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4 December 2018

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Abstract

This paper presents a systematic approach for analyzing the departure-time choice equilibrium (DTCE) problem of a single bottleneck with heterogeneous commuters. The approach is based on the fact that the DTCE is equivalently represented as a linear programming problem with a special structure, which can be analytically solved by exploiting the theory of optimal transport combined with a decomposition technique. By applying the proposed approach to several types of models with heterogeneous commuters, it is shown that the dynamic equilibrium distribution of departure times exhibits striking regularities under mild assumptions regarding schedule delay functions, in which commuters sort themselves according to their attributes, such as desired arrival times, schedule delay functions (value of times), and travel distances to a destination.

Keywords: departure time choice equilibrium, linear programming, optimal transport, sorting

1. Introduction

The modeling and analysis of rush-hour traffic congestion has a rich history, dating back to the seminal work of Vickrey (1969). In the basic model of Vickrey, it is assumed that a fixed number of commuters with homogeneous preferences wish to arrive at a single destination (workplace) at a same preferred time, traveling through a single route that has a bottleneck of fixed capacity. Each commuter chooses the departure time of his/her trip from home to minimize his/her generalized trip cost, including trip time, queuing delay at the bottleneck and schedule delay (i.e., costs of arriving early or late at their destination). The problem is to determine a dynamic equilibrium distribution of departure times where no commuter can reduce his/her generalized cost by changing his/her departure time unilaterally. The importance of the problem in transportation planning and demand management policies has led to various extensions of the basic model to allow for (1) distributed/heterogeneous preferred departure times, (2) heterogeneity in the valuation of travel time and schedule delay, (3) elastic demands and (4) modal/route choices.

☆Acknowledgements.

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Despite these extensive studies, however, the extant analysis approaches have some limitations, and are not necessarily organized into a sufficiently general theory that enables us to systematically understand various extant results. Firstly, most previous studies focus only on a single type of user heterogeneity either in the preferred arrival time (e.g., Hendrickson and Kocur, 1981; Smith, 1984; Daganzo, 1985) or in the valuation of travel time and schedule delay (e.g., Cohen, 1987; Arnott et al., 1988, 1992, 1994; Ramadurai et al., 2010; van den Berg and Verhoef, 2011; Liu et al., 2015; Takayama and Kuwahara, 2017)[1]. As a result, little is known about certain regularities of the equilibrium when there are multiple types of heterogeneity in users' preferences. Secondly, most studies restrict their analysis to a special class of schedule functions (e.g., a piecewise linear function)[2]. This is theoretically problematic since it blurs the critical conditions required for the analysis results (e.g., what conditions are essential for the emergence of "sorting patterns" in the equilibrium? what conditions are required for obtaining an analytical solution?). Finally, for the models with a general schedule delay function and user heterogeneity, systematic and efficient methods for obtaining accurate solutions are still lacking. As shown by Nie and Zhang (2009), straight-forward formulation of the model (equilibrium conditions) results in a variational inequality problem with a non-monotone mapping, which implies that a naïve numerical algorithm neither guarantees convergence nor provides accurate solutions.

This paper presents a systematic approach to analyze a wide variety of models of departure-time choice equilibrium (DTCE) with a single bottleneck. The proposed approach is based on the following two facts: (1) the equilibrium can be obtained by solving a structured linear programming (LP) problem and (2) the special structure of the equivalent LP allows us to apply the theory of optimal transport (Rachev and Rüschendorf, 1998; Burkard, 2007; Villani, 2008), which provides explicit analytical solutions as well as efficient numerical algorithms. More specifically, we first review close relationships between DTCE and dynamic system optimal (DSO) assignment in a single bottleneck network, and it is shown that the DTCE with heterogeneous users can be generally constructed from the solution of the DSO assignment. Thus, the analysis of the DTCE problem is reduced to that of the DSO assignment represented as an LP. We then reveal that the equivalent LPs for various types of DTCE problems have some structural commonalities, to which the theory of optimal transport can be, either directly or indirectly, applied. Through several examples, we demonstrate how the optimal transport theory can be applied to the DTCE problems, and that the proposed approach enables us to systematically analyze basic properties of the equilibrium, such as existence, uniqueness, and some regularity of the flow patterns (e.g., temporal "sorting" patterns).

As a first example of the approach, we demonstrate that the DTCE problem with heterogeneous preferred departure times is analytically solvable and that the sorting property of the equilibrium flow pattern (i.e., the "First-In-First-Work principle" shown by Daganzo, 1985) can be understood from the optimal transport theory as a direct consequence of "submodularity" or the "Monge property" of the schedule cost function. As a second example, we consider the DTCE problem with heterogeneous schedule cost functions (i.e., users are differentiated according to their value of time). For this type of DTCE problem, the straightforward application of the optimal transport theory is not possible because the schedule cost functions do not satisfy the Monge property. Nevertheless, by developing an approach in which the optimal transport

[1] A few exceptions are Newell (1987); and Lindsey (2004); but the former assumes a special class of schedule functions and the latter focuses only on proving the existence and uniqueness of the equilibrium.

[2] Smith (1984) and Daganzo (1985) are a few exceptions; but they assume that all commuters have a same schedule delay function.

theory is applied to the sub-problems generated from Benders decomposition of the original LP, we can show that the DTCE problem is analytically solvable under mild assumptions regarding the schedule cost function. Finally, this approach (of combining Benders decomposition with the optimal transport theory) is further extended to generalized DTCE models with two types of cost heterogeneity.

In the remainder of this article, Section 2 introduces a LP approach to the DTCE with heterogeneous commuters. Section 3 briefly reviews the theory of optimal transport and presents an illustrative example of how it can be applied to analyze the DTCE problem with heterogeneous preferred departure times. Section 4 analyzes the DTCE problems with heterogeneous schedule delay cost functions, in which we provide a new analytical approach of combining Benders decomposition with the optimal transport theory. Section 5 further extends the approach to generalized DTCE models with two types of cost heterogeneity. Concluding remarks are made in Section 6.

2. Linear programming approach to the departure time choice equilibrium problems

2.1. Departure time choice equilibrium

We consider a road network with a single O-D pair connected by one route. Users travel one per vehicle. Users are treated as a continuum, and the total mass is denoted by Q, which is a given constant. The route has a single bottleneck with a capacity (maximum service rate) of μ. The queuing congestion at the bottleneck is described by a point queue model, where a queue is assumed to form vertically at the entrance of the bottleneck. Departure time from the bottleneck $s \in \mathcal{S}$ is assumed to be arrival time at the destination, where $\mathcal{S} \subset \mathbb{R}$ denotes a sufficient arrival time window. The free-flow travel time from the origin to the bottleneck is assumed, without loss of generality, to be zero unless otherwise noted. Users are classified into a finite number K of homogeneous groups. The index set of groups is denoted by $\mathcal{K} = \{1, 2, \ldots, K\}$. Let the mass of users in group $k \in \mathcal{K}$ be Q_k, then, $\sum_{k\in\mathcal{K}} Q_k = Q$.

Each user chooses their departure time from the bottleneck so as to minimize his/her trip cost. Trip cost is assumed to be additively separable in queuing delay cost, schedule delay cost and free-flow travel cost. The queuing delay of users with departure time s is denoted by $b(s)$, and the schedule delay is defined as the difference between actual departure time and preferred departure time (work start time) from the bottleneck. The schedule delay cost function of users in group k measured in (queuing) time unit is denoted as $c_k(s)$, which is assumed to be a continuous function and is specified in the later sections. Then, the trip cost of a user in group k departing from the bottleneck at time s is $b(s) + c_k(s)$.

Under the assumptions mentioned above, the *departure time choice equilibrium* is defined as the state in which no user could reduce his/her trip cost by changing his/her departure time unilaterally. The equilibrium condition is summarized as follows:

2.1.1. Users' optimal choice condition

The first condition is the users' optimal choice condition:

$$\begin{cases} v_k = b(s) + c_k(s) & \text{if} \quad x_k(s) > 0 \\ v_k \leq b(s) + c_k(s) & \text{if} \quad x_k(s) = 0 \end{cases} \qquad \forall k \in \mathcal{K},\ \forall s \in \mathcal{S} \tag{2.1}$$

where v_k represents the minimum (equilibrium) trip cost for users in group k and $x_k(s)$ denotes group k's departure flow rate from the bottleneck at time s.

2.1.2. *Queuing condition*

In the point queue model (for the details, see Appendix A.1), the queuing delay $d(t)$ for a user arriving at the bottleneck at time t can be represented by

$$\dot{d}(t) = \begin{cases} \lambda(t)/\mu - 1 & \text{if} \quad d(t) > 0 \\ \max.[0, \lambda(t)/\mu - 1] & \text{if} \quad d(t) = 0 \end{cases} \tag{2.2a}$$

where $\lambda(t)$ is the arrival flow rate at the bottleneck at time t. This implies that the queuing delay should satisfy

$$\begin{cases} \dot{d}(t) = \lambda(t)/\mu - 1 & \text{if} \quad d(t) > 0 \\ \dot{d}(t) \geq \lambda(t)/\mu - 1 & \text{if} \quad d(t) = 0 \end{cases} \quad \Leftrightarrow \quad 0 \leq d(t) \perp \dot{d}(t) - [\lambda(t)/\mu - 1] \geq 0. \tag{2.2b}$$

In this paper, we employ the complementarity condition (2.2b), instead of (2.2a), as the queuing delay model. This is because it is analytically tractable and its essential features are consistent with the original point queue model (e.g., the FIFO property holds)[3].

Under the assumption of a FIFO service discipline, users departing from the bottleneck at time s arrive at the bottleneck at time $\tau(s) \equiv s - b(s)$. Then, the relationship between the arrival and departure flow rates, $\lambda(\tau(s))$ and $x(s) \equiv \sum_{k \in \mathcal{K}} x_k(s)$, is as follows:

$$x(s) = \mathrm{d}A(\tau(s))/\mathrm{d}s = \lambda(\tau(s)) \cdot \Delta\tau(s), \tag{2.3}$$

where $A(t)$ is the cumulative arrival flow at the bottleneck at time t and Δ denotes the derivative operation with respect to *bottleneck-departure-time* s, i.e.,

$$\Delta\tau(s) \equiv \mathrm{d}\tau(s)/\mathrm{d}s = 1 - \Delta b(s), \ \Delta b(s) \equiv \mathrm{d}b(s)/\mathrm{d}s$$

For users departing from the bottleneck at time s, the queuing delay $b(s)$ is obviously given by $b(s) = d(\tau(s))$, which implies $\Delta b(s) = \dot{d}(\tau(s)) \cdot \Delta\tau(s)$. Combining these with the queuing delay (complementarity) condition (2.2b), we have

$$0 \leq b(s) \perp \mu\,[\Delta\tau(s) + \Delta b(s)] - x(s) \geq 0.$$

Since $\tau(s) + b(s) = s$ and thus $\Delta\tau(s) + \Delta b(s) = 1$, the queuing delay condition finally reduces to the following condition:

$$\begin{cases} \sum_{k \in \mathcal{K}} x_k(s) = \mu & \text{if} \quad b(s) > 0 \\ \sum_{k \in \mathcal{K}} x_k(s) \leq \mu & \text{if} \quad b(s) = 0 \end{cases} \qquad \forall s \in \mathcal{S}. \tag{2.4}$$

2.1.3. *Flow conservation*

For each user group k, the integral of the departure flow rate $x_k(s)$ from the bottleneck must be equal to the total mass Q_k, that is,

$$\int_{\mathcal{S}} x_k(s)\mathrm{d}s = Q_k \qquad \forall k \in \mathcal{K}. \tag{2.5}$$

[3] See Ban et al. (2012), Han et al. (2013) and Jin (2015) for more detailed discussions.

2.2. Equivalent linear programming

The basic idea of the approach employed in this study is that the equilibrium conditions (2.1), (2.4) and (2.5) reduce to an equivalent LP. This approach was first shown by Iryo et al. (2005) and Iryo and Yoshii (2007) in a discrete time setting. We briefly describe the approach below but for a *continuous time* setting.

Consider first the following infinite-dimensional linear programming (LP):

$$[\text{2D-LP}(\mathbf{x})] \qquad \min_{\mathbf{x}\geq\mathbf{0}}.\ Z(\mathbf{x}) \equiv \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} c_k(s)x_k(s)\mathrm{d}s \tag{2.6}$$

$$\text{subject to} \quad \sum_{k\in\mathcal{K}} x_k(s) \leq \mu \qquad \forall s \in \mathcal{S} \tag{2.7}$$

$$\int_{\mathcal{S}} x_k(s)\mathrm{d}s = Q_k \qquad \forall k \in \mathcal{K}, \tag{2.8}$$

and the associated dual problem:

$$[\text{2D-LP}(\mathbf{u},\mathbf{v})] \qquad \max_{\mathbf{u}\geq\mathbf{0},\mathbf{v}}.\ \check{Z}(\mathbf{u},\mathbf{v}) \equiv -\mu \int_{\mathcal{S}} u(s)\mathrm{d}s + \sum_{k\in\mathcal{K}} Q_k v_k \tag{2.9}$$

$$\text{subject to} \quad c_k(s) + u(s) - v_k \geq 0 \qquad \forall k \in \mathcal{K},\ \forall s \in \mathcal{S} \tag{2.10}$$

where $u(s)$ and v_k are Lagrange multipliers for (2.7) and (2.8), respectively.

The problem [2D-LP($\mathbf{x}$)] represents the dynamic system optimal (DSO) assignment problem with *no queuing* in which total schedule delay cost is minimized subject to the capacity constraint and flow conservation. As shown in Appendix B, the strong duality for [2D-LP($\mathbf{x}$)] holds, implying the following complementary slackness (or optimality) conditions:

$$\begin{cases} x_k(s)\{c_k(s) + u(s) - v_k\} = 0 \\ \qquad c_k(s) + u(s) - v_k \geq 0,\ x_k(s) \geq 0 \end{cases} \qquad \forall k \in \mathcal{K},\ \forall s \in \mathcal{S} \tag{2.11}$$

$$\begin{cases} u(s)\{\mu - \sum_{k\in\mathcal{K}} x_k(s)\} = 0 \\ \qquad \mu - \sum_{k\in\mathcal{K}} x_k(s) \geq 0,\ u(s) \geq 0 \end{cases} \qquad \forall s \in \mathcal{S}. \tag{2.12}$$

The optimality conditions (2.11), (2.12), and (2.8) can be interpreted in several different ways. One interpretation supposes that a road manager introduces a dynamic congestion pricing scheme. Comparing the optimality conditions above and the equilibrium conditions (2.1), (2.4), and (2.5), we easily see that replacing the queuing delay $\mathbf{b}$ in the equilibrium condition with the dynamic prices $\mathbf{u}$ leads to the optimality conditions. Thus, we have the following observation.

Observation 1. *(Doan et al., 2011; Daganzo, 2013) Suppose that the dynamic toll pattern for passing the bottleneck is given by the optimal solution* $\mathbf{u}$ *of [2D-LP(*$\mathbf{u},\mathbf{v}$*)] (or the optimal Lagrange multiplier* $\mathbf{u}$ *of [2D-LP(*$\mathbf{x}$*)]). Then the equilibrium under the dynamic pricing scheme achieves the DSO flow pattern* $\mathbf{x}$ *(i.e., the optimal solution of [2D-LP(*$\mathbf{x}$*)]).*

As a variant of the dynamic pricing scheme, we can also consider a time-dependent tradable permit system, which is designed to resolve the problem of congestion during the morning rush hour at a single bottleneck; it consists of the following two parts:

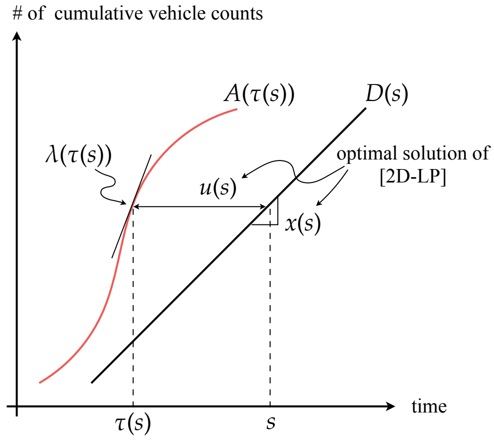

Figure 1: Schematic illustration of construction of cumulative arrival curve from the solution of [2D-LP($\mathbf{x}$)]

a) the road manager issues a right that allows a permit holder to pass through the bottleneck at a pre-specified time period ("bottleneck permits"),

b) a new trading market is established for bottleneck permits differentiated by a pre-specified time

Note here that the arrival flow rate at a bottleneck at any time period is, from the definition of the scheme, equal to or less than the number of permits issued for that time period. This implies that we can completely eliminate the occurrence of queuing congestion by setting the number of permits issued per unit time to be less than or equal to the bottleneck capacity. Under the permit system, the optimality conditions (2.11), (2.12), and (2.8) can be directly interpreted as the equilibrium conditions: (2.11) represents the optimal behavior of users for a given permit prices $\mathbf{u}$, and (2.12) represents the market clearing (demand-supply equilibrium[4]) condition of the permit for departure time s. This leads to the following observation.

Observation 2. *(Akamatsu et al., 2006; Akamatsu, 2007; Wada and Akamatsu, 2010; Akamatsu and Wada, 2017) Competitive market equilibrium prices of the time-dependent tradable permits coincide with the optimal solution* $\mathbf{u}$ *of [2D-LP(*$\mathbf{u}, \mathbf{v}$*)]. Furthermore, the equilibrium under the time-dependent tradable permit system achieves the DSO flow pattern* $\mathbf{x}$ *(i.e., the optimal solution of [2D-LP(*$\mathbf{x}$*)]).*

The two interpretations above implicitly assume for the equilibrium under the pricing scheme (or the DSO flow pattern) that there is no queuing at the bottleneck, which implies that the departure rates $\mathbf{x}$ always coincide with arrival rates at the bottleneck. However, it should be noted that the problem [2D-LP($\mathbf{x}$)] describes no queuing mechanisms (other than the capacity constraint); hence, the arrival rates at the bottleneck need not be equal to the departure rates $\mathbf{x}$. This leads to another interpretation, in which no economic intervention of road managers is assumed.

[4]For each time s, the demand of time period s permit is equal to the departure flow $x(s) = \sum_{k \in \mathcal{K}} x_k(s)$, and the maximum supply of the permit is given by the bottleneck capacity μ.

Observation 3. *(Iryo et al., 2005; Iryo and Yoshii, 2007; Akamatsu et al., 2015) The optimal solution* $(\mathbf{x}, \mathbf{u}, \mathbf{v})$ *of [2D-LP(*$\mathbf{x}$*)] and [2D-LP(*$\mathbf{u}, \mathbf{v}$*)] is consistent with the equilibrium conditions if the Lagrange multiplier* $\mathbf{u}$ *in the LPs can be regarded as the queuing delay* $\mathbf{b}$ *in the equilibrium conditions.*

In order for this interpretation to be valid, arrival flow rate λ at the bottleneck, which is consistent with the queuing conditions (A.1)–(A.3) and is constructed from the optimal solution $(\mathbf{x}, \mathbf{u}, \mathbf{v})$ of [2D-LP($\mathbf{x}$)] (see Figure 1), should be physically feasible (i.e., the arrival flow rate is non-negative and finite). Fortunately, this is true if the schedule cost function satisfies the following condition: $\Delta c_k(s) > -1$ (see Appendix A.2 and Akamatsu et al., 2015), which is consistent with the sufficient condition for the existence of equilibria in departure-time choice models (Smith, 1984; Lindsey, 2004). This condition is assumed to be satisfied throughout this paper.

3. Monge-Kantorovich problem

In order to obtaine a deeper insight into the properties (such as uniqueness, some regularity of flow patterns) of [2D-LP($\mathbf{x}$)] (or the departure time choice equilibrium), the theory of optimal transport (see, Kantorovich, 1942, 1948; Rachev and Rüschendorf, 1998; Burkard, 2007; Villani, 2008) is useful. In this section, we briefly review the role of Monge properties in optimization and show an illustrative example of its application to the departure time choice equilibrium.

3.1. Monge property and analytical solution

The optimal transport problem in a 2-dimensional discrete space setting is the following finite–dimensional LP, which is well known as "*Hitchcock's transportation problem*" in the operations research and transportation fields:

$$\text{[2D-OTP]} \qquad \min_{\mathbf{x}\geq\mathbf{0}} . \; Z_{2D}(\mathbf{x}) \equiv \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} c_{i,k} x_{i,k} \tag{3.1}$$

$$\text{subject to} \quad \sum_{k\in\mathcal{K}} x_{i,k} = S_i \qquad \forall i \in \mathcal{I} = \{1, 2, \ldots, I\} \tag{3.2}$$

$$\sum_{i\in\mathcal{I}} x_{i,k} = Q_k \qquad \forall k \in \mathcal{K} = \{1, 2, \ldots, K\} \tag{3.3}$$

where vectors $\mathbf{S}$ and $\mathbf{Q}$ are given constants satisfying $\sum_{i\in\mathcal{I}} S_i = \sum_{k\in\mathcal{K}} Q_k$.

Before reviewing some useful theorems on the transportation problem, we introduce the following concepts:

Definition. An $I \times K$ real matrix $C = [c_{i,k}]$ is called a *Monge matrix* if C satisfies the following property (*Monge property*)

$$c_{i,k} + c_{i+1,k+1} \leq c_{i,k+1} + c_{i+1,k} \qquad \text{for all } \; 1 \leq i < I, \; 1 \leq k < K \tag{3.4}$$

Also, if the inequality in (3.4) strictly holds, C is termed a *strict Monge matrix*.

Definition. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *submodular* if, and only if,

$$f(x, y) + f(x', y') \leq f(x, y') + f(x', y) \qquad \text{for all } x \leq x', y \leq y' \tag{3.5}$$

Also, if the inequality in (3.5) strictly holds for $x < x', y < y'$, f is called a *strict submodular function*.

This implies that an $I \times K$ matrix C whose elements are given by $c_{i,k} := f(i\Delta x, k\Delta y)$ $(1 \le i \le I,\ 1 \le k \le K)$ is a (strict) Monge matrix if the function $f : \mathbb{R}^2 \to \mathbb{R}$ is (strict) submodular. Thus, we will also term the condition (3.5) the (continuous) Monge property. If the inequalities (3.4) and (3.5) hold in the opposite direction, then the matrix C and function f are said to be an *inverse* Monge matrix and *supermodular*, respectively.

As is well known, a feasible solution to the transportation problem [2D-OTP] can always be determined by a greedy algorithm termed the *northwest corner rule* (Hoffman, 1963).

Northwest corner rule

0. Initialized the indices with $i := 1$ and $k := 1$.
1. Set $x_{i,k} := \min\{S_i, Q_k\}$.
2. Reduce both the supply S_i and demand Q_k by $x_{i,k}$: $S_i := S_i - x_{i,k}$ and $Q_k := Q_k - x_{i,k}$. If some of S_i and Q_k become zero, then these indices are increased by one.
3. If there still exists an unsatisfied constraint, go back to Step 2.

The Monge property further provides the following useful result:

Theorem 3.1. (Hoffman, 1963, 1985): *If the cost matrix* C *of [2D-OTP] has the Monge property, then the northwest corner rule yields an optimal solution for arbitrary supply* **S** *and demand* **Q** *vectors.*

Remarks. It is worthwhile noting that *the cost matrix* C *is not used at all in the northwest corner rule*. This implies that even if we only know that C is a Monge matrix, *we can determine an optimal solution of [2D-OTP] without knowing the explicit values of the cost coefficients*. If the cost coefficients fulfill the *inverse Monge property*, an optimal solution can be found by the *northeast corner rule*[5].

Theorem 3.2. (Dubuc et al., 1999): *If the matrix* $C = [c_{i,k}]$ *in* (3.1) *is a strict Monge matrix, then the optimal solution of [2D-OTP] is unique (i.e., the solution provided by the northwest corner rule is the only solution of [2D-OTP]).*

A continuous analogue of Theorem 3.1 for the continuous transportation problem is as follows. Let $x \in X = \mathbb{R}$ and $y \in Y = \mathbb{R}$ be random variables and let F_1, F_2, and F denote the distribution functions of x, y, and (x, y), respectively. Given a continuous cost function $c : \mathbb{R}^2 \to \mathbb{R}$, the continuous optimal transport problem[6] can be formulated as

$$\text{[2D-COTP]} \qquad \min_{F \in \mathcal{F}(F_1,F_2)} .\ Z_{2D}(F) \equiv \int_{X\times Y} c(x, y)\mathrm{d}F(x, y) \tag{3.6}$$

$$\text{where} \quad \mathcal{F}(F_1, F_2) \equiv \{F(x, \infty) = F_1(x),\ F(\infty, y) = F_2(y), \forall x, y \in \mathbb{R}\} \tag{3.7}$$

Note that $F_1(\infty) = F_2(\infty)$.

[5]In the northeast corner rule, the algorithm begins with the indices with $i := 1$ and $k := K$ (or $i := I$ and $k := 1$), then the index is decreased by one if Q_k (or S_i) becomes zero in Step 2 in the northwest corner rule.

[6]We can easily see the correspondence between the problems [2D-COTP] and [2D-OTP] by rewriting the constraints for supply and demand in [2D-OTP] in terms of *cumulative* supply and demand (see also Subsection 3.2 for an example).

Theorem 3.3. (Cambanis et al., 1976; Dubuc et al., 1999) *Let F_1 and F_2 be distribution functions on $\mathbb{R}$. Furthermore, suppose that $Z_{2D}(F)$ is finite and the cost function $c : \mathbb{R}^2 \to \mathbb{R}$ is a submodular. Then an optimal solution $F^* \in \mathcal{F}(F_1, F_2)$ is given by the so-called Fréchet–Hoeffding distribution*

$$F^*(x, y) = \min\{F_1(x), F_2(y)\} \qquad \forall(x, y) \in \mathbb{R}^2. \tag{3.8}$$

Furthermore, if the function c is a strict submodular, the solution is unique (i.e., $F^*(x, y)$ is the only optimal solution of [2D-COTP]).

We note that the northwest corner rule for the discrete transportation problem can be viewed as explicit rules for calculating the analytical solution (i.e., Fréchet–Hoeffding distribution) in the discrete space setting (Burkard, 2007).

3.2. Illustrative example: Model with heterogeneous preferred departure time

To illustrate the usefulness of the theory of optimal transport in analyzing the departure time choice equilibrium, we here consider the model with heterogeneous preferred departure times. A well known sorting property of the equilibrium flow pattern, the so-called *First-In-First-Work (FIFW) principle* (Daganzo, 1985), can be understood from the theory of optimal transport as a direct consequence of the Monge property of the schedule cost function.

Assumption 3.1. *The schedule delay cost function is assumed to be identical for all users and is given by*

$$c_k(s) \equiv f(\epsilon(\sigma_k, s)), \quad \epsilon(\sigma_k, s) \equiv s - \sigma_k, \tag{3.9}$$

where σ_k is the preferred departure time of users in group k and a function $f : \mathbb{R} \to \mathbb{R}$ is continuous, strictly convex, and has a minimum at $\epsilon = 0$.

Lemma 3.1. *Suppose that user groups are arranged (indexed) in an increasing order of their desired arrival times: $\sigma_1 < \cdots < \sigma_K$. The function $c_k(s)$ defined by (3.9) satisfies the strict Monge property.*

Proof. See Appendix C.1.

In the above setting in which the departure time is continuous but the preferred departure time is discrete (Newell, 1987; Lindsey, 2004), all groups experience queue delay to equilibrate the trip costs of users within each group. This implies that several disjoint departure time windows can exist in equilibrium. Here we assume that a single joint departure time window (i.e., a single rush period) $\hat{\mathcal{S}} \equiv [0, T] \subset \mathcal{S}$ of length $T = Q/\mu$ occurs in equilibrium[7], and we focus on the property of the departure order of users.

For the given equilibrium rush period $\hat{\mathcal{S}} \equiv [0, T]$, the departure time choice equilibrium (i.e., [2D-LP($\mathbf{x}$)]) can be reduced to an instance of the optimal transport problem, where $c_k(s)$ satisfies

[7] This corresponds to a typical assumption that the cumulative number of users who prefer to departure by time σ_k is described by a continuous and differentiable S-shaped curve (e.g., Smith, 1984; Daganzo, 1985).

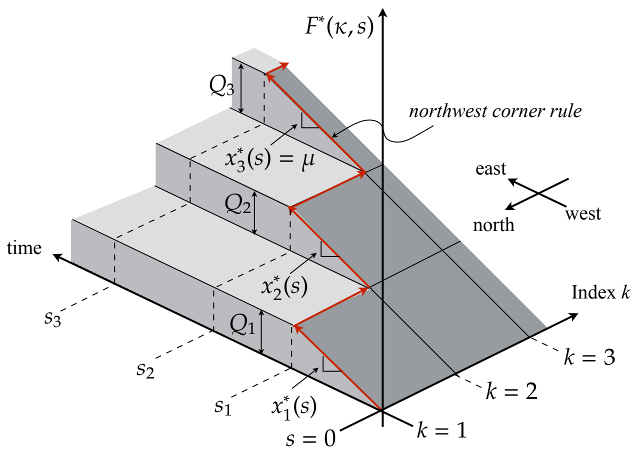

Figure 2: Illustration of the solution to the problem (3.13)

the *strict Monge property*.

$$\min_{\mathbf{x}\geq\mathbf{0}}.\ Z(\mathbf{x}) \equiv \sum_{k\in\mathcal{K}} \int_{s\in\hat{\mathcal{S}}} c_k(s)x_k(s)\mathrm{d}s \tag{3.10}$$

$$\text{subject to} \quad \sum_{k\in\mathcal{K}} x_k(s) = \mu \qquad \forall s \in \hat{\mathcal{S}} \tag{3.11}$$

$$\int_{s\in\hat{\mathcal{S}}} x_k(s)\mathrm{d}s = Q_k \qquad \forall k \in \mathcal{K} \tag{3.12}$$

Its cumulative form is expressed as

$$\min_{F\in\mathcal{F}} \int_{[1,K]\times\hat{\mathcal{S}}} c(\kappa, s)\mathrm{d}F(\kappa, s) \tag{3.13}$$

$$\text{where} \quad \mathcal{F} \equiv \left\{F(\kappa, T) = \textstyle\sum_{k'\leq\kappa} Q_{k'},\ F(K, s) = \mu s,\ \forall\kappa \in [1, K],\ \forall s \in \hat{\mathcal{S}}\right\} \tag{3.14}$$

Note that $F(\kappa, y) \equiv \sum_{k'\leq\kappa} \int_0^s x_{k'}(s)\mathrm{d}s$, which is a step function of κ for a given s. Without loss of generality, $c(\kappa, s)$ is assumed to be a continuous and strict submodular function that satisfies $c(k, s) = c_k(s)$, $\forall k = 1, \ldots, K$, $\forall s \in \hat{\mathcal{S}}$.

According to Theorem 3.3, we have the following proposition:

Proposition 3.1. *Suppose that Assumption 3.1 holds, and user groups are arranged (indexed) in an increasing order of their desired arrival times: $\sigma_1 < \cdots < \sigma_K$. Then, the solution of [2D-LP($\mathbf{x}$)] for the equilibrium rush period $\hat{\mathcal{S}}$ is unique and its cumulative form is given by*

$$F^*(\kappa, s) = \min\left\{\sum_{k'\leq\kappa} Q_{k'}, \mu s\right\} \qquad \forall\kappa \in [1, K],\ \forall s \in \hat{\mathcal{S}} \tag{3.15}$$

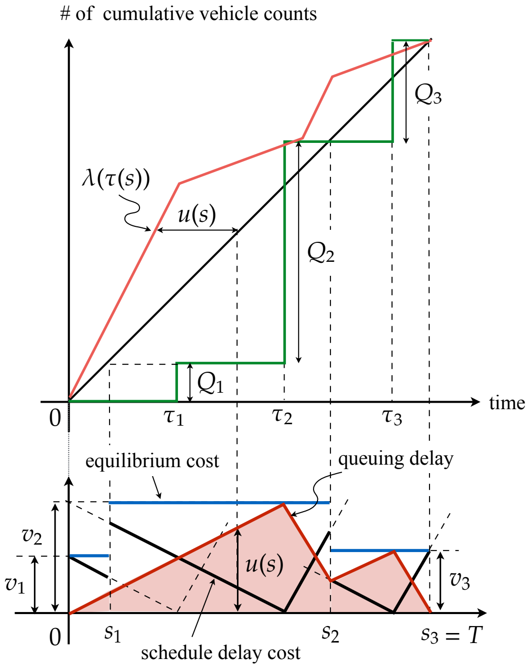

Figure 3: Optimal departure flow pattern, cost pattern, and resulting arrival flow pattern ($K = 3$)

Using this analytical solution (3.15), we can show the regularity of the equilibrium flow pattern. Let

$$s_k \equiv \sum_{k' \leq k} Q_{k'}/\mu \tag{3.16}$$

be the time when the cumulative supply μs is equal to the cumulative demand $\sum_{k' \leq k} Q_{k'}$. Then, the following equation holds (see also Figure 2):

$$F^*(k, s_k) - F^*(k-1, s_{k-1}) = \int_{s_{k-1}}^{s_k} x_k^*(s)\mathrm{d}s = \mu(s_k - s_{k-1}) = Q_k \quad \forall k = 1, 2, \ldots, K. \tag{3.17}$$

where $F(0, s_0) \equiv 0$ and $s_0 = 0$. This equation leads to the following proposition, i.e., the FIFW principle.

Proposition 3.2. *Suppose that the assumptions in Proposition 3.1 hold. Then, the equilibrium flow pattern* $\mathbf{x}^*$ *has the "sorting" property, such that all users in group* k *depart from the bottleneck in a time interval* $[s_{k-1}, s_k]$ *of length* Q_k/μ *and*

$$s_0 \equiv 0 < s_1 < s_2 < \cdots < s_{K-1} < s_K = T. \tag{3.18}$$

The above "sorting" property also implies that the equilibrium cost pattern $\mathbf{v}^*$ and the associated queuing delays $\mathbf{u}^*$ can be uniquely determined. Specifically, the following users' optimal choice condition within each group must hold.

$$v_k^* = u^*(s) + c_k(s) \qquad \forall s \in [s_{k-1}, s_k],\ \forall k \in \mathcal{K}. \tag{3.19}$$

Then we have

$$v_k^* := v_{k+1}^* - c_{k+1}(s_k) + c_k(s_k) \qquad \forall k = 1, \dots, K-1. \tag{3.20}$$

The recursive equation (3.20) together with a boundary condition (e.g., the queuing delay for the last user in group K is zero, $v_K^* = c_K(T)$), can be solved easily. Ultimately, the queuing delay $\mathbf{u}$ can be determined by using Eq.(3.19). In addition, the cumulative arrival curve can be constructed by the optimal solution $(\mathbf{x}^*, \mathbf{u}^*, \mathbf{v}^*)$ above. Figure 3 shows the relationships between the optimal departure flow pattern, the cost pattern, and the resulting arrival flow pattern for the simplest case (i.e., linear schedule delay cost function).

4. Analysis of model with heterogeneous value of time

In this section, we consider the departure time choice equilibrium models in which users have the same preferred departure time σ but are classified into K groups (types) differentiated by schedule delay cost functions. The schedule delay cost for type k users with departure time s from the bottleneck is denoted by $c_k(\epsilon)$, where $\epsilon \equiv s - \sigma$ is the schedule delay. The function $c_k(\epsilon)$ in this section is assumed to have the following form:

$$c_k(s) = \begin{cases} \beta_k f^e(\epsilon) & \text{if} \quad \epsilon \leq 0 \quad \text{(early arrival)} \\ \gamma_k f^l(\epsilon) & \text{if} \quad \epsilon \geq 0 \quad \text{(late arrival)} \end{cases}, \tag{4.1a}$$

$$f^e(0) = f^l(0) = 0 \tag{4.1b}$$

which includes the schedule delay cost functions assumed in conventional models with heterogeneous users (e.g., Cohen, 1987; Arnott et al., 1988, 1992, 1994; van den Berg and Verhoef, 2011; Ramadurai et al., 2010; Liu et al., 2015; Takayama and Kuwahara, 2017) as special cases. For notational simplicity, the preferred departure time σ will be set to zero in the following (i.e., $\epsilon = s$).

4.1. Properties of models with no late arrivals

We first restrict ourselves to the analysis of models with the following assumption:

Assumption 4.1. *$f^e : \mathbb{R} \to \mathbb{R}$ is a continuous decreasing function of ϵ for all $\epsilon \leq 0$ and $f^e(0) = 0$, $f^l(\epsilon) \to +\infty$ for all $\epsilon > 0$ (i.e., late arrival is prohibited).*

Lemma 4.1. *Suppose that Assumption 4.1 holds, and that user groups are arranged (indexed) in a decreasing order of their value of time parameters for early arrivals: $\beta_1 > \beta_2 > \cdots > \beta_K > 0$. Then the function $c_k(s)$ defined by (4.1) satisfies the strict "inverse" Monge property for $s \leq 0$.*

Proof. See Appendix C.2.

As we have seen in Section 2, the departure time choice equilibrium can be obtained by solving the linear programming problem [2D-LP($\mathbf{x}$)], whose objective is to minimize the total schedule delay cost. Since all users have the same preferred departure time σ, the departure times that are closer to σ are chosen at equilibrium. Thus, we obtain the following apparent property of an equilibrium departure time window.

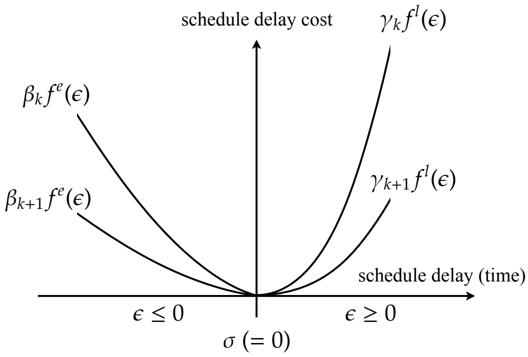

Figure 4: Schedule delay ϵ, heterogeneities k, and schedule delay cost

Lemma 4.2. *Under Assumption 4.1, the optimal solution* $\mathbf{x}^*$ *of [2D-LP(*$\mathbf{x}$*)] satisfies*

$$\begin{cases} \sum_{k\in\mathcal{K}} x_k^*(s) = \mu & \text{if} \quad s \in \hat{\mathcal{S}} \subset \mathcal{S} \\ \sum_{k\in\mathcal{K}} x_k^*(s) = 0 & \text{otherwise} \end{cases} \tag{4.2}$$

where $\hat{\mathcal{S}} \equiv [-T, 0]$ *of length* $T = Q/\mu$.

These lemmas ensure that [2D-LP($\mathbf{x}$)] reduces to an optimal transport problem with a strict *inverse* Monge property. However, to use Theorem 3.3 directly, it is convenient to formulate the optimal transport problem as its cumulative form with a strict Monge property, as in Subsection 3.2. To do this, let us reverse the time direction: $z \equiv -s$; let $c(\kappa, z)$ and $F(\kappa, z) = \sum_{k'\leq\kappa} \int_0^z x_{k'}(z)\mathrm{d}z$ be the cost and distribution functions for the time window $\mathcal{Z} = [0, T] \ni z$, respectively. Without loss of generality, we also assume that $c(\kappa, z)$ is a continuous and strict submodular function that satisfies $c(k, z) = c_k(z)$, $\forall k = 1, \ldots, K$, $\forall z \in \mathcal{Z}$. We then have

$$\min_{F\in\mathcal{F}} \int_{[1,K]\times\mathcal{Z}} c(\kappa, z)\mathrm{d}F(\kappa, z) \tag{4.3}$$

$$\text{where} \quad \mathcal{F} \equiv \{F(\kappa, T) = \textstyle\sum_{k'\leq\kappa} Q_{k'},\ F(K, t) = \mu z,\ \forall\kappa \in [1, K],\ \forall z \in \mathcal{Z}\}. \tag{4.4}$$

We further define the time s_k when the cumulative supply μz is equal to the cumulative demand $\sum_{k'\leq k} Q_{k'}$, i.e.,

$$s_k \equiv \sum_{k'\leq k} Q_{k'}/\mu. \tag{4.5}$$

By applying Theorem 3.3 to this problem, the following proposition is obtained.

Proposition 4.1. *Suppose that Assumption 4.1 holds. Then, the flow pattern of the departure time choice equilibrium with schedule delay cost function (4.1) has the following properties:*

(1) The optimal solution of [2D-LP($\mathbf{x}$*)] (i.e., equilibrium flow pattern* $\mathbf{x}^*$*) is unique and given by the Fréchet–Hoeffding distribution.*

$$F^*(\kappa, z) = \min\left\{\sum_{k'\leq\kappa} Q_{k'}, \mu z\right\} \qquad \forall\kappa \in [1, K],\ \forall z \in \mathcal{Z} \tag{4.6}$$

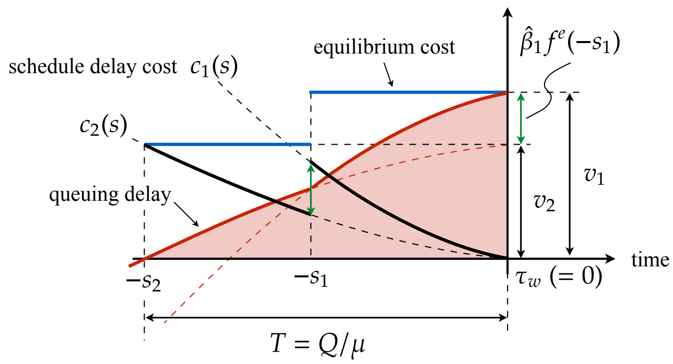

Figure 5: Graphical representation of the strong duality for [2D-LP($\mathbf{x}$)] ($K = 2$, late arrival is prohibited)

(2) The equilibrium flow pattern $\mathbf{x}^*$ *has the "sorting" property, such that all users in group* k *depart from the bottleneck in a time interval* $[-s_k, -s_{k-1}]$ *of length* Q_k/μ *(i.e.,* $s_k = s_{k-1} + Q_k/\mu$*), and*

$$-T = -s_K < -s_{K-1} < \cdots < -s_2 < -s_1 < -s_0 \equiv 0. \tag{4.7}$$

As in the case of Subsection 3.2, the equilibrium cost pattern $\mathbf{v}^*$ and the associated queuing delays $\mathbf{u}^*$ can be uniquely determined (see Figure 5).

Proposition 4.2. *Suppose that Assumption 4.1 holds. Then, the cost pattern of the departure time choice equilibrium with schedule delay cost function (4.1) is unique and is obtained as follows.*

$$v_k^* = \sum_{k' \geq k} \left\{ \hat{\beta}_{k'} f^e(-s_{k'}) \right\} \qquad \forall k \in \mathcal{K} \tag{4.8}$$

$$u^*(s) = \begin{cases} v_k^* - \beta_k f^e(s) & \text{if } s \in [-s_k, -s_{k-1}] \\ 0 & \text{otherwise} \end{cases} \qquad \forall k \in \mathcal{K} \tag{4.9}$$

where $\hat{\beta}_k \equiv \beta_k - \beta_{k+1} > 0$ *and* $\beta_{K+1} \equiv 0$.

Proof. See Appendix D.1.

Figure 5 shows the relationship between costs at equilibrium. Specifically, we can see that the following equation holds.

$$\sum_{k \in \mathcal{K}} (s_k - s_{k-1}) v_k^* - \int_{\mathcal{S}} u^*(s) \mathrm{d}s = \sum_{k \in \mathcal{K}} \int_{-s_k}^{-s_{k-1}} c_k(s) \mathrm{d}s \tag{4.10}$$

The first term of the LHS of Eq.(4.10) shows the total area of the rectangles below the equilibrium trip cost lines (blue bold lines) and the second term shows the red area below the queuing delay curve (red bold line)[8]; the equality holds in equilibrium because the difference in the above-mentioned two areas should be equal to the total area below the schedule delay cost curves

[8] The red dotted lines in Figure 5 are the so-called *isocost queueing curves* in the literature (Lindsey, 2004).

(black bold lines). Furthermore, by recalling that $(s_k - s_{k-1}) = Q_k/\mu$ and multiplying both sides of Eq.(4.10) by μ, we can see the strong duality for [2D-LP($\mathbf{x}$)].

$$Z(\mathbf{x}^*) = \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} c_k(s)x_k^*(s)\mathrm{d}s = \mu \sum_{k\in\mathcal{K}} \int_{-s_k}^{-s_{k-1}} c_k(s)\mathrm{d}s$$
$$= \sum_{k\in\mathcal{K}} Q_k v_k^* - \mu \int_{\mathcal{S}} u^*(s)\mathrm{d}s = \check{Z}(\mathbf{u}^*, \mathbf{v}^*). \tag{4.11}$$

4.2. Properties of models with no early arrivals

In exactly the same manner as we have discussed above, we can consider the "reverse" case, in which the following assumption holds:

Assumption 4.2. *$f^l : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function of ϵ for all $\epsilon \geq 0$ and $f^l(0) = 0$, $f^e \to +\infty$ for all $\epsilon < 0$ (i.e., early arrival is prohibited).*

Although this assumption (no early arrivals) may seem strange, the results for this case together with the previous (no late arrivals) case can be used as fundamental building blocks for analyzing more general cases in Subsection 4.3. As in the case of the previous subsection, the following lemma and propositions hold.

Lemma 4.3. *Suppose that Assumption 4.2 holds and that user groups are arranged (indexed) in decreasing order of their value of time parameters for early arrivals: $\gamma_1 > \gamma_2 > \cdots > \gamma_K > 0$. Then, the function $c_k(s)$ defined by (4.1) satisfies the strict Monge property.*

Proposition 4.3. *Suppose that Assumption 4.2 holds. Then the flow pattern of the departure time choice equilibrium with schedule delay cost function (4.1) has the following properties:*

(1) The optimal solution of [2D-LP($\mathbf{x}$)] (i.e., equilibrium flow pattern $\mathbf{x}^*$) is unique and given by the Fréchet–Hoeffding distribution.

$$F^*(\kappa, s) = \min\left\{\sum_{k'\leq\kappa} Q_{k'}, \mu s\right\} \qquad \forall\kappa \in [1, K],\ \forall s \in \hat{\mathcal{S}} \tag{4.12}$$

where $\hat{\mathcal{S}} \equiv [0, T]$ of length $T = Q/\mu$.

(2) The equilibrium flow pattern $\mathbf{x}^*$ has the "sorting" property, such that all users in group k depart from the bottleneck in a time interval $[s_{k-1}, s_k]$ of length Q_k/μ (i.e., $s_k = s_{k-1} + Q_k/\mu$), and

$$s_0 \equiv 0 < s_1 < s_2 < \cdots < s_{K-1} < s_K = T \tag{4.13}$$

Proposition 4.4. *Suppose that Assumption 4.2 holds. Then, the cost pattern of the departure time choice equilibrium with schedule delay cost function (4.1) is unique and is obtained as follows.*

$$v_k^* = \sum_{k'\geq k} \left\{\hat{\gamma}_{k'} f^l(s_{k'})\right\} \qquad \forall k \in \mathcal{K} \tag{4.14}$$

$$u^*(s) = \begin{cases} v_k^* - \gamma_k f^l(s) & \text{if } s \in [s_{k-1}, s_k] \\ 0 & \text{otherwise} \end{cases} \qquad \forall k \in \mathcal{K} \tag{4.15}$$

where $\hat{\gamma}_k \equiv \gamma_k - \gamma_{k+1} > 0$ and $\gamma_{K+1} \equiv 0$.

4.3. Properties of model with early and late arrivals

We are now in a position to consider a general case in which Assumptions 4.1 and 4.2 are relaxed to allow for both of the late and early arrivals.

Assumption 4.3. *$f^e : \mathbb{R} \to \mathbb{R}$ is a continuous decreasing function of ϵ for all $\epsilon \leq 0$ and $f^e(0) = 0$, $f^l : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function of ϵ for all $\epsilon \geq 0$ and $f^l(0) = 0$.*

Note that this assumption implies that the schedule delay cost function is strictly quasiconvex (or unimodal), which is weaker than Assumption 3.1, under which it is required to be strictly convex.

Let $\mathcal{H} = \{e, l\}$ be the set of indices that indicate either early arrivals (e) or late arrivals (l); X^h_k the total mass of early (if $h = e$) or late (if $h = l$) arrival users in group k; $\mathcal{S}^h$ the time duration $s \leq \sigma$ if $h = e$, $s > \sigma$ if $h = l$. Under Assumption 4.3, the problem [2D-LP($\mathbf{x}$)] can be represented as

$$[\text{2D-LP}(\mathbf{x}, \mathbf{X})] \quad \min_{\mathbf{x}\geq\mathbf{0}, \mathbf{X}\geq\mathbf{0}} . \; Z(\mathbf{x}, \mathbf{X}) \equiv \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} c_k(s)\, x_k(s) \mathrm{d}s \tag{4.16}$$

$$\text{subject to} \quad \sum_{k\in\mathcal{K}} x_k(s) \leq \mu \qquad \forall s \in \mathcal{S} \tag{4.17}$$

$$\int_{\mathcal{S}^h} x_k(s)\mathrm{d}s = X^h_k \qquad \forall k \in \mathcal{K},\ h \in \mathcal{H} \tag{4.18}$$

$$\sum_{h\in\mathcal{H}} X^h_k = Q_k \qquad \forall k \in \mathcal{K} \tag{4.19}$$

Note that the equivalence between [2D-LP($\mathbf{x}$)] and [2D-LP($\mathbf{x}$, $\mathbf{X}$)] is easily confirmed by substituting Eq.(4.18) into Eq.(4.19).

The properties of the problem [2D-LP($\mathbf{x}$, $\mathbf{X}$)] can be analyzed using a decomposition. Specifically, we first transform the problem [2D-LP($\mathbf{x}$, $\mathbf{X}$)] to the following equivalent bi-level program.

$$\min_{\mathbf{X}\geq\mathbf{0}} . \sum_{h\in\mathcal{H}} Z_M(\mathbf{X}^h) \tag{4.20}$$

subject to Eq. (4.19)

$$\text{where} \quad Z_M(\mathbf{X}^h) \equiv \min_{\mathbf{x}\geq\mathbf{0}} . \; Z_S(\mathbf{x} \mid \mathbf{X}^h) \equiv \sum_{k\in\mathcal{K}} \int_{\mathcal{S}^h} c_k(s)\, x_k(s)\mathrm{d}s \tag{4.21}$$

$$\text{subject to} \quad \text{Eqs. (4.17)} \ \forall s \in \mathcal{S}^h \text{ and (4.18).}$$

Since the lower-level (or sub-) problems (for $h = e$ and $h = l$, respectively) are exactly the same problems discussed in the previous subsections, the optimal solution $\mathbf{x}^*$ (for a given $\mathbf{X}^h$) and the associated equilibrium cost (i.e., the optimal Lagrange multiplier) $(\mathbf{u}^*, \mathbf{v}^{h*})$ can be obtained analytically (see Propositions 4.1–4.4). These solutions are useful in the following analysis.

Owing to the strong duality of the subproblems (i.e., Eq.(4.11)), the upper-level (or master) problem reduces to the following problem.

$$\begin{aligned}
\min_{\mathbf{X}\geq\mathbf{0}}.\ \sum_{h\in\mathcal{H}} Z_M(\mathbf{X}^h) &= \sum_{h\in\mathcal{H}} \check{Z}(\mathbf{u}^*, \mathbf{v}^{h*} \mid \mathbf{X}^h) \\
&= \sum_{h\in\mathcal{H}} \left[-\mu \int_{\mathcal{S}^h} u^*(s)\mathrm{d}s + \sum_{k\in\mathcal{K}} X_k^h v_k^{h*} \right] \qquad (4.22)
\end{aligned}$$

subject to Eq. (4.19)

In general, an iterative procedure is required to solve the master problem because the dual solution $(\mathbf{u}^*, \mathbf{v}^{h*})$ is not expressed as an explicit function of the variable $\mathbf{X}^h$. However, thanks to the analytical solution obtained in the previous subsections, we can explicitly transform it to a problem including the decision variable $\mathbf{X}^h$.

Recall that the one-to-one correspondence between the demand distribution $\mathbf{X}^h$ and the equilibrium departure time vector $\mathbf{s}^h$ (Propositions 4.1 and 4.3), i.e.,

$$s_0^h = 0,\ s_1^h = X_1^h/\mu,\ \ldots,\ s_k^h = \sum_{k'\leq k} X_{k'}^h/\mu,\ \ldots,\ s_K^h = T^h \equiv \sum_{k'\leq K} X_{k'}^h/\mu. \qquad (4.23)$$

By using these variables and the relationship (4.11), the master problem can be transformed to the following problem.

$$\min_{\mathbf{s}^h}.\ \sum_{h\in\mathcal{H}} Z_M(\mathbf{s}^h) = \mu \sum_{k\in\mathcal{K}} \int_{-s_k^e}^{-s_{k-1}^e} c_k(s)\mathrm{d}s + \mu \sum_{k\in\mathcal{K}} \int_{s_{k-1}^l}^{s_k^l} c_k(s)\mathrm{d}s \qquad (4.24)$$

$$\text{subject to}\quad 0 \leq s_1^h \leq \cdots \leq s_k^h \leq \cdots \leq s_K^h \qquad \forall h \in \mathcal{H} \qquad (4.25)$$

$$s_k^e + s_k^l = \sum_{k\leq k'} Q_{k'}/\mu \qquad \forall k \in \mathcal{K} \qquad (4.26)$$

As shown in Appendix D.2, this problem is a convex programming problem with a strict convex objective function. We thus obtain the following lemma.

Lemma 4.4. *Suppose that Assumption 4.3 holds. The optimal solution of the master problem is unique.*

Proof. See Appendix D.2.

We finally obtain the following proposition.

Proposition 4.5. *Suppose that Assumption 4.3 holds. Then the flow pattern of the departure time choice equilibrium with schedule cost function (4.1) has the following properties:*

1. *The optimal solution of [2D-LP($\mathbf{x}$)] (i.e., equilibrium flow pattern $\mathbf{x}^*$) is unique.*
2. *The equilibrium flow pattern $\mathbf{x}^*$ has the "sorting" property, such that all users in group k depart from the bottleneck in time intervals $[s_{k-1}^h, s_k^h]$, $\forall h \in \mathcal{H}$, and*

$$-T^e = -s_K^e \leq \cdots \leq -s_1^e \leq 0 \leq s_1^l \leq \cdots \leq s_K^l = T^l. \qquad (4.27)$$

3. *Users in all groups depart from the bottleneck both early and late (i.e., strict inequality holds in condition (4.27)), the equilibrium values of s^e_k and s^l_k are determined by the following equations:*

$$\begin{cases} s^l_k + s^e_k = \sum_{k' \leq k} Q_{k'}/\mu \\ \hat{\beta}_k f^e(-s^e_k) = \hat{\gamma}_k f^l(s^l_k) \end{cases} \qquad \forall k \in \mathcal{K} \tag{4.28}$$

Proof. See Appendix D.3.

Note that the property 3 always holds, for example, if $\gamma_k/\beta_k = \eta$ (constant) for all $k \in \mathcal{K}$ that is the common assumption in the literature (Vickrey, 1973; Arnott et al., 1988, 1992, 1994; van den Berg and Verhoef, 2011; Takayama and Kuwahara, 2017) (See Appendix D.3).

4.4. Relation to an existing semi-analytical approach

Before concluding this section, it is worthwhile mentioning a semi-analytical approach proposed by Liu et al. (2015). Their approach uses the analytical solution of the model with a piece-wise linear schedule delay function (Arnott et al., 1994) to obtain an equivalent variational inequality (VI) problem that allows closed-form trip cost functions (or mappings). Then, the solution existence and uniqueness were examined through the VI problem. In the present framework, their equivalent VI corresponds to the master problem (4.22) or (4.24).

To highlight its similarities and differences with respect to the present approach concretely, let us consider the following VI, which is given as the optimality condition of the master problem (4.22):

Find $\mathbf{X}^* \in \mathcal{X} \equiv \{\mathbf{X} \geq \mathbf{0}$ and Eq.(4.19)$\}$ such that

$$\sum_{h \in \mathcal{H}} \mathbf{v}^{h*}(\mathbf{X}^{h*}) \cdot (\mathbf{X}^h - \mathbf{X}^{h*}) \geq 0 \qquad \mathbf{X} \in \mathcal{X}, \tag{4.29}$$

where the mapping $\mathbf{v}^{h*}(\mathbf{X}^{h*})$ is obtained by applying the envelope theorem to the dual subproblems, that is, the equilibrium trip cost function: $\nabla Z_M(\mathbf{X}^h) = \nabla Z_S(\mathbf{u}^*, \mathbf{v}^{h*} \mid \mathbf{X}^h) = \mathbf{v}^{h*}(\mathbf{X}^h)$, $\forall h \in \mathcal{H}$. This is almost the same as the VI in Liu et al. (2015), except for the measurement unit of trip cost, i.e., the problem (4.29) is *time*-based but that in Liu et al. (2015) is *monetary*-based. Although the monetary- and time-based formulations can be transformed in to each other by changing the scale of schedule penalty coefficients β_k and γ_k for the case of Liu et al. (2015), the mathematical properties of the VIs are quite different[9].

More specifically, the problem (4.29) has desirable properties (see Appendix D.4 for the proofs). The mapping of (4.29) is strictly monotone; if the schedule delay function is differentiable, the mapping of (4.29) is symmetric and thus a scalar potential exists for the vector field $\mathbf{v}^{h*}(\mathbf{X}^{h*})$, i.e.,

$$\sum_{h \in \mathcal{H}} \oint_{\mathbf{0} \to \mathbf{X}^h} \mathbf{v}^{h*}(\boldsymbol{\omega}) \mathrm{d}\boldsymbol{\omega} \left(= \sum_{h \in \mathcal{H}} Z_M(\mathbf{X}^h) \right). \tag{4.30}$$

Therefore, it is easy to obtain the uniqueness of the solutions. On the other hand, the mapping of Liu et al. (2015)'s VI is not monotone; it is hard to establish the uniqueness property of the VI. It

[9] A similar conclusion has been observed in (multiclass) static traffic assignment problems (e.g., Larsson et al., 2002).

should be noted that the proof of the uniqueness property in Liu et al. (2015) is incorrect because the notion of a P-function is applicable to the uniqueness property for special VI problems (i.e., nonlinear complementarity problems) only, and not for general problems such as (4.29) (for more detailed discussions see Harker and Pang, 1990; Facchinei and Pang, 2003). Further note that the monetary-based VI is not integrable (i.e., does not reduce to an equivalent LP problem like [2D-LP($\mathbf{x}$)]); thus, the theory of optimal transport and efficient solution methods of LP are not applicable to it.

In sum, the present LP approach is a more powerful and promising way to address more general problems. As an example, we will analyze an extended problem with more heterogeneities in the next section.

5. Analysis of models with two types of cost heterogeneity

This section demonstrates that the proposed approach is applicable to a case with more general users' heterogeneity. Specifically, we here consider a generalized departure time choice equilibrium model with two types of cost heterogeneities. We then characterize an equilibrium user sorting pattern and obtain an analytical solution by employing both the three-dimensional Monge property and a decomposition of the LP formulation of the model.

5.1. Extension to N-dimensional problem

Before introducing the generalized problem, we briefly show that the Monge property and related useful theorems can be extended to N-dimensional transportation problems. For example, the three-dimensional transportation problem in a discrete space setting is formulated as follows.

$$\text{[3D-OTP]} \qquad \min_{\mathbf{x}\geq\mathbf{0}} . \ Z_{3D}(\mathbf{x}) \equiv \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} c_{i,j,k} x_{i,j,k} \tag{5.1}$$

$$\text{subject to} \quad \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} x_{i,j,k} = S_i \qquad \forall i \in \mathcal{I} \tag{5.2}$$

$$\sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} x_{i,j,k} = R_j \qquad \forall j \in \mathcal{J} \tag{5.3}$$

$$\sum_{j\in\mathcal{J}} \sum_{i\in\mathcal{I}} x_{i,j,k} = Q_k \qquad \forall k \in \mathcal{K} \tag{5.4}$$

where $\mathbf{S}, \mathbf{R}$ and $\mathbf{Q}$ are given constants satisfying $\sum_{i\in\mathcal{I}} S_i = \sum_{j\in\mathcal{J}} R_j = \sum_{k\in\mathcal{K}} Q_k$.

The Monge property for higher-dimensional arrays is as follows.

Definition. Let C be a N-dimensional array of size $m_1 \times m_2 \times \cdots \times m_N$. C is termed a *Monge array*, if for all $i_n = 1, 2, \ldots, m_n$, $j_n = 1, 2, \ldots, m_n$, $n = 1, 2, \ldots, N$,

$$c[s_1, s_2, \ldots, s_N] + c[t_1, t_2, \ldots, t_N] \leq c[i_1, i_2, \ldots, i_N] + c[j_1, j_2, \ldots, j_N] \tag{5.5}$$

where $s_n \equiv \min(i_n, j_n)$, $t_n \equiv \max(i_n, j_n)$ for all $1 \leq n \leq N$.

Note that, as shown in Aggarwal and Park (1988), a N-dimensional array C is a Monge array, if and only if, every two-dimensional submatrix is a Monge matrix. Furthermore, the natural extension of the northwest corner rule solves the problem in a greedy way without explicit values of cost coefficients if C is a Monge array (Bein et al., 1995)[10].

As in the two-dimensional case, the continuous Monge property is characterized by a submodular function.

Definition. A function $f : \mathbb{R}^N \to \mathbb{R}$ is said to satisfy the N-dimensional continuous Monge property, if f is submodular with respect to any two of its arguments:

$$f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y}) \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^N \tag{5.6}$$

where $\mathbf{x} \wedge \mathbf{y}$ and $\mathbf{x} \vee \mathbf{y}$ denote the componentwise maximum/minimum of $\mathbf{x}$ and $\mathbf{y}$, respectively:

$$\mathbf{x} \wedge \mathbf{y} \equiv (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_N, y_N\}) \text{ for all } 1 \leq n \leq N$$
$$\mathbf{x} \vee \mathbf{y} \equiv (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_N, y_N\}) \text{ for all } 1 \leq n \leq N.$$

Then, Tchen (1980) showed that Theorem 3.3 can be generalized for an N-dimensional case. That is, the Fréchet–Hoeffding distribution:

$$F^{*}(x_1, \ldots, x_N) = \min\{F_1(x_1), \ldots, F_N(x_N)\} \qquad \forall (x_1, \ldots, x_N) \in \mathbb{R}^N \tag{5.7}$$

is an optimal solution for the following problem:

$$\min_{F \in \mathcal{F}(F_1, \ldots, F_N)} \int_{\mathbb{R}^N} c(x_1, \ldots, x_N) \mathrm{d}F(x_1, \ldots, x_N) \tag{5.8}$$

if a function $c : \mathbb{R}^N \to \mathbb{R}$ satisfies the continuous Monge property and the objective function is finite.

5.2. *Simultaneous choice equilibrium model of departure time, location and job choices*

We show a simultaneous choice equilibrium model of departure time, location, and job choices as a generalized departure time choice equilibrium model with two types of cost heterogeneity. Consider a long narrow city with a spaceless Central Business District (CBD) where all job opportunities are located. The CBD is located at the edge of the city, and J discrete residential locations are indexed sequentially from the CBD in decreasing order: $J, \ldots, 1$ (see Figure 6). We denote the set of locations using $\mathcal{J} \equiv \{1, \ldots, J\}$. The free flow travel time between location j and the CBD is l_j, and the maximum allowable number of users who live in each location $j \in \mathcal{J}$ is R_j. There are K types of jobs at the CBD, and the labor demand of each job $k \in \mathcal{K} \equiv \{1, \ldots, K\}$ is given by Q_k. The road has a single bottleneck with capacity μ just upstream of the CBD; thus, all users must pass through this bottleneck to commute.

There are Q ex-ante identical users in the city. Each user makes departure time, residential location, and job choices to maximize his/her utility. We assume that a user's preference is quasilinear (i.e., the utility function is linear in *queuing time units*) and let w_k, r_j and $u(s)$ be the wage of job k, land rent at location j and queuing delay at time s, respectively. Then, the utility is defined as follows:

$$w_k - r_j - (c_{j,k}(s) + u(s)) \tag{5.9}$$

[10] Unlike the two-dimensional case, the problem with a *inverse* Monge array cannot be solved in a greedy way.

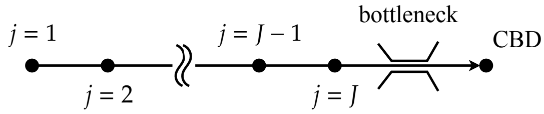

Figure 6: Spatial setting inside city

where $c_{j,k}(s)$ is a "commuting disutility" (a generalization of schedule delay cost) with two types of heterogeneity j and k, which will be specified later. Note that all terms of Equation (5.9) are measured in queuing time units.

By combining a user's optimal choice condition and the equilibrium condition in each market, the equilibrium condition is given as follows:

$$\begin{cases} V = w_k - r_j - u(s) - c_{j,k}(s) & \text{if} \quad x_{j,k}(s) > 0 \\ V \geq w_k - r_j - u(s) - c_{j,k}(s) & \text{if} \quad x_{j,k}(s) = 0 \end{cases} \qquad \forall j \in \mathcal{J},\ \forall k \in \mathcal{K},\ \forall s \in \mathcal{S} \tag{5.10}$$

$$\begin{cases} \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} x_{j,k}(s) = \mu & \text{if} \quad u(s) > 0 \\ \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} x_{j,k}(s) \leq \mu & \text{if} \quad u(s) = 0 \end{cases} \qquad \forall s \in \mathcal{S} \tag{5.11}$$

$$\begin{cases} \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} x_{j,k}(s)\mathrm{d}s = R_j & \text{if} \quad r_j > 0 \\ \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} x_{j,k}(s)\mathrm{d}s \leq R_j & \text{if} \quad r_j = 0 \end{cases} \qquad \forall j \in \mathcal{J} \tag{5.12}$$

$$\begin{cases} \sum_{j\in\mathcal{J}} \int_{\mathcal{S}} x_{j,k}(s)\mathrm{d}s = Q_k & \text{if} \quad w_k > 0 \\ \sum_{j\in\mathcal{J}} \int_{\mathcal{S}} x_{j,k}(s)\mathrm{d}s \geq Q_k & \text{if} \quad w_k = 0 \end{cases} \qquad \forall k \in \mathcal{K} \tag{5.13}$$

$$\sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} x_{j,k}(s)\mathrm{d}s = Q \tag{5.14}$$

where V represents the equilibrium utility level. The first condition (5.10) is the user's optimal choice condition, the second one (5.11) is the queuing delay condition, and the third (5.12) and fourth (5.13) are the market clearing (or demand-supply equilibrium) conditions for land use and job markets, respectively.

The model is a variant of the departure time choice equilibrium models that include the location choice of users, e.g., Arnott (1998); Gubins and Verhoef (2014); Takayama and Kuwahara (2017). To keep the model compact, we here assume that the wage w_k is dependent only on demand-supply conditions (5.13) and abstract from the effects from the productivity of firms or other realistic determinants of the wage. The job-location choice modeling in our formulation corresponds to the celebrated Herbert–Stevens model (Herbert and Stevens, 1960; Wheaton, 1974; Berliant and Tabuchi, 2018) in the urban economics literature. Note that it might be considered that the simultaneous equilibrium model of the different time scales choices is unrealistic, even though theie interactions of them are important factors in relation to appropriate long-term policies (see Osawa et al., in press, for an example). However, as shown in the next subsection, we can convert the problem to a bi-level problem where the upper-level (or *long-run*) problem is the job-location choice problem and the lower-level (or *short-term*) problem is the departure time choice problem; as such, the two problems can be analyzed *sequentially*.

In parallel with the discussion in the previous sections, this equilibrium model is equivalently written as the following LP:

$$[\text{3D-LP}(\mathbf{x})] \qquad \min_{\mathbf{x}\geq\mathbf{0}}.\ Z(\mathbf{x}) \equiv \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}}\int_{\mathcal{S}} c_{j,k}(s)x_{j,k}(s)\mathrm{d}s \tag{5.15}$$

$$\text{subject to} \quad \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}} x_{j,k}(s) \leq \mu \qquad \forall s \in \mathcal{S} \tag{5.16}$$

$$\sum_{k\in\mathcal{K}}\int_{\mathcal{S}} x_{j,k}(s)\mathrm{d}s \leq R_j \qquad \forall j \in \mathcal{J}, \tag{5.17}$$

$$\sum_{j\in\mathcal{J}}\int_{\mathcal{S}} x_{j,k}(s)\mathrm{d}s \geq Q_k \qquad \forall k \in \mathcal{K}. \tag{5.18}$$

with $\sum_{j\in\mathcal{J}} R_j = \sum_{k\in\mathcal{K}} Q_k = Q$, where the price variables $\mathbf{u}$, $\mathbf{r}$, and $\mathbf{w}$ are determined as the Lagrange multipliers for the constraints (5.16), (5.17) and (5.18), respectively.

We see that the equivalent optimization problem [3D-LP($\mathbf{x}$)] of the generalized departure time choice equilibrium has almost the same form as the three–dimensional transportation problem [3D-OTP]. Therefore, if the commuting disutility has the Monge property, the equilibrium sorting property would be established by the theory of optimal transport. In the next subsection, we will show an example of such disutility functions.

5.3. Analytical solution and sorting pattern

For the simplicity of exposition, we assume in the following that all users have a common preferred CBD arrival time, $\sigma = 0$, and "late arrival" is prohibited. We then consider the following commuting disutility function.

Assumption 5.1. *The commuting disutility for a type (j, k) user arriving at the destination at time s is given by*

$$c_{j,k}(s) = c^{(1)}_{j,k} + c^{(2)}_{k}(s) + c^{(3)}_{j,k}(s) \tag{5.19a}$$

$$\text{where} \quad c^{(1)}_{j,k} \equiv \alpha_k l_j,\ c^{(2)}_{k}(s) \equiv \beta_k f(s),\ c^{(3)}_{j,k}(s) \equiv \gamma_k g(s - l_j) \tag{5.19b}$$

where $f(s)$ is a continuous function of "schedule delay" $s\ (= s - \sigma)$ satisfying

$$\begin{cases} f(s) \geq 0 & \text{for} \quad s \leq 0 \\ f(s) \to \infty & \text{for} \quad s > 0 \end{cases}, \quad , f'(s) < 0 \quad \forall s \leq 0.$$

The function $g(t)$ denotes the "early bird cost"[11] *for a user departing from his/her home at time $t(s, l_j) \equiv s - l_j \leq -l_J$, which is a continuous function with the following properties:*

$$g(t(s, l_j)) > 0, \quad g'(t(s, l_j)) < 0, \quad g''(t(s, l_j)) < 0 \quad \forall t \leq -l_J$$

where the coefficients α_k, β_k, and γ_k are values of travel time, schedule delay and activity at home, respectively; they are arranged in an increasing order: $\alpha_1 < \cdots < \alpha_K$, $\beta_1 < \cdots < \beta_K$, $\gamma_1 < \cdots < \gamma_K$.

[11] This type of disutility has been considered for expressing the heterogeneity in the residential location in the city (Fosgerau and de Palma, 2012) and in the trip length within the urban network (Fosgerau, 2015).

The present model can be viewed as a departure time choice equilibrium model with two types of cost heterogeneity: the values of travel time, schedule delay, and activity at home depend on the job k[12], and the early bird cost depends on the residential location j. The main difference between the conventional and present models related to whether the heterogeneity are given exogenously or not. Specifically, in a conventional model (with heterogeneous preferred departure times and value of times) such as Newell (1987) and Lindsey (2004), a joint distribution of cost heterogeneities is given exogenously. In the present model, the marginal distributions of cost heterogeneities, $\{R_j\}$ and $\{Q_k\}$, are given exogenously but the joint distribution $\{X_{j,k}\}$ is determined as a result of job-location choice.

Lemma 5.1. *Suppose that Assumption 5.1 holds. Then the function $c_{j,k}(s)$ defined by (5.19) satisfies the strict 3D-Monge property.*

Proof. *See Appendix C.3.*

As in Lemma 4.2, combining the fact that [3D-LP($\mathbf{x}$)] is the total commuting disutility minimization problem with the functional form of $c_{j,k}(s)$ leads to the following property of an equilibrium departure time window.

Lemma 5.2. *Under Assumption 5.1, the optimal solution $\mathbf{x}^*$ of [3D-LP($\mathbf{x}$)] satisfies*

$$\begin{cases} \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}} x^*_{j,k}(s) = \mu & \text{if} \quad s \in \hat{\mathcal{S}} \subset \mathcal{S} \\ \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}} x^*_{j,k}(s) = 0 & \text{otherwise} \end{cases} \tag{5.20}$$

where $\hat{\mathcal{S}} \equiv [-T, 0]$ of length $T = Q/\mu$.

Then the problem [3D-LP($\mathbf{x}$)] reduces to a three-dimensional optimal transport problem with a strict 3D-Monge property. That is

$$\min_{F\in\mathcal{F}} \int_{[1,J]\times[1,K]\times\hat{\mathcal{S}}} c(\zeta,\kappa,s)\mathrm{d}F(\zeta,\kappa,s) \tag{5.21}$$

$$\text{where} \quad \mathcal{F} \equiv \Big\{F(\zeta,K,0) = \textstyle\sum_{j'\leq\xi} R_{j'},\ F(J,\kappa,0) = \sum_{k'\leq\kappa} Q_{k'},$$
$$F(J,K,s) = \mu(T+s),\ \forall\xi\in[1,J],\ \forall\kappa\in[1,K],\ \forall s\in\hat{\mathcal{S}}\Big\}. \tag{5.22}$$

where $c(\zeta,\kappa,s)$ is a continuous and submodular function that satisfies $c(j,k,s) = c_{j,k}(s)$, $\forall j = 1,\dots,J$, $\forall k = 1,\dots,K$, $\forall s \in \hat{\mathcal{S}}$.

Based on the theory of optimal transport in Section 5.1, we have the following proposition.

Proposition 5.1. *Suppose that the commuting disutility, $c_{j,k}(s)$, satisfies Assumption 5.1. Then, the solution of the DTCE problem (5.21) (or [3D-LP($\mathbf{x}$)]) is unique and given by the Fréchet–Hoeffding distribution.*

$$F^*(\zeta,\kappa,s) = \min\left\{\sum_{j'\leq\zeta} R_{j'}, \sum_{k'\leq\kappa} Q_{k'}, \mu(T+s)\right\}. \tag{5.23}$$

To interpret the analytical solution, let us introduce a new index $i \in \mathcal{I}$ and a joint distribution $\{X_i\}$ of $\{R_j\}$ and $\{Q_k\}$, which are determined by the northwest corner rule:

[12] This is consistent with the assumption in Takayama and Kuwahara (2017) of a positive correlation between the value of time of users and their wage.

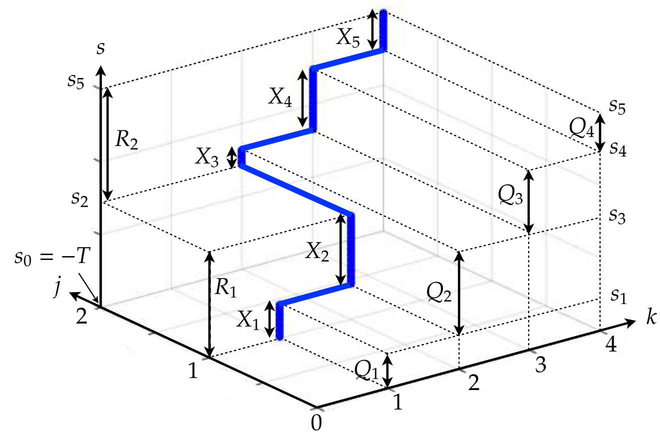

Figure 7: Optimal path in three–dimensional index space

0. Initialized the indices with $i := 1$, $j := 1$ and $k := 1$.
1. Set $X_i = \min\left\{R_j, Q_k\right\}$.
2. Reduce both R_j and Q_k by X_i: $R_j := R_j - X_i$ and $Q_k := Q_k - X_i$. If any of R_j and Q_k become zero, then these indices are increased by one.
3. If $\sum_{i' \leq i} X_{i'} = Q$ then $I = i$ and stop; otherwise $i := i + 1$ and go back to Step 2.

This procedure corresponds to solving the minimization problem of the first two terms of the analytical solution (5.23). By using the unique joint distribution, the analytical solution reduces to the following form.

$$F^*(\zeta, \kappa, s) = \min\left\{\sum_{i' \leq i} X_{i'}, \mu(T + s)\right\} \tag{5.24}$$

The joint distribution $\{X_i\}$ represents the demand distribution of the departure time choice problem. As in the previous sections, let us define the time s_i so that the cumulative capacity supply $\mu(T + s)$ is equal to the cumulative demand $\sum_{i' \leq i} X_{i'}$ as follows.

$$s_i \equiv \sum_{i' \leq i} X_{i'}/\mu - T \tag{5.25}$$

Then, the regularities of the equilibrium flow pattern can be summarized as the following proposition.

Proposition 5.2. *The equilibrium flow pattern* $\mathbf{x}^*$ *has the "sorting" property such that all users in group i depart from the bottleneck in a time interval* $[s_{i-1}, s_i]$ *of length* X_i/μ *(i.e.,* $s_i = s_{i-1} + X_i/\mu$*), and*

$$s_0 \equiv -T < s_1 < s_2 < \cdots < s_{I-1} < s_I = 0 \tag{5.26}$$

From the determination process of the joint distribution, we see that job and location groups are sorted in the increasing order of their indices, respectively. Hence, the proposition means that users who live in locations closer to the CBD and have jobs with higher VOTs depart from the bottleneck at times closer to the desired time. Figure 7 shows an example of the optimal index path of the problem with $J = 2, K = 4$ and $\mu = 1$. We see that users depart from the bottleneck in the following order along time s: $(j, k) = (1, 1), (1, 2), (2, 2), (2, 3), (2, 4)$.

Some final remarks about the analytical solution are in order. First, the interpretation of the analytical solution provides a clear relationship between short-run and long-run choice problems. It is easily seen that the determination of the joint distribution $\{X_i\}$ corresponds to the long-run job-location choice problem. For given this demand distribution, the problem (5.24) determines the short-run departure time choice problem. Second, the solution can be understood from the combination of the two 2D-Monge properties. To see this definitively, we define a new variable $X_{j,k}$ as follows.

$$X_{j,k} \equiv \int_{\hat{\mathcal{S}}} x_{j,k}(s)\mathrm{d}s. \tag{5.27}$$

Substituting the commuting disutility function (5.19) into the objective function of [3D-LP($\mathbf{x}$)], we then see that $Z(\mathbf{x})$ can be decomposed into the following two terms:

$$Z(\mathbf{x}) = Z_M(\mathbf{X}) + Z_S(\mathbf{x}) \tag{5.28}$$

where

$$Z_M(\mathbf{X}) \equiv \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}} c^{(1)}_{j,k} X_{j,k} \tag{5.29}$$

$$Z_S(\mathbf{x}) \equiv \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}} \int_{\hat{\mathcal{S}}} \left(c^{(2)}_k(s) + c^{(3)}_{j,k}(s)\right) x_{j,k}(s)\mathrm{d}s \tag{5.30}$$

This implies that [3D-LP($\mathbf{x}$)] allows the following decomposition:

$$\min_{\mathbf{X}\geq\mathbf{0}} .\ Z_M(\mathbf{X}) + Z^*_S(\mathbf{X}) \tag{5.31}$$

$$\text{subject to} \quad \sum_{k\in\mathcal{K}} X_{j,k} = R_j \qquad \forall j \in \mathcal{J} \tag{5.32}$$

$$\sum_{j\in\mathcal{J}} X_{j,k} = Q_k \qquad \forall k \in \mathcal{K} \tag{5.33}$$

$$\text{where} \quad Z^*_S(\mathbf{X}) \equiv \min_{\mathbf{x}\geq\mathbf{0}} .\ Z_S(\mathbf{x} \mid \mathbf{X}) \tag{5.34}$$

$$\text{subject to} \quad \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}} x_{j,k}(s) = \mu \qquad \forall s \in \hat{\mathcal{S}} \tag{5.35}$$

$$\int_{\hat{\mathcal{S}}} x_{j,k}(s)\mathrm{d}s = X_{j,k} \qquad \forall j \in \mathcal{J},\ \forall k \in \mathcal{K} \tag{5.36}$$

The subproblem (or lower-level problem) determines the short-run departure time choice equilibrium for a given joint distribution of $\{R_j\}$ and $\{Q_k\}$. The master (or upper-level) problem determines the long-run job-location choice equilibrium based on the cost including the short-run effect $Z^*_S(\mathbf{X})$. While the master and sub- problems should generally be solved iteratively in general, the analytical solution implies that these problems can be solved sequentially. Specifically, the master problem can be solved by the northwest corner rule (i.e., the objective function (5.31) satisfies the 2D-Monge property with respect to j and k), and the subproblem can also be solved in a similar manner for a given optimal solution $\{X_{j,k}\}$ or equivalently $\{X_i\}$ (i.e., the objective function (5.34) satisfies the 2D-Monge property with respect to s and i).

6. Concluding remarks

This paper has presented a systematic approach for analyzing the departure-time choice equilibrium (DTCE) problem of a single bottleneck with heterogeneous commuters. The approach is based on the fact that the DTCE is equivalently represented as a linear programming problem with a special structure, which can be analytically solved by exploiting the theory of optimal transport combined with Benders decomposition technique. Through applying the proposed approach to several types of model with heterogeneous commuters, it is revealed that the dynamic equilibrium distribution of departure times exhibits striking regularities, in which commuters sort themselves according to their attributes, such as desired arrival times, schedule delay functions (value of times), and travel distances to a destination.

A straightforward extension of this approach would be to analyze models of simultaneous departure time and route choice in a single bottleneck per route network (e.g., a single O-D network with multiple parallel routes) (e.g., Kuwahara and Newell, 1987; Arnott et al., 1992; Iryo et al., 2005; Iryo and Yoshii, 2007; Liu et al., 2015). Since the equilibrium condition can be equivalently represented as a structured LP problem, the approach proposed herein could be applied without significant modification.

Another interesting direction would be to extend the proposed approach to analyze the DTCE problem in a corridor network with multiple bottlenecks (for a specific instance of the corridor problem, see Akamatsu et al., 2015). Although a straightforward formulation of the problem in a corridor network does not reduce to a LP problem, but instead just a linear complementarity problem, the recent study by Fu et al. (2018) reveals that the solution of the DTCE assignment with homogeneous commuters can be analytically constructed from that of the DSO assignment formulated as a structured LP. Therefore, it would be expected that the solution of the DTCE problem with heterogeneous commuters would also be obtained from an LP, which can be analyzed by extending the approach presented herein. The partial (positive) answer to this conjecture is provided by Osawa et al. (in press), and a full treatment is planned for future work.

Appendix A. Point queue model

Appendix A.1. Formulation

According to Kuwahara and Akamatsu (1993, 1997), the point queue model can be represented by the following three conditions. First, the state equation for the number of users queuing at the bottleneck $E(t)$ is

$$E(t) = A(t) - D(t) \tag{A.1a}$$

$$\mathrm{d}E(t)/\mathrm{d}t = \lambda(t) - \kappa(t) \tag{A.1b}$$

where $D(t)$ denotes the cumulative departure flow at the bottleneck at time t, and its derivative respect to time t is $\kappa(t) \equiv \mathrm{d}D(t)/\mathrm{d}t$.

The second condition is the exit flow model. That is, the departure flow rate from the bottleneck at time t is

$$\kappa(t) = \begin{cases} \mu & \text{if} \quad E(t) > 0 \\ \min.[\lambda(t), \mu] & \text{if} \quad E(t) = 0 \end{cases}. \tag{A.2}$$

The final condition concerns the relationship between the state variables and the queuing delay. In the point queue model, the queuing delay $d(t)$ for a user arriving at the bottleneck at time t is the horizontal distance between the cumulative arrival curve $A(t)$ and the cumulative departure curve $D(t)$. That is,

$$d(t) = E(t)/\mu \tag{A.3}$$

By combining these conditions, (A.1)–(A.3), we can obtain the condition (2.2a).

Appendix A.2. Non-negative and finiteness of arrival flow rate at the bottleneck

To examine the non-negative and finiteness of arrival flow rate at the bottleneck, it is sufficient to check the condition $\Delta u(s) < 1$ for time s when $x(s) > 0$ because the arrival flow rate $\lambda(\tau(s))$ is given as $x(s)/\Delta\tau(s)$ (i.e., Eq.(2.3)) and $\Delta\tau(s) = 1 - \Delta u(s)$ holds. Consider a time interval $\mathcal{S}_k \subset \mathcal{S}$ when there is at least one group, say k, that has a positive departure rate, i.e., $x_k(s) > 0$. Then, the condition (2.11) can be written as

$$c_k(s) + u(s) - v_k = 0 \qquad \forall s \in \mathcal{S}_k \tag{A.4}$$

This implies that

$$\Delta u(s) = -\Delta c_k(s) \qquad \forall s \in \mathcal{S}_k \tag{A.5}$$

Therefore, $\Delta c_k(s) > -1$ asserts that $\Delta u(s) < 1$ and $\lambda(\tau(s)) > 0$ hold.

Appendix B. Strong duality and complementarity slackness for the problem [2D-LP(x)]

Appendix B.1. Strong duality

We first prove the strong duality of [2D-LP($\mathbf{x}$)] and [2D-LP($\mathbf{u}, \mathbf{v}$)], both of which have infinite-dimensional decision variables. We first observe the weak duality.

Theorem B.1. *Let $\mathbf{x}$ and $(\mathbf{u}, \mathbf{v})$ be arbitrary feasible solutions of [2D-LP($\mathbf{x}$)] and [2D-LP($\mathbf{u}, \mathbf{v}$)], respectively. Then, we have $Z(\mathbf{x}) \geq \check{Z}(\mathbf{u}, \mathbf{v})$.*

Proof. We have

$$\begin{aligned} Z(\mathbf{x}) - \check{Z}(\mathbf{u}, \mathbf{v}) &= \sum_{k \in \mathcal{K}} \int_{\mathcal{S}} c_k(s) x_k(s) \mathrm{d}s + \mu \int_{\mathcal{S}} u(s) \mathrm{d}s - \sum_{k \in \mathcal{K}} Q_k v_k \\ &= \sum_{k \in \mathcal{K}} \int_{\mathcal{S}} c_k(s) x_k(s) \mathrm{d}s + \mu \int_{\mathcal{S}} u(s) \mathrm{d}s - \sum_{k \in \mathcal{K}} v_k \int_{\mathcal{S}} x_k(s) \mathrm{d}s \\ &= \int_{\mathcal{S}} \left[\sum_{k \in \mathcal{K}} x_k(s) \left(c_k(s) + u(s) - v_k \right) + u(s) \left(\mu - \sum_{k \in \mathcal{K}} x_k(s) \right) \right] \mathrm{d}s \\ &\geq 0 \end{aligned} \tag{B.1}$$

where the inequality is due to (2.7), (2.8), and (2.10). We thus have $Z(\mathbf{x}) \geq \check{Z}(\mathbf{u}, \mathbf{v})$. $\square$

Next, we divide the interval $\mathcal{S}$ to several pieces. Let N be an arbitrary positive integer, and let the sub-interval be defined by

$$\mathcal{S}_n \equiv [\underline{s} + (n-1)\Delta S, \underline{s} + n\Delta S) \qquad n = 1, 2, \ldots, N$$

with $\Delta S \equiv (\overline{s} - \underline{s})/N$. Moreover, let $\overline{c}_k$ and $\underline{c}_k$ be piece-wise constant (step/staircase) functions defined as

$$\overline{c}_k \equiv \begin{cases} c_{k,1}^{\max} & (s \in \mathcal{S}_1) \\ \vdots & \\ c_{k,N}^{\max} & (s \in \mathcal{S}_N) \end{cases}, \qquad \underline{c}_k \equiv \begin{cases} c_{k,1}^{\min} & (s \in \mathcal{S}_1) \\ \vdots & \\ c_{k,N}^{\min} & (s \in \mathcal{S}_N) \end{cases}$$

with

$$c_{k,n}^{\max} \equiv \sup\{c_k(s) \mid s \in \mathcal{S}_n\}, \qquad c_{k,n}^{\min} \equiv \inf\{c_k(s) \mid s \in \mathcal{S}_n\}.$$

Then, both functions belong to the following function set

$$\mathcal{PC}_N \equiv \left\{ \text{function } f : \mathcal{S} \to \mathbb{R} \;\middle|\; f(s) \text{ is constant over } \mathcal{S}_n \text{ for } n = 1, \ldots, N \right\}$$

Now, consider the following problems:

- [2D-LP($\mathbf{x}$)-u]: a problem analogous to [2D-LP($\mathbf{x}$)] with $c_k(s)$ replaced by $\overline{c}_k$ and the constraint $x_k(s) \in \mathcal{PC}_N$ is added.
- [2D-LP($\mathbf{x}$)-l]: a problem analogous to [2D-LP($\mathbf{x}$)] with $c_k(s)$ replaced by $\underline{c}_k$ and the constraint $x_k(s) \in \mathcal{PC}_N$ is added.
- [2D-LP($\mathbf{u}, \mathbf{v}$)-u]: a problem analogous to [2D-LP($\mathbf{u}, \mathbf{v}$)] with $c_k(s)$ replaced by $\overline{c}_k$ and the constraint $u(s) \in \mathcal{PC}_N$ is added.
- [2D-LP($\mathbf{u}, \mathbf{v}$)-l]: a problem analogous to [2D-LP($\mathbf{u}, \mathbf{v}$)] with $c_k(s)$ replaced by $\underline{c}_k$ and the constraint $u(s) \in \mathcal{PC}_N$ is added.

Notice that those four problems can be regarded as LPs with finite-dimensional decision variables since $x_k(s)$ and $u(s)$ are piece-wise constant functions. Therefore, the strong duality must hold between [2D-LP($\mathbf{x}$)-u] and [2D-LP($\mathbf{u}, \mathbf{v}$)-u], and between [2D-LP($\mathbf{x}$)-l] and [2D-LP($\mathbf{u}, \mathbf{v}$)-l].

Now, let val[$\cdot$] and $\mathcal{F}$[$\cdot$] be the optimal value and feasible set of problem [$\cdot$]. Then we have the following lemma.

Lemma B.1. *For any $N > 0$, we have*

$$\begin{aligned} 0 \leq \text{val[2D-LP}(\mathbf{x})] - \text{val[2D-LP}(\mathbf{u}, \mathbf{v})] &\leq \text{val[2D-LP}(\mathbf{x})\text{-u]} - \text{val[2D-LP}(\mathbf{u}, \mathbf{v})\text{-l]} \\ &= \text{val[2D-LP}(\mathbf{x})\text{-u]} - \text{val[2D-LP}(\mathbf{x})\text{-l]} \end{aligned}$$

Proof. The first inequality is due to the weak duality, and the equality follows since val[2D-LP($\mathbf{u},\mathbf{v}$)-l] and val[2D-LP($\mathbf{x}$)-l] are the primal-dual pair of *finite-dimensional* LPs. Thus, we only show the second inequality.

We first show val[2D-LP($\mathbf{x}$)] $\le$ val[2D-LP($\mathbf{x}$)-u]. Let $\overline{Z}(\mathbf{x})$ be the objective function of [2D-LP($\mathbf{x}$)-u]. Then, from $c_k(s) \le \bar{c}_k(s)$, we have $Z(\mathbf{x}) \le \overline{Z}(\mathbf{x})$ for any $\mathbf{x} \ge \mathbf{0}$. Moreover, we have $\mathcal{F}$[2D-LP($\mathbf{x}$)-u] $\subseteq \mathcal{F}$[2D-LP($\mathbf{x}$)] since [2D-LP($\mathbf{x}$)-u] has additional constraints $x_k(s) \in \mathcal{PC}_N (k \in \mathcal{K})$. Thus we have val[2D-LP($\mathbf{x}$)] $\le$ val[2D-LP($\mathbf{x}$)-u].

Next, we show val[2D-LP($\mathbf{u},\mathbf{v}$)] $\ge$ val[2D-LP($\mathbf{u},\mathbf{v}$)-l]. Notice that $\check{Z}(\mathbf{u},\mathbf{v})$ serves as the objective function of [2D-LP($\mathbf{u},\mathbf{v}$)-l] as well as [2D-LP($\mathbf{u},\mathbf{v}$)]. Moreover, we have $\mathcal{F}$[2D-LP($\mathbf{u},\mathbf{v}$)-l] $\subseteq \mathcal{F}$[2D-LP($\mathbf{u},\mathbf{v}$)] since $\mathcal{F}$[2D-LP($\mathbf{u},\mathbf{v}$)-l] has additional constraints $x_k(s) \in \mathcal{PC}_N (k \in \mathcal{K})$ and it follows $\underline{c}_k(s) + u(s) - v_k \ge 0 \;\Rightarrow\; c_k(s) + u(s) - v_k \ge 0$. Hence we have val[2D-LP($\mathbf{u},\mathbf{v}$)] $\ge$ val[2D-LP($\mathbf{u},\mathbf{v}$)-l]. This completes the proof. □

Using this lemma, we can show the strong duality of [2D-LP($\mathbf{x}$)] and [2D-LP($\mathbf{u},\mathbf{v}$)].

Theorem B.2. *Suppose that $c_k(s)$ is continuous over $\mathcal{S}$. Then we have* val[2D-LP($\mathbf{x}$)]= val[2D-LP($\mathbf{u},\mathbf{v}$)].

Proof. First we fix $N > 0$ arbitrarily. Let $\overline{\mathbf{x}}^N$ and $\underline{\mathbf{x}}^N$ be the optima of [2D-LP($\mathbf{x}$)-u] and [2D-LP($\mathbf{x}$)-l], respectively. Noting $\bar{x}_k^N(s), \underline{x}_k^N(s) \in \mathcal{PC}_N$, we denote the n-th constant of $\bar{x}_k^N(s)$ and $\underline{x}_k^N(s)$ by $\bar{x}_{k,n}^N$ and $\underline{x}_{k,n}^N$, respectively, for each $k \in \mathcal{K}$. Moreover, let $\delta^N \ge 0$ be defined by

$$\begin{aligned}
\delta^N &\equiv \max\left\{\bar{c}_k(s) - \underline{c}_k(s) \,\middle|\, k \in \mathcal{K},\ s \in \mathcal{S}\right\}\\
&= \max\left\{c_{k,n}^{\max} - c_{k,n}^{\min} \,\middle|\, k \in \mathcal{K},\ n \in \{1, 2, \dots, N\}\right\}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\text{val[2D-LP}(\mathbf{x})\text{-u]} - \text{val[2D-LP}(\mathbf{x})\text{-l]}\\
&\quad= \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} \bar{c}_k(s)\bar{x}_k^N(s)\mathrm{d}s - \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} \underline{c}_k(s)\underline{x}_k^N(s)\mathrm{d}s\\
&\quad\le \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} \bar{c}_k(s)\underline{x}_k^N(s)\mathrm{d}s - \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} \underline{c}_k(s)\underline{x}_k^N(s)\mathrm{d}s\\
&\quad\le \sum_{k\in\mathcal{K}} \int_{\mathcal{S}} \delta^N \underline{x}_k^N(s)\mathrm{d}s\\
&\quad= \delta^N \sum_{k\in\mathcal{K}} Q_k
\end{aligned} \tag{B.2}$$

where the first inequality follows since $\overline{\mathbf{x}}^N$ is the optimum of [2D-LP($\mathbf{x}$)-u] and $\underline{\mathbf{x}}^N$ belongs to $\mathcal{F}$[2D-LP($\mathbf{x}$)-u]. Since $c_k(s)$ is continuous, we have $\lim_{N\to\infty} \delta^N = 0$. This together with (B.2) and **Lemma B.1** yields val[2D-LP($\mathbf{x}$)]= val[2D-LP($\mathbf{u},\mathbf{v}$)]. This completes the proof. □

In the above theorem, we assumed that $c_k(s)$ is continuous over $\mathcal{S}$. However, even when $c_k(s)$ is discontinuous at a finite number of points on $\mathcal{S}$, we can obtain the same result.

Corollary B.1. *Suppose that there exist $\hat{s}_m$ ($m = 1, 2, \dots, M$) such that $c_k(s)$ is continuous over $\mathcal{S} \setminus \{\hat{s}_1, \dots, \hat{s}_M\}$. Then we have* val[2D-LP($\mathbf{x}$)]= val[2D-LP($\mathbf{u},\mathbf{v}$)].

Proof. Considering the finitely many sub-intervals $[\underline{s}, \hat{s}_1)$, $[\hat{s}_1, \hat{s}_2), \dots,$ $[\hat{s}_M, \bar{s}]$, we can prove the corollary in a similar manner.

Appendix B.2. Equivalence of complementarity slackness conditions

Notice that (B.1) implies

$$Z(\mathbf{x}) - \check{Z}(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{S}} \left[\sum_{k \in \mathcal{K}} x_k(s) \left(c_k(s) + u(s) - v_k \right) + u(s) \left(\mu - \sum_{k \in \mathcal{K}} x_k(s) \right) \right] \mathrm{d}s.$$

Hence, if $\mathbf{x}$ and $(\mathbf{u}, \mathbf{v})$ satisfy (2.11)–(2.12), then we have $Z(\mathbf{x}) - \check{Z}(\mathbf{u}, \mathbf{v}) = 0$; that is, $\mathbf{x}$ and $(\mathbf{u}, \mathbf{v})$ are optima of [2D-LP($\mathbf{x}$)] and [2D-LP($\mathbf{u}, \mathbf{v}$)], respectively.

On the contrary, if $\mathbf{x}$ and $(\mathbf{u}, \mathbf{v})$ optimize [2D-LP($\mathbf{x}$)] and [2D-LP($\mathbf{u}, \mathbf{v}$)], respectively, then we have $Z(\mathbf{x}) - \check{Z}(\mathbf{u}, \mathbf{v}) = 0$; that is, (2.11)–(2.12) are satisfied[13].

Appendix C. Proofs of Monge property

Appendix C.1. Proof of Lemma 3.1

We show that the function $c : \mathcal{K} \times \mathcal{S} \to \mathbb{R}$ defined by (3.9) is a strict submodular:

$$c_k(s) + c_{k+1}(s') < c_{k+1}(s) + c_k(s') \qquad \text{for all } s < s',\ 1 \leq k < K. \tag{C.1}$$

The condition (C.1) can be rewritten using $f(\epsilon)$ as follows.

$$\frac{f(\epsilon(\sigma_{k+1}, s) + \delta s) - f(\epsilon(\sigma_{k+1}, s))}{\delta s} < \frac{f(\epsilon(\sigma_k, s) + \delta s) - f(\epsilon(\sigma_k, s))}{\delta s} \tag{C.2}$$

where $\delta s \equiv s' - s > 0$. The strict convexity of the function f implies that the slope monotonically increases with the increase of ϵ. Combining this property with $\epsilon(\sigma_{k+1}, s) < \epsilon(\sigma_k, s)$ ($\because\ \sigma_k < \sigma_{k+1}$), we conclude that Eq.(C.2), and thus Eq.(C.1), hold true. □

Appendix C.2. Proof of Lemma 4.1

We show that the function $c : \mathcal{K} \times \mathcal{S} \to \mathbb{R}$ defined by (4.1) is a strict supermodular for $s \leq 0$:

$$c_k(s) + c_{k+1}(s') > c_{k+1}(s) + c_k(s') \qquad \text{for all } s < s' \leq 0,\ 1 \leq k < K. \tag{C.3}$$

The condition (C.3) can be rewritten as follows.

$$(\beta_k - \beta_{k+1}) \cdot (f^e(s) - f^e(s')) > 0 \tag{C.4}$$

This holds true because $\beta_k - \beta_{k+1} > 0$ and $f^e(\cdot)$ is a decreasing function of s (or ϵ). □

[13] In this case, the equalities in (2.11) and (2.12) hold almost everywhere over $\mathcal{S}$.

Appendix C.3. Proof of Lemma 5.1

We prove that the function c: $\mathcal{J}\times\mathcal{K}\times\mathcal{S} \to \mathbb{R}$ defined by (5.19) satisfies the three-dimensional Monge property. To do so, we show that the function c is submodular with respect to any two of its arguments:

$$c_{j,k}(s) + c_{j,k+1}(s') < c_{j,k+1}(s) + c_{j,k}(s') \qquad \text{for all } s \le s' \le 0,\ 1 \le k < K,\ \forall j \in \mathcal{J} \tag{C.5a}$$

$$c_{j,k}(s) + c_{j+1,k}(s') < c_{j+1,k}(s) + c_{j,k}(s') \qquad \text{for all } s \le s' \le 0,\ 1 \le j < J,\ \forall k \in \mathcal{K} \tag{C.5b}$$

$$c_{j,k}(s) + c_{j+1,k+1}(s) < c_{j+1,k}(s) + c_{j,k+1}(s) \qquad \text{for all } 1 \le j < J,\ 1 \le k < K,\ \forall s \le 0 \tag{C.5c}$$

By a straightforward manipulation, we have the following equivalent conditions:

$$(\beta_k - \beta_{k+1})\frac{f(s') - f(s)}{\delta s} + (\gamma_k - \gamma_{k+1})\frac{g(s' - l_j) - g(s - l_j)}{\delta s} > 0 \tag{C.6a}$$

$$-\frac{1}{\delta l}\left(\frac{g(s' - l_{j+1}) - g(s - l_{j+1})}{\delta s} - \frac{g(s' - l_j) - g(s - l_j)}{\delta s}\right) > 0 \tag{C.6b}$$

$$-(\alpha_k - \alpha_{k+1}) + (\gamma_k - \gamma_{k+1})\frac{g(s - l_{j+1}) - g(s - l_j)}{\delta l} > 0 \tag{C.6c}$$

where $\delta s \equiv s' - s > 0$ and $\delta l \equiv l_j - l_{j+1} > 0$. The first line holds true because $\beta_k - \beta_{k+1} < 0, \gamma_k - \gamma_{k+1} < 0$ and $f'(s), g'(t) < 0$; the second line holds true because $g''(t) < 0$; and the third line holds true because $\alpha_k - \alpha_{k+1} < 0$. □

Appendix D. Other proofs in Section 4

Appendix D.1. Proof of Proposition 4.2

By solving the following recursive equation:

$$v_k^* := v_{k+1}^* - c_{k+1}(-s_k) + c_k(-s_k) = v_{k+1}^* + \hat{\beta}_k f^e(-s_k) \qquad \forall k = 1, \dots, K-1. \tag{D.1}$$

with the boundary condition $v_K^* = c_K(-T) = \hat{\beta}_K f^e(-s_K)$, we can easily see that the analytical solution (4.8) is obtained. By substituting $\mathbf{v}^*$ into a user's optimal choice condition, we have

$$u^*(s) = v_k^* - \beta_k f^e(s) \qquad \forall s \in [-s_k, -s_{k-1}],\ \forall k \in \mathcal{K} \tag{D.2}$$

Combining the queuing condition (2.4) with Lemma 4.2 yields

$$\sum_{k\in\mathcal{K}} x_k^*(s) = 0 < \mu \ \Rightarrow\ u^*(s) = 0 \qquad \forall s \in \mathcal{S} \setminus \hat{\mathcal{S}}. \tag{D.3}$$

We thus conclude that Eq.(4.9) holds. □

Appendix D.2. Proof of Lemma 4.4

Suppose that the objective function of the master problem is convex. Then the optimality condition of the master problem (4.24) is given as the following variational inequality (VI) problem.

Find $\mathbf{s}^* \in \mathbf{\Omega} \equiv$ {Eqs.(4.25) and (4.26)} such that

$$\sum_{h\in\mathcal{H}} \nabla Z_M(\mathbf{s}^{h*}) \cdot (\mathbf{s}^h - \mathbf{s}^{h*}) \ge 0 \qquad \mathbf{s}^* \in \mathbf{\Omega}. \tag{D.4}$$

where the element of the gradient of the objective function $\nabla Z_M(\mathbf{s}^{h*})$ is given by

$$\frac{\partial Z_M(\mathbf{s}^h)}{\partial s_k^h} = \begin{cases} \mu\hat{\beta}_k f^e(-s_k^e) & \text{if} \quad h = e \\ \mu\hat{\gamma}_k f^l(s_k^l) & \text{if} \quad h = l \end{cases} \tag{D.5}$$

We then show the strict monotonicity of $\nabla Z_M(\mathbf{s}^{h*})$. For any two nonnegative vectors $\mathbf{s}^h$ and $\tilde{\mathbf{s}}^h$ ($\mathbf{s}^h \neq \tilde{\mathbf{s}}^h$), we have

$$(\nabla Z_M(\mathbf{s}^e) - \nabla Z_M(\tilde{\mathbf{s}}^e)) \cdot (\mathbf{s}^e - \tilde{\mathbf{s}}^e) = \mu \sum_{k\in\mathcal{K}} \hat{\beta}_k \{f^e(-s_k^e) - f^e(-\tilde{s}_k^e)\}(s_k^e - \tilde{s}_k^e) > 0 \tag{D.6a}$$

$$(\nabla Z_M(\mathbf{s}^l) - \nabla Z_M(\tilde{\mathbf{s}}^l)) \cdot (\mathbf{s}^l - \tilde{\mathbf{s}}^l) = \mu \sum_{k\in\mathcal{K}} \hat{\gamma}_k \{f^l(s_k^l) - f^l(\tilde{s}_k^l)\}(s_k^l - \tilde{s}_k^l) > 0 \tag{D.6b}$$

The last inequalities of both equations follow from the facts that $f^e(s)$ is a decreasing function of s and $f^l(s)$ is an increasing function of s. Thus, we conclude that the master problem (4.24) is actually a convex programming problem and the optimal solution of it (or the VI problem (D.4)) is unique. □

Appendix D.3. Proof of Proposition 4.5

Because Properties 1 and 2 of Proposition 4.5 are trivial, we show the proof of the property 3 only. If the strict inequality holds in constraint (4.25) or condition (4.27) (i.e., an optimal solution of the problem (4.24) is in the interior of the feasible region), the equilibrium costs of early and late arrival users in group k must be identical (i.e., $v_k^{e*} = v_k^{l*}$):

$$\sum_{k'\geq k} \left\{\hat{\beta}_{k'} f^e(-s_{k'}^e)\right\} = \sum_{k'\geq k} \left\{\hat{\gamma}_{k'} f^l(s_{k'}^l)\right\} \qquad \forall k \in \mathcal{K} \tag{D.7}$$

which implies

$$\hat{\beta}_k f^e(-s_k^{e*}) = \hat{\gamma}_k f^l(s_k^{l*}) \qquad \forall k \in \mathcal{K} \tag{D.8}$$

We thus have Eq.(4.28). □

We give an example that an interior optimal solution is always feasible and thus exists. Assume that $\gamma_k/\beta_k = \eta$ (constant) for all $k \in \mathcal{K}$ and that Eq.(D.8) is satisfied for all $k \in \mathcal{K}$, i.e.,

$$\begin{cases} \mu(s_k^e + s_k^l) = \sum_{k'\leq k} Q_{k'} \\ f^e(-s_k^e) = \eta f^l(s_k^l) \end{cases} \qquad \forall k \in \mathcal{K} \tag{D.9}$$

From these equations, we first see that s_k^e and s_k^l are positive for $\sum_{k'\leq k} Q_{k'} > 0$. This is because both $f^e(-s)$ and $f^l(s)$ are zero at $s = 0$ and are strictly increasing functions of $s > 0$. We then consider the consecutive groups k and $k+1$ and assume that $s_k^e \geq s_{k+1}^e$. From the second condition of Eq.(D.9), we have

$$f^l(s_k^l) \geq f^l(s_{k+1}^l) \quad \Leftrightarrow \quad s_k^l \geq s_{k+1}^l. \tag{D.10}$$

We thus conclude that

$$s_k^e + s_k^l \geq s_{k+1}^e + s_{k+1}^l. \tag{D.11}$$

However, this condition contradicts the first condition of Eq.(D.9) (i.e., flow conservation), i.e., $s^e_k < s^e_{k+1}$ must hold. By the same logic, we can have $s^l_k < s^l_{k+1}$. Hence we finally obtain

$$0 < s^h_1 < s^h_2 < \cdots < s^h_{K-1} < s^h_K \qquad \forall h \in \mathcal{H} \tag{D.12}$$

which shows the existence of an interior optimal solution.

Appendix D.4. Proof of the strict monotonicity of $\mathbf{v}^{h*}(\mathbf{X}^h)$

We show that the strict monotonicity of $\mathbf{v}^{h*}(\mathbf{X}^h)$ for the case $h = e$ (we omit the proof for the case $h = l$ because almost the same discussion holds). From the analytical solution (4.8) and the definition of s_k, the equilibrium trip cost can be expressed as an explicit function of $\mathbf{y}^e$:

$$v^{*e}_k(\mathbf{X}^e) = \sum_{k' \geq k} \hat{\beta}_{k'} \cdot f^e(-\textstyle\sum_{m \leq k'} X^e_m/\mu) \qquad \forall k \in \mathcal{K} \tag{D.13}$$

or its vector-matrix form of

$$\mathbf{v}^*(\mathbf{X}^e) = \mathbf{L}^{\mathrm{T}}\mathbf{c}(-\mathbf{s}^e(\mathbf{X}^e)) \tag{D.14}$$

where $\mathbf{L}$ ($\mathbf{L}^{\mathrm{T}}$) is the lower (upper) triangle matrix, $\mathbf{c}(-\mathbf{s}^e(\mathbf{X}^e)) \equiv [\hat{\beta}_1 \cdot f^e(-s^e_1), \ldots, \hat{\beta}_K \cdot f^e(-s^e_K)]^{\mathrm{T}}$ and $\mathbf{s}^e(\mathbf{X}^e) \equiv [X^e_1/\mu, \ldots, \sum_{k' \leq K} X^e_{k'}/\mu]^{\mathrm{T}} = (1/\mu)\mathbf{L}\mathbf{X}^e$. We thus have, for any two nonnegative vectors $\mathbf{X}^e$ and $\tilde{\mathbf{X}}^e$,

$$\begin{aligned}
(\mathbf{v}^*(\mathbf{X}^e) - \mathbf{v}^*(\tilde{\mathbf{X}}^e)) \cdot (\mathbf{X}^e - \tilde{\mathbf{X}}^e) &= \{\mathbf{L}^{\mathrm{T}}(\mathbf{c}(-\mathbf{s}^e(\mathbf{X}^e)) - \mathbf{c}(-\mathbf{s}^e(\tilde{\mathbf{X}}^e)))\}^{\mathrm{T}}(\mathbf{X}^e - \tilde{\mathbf{X}}^e) \\
&= (\mathbf{c}(-\mathbf{s}^e(\mathbf{X}^e)) - \mathbf{c}(-\mathbf{s}^e(\tilde{\mathbf{X}}^e))^{\mathrm{T}}\mathbf{L}(\mathbf{X}^e - \tilde{\mathbf{X}}^e) \\
&= \mu \sum_{k \in \mathcal{K}} \hat{\beta}_k \{f^e(-s^e_k) - f^e(-\tilde{s}^e_k)\}(s^e_k - \tilde{s}^e_k) > 0
\end{aligned}$$

The last inequality follows from the fact that $f^e(s)$ is a decreasing function of s. Thus, we conclude that the variational inequality problem (4.29) is a monotone problem. Furthermore, if the function $f^e(\cdot)$ is differentiable, we have

$$\frac{\partial v^*_k(\mathbf{X}^e)}{\partial X^e_{k'}} = \begin{cases} (-1/\mu) \sum_{k' \geq k} \hat{\beta}_{k'} \cdot f'^e(-s^e_{k'}) & \text{if} \quad k = k' \\ (-1/\mu) \sum_{m \geq \max\{k,k'\}} \hat{\beta}_m \cdot f'^e(-s^e_m) & \text{if} \quad k \neq k' \end{cases}. \tag{D.15}$$

This shows that $\nabla\mathbf{v}^*(\mathbf{X}^e)$ is symmetric and $\mathbf{v}^*(\mathbf{X}^e)$ is integrable.

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