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ABSTRACT

Under multi-stage budgeting, the consumer allocates his income first to groups of goods (the highest stage), then for each group the expenditure to subgroups (the next-highest stage), etc., until finally the expenditure on the goods has been allocated (the lowest stage). This paper derives expressions that relate the price and income elasticities and the elasticities of substitution of the demand for goods to the corresponding elasticities of the demand for groups at each stage. In particular, it is shown that the elasticity of substitution between two goods is equal to a weighted sum of the elasticities of substitution at the stages, modified for within-stage income effects.

1. Introduction

Under multi-stage budgeting, the consumer allocates his income first to groups of goods (the highest stage), then for each group the expenditure to subgroups (the next-highest stage), etc., until finally the expenditure on the goods has been allocated (the lowest stage). Multi-stage budgeting is a generalization of two-stage budgeting, where the consumer allocates first his income to groups of goods and then allocates for each group the expenditure to the goods that belong to the group.[1] It is an attractive way to model demand and supply in terms of aggregates of goods, and it underlies, implicitly or explicitly, many empirical studies of consumer behavior; see e.g. Deaton (1975, chapter 6) and Blackorby, Boyce and Russell (1978) for explicit use of two-stage budgeting. It has also been used in analyses of trade (e.g. Armington, 1969) and of price formation (Zeelenberg, 1986), and in applied general equilibrium analysis (e.g. Keller, 1980 and Ballard, Fullerton, Shoven and Whalley, 1986).

Here I will analyze multi-stage budgeting under two additional constraints. First, it is required that the multi-stage procedure is consistent, i.e. that it gives the same demand functions as the one-stage procedure where the demand functions for the goods are determined directly. Second, it is required that the allocations of the upper stages can be carried out with knowledge only of a price index for each group. It is well known that these constraints impose restrictions on the functional forms of preferences, and thereby on those of the demand functions (Gorman, 1959, or Green, 1964, chapter 3).

The purpose of this paper is to derive expressions for the income, price and substitution elasticities under multi-stage budgeting, and to relate them to the corresponding elasticities of the demand at the upper stages. It will appear that these formulae are relatively simple and that they can be used with any specific demand systems for the upper stages, provided these satisfy the constraints of consistency and the existence of price indices. For the several stages one may even specify demand systems that

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[1] See Deaton and Muellbauer (1980, section 5.1) for an introduction to two-stage budgeting.

cannot be solved in a closed form for the demand functions of the goods.

In section 2 multi-stage budgeting is formally presented; section 3 gives the conditions for the existence of price indices; section 4 derives the formulae for the income and price elasticities; and in section 5 two special cases are analyzed. The appendices give proofs of some statements. The methods used in this paper are an extension of Zeelenberg (1986, appendix A), who analyzes homogeneous two-stage budgeting and of Keller (1976) who analyzes a multi-level CES function.

2. Multi-stage budgeting

The consumer's allocation problem is to maximize the utility function subject to the budget constraint:

$$\left.\begin{array}{l} \max\ u(q) \\ \text{subject to } \sum_{i=1}^{N} p_i q_i = y, \end{array}\right\} \qquad (2.1)$$

where u is the utility function, p_i is the price of good i, q_i is the quantity of good i, N is the number of goods and y is the total, given, budget ('income'). I assume that the utility function is well-behaved, i.e. twice continuously-differentiable, strictly quasi-concave, and strictly increasing in the quantities.

Figure 1 gives an illustration of multi-stage budgeting, with the number of stages equal to 3. There are four levels of goods and composite goods; the allocation at stage ℓ consists of the allocation from level ℓ to level ℓ - 1. At stage 3, the consumer allocates his income to two groups, food and shelter. At stage 2, the expenditure on the two groups is allocated to subgroups: the expenditure on food is allocated to meat and drink and the expenditure on shelter is allocated to housing and energy. At stage 1, the expenditures determined in the second stage are allocated to 8 goods, respectively fresh and preserved meat, milk and wine, rent and furniture and electricity and gas.

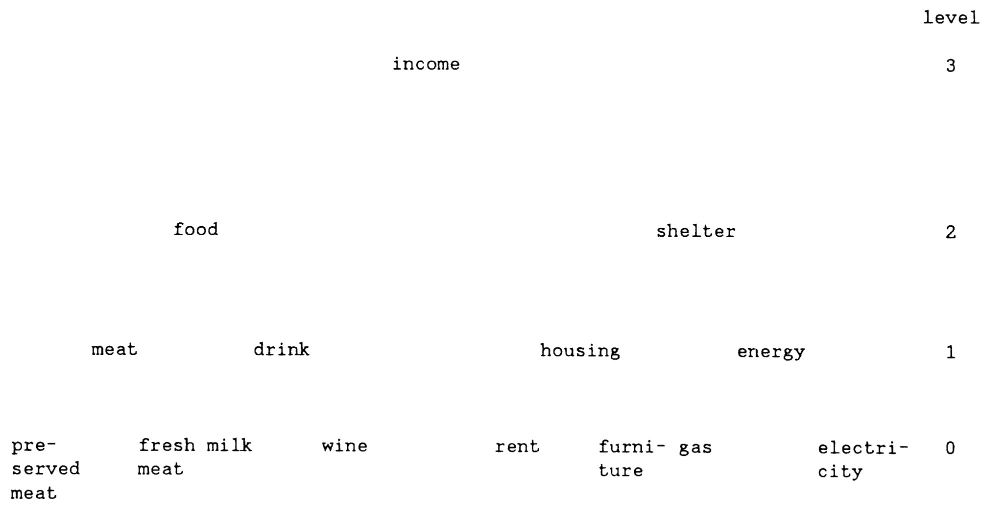

Figure 1. Multi-stage budgeting

It can be shown that, if multi-stage budgeting is to be possible, the preferences of the consumer must be separable at each stage, i.e. they can be represented by a utility tree. To formalize this condition, we must first introduce some concepts and notations; these are derived from Keller (1976, section 2). The levels of the utility tree are numbered with the lowest level equal to 0 (see figure 1). We define the *composite good at level* ℓ of a good as the group at level ℓ to which that good belongs; thus shelter is the composite good at level 2 of rent as well as the composite good at level 2 of gas. Let i and j be two goods at level 0. The composite goods of i and j at higher levels will also be denoted by i, respectively j; this will allow an easy, though not unique, notation. If no confusion can arise, then the composite good i at level ℓ will also be called good i at level ℓ. We say that good j *belongs at level* ℓ to good i if the utility function representing i at level ℓ is a function of amongst others good j at level ℓ - 1; this will be denoted by $j \in i$. For example, in figure 1, fresh meat belongs to meat at level 1, meat belongs to food at level 2 and food belongs to income at level 3. Then, under separability, at each level the utility functions are functions of composite goods at the next lower level; e.g. in figure 1 the utility function representing food is a function of the composite goods meat and drink. Formally, the preferences can be recursively defined by

$$u_i^\ell = u_i^\ell(u_j^{\ell-1}: j \in i), \qquad \ell = 1, 2, \ldots, L, \quad i = 1, 2, \ldots, N,$$
$$u_i^0 = q_i, \quad i = 1, 2, \ldots, N,$$

where L is the number of levels and u_i^ℓ ($\ell = 1, 2, \ldots, L$) represents the preferences of good i at level ℓ; I assume that the u_i^ℓ are well-behaved utility functions. The notation is not necessarily unique; e.g. in figure 1, there holds $u_1^1 \equiv u_2^1$, $u_1^2 \equiv u_3^2$, u_i^3 is the same for all i, etc. The function u_i^ℓ is called the macro-utility function at level ℓ and the $u_j^{\ell-1}$ the subutility functions at level $\ell - 1$; note that the $u_j^{\ell-1}$ are macro-utility functions at level $\ell - 1$.

We define *the lowest common level* C_{ij} of two goods i and j as the level at which their branches first meet:

$$C_{ij} = \ell \text{ iff. } (u_i^\ell \equiv u_j^\ell,\ \ell \geq C_{ij}) \wedge (u_i^\ell \not\equiv u_j^\ell,\ \ell < C_{ij}).$$

Thus $j \in i$ at level ℓ if and only if $C_{ij} = \ell$. The subscripts i and j will be deleted if it is obvious to which two goods C_{ij} refers.

3. Existence of price indices

To carry out the allocation at a stage one needs for each composite good at the lower level of the stage a price index that is a function only of the price indices of the composite goods that belong to the group; formally the allocation at stage $\ell + 1$ ($\ell = 0, 1, \ldots, L - 1$) can be written as

$$\left.\begin{aligned} &\max\ u_i^{\ell+1} \\ &\text{subject to} \quad \sum_{k \in i} p_k^\ell q_k^\ell = y_i^{\ell+1}, \end{aligned}\right\} \tag{3.1}$$

where the p_k^ℓ are the price indices, $y_i^{\ell+1}$ is the group budget, and the q_k^ℓ are quantity indices. Solution of (3.1) gives the optimal quantities at level ℓ as a function of the prices and the budget:

$$q_j^\ell = f_j^\ell(y_i^{\ell+1}, P_i^\ell), \quad \ell = 0, 1, \ldots, L-1, \tag{3.2}$$

where $P_i^\ell = (p_k^\ell: k \in i)$ is the vector with the price indices of the composite goods that belong to i at level $\ell + 1$, and f_j^ℓ is homogeneous of degree zero in the price indices p_k^ℓ and the budget $y_i^{\ell+1}$. Note that $y_i^{\ell+1} = y_j^{\ell+1}$ $(j \in i) = p_i^{\ell+1} q_i^{\ell+1} = \Sigma_{k\in i} p_k^\ell q_k^\ell = \Sigma_{k\in i} y_k^\ell$ and that $y_i^L = y$. The function f_j^ℓ can be regarded as the within-group demand for composite good j at level ℓ.

We require that the price indices p_j^ℓ are functions of only the price indices of the goods that belong to good j: $p_j^\ell = p_j^\ell(P_j^{\ell-1}) = p_j^\ell(p_k^{\ell-1}: k \in j)$. For the allocation at the lowest stage we can of course use the prices of the goods as the price indices, i.e. $p_j^0 = p_j$. For the other stages, however, we must impose restrictions on the preferences if price indices are to exist. Gorman (1959) has shown that[2] there exist price indices which are functions of only the price indices of the goods at the next lower level if and only if it is possible to divide the goods that belong to i at level $\ell + 1$ into two disjoint sets $A_i^{\ell+1}$ and $H_i^{\ell+1}$ such that

(i) the macro-utility function can be written (possibly after a monotone increasing transformation) as

$$u_i^{\ell+1} = \sum_{j\in A_i^{\ell+1}} u_j^\ell + g_i^{\ell+1}(u_j^\ell: j \in H_i^{\ell+1}), \quad \ell = 1, 2, \ldots, L-1, \tag{3.3}$$

and

(ii) the indirect utility function corresponding to the subutility function u_j^ℓ can be written as

$$\psi_j^\ell(y_j^\ell, P_j^{\ell-1}) = F_j^\ell\left[\frac{y_j^\ell}{b_j^\ell(P_j^{\ell-1})}\right] + a_j^\ell(P_j^{\ell-1}), \quad \ell = 1, 2, \ldots, L-1, \tag{3.4}$$

with F_j^ℓ monotonically increasing, b_j^ℓ linearly homogeneous, a_j^ℓ homogeneous of degree zero, and $a_j^\ell \equiv 0$ for $j \in H_i^{\ell+1}$ (i.e. the

[2] Apart from some special cases, e.g. a macro-utility function with only two arguments.

subutility functions u_j^ℓ are homothetic for $j \in H_i^{\ell+1}$).

Thus either the macro-utility function has to be additive and the sub-utility functions must have the socalled Gorman generalized polar form (GGPF) (3.4) or the subutility functions have to be homothetic. It can be shown that the functions b_j^ℓ are the price indices p_j^ℓ (see also appendix A).

The allocation at the lowest stage consists simply of maximizing the subutility functions u_i^1 subject to the constraint that total expenditure on the group equals the expenditure on the group determined in the previous stage:

$$\max u_i^1(q_i^0)$$

$$\text{subject to} \quad \sum_{j \in i} p_j q_j = y_i^1 .$$

The solution of this maximization gives demand functions q_j^0 that are functions of the prices p_k ($k \in i$) and the group budget y_i^1:

$$q_j^0 = f_j^0(y_i^1, P_i^0). \tag{3.5}$$

4. Elasticities under multi-stage budgeting

4.1. Income elasticities

The income elasticity of good i is from (3.5) and (3.2)

$$\frac{\partial \log q_i^0}{\partial \log y} = \frac{\partial \log f_i^0}{\partial \log y_i^1} \frac{\partial \log y_i^1}{\partial \log y_i^2} \cdots \frac{\partial \log y_i^{L-1}}{\partial \log y} .$$

Therefore in elasticity notation we have

$$\eta_i = \eta_i^0 \eta_i^1 \ \ldots \ \eta_i^{L-1} = \prod_{\ell=0}^{L-1} \eta_i^\ell , \tag{4.1}$$

where $\eta_i^\ell = \partial \log f_i^\ell \,/\, \partial \log y_i^{\ell+1}$ is the within-group income elasticity of good i at level ℓ. Thus the income elasticity of a good is equal to the product of the corresponding within-group income elasticities at all levels. Note that the subutility functions for $j \in H_i^{\ell+1}$ are homothetic, and thus the within-group income elasticities are equal to 1:

$$\eta_j^{\ell-1} = 1, \quad j \in H_i^{\ell+1}. \tag{4.2}$$

4.2. Price elasticities

To obtain the price elasticities we differentiate (3.5) logarithmically:

$$\varepsilon_{ij} = \frac{\partial \log q_i^0}{\partial \log p_j} = \frac{\partial \log q_i^0}{\partial \log y_i^1} \frac{\partial \log y_i^1}{\partial \log y_i^2} \cdots \frac{\partial \log y_i^{C-2}}{\partial \log y_i^{C-1}} \times$$

$$\left(\frac{\partial \log y_i^{C-1}}{\partial \log p_j^{C-1}} \frac{\partial \log p_j^{C-1}}{\partial \log p_j} + \frac{\partial \log y_i^{C-1}}{\partial \log y_i^C} \frac{\partial \log y_i^C}{\partial \log p_j} \right) =$$

$$\eta_i^0 \eta_i^1 \ \ldots \ \eta_i^{C-2} (\varepsilon_{ij}^{C-1} \pi_{jj}^{C-1} + \eta_j^{C-1} \frac{\partial \log y_j^C}{\partial \log p_j}) =$$

$$\left(\prod_{\ell=0}^{C-2} \eta_i^\ell\right) [\varepsilon_{ij}^{C-1} \pi_{jj}^{C-1} + \sum_{\ell=C}^{L-1} \{(1 + \varepsilon_{jj}^\ell) \pi_{jj}^\ell (\prod_{n=C-1}^{\ell-1} \eta_j^n)\}], \tag{4.3}$$

where C is the lowest common level of i and j, $\pi_{jj}^\ell = \partial \log p_j^\ell \, / \, \partial \log p_j$ is the elasticity of the price index of composite good j at level ℓ with respect to the price of good j, and $\varepsilon_{ij}^\ell = \partial \log f_i^\ell \, / \, \partial \log p_j^\ell$ is the within-group elasticity of demand for good i at level ℓ with respect to the price of good j at level ℓ. Note that the second equality sign follows because $y_i^\ell = y_j^\ell$ for $\ell \geq C$. The proof of the last equality sign involves a recursive equation and is given in appendix B. In (4.3) we have adopted the convention that a sum is equal to zero and a product equal to one if the upper bound is strictly smaller than the lower bound.

It is shown in appendix A that, under the additive-GGPF/homothetic preferences of section 3, the elasticity of the price index with respect to the price of a good is

$$\pi^{\ell}_{jj} = \frac{\partial \log p^{\ell}_j}{\partial \log p_j} = \frac{w_{0j}}{w_{\ell j}} \prod_{n=1}^{\ell} \frac{1 + \varphi^{n+1}_j \eta^n_j \eta^{n-1}_j}{1 + \varphi^{n+1}_j \eta^n_j} , \quad \ell = 1, 2, \ldots, L - 1, \tag{4.4}$$

where $w_{\ell i} = y^{\ell}_i / y$ is the share of composite good i at level ℓ in the total budget y and φ^{ℓ}_j is a parameter of the preferences of composite good j at level ℓ [the 'income flexibility' (the inverse of the income elasticity of the 'marginal utility of income') or minus the 'overall elasticity of substitution'; see Sato, 1972].

5. Two special cases

In this section we will for two special cases work out equation (4.4) for the elasticity of the price index and use the result to derive an expression for the elasticities of substitution.

5.1. All subutility functions are homothetic

Suppose that all subutility functions are homothetic. Then from (4.2) and (4.4) we get $\pi^{\ell}_{jj} = w_{0j}/w_{\ell j}$. Then from (4.3) we obtain, using (4.2),

$$\varepsilon_{ij} = \varepsilon^{C-1}_{ij} \frac{w_{0j}}{w_{C-1,j}} + \sum_{\ell=C}^{L-1} (1 + \varepsilon^{\ell}_{jj}) \frac{w_{0j}}{w_{\ell j}} . \tag{5.1}$$

Using the Slutsky equations $\varepsilon_{ij} = (\sigma_{ij} - \eta_i) w_j$ and $\varepsilon^{\ell}_{ij} = (\sigma^{\ell}_{ij} - \eta^{\ell}_i) w^{\ell}_j$, where σ denotes the elasticity of substitution and $w^{\ell}_j = y^{\ell}_j / y^{\ell+1}_j$ is the within-group budget share of composite good j at level ℓ, we get after some algebra

$$\sigma_{ij} = \frac{1}{w_{Ci}} \sigma^{C-1}_{ij} + \sum_{\ell=C}^{L-1} \frac{1}{w_{\ell+1,i}} \sigma^{\ell}_{jj} . \tag{5.2}$$

Thus the elasticity of substitution between two goods is equal to a weighted sum of the within-group elasticities of substitution, with the weights a declining series (since $w_{\ell i} \leq w_{\ell+1,i}$); the largest weight is given to the elasticity of substitution of the level at which the two goods first meet.

5.2. All subutility functions have the Gorman generalized polar form and all macro-utility functions are additive

If all functions u_i^ℓ ($\ell = 1, 2, \ldots, L - 1$) have the Gorman generalized polar form and all functions u_i^ℓ ($\ell = 2, 3, \ldots, L$) are additive, then there holds (see appendix A, equation A.8)

$$\pi_{jj}^\ell = \frac{\partial \log p_i^\ell}{\partial \log p_j} = \frac{w_{0j}}{w_{\ell j}} \frac{1 + \varphi_j^{\ell+1} \Pi_{n=0}^{\ell} \eta_j^n}{1 + \varphi_j^{\ell+1} \eta_j^\ell} \; . \tag{5.3}$$

Using again the Slutsky equations and the fact that for $\ell = 1, 2, \ldots,$ $L - 1$, there holds $\sigma_{ij}^\ell = -\varphi_i^{\ell+1}\eta_i^\ell\eta_j^\ell$ ($i \neq j$) and $\sigma_{ii}^\ell = -\varphi_i^{\ell+1}(\eta_i^\ell)^2 + \varphi_i^{\ell+1}\eta_i^\ell/w_i^\ell$ (cf. Deaton and Muellbauer, 1980, p. 138, equation 3.5), we obtain after some algebra

$$\sigma_{ij} = \frac{1}{w_{Ci}}\Big(\prod_{\ell=0}^{C-2} \eta_i^\ell\Big)\sigma_{ij}^{C-1}\Big(\prod_{\ell=0}^{C-2} \eta_j^\ell\Big) + \sum_{\ell=C}^{L-1} \Big\{\frac{1}{w_{\ell+1,i}}\Big(\prod_{n=0}^{\ell-1} \eta_j^n\Big)\sigma_{jj}^\ell\Big(\prod_{n=0}^{\ell-1} \eta_j^n\Big)\Big\}, \tag{5.4}$$

Thus the elasticity of substitution between two goods is equal to a weighted average of the within-group elasticities of substitution, modified for within-group income effects. Note that (5.2) can be considered to be a special case of (5.4). It is easily shown that (5.4) also holds for the case where the u_i^ℓ are homothetic for $\ell = 2, 3, \ldots, L$, u_i^2 is additive and u_i^1 has the Gorman generalized polar form. It is my conjecture that (5.4) also holds for preferences that are more general mixtures of homothetic and additive-GGPF preferences.

Appendix A. The elasticity of the price index

A.1. Proof of equation (4.4)

This appendix gives the proof of (4.4), i.e.

$$\pi^{\ell}_{jj} = \frac{\partial \log p^{\ell}_j}{\partial \log p_j} = \frac{w_{0j}}{w_{\ell j}} \prod_{n=1}^{\ell} \frac{1 + \varphi^{n+1}_j \eta^n_j \eta^{n-1}_j}{1 + \varphi^{n+1}_j \eta^n_j} .$$

There holds

$$\pi^{\ell}_{jj} = \frac{\partial \log p^{\ell}_j}{\partial \log p_j} = \prod_{n=1}^{\ell} \frac{\partial \log p^n_j}{\partial \log p^{n-1}_j} .$$

It will be shown that

$$\frac{\partial \log p^{\ell}_j}{\partial \log p^{\ell-1}_j} = w^{\ell-1}_j \frac{1 + \varphi^{\ell+1}_j \eta^{\ell}_j \eta^{\ell-1}_j}{1 + \varphi^{\ell+1}_j \eta^{\ell}_j} , \tag{A.1}$$

from which (4.4) follows, since $\Pi^{\ell}_{n=1} w^{n-1}_j = y_{0j}/y_{\ell j} = w_{0j}/w_{\ell j}$.

Define the price indices

$$p^{\ell}_j(P^{\ell-1}_j) = b^{\ell}_j(P^{\ell-1}_j), \quad \ell = 1, 2, \ldots, L, \tag{A.2}$$

where the b^{ℓ}_j are the deflators of the budget $y^{\ell+1}_i$ in (3.4). Making the substitution $q^{\ell}_j = y^{\ell}_j/p^{\ell}_j = e^{\ell}_j/p^{\ell}_j = \theta^{\ell}_j(u^{\ell}_j - a^{\ell}_j)$, where e^{ℓ}_j is the expenditure function and θ^{ℓ}_j is the inverse of F^{ℓ}_j, and using (3.3), we can write the allocation problem at stage $\ell + 1$ (cf. equation 3.1) as[3]

$$\left.\begin{array}{l} \max \sum\limits_{j \in A_i^{\ell+1}} F_j^\ell(q_j^\ell) + g_i^{\ell+1}[F_j^\ell(q_j^\ell): j \in H_i^{\ell+1})] + \sum\limits_{j \in A_i^{\ell+1}} a_j^\ell(P_j^{\ell-1}) \\ \text{subject to } \sum\limits_{j \in i} p_j^\ell q_j^\ell = y_i^{\ell+1}. \end{array}\right\} \qquad (A.3)$$

Solution of (A.3) gives the optimal quantities q_j^ℓ and the optimal expenditures $y_j^\ell = p_j^\ell q_j^\ell$. Note that the terms $a_j^\ell(P_j^{\ell-1})$ are independent of the q_j^ℓ and are thus irrelevant to the solution of (A.3).

For $j \in H_i^{\ell+1}$ it follows from (3.4) that $p_j^\ell = e_j^\ell/\theta_j^\ell(u_j^\ell)$; thus by Shephard's Lemma

$$\frac{\partial \log p_j^\ell}{\partial \log p_j^{\ell-1}} = \frac{p_j^{\ell-1}q_j^{\ell-1}}{y_j^\ell} = \frac{y_j^{\ell-1}}{y_j^\ell} = w_j^{\ell-1}, \qquad j \in H_i^{\ell+1},$$

which proves (A.1), since $\eta_j^{\ell-1} = 1$ [see (4.2)].

For $j \in A_i^{\ell+1}$ the proof consists of two parts. First I will derive an expression for the income flexibility $\varphi_i^{\ell+1}$ in terms of the indirect utility functions ψ_j^ℓ. The second part consists of deriving expressions for the within-group budget shares and the within-group marginal budget shares of the composite goods at level ℓ - 1.

The first-order conditions for the maximization problem (A.3) are

$$F_j^{\ell\,\prime} = \lambda_i^{\ell+1}p_j^\ell, \qquad j \in A_i^{\ell+1},$$

$$\frac{\partial g_i^{\ell+1}}{\partial q_j^\ell} = \lambda_i^{\ell+1}p_j^\ell, \qquad j \in H_i^{\ell+1},$$

where a prime denotes a derivative, and $\lambda_i^{\ell+1}$ is the Lagrange-multiplier (the 'marginal utility of income'). Differentiation of the conditions for $j \in H_i^{\ell+1}$ with respect to $y_i^{\ell+1}$ gives

[3] See Deaton and Muellbauer (1980, section 5.2).

$$F_j^{\ell\,\prime\prime}\frac{\partial q_j^\ell}{\partial y_i^{\ell+1}} = p_j^\ell\frac{\partial \lambda_i^{\ell+1}}{\partial y_i^{\ell+1}}, \quad j \in A_i^{\ell+1}.$$

In elasticity notation this can be written as

$$\frac{q_j^\ell}{\lambda_i^{\ell+1}}\frac{F_j^{\ell\,\prime\prime}}{p_j^\ell} = \frac{\omega_i^{\ell+1}}{\eta_j^\ell}, \qquad j \in A_i^{\ell+1},$$

where $\omega_i^{\ell+1} = \partial \log \lambda_i^{\ell+1}/\partial \log y_i^{\ell+1}$ is the income elasticity of the marginal utility of income. Since $\lambda_i^{\ell+1}p_j^\ell = F_j^{\ell\,\prime}$, this can be written as

$$\frac{F_j^{\ell\,\prime}}{F_j^{\ell\,\prime\prime}} = \varphi_i^{\ell+1}\eta_j^\ell q_j^\ell, \quad j \in A_i^{\ell+1}, \tag{A.4}$$

where $\varphi_i^{\ell+1} = 1/\omega_i^{\ell+1}$ is the income flexibility.

Application of Roy's Identity to the indirect utility function gives the within-group demand for good j:

$$q_j^{\ell-1} = -\frac{\partial\psi_j^\ell/\partial p_j^{\ell-1}}{\partial\psi_j^\ell/\partial y_j^\ell} = \frac{y_j^\ell}{b_j^\ell}\frac{\partial b_j^\ell}{\partial p_j^{\ell-1}} - \frac{b_j^\ell}{F_j^{\ell\,\prime}}\frac{\partial a_j^\ell}{\partial p_j^{\ell-1}}. \tag{A.5}$$

It follows from (A.5) that the within-group budget share of good j is

$$w_j^{\ell-1} = \frac{p_j^{\ell-1}q_j^{\ell-1}}{y_j^\ell} = \frac{\partial \log b_j^\ell}{\partial \log p_j^{\ell-1}} - \frac{p_j^{\ell-1}b_j^\ell}{F_j^{\ell\,\prime}y_j^\ell}\frac{\partial a_j^\ell}{\partial p_j^{\ell-1}}, \tag{A.6}$$

It also follows from (A.5) that the within-group marginal budget shares are

$$\mu_j^{\ell-1} = \eta_j^{\ell-1}w_j^{\ell-1} = p_j^{\ell-1}\frac{\partial q_j^{\ell-1}}{\partial y_j^\ell} = \frac{\partial \log b_j^\ell}{\partial \log p_j^{\ell-1}} + \frac{F_j^{\ell\,\prime\prime}}{(F_j^{\ell\,\prime})^2}p_j^{\ell-1}\frac{\partial a_j^\ell}{\partial p_j^{\ell-1}}. \tag{A.7}$$

Using (A.4), (A.6), and (A.7) one easily shows that

$$\frac{\partial \log b_j^\ell}{\partial \log p_j^{\ell-1}} = w_j^{\ell-1}\frac{1+\varphi_i^{\ell+1}\eta_j^\ell\eta_j^{\ell-1}}{1+\varphi_i^{\ell+1}\eta_j^\ell}, \quad j \in A_i^{\ell+1},$$

from which (A.1) follows, because $u_i^{\ell+1} \equiv u_j^{\ell+1}$, and thus $\varphi_i^{\ell+1} = \varphi_j^{\ell+1}$.

Finally we have to prove (5.3). This will be done by showing that $\varphi_j^\ell = \varphi_j^{\ell+1}\eta_j^\ell$, which, when substituted into (4.4), gives after some algebra (5.3). The marginal utility of income for the preferences represented by the indirect utility function (3.4) is

$$\lambda_j^\ell = \frac{\partial\ \psi_j^\ell}{\partial\ y_j^\ell} = F_j^{\ell\,\prime}\frac{1}{b_j^\ell}\ .$$

It follows that the inverse of the income elasticity of the marginal utility of income is

$$\varphi_j^\ell = \left(\frac{\partial\ \log\ \lambda_j^\ell}{\partial\ \log\ y_j^\ell}\right)^{-1} = \frac{y_j^\ell}{b_j^\ell}\ \frac{F_j^{\ell\,\prime}}{F_j^{\ell\,\prime\prime}} = \varphi_j^{\ell+1}\eta_j^\ell\ , \qquad \text{(A.8)}$$

where the last equality sign follows from (A.4).

A.2. A special case

A special case arises when the group preferences have the Gorman polar form

$$\psi_j^\ell(y_j^\ell,\ P_j^{\ell-1}) = \frac{y_j^\ell - d_j^\ell(P_j^{\ell-1})}{b_j^\ell(P_j^{\ell-1})},$$

where d_j^ℓ is linearly homogeneous in the prices $p_k^{\ell-1}$ $(k \in j)$; this form corresponds to (3.4) with $F_j^{\ell\prime} = 1$, $F_j^{\ell\prime\prime} = 0$, and $a_j^\ell = -d_j^\ell/b_j^\ell$. Such a form occurs for example if for each composite good there exists a base quantity $\overline{q}_k^{\ell-1}$ and the subutility functions are linearly homogeneous in the excess quantities $q_k^{\ell-1} - \overline{q}_k^{\ell-1}$ (Keller, 1976); then $d_j^\ell(P_j^{\ell-1}) = \Sigma_{k\in j} p_k^{\ell-1}\overline{q}_k^{\ell-1}$.

Then from (A.4) we have

$$\frac{\partial \log b_j^\ell}{\partial \log p_j^{\ell-1}} = \mu_j^{\ell-1}.$$

Thus the weights in the price index are equal to the within-group marginal budget shares; i.e. the price indices are Frisch price indices. On the other hand we have from (A.3)

$$\frac{\partial \log b_j^\ell}{\partial \log p_j^{\ell-1}} = \frac{p_j^{\ell-1}(q_j^{\ell-1} - \partial d_j^\ell/\partial p_j^{\ell-1})}{y_j^\ell - d_j^\ell(p_j^{\ell-1})} .$$

If $d_j^\ell(P_j^\ell)$ is interpreted as base expenditure and $\partial d_j^\ell/\partial p_j^{\ell-1}$ as the base quantity for good j, then one can say that the marginal budget shares are equal to the shares of the excess quantities in excess income (the 'excess budget shares').

Appendix B. Proof of equation (4.3)

This appendix gives a proof of the last equality sign in (4.3). There holds

$$\frac{\partial \log y_j^\ell}{\partial \log p_j} = \frac{\partial \log p_j^\ell}{\partial \log p_j} + \frac{\partial \log q_j^\ell}{\partial \log p_j^\ell}\frac{\partial \log p_j^\ell}{\partial \log p_j} + \frac{\partial \log q_j^\ell}{\partial \log y_j^{\ell+1}}\frac{\partial \log y_j^{\ell+1}}{\partial \log p_j} =$$

$$= (1 + \varepsilon_{jj}^\ell)\pi_{jj}^\ell + \eta_j^\ell \frac{\partial \log y_j^{\ell+1}}{\partial \log p_j}, \quad \ell = 1, 2, \ldots, L - 1. \tag{B.1}$$

Since $y_j^L = y$, there holds $\partial \log y_j^L / \partial \log p_j = 0$ and thus

$$\frac{\partial \log y_j^{L-1}}{\partial \log p_j} = (1 + \varepsilon_{jj}^{L-1})\pi_{jj}^{L-1}. \tag{B.2}$$

Equation (B.1) is a recursive equation with known endpoint (B.2). Repeatedly substituting (B.1) into itself and using (B.2), one can show that the solution is

$$\frac{\partial \log y_j^\ell}{\partial \log p_j} = (1 + \varepsilon_{jj}^\ell)\pi_{jj}^\ell + \sum_{m=\ell+1}^{L-1} \{(1 + \varepsilon_{jj}^m \pi_{jj}^m)(\prod_{n=\ell}^{m-1} \eta_j^n)\}.$$

Using this equation one easily proves the last equality sign in equation (4.3).

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