set of their expectations. If there is some non-zero minimal variance of the data such that the inequality (3) is true for the data of any series of the set, then there exist forbidden zones (4) (near the boundaries of the interval) for the set of the expectations of the series.

The expectations of the data of the measurements cannot indeed be located inside the forbidden zones. Therefore, they cannot be located closer to the boundaries of the interval than the width of the forbidden zone. So the above forbidden zones can produce biases in the expectations of the data of the measurements.

These biases have the following features:

1) They are directed from the boundaries to the middle of the interval.

2) They have opposite signs near the opposite boundaries of the interval.

3) Their moduli decrease from the boundaries to the middle of the interval.

When the minimal variance of the data is equal to zero (that is when 3 is not true), then the expectations of the data of measurements can touch the boundaries of the interval. When the above forbidden zones exist and are not taken into the consideration, then the predicted results are located closer to the boundaries than in the real case. Hence the predicted results are biased in the comparison with the real ones.

We will now look at a particular example of these biases. If the minimal variance (3) of the data $\sigma^2_{\mathrm{Min}}$ in is non-zero, that is if $\sigma^2 > \sigma^2_{\mathrm{Min}} > 0$ is true, then the theorem predicts (5) that near the boundaries of intervals the absolute value $|\Delta_{\mathrm{Bias}}|$ of the biases is

$$|\Delta_{Bias}| \geq \frac{\sigma^2_{Min}}{b-a}.$$

So, the presented theorem and method, their consequences and applications can be considered as a preliminary step to the general mathematical description of the biases of measurement data near the boundaries of finite intervals.

# 8. Conclusions and discussion

## 8.1. The problematic that motivated this research

### 8.1.1. Problems of behavioral economics

There are some well-known and generic problems of behavioral economics (see, e.g., [3], [31], [32], [49]). Their essence can be formulated as: the choices of the subjects (people) don't correspond to the probabilistic expectations of the games.

Some of the typical problems consist in the comparison of sure and uncertain games (see, e.g., [32], [49]). These are most pronounced near the boundaries of intervals. Some of them have opposite solutions for different domains. For example, [49] states (the italics are my own):

"We observe a pattern that was frequently displayed: subjects were *risk averse in the domain of gains* but *risk seeking in the domain of losses*."

These problems can be represented in the simplified and demonstrable form by the special qualitative problems (that is by the problems of the equal expectations for the uncertain and sure games) considered in the present article similar to [25]:

**First domain. Gains**. Choose between a sure game and an uncertain one:

A) A sure gain of $99.

B) A 99% chance to gain $100 and a 1% chance to gain or lose nothing.

The expectations are

$$\$99 \times 100\% = \$99 = \$99 = \$100 \times 99\% .$$

Second domain. Losses. Choose between a sure game and an uncertain one:

A) A sure loss of $99.

B) A 99% chance to lose $100 and a 1% chance to lose or gain nothing.

The expectations are

$$-\$99 \times 100\% = -\$99 = -\$99 = -\$100 \times 99\% .$$

The expectations of games are exactly equal to each other in both domains. A wealth of experiments (see, e.g., [32], [46], [49]) proves nevertheless that the choices of the subjects are essentially biased. Moreover, they are biased in the opposite directions for gains and losses (see, e.g., [49]). These are well-known and fundamental problems that are usual in behavioral and social sciences.

### 8.1.2. Analysis of the problems. Need for theoretical support

A new analysis (see, e.g., [25]) of the above problems was developed in recent years. It is founded on the idea of the non-zero forbidden zones studied here.

The analysis explains, at least partially or qualitatively, the underweighting of high and the overweighting of low probabilities, risk aversion, risk premium, Allais paradox, etc. It provides also a uniform explanation (at least partial or qualitative) for the above opposite solutions in more than one domain.

Nevertheless the analysis has not until now had a sufficient theoretical support.

## 8.2. Four main contributions of the article

### 8.2.1. Mathematical support for the analysis

A theorem regarding certain inequalities and forbidden zones is proven here.

Consider a set $\{X_i\}$, $i = 1, \dots, n$, of random variables $X_i$ whose values lie within an interval $[a, b]$. If $0 < (b\text{-}a) < \infty$ holds for $[a, b]$, and if $\sigma^2_i \geq \sigma^2_{\text{Min}} > 0$ holds for their variances $\sigma^2_i$, then their expectations $\mu_i$ are separated from the boundaries $a$ and $b$ of the interval $[a, b]$ by forbidden zones of non-zero width,

$$a < \left( a + \frac{\sigma^2{}_{Min}}{b-a} \right) \leq \mu_i \leq \left( b - \frac{\sigma^2{}_{Min}}{b-a} \right) < b .$$

In other words, the theorem proves the possibility of the existence of non-zero forbidden zones for the expectations of the measurement data that were used in the above analysis. This proof evidently supports the above analysis.

8.2.2. General mathematical method (approach) for the analysis

The general mathematical method (approach) of the biases of the expectations (MMBE) is founded on the theorem and is to explain not only the objective situations but also and mainly the subjective behavior and choices of subjects.

The two main presuppositions of the method are:

1. The subjects make their choices (at least to a considerable degree) as if there were some biases of the expectations of the games.
(This presupposition of MMBE can be supported, at least formally: such biases may be proposed and tested even only from the purely formal point of view)

2. These biases (real biases or subjective reactions and choices of the subjects) can be explained (at least to a considerable degree) with the help of the theorem.

The supposed general mathematical relations of MMBE can be collected into three groups (partially corresponding to the main presuppositions):

1) Inequality (6) of the non-zero difference between the biases in the choices

$$\exists\, d_{Choice} : |\, d_{Choice} |> 0 \quad \text{or} \quad \exists\, d_{Choice} : \operatorname{sgn} d_{Choise} \neq 0 .$$

2) Relations (7) of the theorem and biases of the choices

$$\sigma^2 \geq \sigma^2{}_{Min} > 0 \text{ and either } d_{Theorem} \approx d_{Choice} \text{ or at least } d_{Theorem} = O(d_{Choice}) .$$

3) Inequality (8) of the choices for the sure and uncertain games

$$|\, \Delta_{Uncert} |>| \Delta_{Sure} | \quad \text{or} \quad \operatorname{sgn} d_{Choice} = \operatorname{sgn} \Delta_{Uncert} .$$

Here $\Delta_{\text{Uncert}}$, $\Delta_{\text{Sure}}$ and $d_{\text{Choice}} \equiv \Delta_{Uncert}$ - $\Delta_{Sure}$ are appropriately the presupposed biases of the expectations of the data for the uncertain and sure games and also their difference that is required to obtain the data corresponding to these choices; $d_{\text{Theorem}}$ is the difference that can be obtained by the theorem.

The first stage of the approach (method) consists in the qualitative mathematical explanation of the qualitative problems by qualitative mathematical models.

### 8.2.3. Special qualitative mathematical model for the analysis

**Basics of the general model**.

The basics of the general formal preliminary qualitative mathematical model have been developed here.

The supposed general inequalities (9) and (10) additional to the method are

$$\operatorname{sgn} d_{Choice} \neq \operatorname{sgn} d_{\mu} \quad \text{and} \quad |d_{Choice}| \geq |d_{\mu}|,$$

where $d_\mu \equiv \mu_{\text{Uncert}} - \mu_{\text{Sure}}$ is the difference between the real expectations.

The general model enables formal solutions of the qualitative problems considered here, but the limits of its applicability need additional research.

**Special model**.

The special practical qualitative mathematical model (SQM or SPQMM) is intended for the practical analysis of the special cases when the expectations for the uncertain and sure games are exactly equal to each other.

For these special cases, we have the additional inequality (11)

$$\operatorname{sgn} d_{\mu} = 0 \quad \text{or} \quad d_{\mu} = 0 \quad \text{or} \quad \mu_{Uncert} = \mu_{Sure}.$$

SQM can be considered as the first step of the first stage of the approach.

SQM implies the application of the theorem, method, and basics of the general model under the following additional facilitating supposition:

Due to inequality (8), the bias for the uncertain games $|\Delta_{\text{Uncertain}}| > 0$ should be non-zero, but, due to (11), it can be as small as possible. Therefore the minimal variance of the measurement data for the uncertain games can be supposed to be equal to an arbitrary non-zero value that is as small as possible to be evidently explainable in the presence of a common noise and scattering of the data.

8.2.4. Successful uniform application of SQM in two different domains

In the scope of SQM, suppose that the biases of the expectations are equal, for example, to $\Delta_{\text{Uncertain}}$ = \$2 for the uncertain games and $\Delta_{\text{Sure}}$ = \$1 for the sure games. Then we have:

**1. First domain. Gains**. In the case of gains we have

$$\$99 \times 100\% - \Delta_{Sure} = \$99 - \$1 = \$98,$$

$$\$100 \times 99\% - \Delta_{Uncert} = \$99 - \$2 = \$97.$$

The expected value \$97 of the uncertain gain is biased more than that \$98 of the sure one. The biases are directed from the boundary to the middle of the interval, decrease the moduli of both values and, due to their positive signs, decrease both values. Hence the biased expectation for the sure gain is more than that for the uncertain one:

$$\$98 > \$97.$$

So, the sure gain (game) is evidently more preferable than the uncertain one and this choice is supported by a wealth of experiments.

**2. Second domain. Losses**. In the case of losses we have

$$-\$99 \times 100\% + \Delta_{Sure} = -\$99 + \$1 = -\$98,$$

$$-\$100 \times 99\% + \Delta_{Uncert} = -\$99 + \$2 = -\$97.$$

The expected value -\$97 of the uncertain loss is biased more than that -\$98 of the sure one. The biases are directed from the boundary to the middle of the interval, decrease the moduli of the values but, due to their negative signs, increase both values. Hence the biased expectation of the sure loss is less than that of the uncertain one:

$$-\$98 < -\$97.$$

So, the uncertain loss (game) is evidently more preferable than the sure one and this choice is supported by a wealth of experiments.

So, SQM enables a qualitative analysis and qualitative explanation for the above special qualitative problems in more than one domain.

In spite of its seeming simplicity, the successful natural and uniform application of the special practical qualitative mathematical model in more than one domain is an important one. Such an application has not received any mention in the literature as well. Hence it belongs to the main contributions of the present article.

### 8.3. Inequalities and zones

Some general relations between inequalities and allowed and forbidden zones have been considered in the present article. The cases of one inequality and two inequalities for one and two points and their mutual correspondence with semi-infinite, finite and point allowed and forbidden zones have been analyzed.

Two variants of the main theorem of the present article for inequalities and zones have been given and compared. Inequalities and the corresponding zones have been successfully used for the consequences of the theorem, namely for mathematical method, general and special qualitative mathematical models.

### 8.4. Main future questions

The first main question for future research is to analyze the widths of the forbidden zones for various types of distributions both at low and high minimal variances.

The second group of questions is concerned with noise. In particular, it includes rigorous definition of the term "non-negligible noise" and proof that any such noise of measurements causes some non-zero minimal variance 3 of the measurement data or, at least, to rigorously determine such types of noise.

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## A. Appendix. Lemmas

Preliminaries

Let us prove three lemmas for the theorem. Namely let us prove that the maximal variance of a random variable is equal to the variance of the discrete random variable whose probability mass function has only two non-zero values, which are located at the boundaries of the interval.

The proof for discrete distributions is given in [8]. Let us give an alternate proof that is uniform for the general case.

In the general case, we have for the random variable of subsection 3.1

$$
\begin{aligned}
E[X-\mu]^2 &= \sum_{x_k \in [a,b]} (x_k-\mu)^2 p_X(x_k) + \int_a^b (x-\mu)^2 f_X(x)dx, \\
&\equiv E_{Discr}[X-\mu]^2 + E_{Contin}[X-\mu]^2
\end{aligned}
$$

under condition (1) that either the probability mass function or probability density function or both of them are not identically equal to zero and their total norm is

$$\sum_{x_k \in [a,b]} p_X(x_k) + \int_a^b f_X(x)dx = 1 .$$

We will transform every value of the probability mass and probability density functions. Namely, divide them into pairs of values in the following manner:

Every value $p_X(x_k)$ is divided into two values located at $a$ and $b$

$$p_X(x_k)\frac{b-x_k}{b-a} \quad \text{and} \quad p_X(x_k)\frac{x_k-a}{b-a} .$$

The total value of these two parts is evidently equal to $p_X(x_k)$. The center of gravity of these two parts is evidently equal to $x_k$.

Every value of $f_X(x)$ is also divided into the two values located at $a$ and $b$

$$f_X(x)\frac{b-x}{b-a} \quad \text{and} \quad f_X(x)\frac{x-a}{b-a} .$$

The total value of these two parts is evidently equal to $f_X(x)$. The center of gravity of these two parts is evidently equal to $x$.

So, neither of these divisions change the expectation of the random variable.

We prove that the variances of the divided parts are not less than those of the initial parts.

## A1. Lemma 1: Discrete case

**Lemma 1. Discrete case**. If the values of a random variable $X$ lie within an interval $[a, b] : 0 < (b\text{-}a) < \infty$ and its variance can be represented as

$$E[X-\mu]^2 = \sum_{x_k \in [a,b]} (x_k-\mu)^2 p_X(x_k) + \int_a^b (x-\mu)^2 f_X(x)dx$$

and (1) holds, then

$$\sum_{x_k \in [a,b]} \left[ (\mu-a)^2 \frac{b-x_k}{b-a} + (b-\mu)^2 \frac{x_k-a}{b-a} \right] p_X(x_k) \geq \sum_{x_k \in [a,b]} (x_k-\mu)^2 p_X(x_k).$$

**Proof**. Let us consider separately the difference between these transformed and initial expressions for the discrete part of the variance for the cases $x_k \geq \mu$ and $x_k \leq \mu$.

A.1.1. Case $x_k \geq \mu$

If $a \leq \mu \leq x_k \leq b$, then

$$\left[ (a-\mu)^2 \frac{b-x_k}{b-a} + (b-\mu)^2 \frac{x_k-a}{b-a} \right] - (x_k-\mu)^2 \geq$$

$$\geq (b-\mu)^2 \left[ \frac{x_k-a}{b-a} - \left( \frac{x_k-\mu}{b-\mu} \right)^2 \right]$$

and

$$0 \leq \frac{x_k-\mu}{b-\mu} \leq 1, \quad \text{and} \quad \left( \frac{x_k-\mu}{b-\mu} \right)^2 \leq \frac{x_k-\mu}{b-\mu}.$$

Then

$$\frac{x_k-a}{b-a} - \left( \frac{x_k-\mu}{b-\mu} \right)^2 \geq \frac{x_k-a}{b-a} - \frac{x_k-\mu}{b-\mu} \equiv \frac{(x_k-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)} - \frac{x_k-\mu}{b-\mu}$$

and we have

$$(b-\mu)^2 \left[ \frac{x_k-a}{b-a} - \left( \frac{x_k-\mu}{b-\mu} \right)^2 \right] \geq 0.$$

So in the case when $x_k \geq \mu$, the difference between the transformed and initial expressions for the discrete part of the variance is non-negative.

A.1.2. Case $x_k \le \mu$

If $a \le x_k \le \mu \le b$, then, analogously to the above case,

$$\left[(a-\mu)^2 \frac{b-x_k}{b-a} + (b-\mu)^2 \frac{x_k-a}{b-a}\right] - (x_k-\mu)^2 \ge$$

$$\ge (\mu-a)^2 \left[\frac{b-x_k}{b-a} - \left(\frac{\mu-x_k}{\mu-a}\right)^2\right]$$

and

$$0 \le \frac{\mu-x_k}{\mu-a} \le 1 \quad \text{and} \quad \left(\frac{\mu-x_k}{\mu-a}\right)^2 \le \frac{\mu-x_k}{\mu-a}.$$

Then

$$\frac{b-x_k}{b-a} - \frac{\mu-x_k}{\mu-a} \equiv \frac{(b-\mu)+(\mu-x_k)}{(b-\mu)+(\mu-a)} - \frac{\mu-x_k}{\mu-a}$$

and we have

$$(\mu-a)^2 \left[\frac{b-x_k}{b-a} - \left(\frac{\mu-x_k}{\mu-a}\right)^2\right] \ge 0.$$

So in the case when $x_k \le \mu$, the difference between the transformed and initial expressions for the discrete part of the variance is non-negative as well.

So the difference

$$\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p_X(x_k)-(x_k-\mu)^2 p_X(x_k)$$

is non-negative for any $x_k$ such that $a \le x_k \le b$.

Let us estimate the difference between the transformed and initial expressions for the discrete part of the variance

$$E_{Discr.Transformed}[X-\mu]^2 - E_{Discr.Initial}[X-\mu]^2 =$$

$$= \sum_{x_k \in [a,b]}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]p_X(x_k).$$

Every member of the sum is non-negative. Hence the total sum is non-negative as well. Lemma 1 has been proven.

So, the variance of any discrete random variable whose values lie within a finite interval is not more than the variance of the discrete random variable (with the same expectation) which has only two values, located at the two boundary points of the interval. And the discrete part of the variance of $X$ is not more than the variance for the probability mass function (with the same norm and expectation as for this discrete part) which has only two values, located at $a$ and $b$.

## A2. Lemma 2: Continuous case

**Lemma 2. Continuous case**. If the values of a random variable $X$ lie within an interval $[a,b]$: $0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2 = \sum_{x_k \in [a,b]} (x_k-\mu)^2 p_X(x_k) + \int_a^b (x-\mu)^2 f_X(x)dx$$

and (1) holds, then

$$\int_a^b \left[ (\mu-a)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} \right] f_X(x)dx \geq \int_a^b (x-\mu)^2 f_X(x)dx .$$

**Proof**. Let us find the difference between these transformed and initial expressions for the continuous part of the variance. Let us consider separately the cases $x \geq \mu$ and $x \leq \mu$.

A.2.1. Case $x \geq \mu$

If $a \leq \mu \leq x \leq b$, then, analogously to the proof of the previous lemma,

$$\left[ (\mu-a)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} \right] - (x-\mu)^2 \geq (b-\mu)^2 \left[ \frac{x-a}{b-a} - \left( \frac{x-\mu}{b-\mu} \right)^2 \right]$$

and

$$\frac{x-a}{b-a} - \left( \frac{x-\mu}{b-\mu} \right)^2 \geq \frac{x-a}{b-a} - \frac{x-\mu}{b-\mu} \equiv \frac{(x-\mu)+(\mu-a)}{(b-\mu)+(\mu-a)} - \frac{x-\mu}{b-\mu}$$

and we have

$$(b-\mu)^2 \left[ \frac{x-a}{b-a} - \left( \frac{x-\mu}{b-\mu} \right)^2 \right] \geq 0 .$$

So in the case when $x \geq \mu$, the difference between the transformed and initial expressions for the variance is non-negative.

### A.2.2. Case $x \le \mu$

If $a \le x \le \mu \le b$, then considerations that are entirely analogous to the above cases lead to the conclusion

$$\left[(\mu - a)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a}\right] - (\mu - x)^2 \ge 0 .$$

So in the case when $x \le \mu$ the difference between the transformed and initial expressions for the variance is non-negative as well.

### A.2.3. Maximality

Let us estimate the difference between the transformed and initial expressions of the continuous part of the variance

$$E_{Contin.Transformed}[X-\mu]^2 - E_{Contin.Initial}[X-\mu]^2 =$$

$$= \int_a^b \left[(a-\mu)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} - (x-\mu)^2\right] f_X(x)dx \, .$$

Since the integrand of the integral is non-negative for every point in the scope of the limits of integration, the integral is non-negative as well. The difference between the expressions is therefore non-negative. Lemma 2 has been proven.

So, the variance of any continuous random variable whose values lie within a finite interval is not more than the variance of the discrete random variable which has only two values, located at the two boundary points of the interval. And the continuous part of the variance of $X$ is not more than the variance for the probability density function (with the same norm and expectation as for this continuous part) which has only two values, located at $a$ and $b$.

## A3. Lemma 3: General mixed case

**Lemma 3. General mixed case**.

If the values of a random variable $X$ lie within an interval $[a,b]: 0<(b-a)<\infty$ and its variance can be represented as

$$E[X-\mu]^2 = \sum_{x_k\in[a,b]}(x_k-\mu)^2 p_X(x_k)+\int_a^b (x-\mu)^2 f_X(x)dx$$

and (1) holds, then

$$\sum_{x_k\in[a,b]}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p_X(x_k)+$$
$$+\int_a^b\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f_X(x)dx\geq \quad .$$
$$\geq \sum_{x_k\in[a,b]}(x_k-\mu)^2 p_X(x_k)+\int_a^b (x-\mu)^2 f_X(x)dx$$

Proof. The conclusions that have been drawn about the discrete and continuous parts of this general mixed case, under condition 1, are true for their sum as well. Lemma 3 has been proven.

So in any case, the variance is maximal for the probability mass function that has only two values, located at the two boundary points $a$ and $b$ of the interval.

The considered transformations do not change the expectation of the random variable. The expectation for the probability mass function of these two boundary points is therefore equal to that of the initial random variable. Any two-point probability mass function $p_{ab}$ is determined by its expectation (and these two points). So

$$p_{ab}(a)=\frac{b-\mu}{b-a} \quad \text{and} \quad p_{ab}(b)=\frac{\mu-a}{b-a}$$

and the variance is

$$E_{ab}[X-\mu]^2=(\mu-a)^2\frac{b-\mu}{b-a}+(b-\mu)^2\frac{\mu-a}{b-a}=(\mu-a)(b-\mu).$$

This expression agrees naturally with the result of [8] for discrete variables and Lemma 1 can be treated as another version of this result.

So the variance of any random variable whose values lie within a finite interval $[a, b]$ is not more than that in inequality (2), that is,

$$E[X-\mu]^2\leq(\mu-a)(b-\mu).$$

Detailed contents