Patterns of Competition with Captive Customers

Mark Armstrong and John Vickers

Department of Economics, University of Oxford

3 December 2018

Patterns of Competition with Captive Customers

Mark Armstrong John Vickers*

December 2018

Abstract

We study mixed-strategy equilibrium pricing in oligopoly settings where consumers vary in the set of suppliers they consider for their purchase—some being captive to a particular firm, some consider two particular firms, and so on. In the case of "nested reach" we find equilibria, unlike those in more standard models, in which firms are ranked in terms of the prices they might charge. We characterize equilibria in the three-firm case, and contrast them with equilibria in the parallel model with capacity constraints. A theme of the analysis is how patterns of consumer interaction with firms matter for competitive outcomes.

1 Introduction

In settings where consumers vary in the set of suppliers they consider for their purchase, how do competitive outcomes depend on the distribution of consideration sets in the consumer population? The simplest situation in which this question arises is a duopoly in which each firm has some captive customers, while non-captive customers are able to choose whichever firm's offer they like best. With more than two firms, richer patterns of consideration become possible. Some consumers may be captive to particular firms, some might consider the offers of all firms, while others can choose among the offers of various subsets of firms. Competitive outcomes then depend not only on the number and firms and their relative sizes, but also upon the *pattern* of consumer consideration of firms. The main aim of this paper is to explore this issue in an otherwise standard setting where firms compete in prices using mixed strategies.

There are various reasons why some consumers have more choices open to them than others. Perhaps following a prior stage of advertising by firms or search by consumers,

*Both authors at Department of Economics and All Souls College, University of Oxford. We are grateful to Massimo De Francesco, Jon Levin, Vlad Nora, Neri Salvadori, and Jidong Zhou for helpful comments on this draft.

some might know about more suppliers than do other consumers. For instance, Honka *et al.* (2017, Table 1) document different levels of consumer awareness of various retail banks in a local market. Alternatively, as in Spiegler (2006), there might be horizontal product differentiation such that each consumer considers only a subset of products to be suitable. The set of firms who are currently active in the market might be uncertain (Janssen and Rasmusen (2002)) or the set of firms who choose to post price on a comparison website might be uncertain (Baye and Morgan (2001)). Or some consumers might be constrained in their choices by location, transport costs or switching costs. Consumers might also differ in their ability to make comparisons between offers, with confused consumers choosing randomly between suppliers or buying from a default seller (Piccione and Spiegler (2012), Chioveanu and Zhou (2013)). Our analysis does not take a view on the underlying reason why some consumers have limited options. Rather, it takes the distribution of consideration sets in the consumer population as given, and explores the consequences for competition.

A considerable literature has explored aspects of this question, and some settings are well understood—the case with symmetric sellers considered randomly, the case of independent reach, and duopoly. As to the first of these, Rosenthal (1980) and Varian (1980) considered the situation in which some consumers are randomly, hence symmetrically, captive to particular firms, while others compare the offerings of *all* firms and buy from the cheapest. There is a symmetric equilibrium with price dispersion, in which all firms choose prices according to the same mixed strategy. Burdett and Judd (1983) analyze a more general symmetric model, in which arbitrary fractions of consumers consider one random firm, two random firms, and so on. Provided some consumers consider just one firm and some consider more than one, the symmetric equilibrium involves price dispersion, and industry profit is proportional to the number of captive consumers who consider just one firm.

With independent reach, a consumer considering one firm does not affect the probability she considers any other firm. It follows that all possible subsets of firms are seen by some consumers, in contrast to the "all or nothing" information structure in the Varian-type models. Ireland (1993) and McAfee (1994) study a model in which firms have asymmetric independent reach.[1] When reach is independent, the firm that reaches the most con-

[1] Manzini and Mariotti (2014) study a theoretical choice model, where an agent is aware of a particular option with specified independent probability. In an empirical study of the personal computer market, Sovinsky Goeree (2008) assumes that the reach of the various products is independent.

sumers also has the largest proportion of captive consumers among the consumers within its reach—i.e., the highest captive-to-reach ratio. In the unique equilibrium they construct, all firms have the same minimum price p_0 in the range of prices that they might charge, but the maximum price charged is lower for smaller firms. In the case of unit demands, each firm's profit is equal to its reach times p_0, and p_0 is equal to the captive-to-reach ratio of the largest firm. The same is true in duopoly, as analyzed by Narasimhan (1988).[2] In these situations with symmetry, independent reach or duopoly, firms compete "head-to-head" in price, in the sense that there are prices which all firms choose.

The aim of the present paper is to take further the analysis of asymmetric cases.[3] In doing so, we discover equilibria with quite different characteristics from those in the literature. In the case of *nested* reach, in which only the largest firm has any captive customers, we find equilibria with an "overlapping duopoly" property if the increments between successive firm sizes are non-decreasing. There is a decreasing sequence of prices $\{p_k\}$ such that the range of prices that the kth largest firm might charge is an interval $[p_{k-1}, p_{k+1}]$. Hence small firms charge low prices while large firms charge high prices, so that price competition is segmented instead of being head-to-head. It is no longer the case that a firm's profit is proportional to its reach.

The paper then provides a general analysis of the three-firm case. With triopoly, a consumer who considers at least one firm could be in one of seven situations, there being three one-firm possibilities, three two-firm possibilities, and one all-firm possibility. A wide variety of patterns of consumer consideration is therefore possible. We state conditions under which, as in the cases examined in the existing literature (e.g., independent reach), each firm's profit equals its reach multiplied by the captive-to-reach ratio of the largest firm. (In some of those cases, however, we find the novel feature that the price support of one firm might not be an interval—the firm might price high and low but not in an intermediate range.) When, and only when, those conditions do not hold, we find equilibria with the "overlapping duopoly" property—one firm prices low, one high, and one across the full price range. The triopoly case also allows analysis of the competitive effects of entry.

[2] In a companion paper, Armstrong and Vickers (2018), we use Narasimhan's duopoly framework to investigate the impact of firms being able to offer different deals to captive and contested customers.

[3] An important early example of an asymmetric model is Baye *et al.* (1992, Section V), where consumers either consider a single firm or all firms, but firms have different numbers of captive customers. They show that all but the two smallest firms choose the monopoly price for sure, while the two smallest firms compete using mixed strategies as in the Narasimhan duopoly model. This is an extreme case of the situation where large firms choose only high prices, which we discuss further at several points in the analysis to follow.

While entry pushes down prices in some cases, there are natural patterns of competitive interaction where, counter-intuitively, the opposite happens and consumers are harmed by entry.

Another setting in which firms have "limited reach" is when they have capacity constraints, as in the classic Bertrand-Edgeworth model. For completeness, and for comparison with the model with consideration sets, section 5 presents the solution to the triopoly version of the Bertrand-Edgeworth model in a simplified setting with unit demand. The capacity model is somewhat easier to solve, since there is a clear-cut ordering of the firms which is not always possible within the consideration set framework. When all firms have some captive demand, each firm obtains the same profit per unit of capacity. Otherwise the equilibrium involves the smallest firm choosing prices in an intermediate range, and obtaining higher profit per unit of capacity than its larger rivals. In contrast to the consideration set model, here price competition is always head-to-head. In addition, in the capacity model it is not possible for entry by a third firm to harm consumers.

An extensive literature has explored equilibria in the Bertrand-Edgeworth model, which often involve mixed strategies for prices—see, for example, Vives (1999, section 5.2). The duopoly case with unit demands is analysed in detail by Acemoglu *et al.* (2009, section 5), who also obtain bounds on equilibria in the n-firm case. In a richer framework with downward-sloping demand, De Francesco and Salvadori (2013) study the situation where all firms have captive demand. The closest papers to our triopoly analysis are Hirata (2009) and De Francesco and Salvadori (2015), who showed how a small firm with small capacity might be unwilling to price as low as larger firms, and obtain a higher profit per unit of capacity than them.

The next section of the paper sets out the general model and shows how it operates in the cases of duopoly, symmetry, and independent reach. Sections 3 and 4, respectively, address the cases of nested reach and triopoly. Section 5 considers how our model of consideration sets compares with the Bertrand-Edgeworth model of competition between capacity-constrained firms. The main theme of the conclusion, in section 6, is that *patterns* of consumer interaction with firms, and not just the number and sizes of firms, are important for market outcomes. Possibilities for further work are also discussed.

2 A framework

There are n firms that costlessly supply a homogeneous product. Consumers have unit demand, and are willing to pay up to 1 to buy a unit of the product.[4] Consumers differ according to which firms they consider, and for each subset $S \subset \{1, ..., n\}$ of firms suppose that the fraction of consumers who consider exactly the subset S is α_S. (We slightly abuse notation, and write α_{12} for the fraction of consumers who consider firms 1 and 2, α_1 for the fraction who consider only firm 1, and so on.) When there are only few firms the pattern of consideration sets can be illustrated using a Venn diagram, and Figure 1 depicts a market with three firms.[5] (Here, a consumer considers a particular subset of firms if she lies inside the "circle" of each of those firms. For instance, a fraction α_{12} of consumers consider the two firms 1 and 2.)

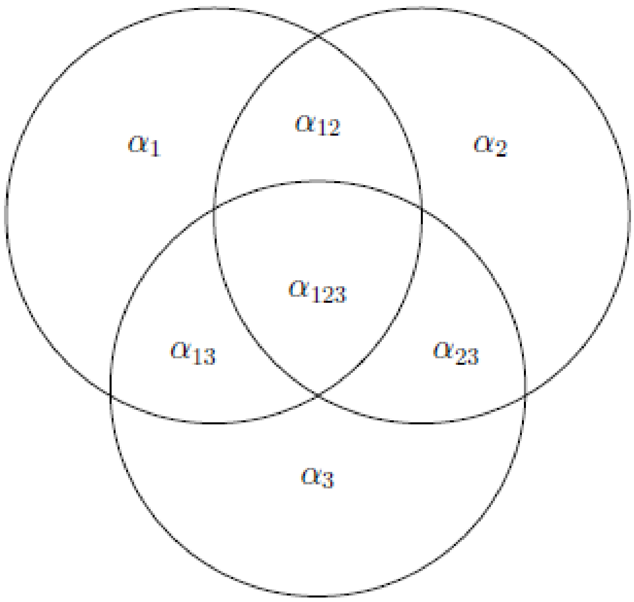

Figure 1: Consideration sets with three firms

[4]The positive analysis which follows is not affected if each consumer has a downward-sloping demand function $q(p)$, provided revenue $R(p) = pq(p)$ is an increasing function up to the monopoly price. However, welfare analysis (for instance in our discussion of entry) requires adjustment with downward-sloping demand. See Armstrong and Vickers (2018) for a welfare analysis with downward-sloping demand in the duopoly context.

[5]With more firms, consideration sets can be conveniently depicted using a bipartite graph, where the two groups in the graph are the consumers and the firms, and a line connecting a consumer to a firm corresponds to the former considering the latter.

A consumer is *captive* to firm i if she considers i but no other other firm, and there are α_i such consumers. The *reach* of firm i, denoted σ_i, is the fraction of consumers who consider i; formally this is

$$\sigma_i = \sum_{S|i \in S} \alpha_S \ .$$

Finally, the *captive-to-reach* ratio of firm i is denoted ρ_i, where

$$\rho_i = \frac{\alpha_i}{\sigma_i} \ .$$

At points in the following analysis we will discuss entry by a new firm. To model the impact of entry on the pattern of consideration sets, we will assume the entrant has reach which is a new "circle" superimposed onto the existing Venn diagram. That is, entry does not affect how consumers consider the incumbent firms, and the reach of an incumbent firm is unaffected by entry, although its number of captive customers will weakly fall.[6] Likewise, when a firm's reach increases we mean that its "circle" expands, so that a larger subset of consumers consider it, keeping the other firms' reaches unchanged.

Firms compete in one-shot Bertrand manner, and a consumer buys from the firm she considers with the lowest price (provided this price is no greater than 1). We assume that at least one firm has some captive customers, for otherwise the unique equilibrium has all firms choosing the competitive price $p = 0$. Typically, there is no pure strategy equilibrium in prices, and at least some firms will employ a mixed strategy for their prices.

When firm i chooses price $p \leq 1$ it will sell to a consumer when that consumer is within its reach and when none of the other firms the consumer considers offers a lower price. Therefore, when rival firms $j \neq i$ choose price according to the CDF $F_j(p)$, firm i's expected demand with price $p \leq 1$ is

$$q_i(p) \equiv \sum_{S|i \in S} \alpha_S \left(\prod_{j \in S/i} (1 - F_j(p)) \right) \ . \tag{1}$$

Equilibrium occurs when each firm i obtains profit π_i, chooses price according to the CDF $F_i(p)$, and firm i's profit $pq_i(p)$ is equal to π_i for every price in firm i's support and no higher than π_i for any price outside its support. Since industry profit is a continuous function of the vector of prices chosen, Theorem 5 in Dasgupta and Maskin (1986) shows that such an equilibrium exists.

[6] In particular, there is no danger of "choice overload", whereby the fraction of consumers who compare prices falls when there are more firms, as discussed for instance in Spiegler (2011, page 150).

The following result collects a number of observations about the nature of equilibrium which are mostly familiar from the existing literature.[7]

Lemma 1 *In equilibrium:*
(i) firm i obtains profit $\pi_i \geq \alpha_i$, with equality for at least one firm;
(ii) firm i chooses prices with support contained in the interval $[\rho_i, 1]$ and its distribution for choosing price is continuous (that is, it has no "atoms") in the half-open interval $[\rho_i, 1)$;
(iii) the minimum price chosen by any firm, say p_0, lies weakly between the second lowest ρ_i and the highest ρ_i, and
(iv) each price in the interval $[p_0, 1]$ is chosen by at least two firms.

Sketch proof. In an equilibrium we have $\pi_i \geq \alpha_i$, since a firm can always obtain at least this profit by choosing price equal to 1 and serving its captive customers. For this reason, no firm would ever offer a price below ρ_i, its captive-to-reach ratio, since if it did so it would obtain profit below α_i even if it managed to sell to its entire reach. Since no firm would choose a price above the reservation price 1, the price support for firm i lies in the interval $[\rho_i, 1]$.

To see that each firm's distribution for price is continuous on the half-open interval $[\rho_i, 1)$, suppose by contrast firm i had an atom at some price $p < 1$ in its support. Then rivals would have a jump in their demand across the price p and so none of them set a price in some interval above p, and in that case firm i could increase its price somewhat above p without losing any demand. This completes the proof for part (ii).

If a price is in one firm's support it must be in the support of at least two firms. This is because if only a single firm was active over a range of prices, say in the interval $[p, p']$, then it loses no demand if it chooses the higher price p' rather than p, and so this cannot be part of an equilibrium. Similarly, if p_0 is the minimum price ever chosen in the market, then all prices in the interval $[p_0, 1]$ are sometimes chosen: if no firm is active in the interval $(p, p') \subset [p_0, 1]$, then a firm which sometimes chose p loses no demand if it instead chooses price p', and this cannot occur in equilibrium. (This latter argument makes use of the previous observation that no firm has an atom has price p.) This proves part (iv).

Firms can have an atom at the reservation price $p = 1$. However, if firm i has an atom at $p = 1$ its profit is $\pi_i = \alpha_i$, i.e., when it chooses price $p = 1$ it sells only to its captive

[7]For instance, see McAfee (1994, page 28).

customers and there can be no "ties" with another firm who also has an atom at $p = 1$. For instance, if both firms i and j have an atom at $p = 1$ then we must have $\alpha_{ij} = 0$ so that no consumers consider just these two firms, otherwise either firm has an incentive to set a price strictly below 1. If no firm has an atom at $p = 1$ then any firm with $p = 1$ in its support (and there must be at two such firms from part (iv)) has profit equal to α_i. This completes the proof for part (i).

Let firm j be a firm which obtains profit equal to α_j. Then the minimum price ever chosen, p_0, must be no higher than ρ_j (for otherwise firm j could obtain more profit by choosing $p = p_0$), and so p_0 cannot be strictly greater than the highest ρ_i. Since no firm sets a price below its ρ_i, the minimum price p_0 (which from part (iv) is sometimes chosen by at least two firms) must be weakly above the second lowest ρ_i. This proves part (iii) and completes the proof of the lemma.

As discussed in the introduction, previous work has studied the special cases of duopoly, symmetry arising from random consideration, and independent reach, and we describe those cases here for future reference. In the latter two situations we provide generalizations to the existing analysis.

Duopoly: Lemma 1 essentially determines the unique equilibrium when there are two firms, the situation studied by Narasimhan (1988). Suppose that firm 1 is the larger firm in the sense that $\alpha_1 \geq \alpha_2$ (which implies that $\sigma_1 \geq \sigma_2$ and $\rho_1 \geq \rho_2$). Then both firms have the same support for prices, $[p_0, 1]$, where $p_0 = \rho_1$, and firm $i = 1, 2$ has profit $\pi_i = \sigma_i \rho_1$. Note that the smaller firm's profit weakly exceeds its captive profit α_2. The larger firm's profit necessarily increases when its reach increases, as its profit is equal to its number of captive customers which weakly increases. However, the smaller firm's profit could fall with wider reach, for instance if its captive base does not change but it expands sufficiently into the rival's reach to become the larger firm.

Industry profit in equilibrium is

$$\Pi = (\sigma_1 + \sigma_2)\rho_1 = \sigma_1 + \sigma_2 - \alpha_{12} - \alpha_{12}\frac{\sigma_2}{\sigma_1} \ . \tag{2}$$

Total welfare is the number of consumers reached, $W = \sigma_1 + \sigma_2 - \alpha_{12}$, which must increase whenever a firm's reach expands. Consumer surplus therefore

$$CS = W - \Pi = \alpha_{12} \frac{\sigma_2}{\sigma_1} \ .$$

Thus, keeping reaches constant, consumer surplus increases when the overlap α_{12} is larger, even though fewer consumers are then served. Likewise, consumer surplus decreases when the larger firm's set of captive customers expands, keeping the other regions of the Venn diagram unchanged, even though more consumers are served.

Symmetric firms: Burdett and Judd (1983, section 3.3) study a market with $n \geq 2$ symmetric firms, where consumers consider firms at random (a specified fraction consider one random firm, another fraction consider two random firms, and so on). This model can be generalised so that firms are symmetric but consideration sets need not be random. Specifically, suppose that each firm reaches a_0 captive customers, a_1 consumers who consider exactly one other firm (not necessarily random), and in general for $m \leq n-1$ each firm reaches a_m consumers who consider m other firms. Thus, the reach of each firm is $\sigma = a_0 + ... + a_{n-1}$, and the captive-to-reach ratio is $\rho = a_0/\sigma$ which is therefore the minimum price offered in the market. Each firm obtains equilibrium profit equal to a_0.

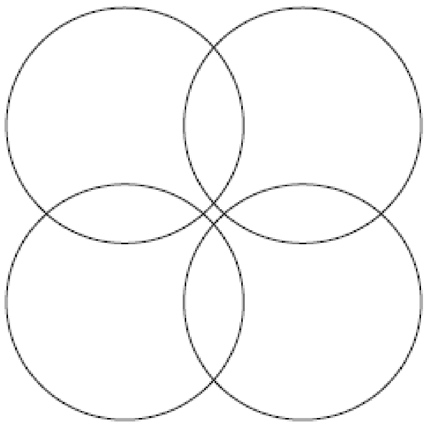

Figure 2: Local competition

To illustrate, suppose there are four symmetric firms with a pattern of consideration as depicted on Figure 2. Thus, no consumers consider three or four firms (i.e., $a_2 = a_3 = 0$), each firm has the same number of captive customers, and a firm overlaps with two neighbours but not with the third more distant rival. This situation could be described as one with local competition between firms, and there is no head-to-head competition for a pool of consumers who consider all firms.

In a symmetric market, the symmetric equilibrium where each firm uses the same CDF for its price, $F(p)$, is derived as follows. Let

$$\phi(x) \equiv a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1}$$

be the probability generating function associated with the random variable m, the number of rivals faced by a firm. Here, $\phi(x)$ is convex and increasing, the number of captive customers for each firm is $\phi(0)$, each firm has reach is $\sigma = \phi(1)$ and captive-to-reach ratio $\rho = \phi(0)/\phi(1)$. The probability a firm which chooses price p will sell to a consumer is therefore

$$a_0 + a_1(1 - F(p)) + ... + a_{n-1}(1 - F(p))^{n-1} = \phi(1 - F(p)) \ .$$

Therefore, since each firm makes profit $\phi(0)$, the symmetric equilibrium CDF satisfies

$$\phi(1 - F(p)) \equiv \frac{\phi(0)}{p} \ , \tag{3}$$

where $F(p)$ is a function which strictly increases from 0 to 1 as p increases from ρ to 1.

The models in Rosenthal (1980) and Varian (1980) are a special case of this symmetric framework, where consumers either consider one random firm or consider all firms, so that $a_m = 0$ for $1 \le m \le n - 2$. With this "all-or-nothing" pattern of consideration, Baye, Kovenock, and De Vries (1992) show that when $n \ge 3$ there are multiple equilibria (all of which involve the same profit for firms). For instance, all but two firms might choose $p = 1$ for sure, selling only to their captive customers, while the remaining two firms choose prices on the interval $[\rho, 1]$. If $a_1 > 0$, however, so that a firm sometimes has exactly one rival, one can adapt the proof of Proposition 1 in Spiegler (2006) to show that the strategy in (3) is the only equilibrium.

In general, the symmetric model is not well-suited to study the impact of entry. However, if entry occurs in such a way as to preserve symmetry between firms, then it cannot harm consumers. The entrant will obtain profit equal to its captive customers (i.e., its customers which did not previously consider any incumbent), which in turn is equal to the gain in welfare due to entry. Therefore, the gain in consumer surplus due to entry is equal to the reduction in *incumbent* profit, and incumbents cannot gain since entry must weakly shrink each incumbent's set of captive customers.

Independent reach: Ireland (1993) and McAfee (1994) study the situation where each firm has an independent chance of being considered by a consumer. Specifically, firm i is considered by an independent fraction σ_i of the consumer population, where firms are labelled so that $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$. The fraction of consumers who are captive to firm i is $\alpha_i = \sigma_i \Pi_{j \neq i}(1 - \sigma_j)$ and so this firm's captive-to-reach ratio is $\rho_i = \Pi_{j \neq i}(1 - \sigma_j)$. Thus, as with duopoly, the firm with the largest reach is also the firm with the highest captive-to-reach ratio.

If firm j chooses its price with the CDF $F_j(p)$, firm i sells to a consumer if it reaches that consumer (which occurs with probability σ_i) and no rival reaches that consumer with a lower price. The probability that firm j does reach the consumer with a lower price is $\sigma_j F_j(p)$. Therefore, firm i's demand with price p takes the multiplicatively separable form

$$q_i(p) = \sigma_i \prod_{j \neq i} (1 - \sigma_j F_j(p)) \ . \tag{4}$$

Ireland (1993) and McAfee (1994) show that the equilibrium is such that all firms have the same minimum price p_0, which is therefore equal to $\rho_1 = \Pi_{j=2}^{n}(1 - \sigma_j)$, and the profit of firm i is $\pi_i = \sigma_i p_0$. In particular, unless it is the largest firm, a firm's equilibrium profit decreases with its reach σ_i when $\sigma_i \geq 1/2$.

Thus, firms' profits are proportional to their reaches, the profit of the largest firm is equal to its number of captive consumers, while the profit of smaller firms is weakly greater than their number of captive consumers. The CDFs which support these equilibrium profits are such that firm i chooses its price with interval support $[p_0, p_i]$, where firm i's maximum price p_i is smaller for smaller firms. The two largest firms choose prices with support $[p_0, 1]$.[8] Thus smaller firms only offer low prices, while larger firms offer the full range of prices.

With independent reach, industry profit is

$$\Pi = \left(\sum_{i=1}^{n} \sigma_i \right) p_0 = \left(\sum_{i=1}^{n} \sigma_i \right) \prod_{i=2}^{n} (1 - \sigma_i) \ . \tag{5}$$

Total welfare is the fraction of consumers who see at least one firm, which is

$$W = 1 - \prod_{i=1}^{n} (1 - \sigma_i) \ . \tag{6}$$

[8] This equilibrium was subsequently shown by Szech (2011) to be unique.

The difference between W and Π is consumer surplus, which is therefore

$$CS = 1 - \left(1 + \sum_{i=2}^{n} \sigma_i\right) \prod_{i=2}^{n} (1 - \sigma_i) \ . \tag{7}$$

Expression (7) can be interpreted as an index of the "competitiveness" of the market in this context. Consumer surplus does not depend on the reach of the largest firm, σ_1, but increases with the reach of each smaller firm. One can readily verify that entry by a new firm, also with independent reach, will necessarily increase consumer surplus in (7).

This analysis of Ireland and McAfee can be extended to situations where reach is "conditionally independent" in the following sense. Suppose some consumers are systematically harder to reach than others, and one firm being considered by a given consumer makes it more likely that another firm is also considered by that consumer. Specifically, suppose that consumers differ according to a scalar parameter β, interpreted as the "reachability" of the consumer, and the type-β consumer considers firm i with independent probability $\beta\gamma_i$, where $\gamma_1 \geq \gamma_2 \geq ... \geq \gamma_n$. Thus the reach of firm i is $\sigma_i = \gamma_i \times \mathbb{E}_\beta[\beta]$. Then similarly to expression (4), firm i's demand with price p is

$$q_i(p) = \mathbb{E}_\beta \left\{ \beta\gamma_i \prod_{j \neq i} [1 - \beta\gamma_j F_j(p)] \right\} \ .$$

The construction of Ireland and McAfee continues to be valid, and in equilibrium all firms have the same minimum price (equal to the captive-to-reach ratio of the largest firm), and smaller firms progressively drop out with higher prices. (See Proposition 2 for further details in the case of three firms.)

In each of these cases of duopoly, symmetry and independence, the format of the equilibrium is similar: all firms choose their price from an interval support, all firms have the same minimum price p_0, and as a result a firm's profit is proportional to its reach. All firms compete "head-to-head" in prices, in the sense that there is a range of prices that all firms choose. This is so even if they do not compete "head-to-head" in terms of consumer consideration, as with local competition in Figure 2. In the remainder of the paper we show that richer possibilities exist outside these special cases. We start in the next section by describing a radically different kind of equilibrium when firms have *nested* reach.

3 Nested reach

The situation with independent reach has all consumers being equally likely to be reached by a firm, regardless of which other firms they consider. At the other extreme one could envisage consideration sets as being *nested*, in the sense that if firm i reaches a greater number of consumers than firm j, all firm j's consumers also consider firm i. For example, an entrant's reach lies inside an incumbent's reach if only a subset of latter's existing customers are willing to consider buying from the entrant. Likewise, if consumers consider options in an ordered fashion, as may be the case with internet search results (where some consumers just consider the first result, others consider the first two, and so on), then the reach of a lower ranked option is nested inside that of a higher ranked option. Alternatively, if consumers only consider the firms whose product they find suitable, then low-quality firms could supply a product which is found suitable by only a subset of the consumers who like the product of a higher-quality firm. With nested reach only the largest firm has any captive consumers, and a smaller firm only has positive demand if its price is below all the prices of larger firms.

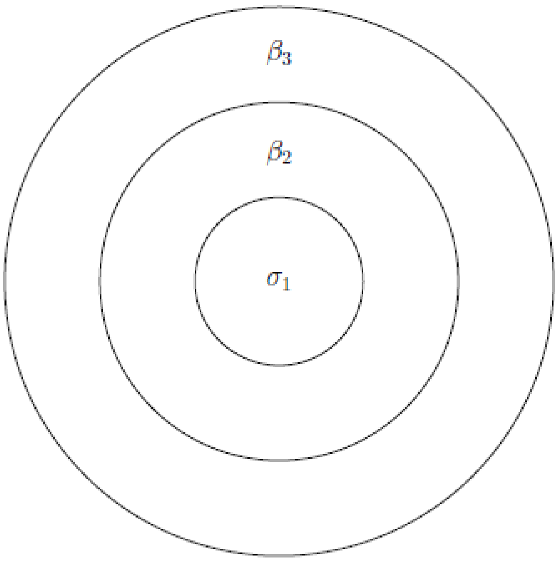

Figure 3: Three firms with nested reach

As depicted on Figure 3, suppose there are $n \geq 3$ firms with nested reach, let firm i have reach σ_i, where firms are ordered as $\sigma_1 < \sigma_2 < ... < \sigma_n$, and for $i \geq 2$ write $\beta_i = \sigma_i - \sigma_{i-1}$ for the incremental reach of firm i. While it is hard to find the equilibrium in all nested situations, the following result describes equilibrium in a broad class of cases.

Proposition 1 *Suppose $n \geq 3$ firms have nested reach such that*

$$0 < \beta_2 \leq \ \dots \ \leq \beta_n \ . \tag{8}$$

Then there are price thresholds $p_1 < p_2 < \dots < p_{n-1} < p_n = 1$ such that only sellers i and $i+1$ (where $1 \leq i < n$) choose prices in the interval $[p_i, p_{i+1}]$. The thresholds are determined recursively by $p_2 = \frac{\sigma_1+\beta_2}{\beta_2} p_1$ and for $1 < i < n$

$$p_{i+1} = p_i + \frac{\beta_i}{\beta_{i+1}} p_{i-1} \tag{9}$$

and the profit of firm i is $\pi_i = \beta_i p_i$.

Proof. This and subsequent proofs are contained in the appendix. ■

Thus, in this equilibrium smaller firms only choose low prices while larger firms only choose high prices.[9] In this sense there is segmented price competition rather than head-to-head price competition, even though there is head-to-head competition in terms of consumer consideration (as firm 1's potential customers consider all firms). Nevertheless, the presence of large firms affects the profits of smaller firms, and (except for the very largest firm) *vice versa.* To illustrate, suppose that $\sigma_1 = \beta_2 = \dots = \beta_n \equiv \beta$ so that reach is equally spaced. Then expression (9) implies that $p_{i+1} = p_i + p_{i-1}$, so that $p_i = p_1 \times \varphi_i$ where φ_i is the i^{th} number in the Fibonacci sequence (as given by 1, 2, 3, 5, 8, 13,...). Since $p_n = 1$, it follows that the lowest price is $p_1 = 1/\varphi_n$, in which case $p_i = \varphi_i/\varphi_n$ and the profit of firm i is $\pi_i = \beta\varphi_i/\varphi_n$.

Proposition 1 describes equilibrium only for cases where incremental reach increases. As shown in the next section, other pricing patterns are found in other configurations of nested reach, including patterns akin to independent reach where all firms have the same minimum price and have profit proportional to reach. In all the asymmetric cases considered so far (duopoly, independent reach, and nested reach) there is a clear-cut ordering of the firms, in the sense that a firm with a larger reach also has a weakly higher captive-to-reach

[9] A similar pattern of segmented pricing is seen in Bulow and Levin (2006). They study a matching model where n heterogeneous firms each wish to hire a single worker from a pool with n heterogeneous workers, where the payoff from a match is (in the simplest version of their model) the product of qualities of the firm and worker. Firms choose wages which they must pay regardless of the quality of the worker eventually hired, workers care only about their wage, and higher quality workers choose their employer first. In equilibrium, firms offer wages according to mixed strategies, where higher quality firms offer wages in a higher range than lower quality firms.

ratio. However, more generally the two ways to order firms need not always coincide. For instance, a "niche" firm could have limited reach but have a high proportion of its reach being captive. In the next section we allow for general patterns of awareness in the context of triopoly.

4 Triopoly

In this section we analyze equilibria in all situations when there are three firms in the market. As shown on Figure 1, firm $i = 1, 2, 3$ has α_i captive consumers, α_{ij} consumers consider firms i and j (but not k), while to save on notation we say that α consumers consider all three firms. Unless explicitly stated, all parts of this Venn diagram have positive numbers of consumers. When firms use the CDFs $\{F_1(\cdot), F_2(\cdot), F_3(\cdot)\}$ for their prices, expression (1) implies that firm i's expected demand with price $p \leq 1$ is

$$q_i(p) = \alpha_i + \alpha_{ij}(1 - F_j(p)) + \alpha_{ik}(1 - F_k(p)) + \alpha(1 - F_j(p))(1 - F_k(p)) \tag{10}$$

where i, j and k are distinct. If π_i is firm i's equilibrium profit, then for a price p in firm i's support we require that $pq_i(p) \equiv \pi_i$, and that $pq_i(p) \leq \pi_i$ for all prices outside its support.

We will show that equilibria in this market take one of three broad formats, as presented in the next proposition. In the first, all firms are active for low prices (where the lowest price is equal to the highest captive-to-reach ratio), and above a threshold price only two firms are active. (The case with independent reach falls into this format.) In the second format we again have all firms active for low prices, but now one firm has a gap in its price support and does not choose prices in an intermediate range. In the third format, two firms are willing to choose prices which are below what the other firm is willing to charge, and profits are no longer proportional to reach. This is the situation with "overlapping duopolies", already seen with nested reach in Proposition 1.

To characterise these equilibria is it useful to define the three parameters

$$t_i = (\alpha + \alpha_{jk})\sigma_i \ , \tag{11}$$

where, in words, t_i is the probability that a consumer considers at least firms j and k multiplied by the probability she considers firm i. In rough terms, the format of equilibrium depends on how close together the parameters $\{t_1, t_2, t_3\}$ are. To illustrate, with independent reach we have

$$t_1 = t_2 = t_3 = \sigma_1\sigma_2\sigma_3 \ , \tag{12}$$

with nested reaches $\sigma_1 > \sigma_2 > \sigma_3$ we have

$$t_1 = \sigma_1\sigma_3 \ ; \ t_2 = t_3 = \sigma_2\sigma_3 \ , \tag{13}$$

while in the model in Baye *et al.* (1992, Section V), where no consumer considers exactly two firms, we have

$$t_1 = \alpha\sigma_1 \ ; \ t_2 = \alpha\sigma_2 \ ; \ t_3 = \alpha\sigma_3 \ . \tag{14}$$

The following result characterizes the equilibria for all parameter values in Figure 1. (For a precise description of the threshold prices in the statement of the result, see the proof in the appendix.)

Proposition 2 *Suppose firms are labelled so that firm 1 has the highest captive-to-reach ratio, ρ_1, while firms 2 and 3 are labelled so that*

$$\alpha_{12}\sigma_3(\rho_1 - \rho_3) \geq \alpha_{13}\sigma_2(\rho_1 - \rho_2) \ . \tag{15}$$

(i) If

$$\frac{\alpha + \alpha_{12}}{\alpha_{12}} |t_1 - t_2| \leq \alpha_{12}\sigma_3 \tag{16}$$

there is an equilibrium where all firms choose the same minimum price $p_0 = \rho_1$, each firm i's profit is $\sigma_i \times p_0$, and there is a price p_1, with $p_0 < p_1 \leq 1$, such that all firms choose prices in the lower range $[p_0, p_1]$ and only firms 1 *and* 2 *choose prices in the upper range $[p_1, 1]$.*

(ii) If

$$|t_1 - t_2| < \alpha_{12}\sigma_3 < \frac{\alpha + \alpha_{12}}{\alpha_{12}} |t_1 - t_2| \tag{17}$$

there is an equilibrium where all firms choose the minimum price $p_0 = \rho_1$, firm i's profit is $\sigma_i \times p_0$, and there are prices $\hat{p}$ and p_1, with $p_0 < \hat{p} < p_1 < 1$, such that all firms choose prices in the low range $[p_0, \hat{p}]$, firm 3 *and one of the other firms (whichever of firms 1 and 2 has the lower t_i) choose prices in the middle range $[\hat{p}, p_0]$, and firms* 1 *and* 2 *choose prices in the upper range $[p_1, 1]$.*

(iii) If

$$\alpha_{12}\sigma_3 \leq |t_1 - t_2| \tag{18}$$

there is an equilibrium with prices p_0 and p_1, where $p_0 < p_1 \leq 1$ and p_0 is below ρ_1, such that firm 3 *and one of the other firms (whichever has the lower t_i) choose prices in the lower range $[p_0, p_1]$ and firm 1 and one of the other firms (firm 2 if $t_2 \geq t_1$ and otherwise the firm with the larger σ_i) choose prices in the upper range $[p_1, 1]$.*

Clearly independent reach, where we have (12), is covered by part (i) of this result. More generally, part (i) applies whenever $t_1 = t_2 = t_3$, as is so with the "conditional independence" discussed in section 2 when we have

$$t_1 = t_2 = t_3 = \gamma_1\gamma_2\gamma_3\left(\mathbb{E}[\beta]\mathbb{E}[\beta^2]\right) = \sigma_1\sigma_2\sigma_3\frac{\mathbb{E}[\beta^2]}{(\mathbb{E}[\beta])^2} .$$

As the t_i parameters become less similar, part (ii) of the result will apply. In technical terms, when the t_i parameters are relatively far apart, when one attempts to construct the equilibrium CDFs as in part (i) of the result, the candidate CDF of the firm with the largest t_i will be decreasing over an intermediate range, in which case the appropriate CDF for this firm will be the "ironed" version of the candidate CDF, and the flat portion corresponds to the high-t_i firm ceasing to choose intermediate prices. Note that part (ii) of the result cannot apply if $\alpha = 0$, and so there must be head-to-head competition for a pool of consumers who consider all firms for this pricing pattern to emerge.

As the t_i parameters become further apart, this intermediate range of prices becomes wider and eventually this high-t_i firm ceases to offer low prices at all, and the equilibrium format has one firm only offering low prices, one firm only offering high prices (and the remaining firm choosing prices throughout the whole range). This regime is covered by part (iii). For instance, with nested reach $\sigma_1 > \sigma_2 > \sigma_3$ and (13), firms are labelled as in the statement of the Proposition, and part (iii) applies if (18) holds, i.e., if the incremental reach of the largest firm is at least as great as that of the medium firm, thus verifying Proposition 1. Otherwise, the nested case has all three firms choosing the same minimum price.[10] Another case which is covered by part (iii) of the result is the model in Baye *et al.* (1992, Section V), where the largest firm chooses price $p = 1$ for sure (i.e., in the statement of the result we have $p_1 = 1$), and the two smaller firms compete in the range $[\rho_2, 1]$.

The impact of entry: As an application of this analysis, consider the following simple entry scenario. Initially, there are two firms, 1 and 2, who together cover the market, i.e., all consumers consider one or both of these firms. Suppose firm i reaches a proportion σ_i of the consumers, where firms are labelled so that $\sigma_1 \geq \sigma_2$. Since they cover the market, $\sigma_1 + \sigma_2 - 1 \geq 0$ consumers consider both firms, while firm i has $1 - \sigma_j$ captive customers.

[10] Part (ii) applies if the nested reaches satisfy $(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) < (\sigma_2 - \sigma_3)^2 < \sigma_2(\sigma_1 - \sigma_2)$, as is the case in the example with reaches proportional to $\sigma_1 = 12$, $\sigma_2 = 9$ and $\sigma_3 = 5$.

Expression (2) shows that industry profit is

$$(\sigma_1 + \sigma_2)\frac{1-\sigma_2}{\sigma_1} \ ,$$

while consumer surplus is one minus this profit.

Now suppose a third firm, firm 3, enters this market with independent reach σ, i.e., it is considered with probability σ by all consumers, regardless of whether they already consider firm 1 or firm 2 or both incumbents. Since firm 3 has no captive customers, firm 1 has the highest captive-to-reach ratio. One can calculate that

$$t_1 = t_2 = \sigma\sigma_1\sigma_2 \ ; \ t_3 = \sigma(\sigma_1 + \sigma_2 - 1) \ ,$$

so that the entrant has the smallest t_i. One can check that either part (i) or (ii) of Proposition 2 applies, and the minimum price is $p_0 = \frac{(1-\sigma_2)(1-\sigma)}{\sigma_1}$. Therefore, the change in industry profit due to entry is

$$(\sigma_1 + \sigma_2 + \sigma)\frac{(1-\sigma_2)(1-\sigma)}{\sigma_1} - (\sigma_1 + \sigma_2)\frac{1-\sigma_2}{\sigma_1} = -\frac{\sigma}{\sigma_1}(1-\sigma_2)(\sigma + \sigma_1 + \sigma_2 - 1) < 0 \ .$$

Thus an entrant with independent reach intensifies competition, so that industry profit falls and consumers benefit.

Alternatively, suppose that the entrant is considered only by those consumers who already consider both incumbents, as illustrated by the Venn diagram on Figure 4. This consideration pattern is reasonable if only "savvy" consumers consider buying from the entrant, and these are the consumers who are already willing to consider both incumbents. In many cases, but not all, this form of entry will *increase* industry profit: by construction, the entrant does not cut into the incumbents' captive markets, and so the incumbents will often be able to maintain their profits, while the entrant makes positive profit. When this is so, consumers are *harmed* by this pattern of entry.

This is true if the incumbents are symmetric, when part (i) applies to the post-entry market. The minimum price is equal to an incumbent's captive-to-reach ratio, which is unchanged with entry. Since the sum of the firms' reaches increases with entry, it follows the industry profit rises and consumer surplus falls. Although their profit is unaffected by entry, incumbents adjust their pricing strategy to focus more on exploiting their captive base, so that captive customers pay higher prices after entry. In addition, it is perfectly possible that even the contested consumers are harmed by this form of entry, despite being able to choose among more firms, as the higher prices offered by incumbents leave the entrant relatively unconstrained to set high prices too.[11]

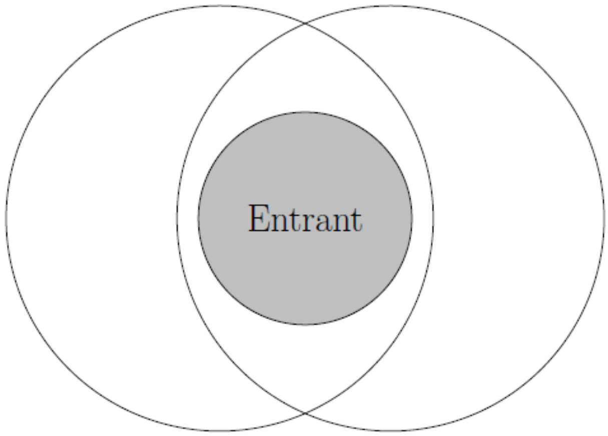

Figure 4: Entry into the contested market

This result is related to Rosenthal (1980), where entry by a new firm causes the average price paid by both captive and informed consumers to rise. However, in his model the entrant arrives with its own pool of captive customers, whereas the effect arise in our model despite the entrant having none.[12]

5 Capacity constraints

As discussed in the introduction, another way in which firms have limited reach is when they have capacity constraints, as in the Bertrand-Edgeworth model of competition. A natural question is how equilibria in this scenario compare with equilibria in our main model with consideration sets. To address this question in the most direct way we assume that consumers have unit demands, which avoids the need to posit a rationing rule. As we

[11] To calculate expected prices paid by different consumers in the Venn diagram requires detailed analysis using the equilibrium CDFs for the firms' mixed pricing strategies. To illustrate, suppose symmetric incumbents each have equal numbers of captive and contested customers. Then before entry the average captive price is about 0.69 and the average price paid by a contested customer is 0.61. If entry occurs and the entrant is considered by all the contested customers, the captive customers now pay an average price 0.91 while the contested customers pay 0.67.

[12] Relatedly, in a setting with differentiated products, Chen and Riordan (2008) show how entry to a monopoly market can induce the incumbent to raise its price. For instance, entry by generic pharmaceuticals might cause a branded incumbent to raise its price, as it prefers to focus on those "captive customers" who care particularly about its brand. Closer to the consideration set framework is Chen and Riordan (2007), who study a model with symmetric firms, where consumers either consider a single random firm or consider a random pair of firms. Among other results, they show that the equilibrium price can increase when an additional firm enters.

explain, for some configurations of capacities, equilibria in the Bertrand-Edgeworth model resemble those that arise in a model with consideration sets. But for other configurations they are quite unlike any such equilibria.

Suppose there is a continuum population of identical consumers of measure 1 who each consider all prices and are willing to pay 1 for a unit of homogeneous product. There are $n \geq 2$ firms, where firm i can costlessly supply any quantity up to its capacity κ_i (but cannot supply beyond this capacity), and where firms are labelled $\kappa_1 \geq ... \geq \kappa_n$. A consumer tries to buy at the lowest available price, but is not always able to do so: once the capacity of the cheapest firm is exhausted, remaining consumers then try to buy from the second cheapest firm, and so on. Write

$$\mu = \sum_{i=1}^{n} \kappa_i - 1$$

for the excess of total capacity over demand. We assume $\mu > 0$, otherwise there is no competition between firms and the equilibrium price for each firm is $p = 1$. Here, $\min\{\kappa_i, 1\}$ is firm i's supply when it offers a price below all its rivals, and so plays the role "reach" did in our main model with consideration sets. Firm i's supply if it offers a higher price than all its rivals is $1 - \Sigma_{j \neq i} \kappa_j$ if this is positive, and this represent the firm's captive customers. Firm i has captive customers if and only if $\kappa_i > \mu$, and we assume from now on that $\kappa_1 > \mu$ (otherwise equilibrium involves all firms choosing the competitive price $p = 0$). Unlike our framework with consideration sets, here firms are necessarily ordered so that firms with large reach also have a large captive-to-reach ratio.

With appropriate interpretation of the captive-to-reach ratio, Lemma 1 continues to hold. Equilibrium in this capacity framework is easily derived in the special case where all firms weakly have captive customers, i.e., when $\kappa_n \geq \mu$.[13] (An example of this case is when firms are symmetric and each has capacity κ, where $(n-1)\kappa < 1 < n\kappa$.) In this situation, the only way that a firm cannot supply its entire capacity is when all rivals choose a lower price. (If firm j sets a higher price that firm i, the latter firm's residual demand is at least $1 - \Sigma_{k \neq i,j} \kappa_k$ which exceeds κ_i given that firm j has captive demand.) Therefore, when firms use CDFs $\{F_1(\cdot), ..., F_n(\cdot)\}$, firm i's supply when it chooses price p is

$$s_i(p) = \kappa_i - \mu \prod_{j \neq i} F_j(p) \ . \tag{19}$$

[13] This case is discussed in detail in De Francesco and Salvadori (2013).

It follows that all firms have the same minimum price p_0 in equilibrium. To see this, suppose that firm i has the smallest minimum price of the n firms, p_0, and so makes profit $p_0\kappa_i$. Then for another firm j's lowest price, m say, we have

$$p_0\kappa_i = p_0 s_i(p_0) \geq m s_i(m) = m[\kappa_i - \mu \prod_{k\neq i} F_k(m)] = m\kappa_i$$

where the inequality follows since firm i could choose price m but this cannot increase its profit above $p_0\kappa_i$, while the final equality follows since m is firm j's lowest price. We deduce that all firms have the same minimum price, which is necessarily equal to firm 1's captive-to-reach ratio, $p_0 = (\kappa_1 - \mu)/\kappa_1$. This then determines the firms' profits uniquely as $\pi_i = \kappa_i p_0$ for $i = 1, ..., n$. As with the case of independent reach discussed in section 2, which closely resembles this situation with capacity constraints, these equilibrium profits are supported by CDFs such that firm i chooses its price with interval support $[p_0, p_i]$, where firm i's maximum price p_i is smaller for smaller firms. The two largest firms choose prices with support $[p_0, 1]$. Thus smaller firms only offer low prices, while larger firms offer the full range of prices. (See part (i) of Proposition 3 below for derivation of this in the case of triopoly.)

Another straightforward situation is when $\kappa_1 \geq 1$, so that one firm on its own has sufficient capacity to serve all demand.[14] In this case, a smaller firm either has no demand (if firm 1 offers a lower price) or can supply its entire capacity (if firm 1's price is higher). It follows that when firms use CDFs $\{F_1(\cdot), ..., F_n(\cdot)\}$, the supply functions are

$$\begin{aligned} s_1(p) &= 1 - \kappa_2 F_2(p) - ... - \kappa_n F_n(p) \\ s_i(p) &= \kappa_i(1 - F_1(p)) \text{ for } i = 2, ..., n \ . \end{aligned}$$

As in the previous situation where all firms have captive customers, each firm obtains profit $\pi_i = \kappa_i p_0$, where $p_0 = \kappa_1 - \mu$ is the largest firm's captive-to-reach ratio.[15]

Other situations are significantly more complicated, and to make further progress suppose as in section 4 there are three firms. Suppose also that $\kappa_1 < 1$ so that no single firm

[14] This situation is discussed as Case 1 in Hirata (2009)

[15] Even if $\kappa_1 > 1$ the large firm can only ever supply quantity 1, and so its reach is 1. Although profits are uniquely determined, the CDFs of the smaller firms are not pinned down uniquely, and together merely need to satisfy

$$\kappa_2 F_2(p) + ... + \kappa_n F_n(p) = 1 - \frac{\kappa_1 - \mu}{p} \ .$$

This scenario where one firm has enough capacity serve all demand is isomorphic to a model with considerations sets, where one firm is considered by all consumers and smaller firms do not overlap in their reach (i.e., all consumers either consider only the large firm or the large firm and one other firm).

can supply all demand. Then firm i's expected supply with price $p \leq 1$ is

$$\begin{aligned} s_i(p) \;\; &\equiv \;\; F_j(p)F_k(p)\min\{0, 1-\kappa_j-\kappa_k\} + (1-F_j(p))(1-F_k(p))\kappa_i \\ &+(1-F_j(p))F_k(p)\min\{\kappa_i, 1-\kappa_k\} + F_j(p)(1-F_k(p))\min\{\kappa_i, 1-\kappa_j\} \;, \end{aligned} \tag{20}$$

where i, j and k are distinct and F_j and F_k are the CDFs of its two rivals. For instance, if firm j undercuts firm i and firm k does not, firm i can supply the residual demand $1-\kappa_j$ or its capacity κ_i, whichever is the smaller. Note that $\kappa_i \leq 1-\kappa_j$ if and only if firm k (weakly) has some captive customers. If π_i is firm i's equilibrium profit, equilibrium requires that $ps_i(p) = \pi_i$ for any price in firm i's support, and $ps_i(p) \leq \pi_i$ for any price outside its support.

The proof of the following result describes equilibrium for all parameter values. (For a precise description of the threshold prices in the statement of the result, see the proof in the appendix.)

Proposition 3 *Assume three firms have capacities* $0 < \kappa_3 \leq \kappa_2 \leq \kappa_1 < 1$.
(i) If $\kappa_3 \geq \mu$ *or if* $\kappa_2 = \kappa_3$ *then all firms have the same minimum price* p_0 *and obtain profits* $\pi_i = p_0\kappa_i$, *where*

$$p_0 = 1 - \frac{\mu}{\kappa_1} \tag{21}$$

is firm 1's captive-to-reach ratio. Firms 1 and 2 choose prices in the interval $[p_0, 1]$ *while firm 3 chooses price in the lower interval* $[p_0, p_1]$ *where* $p_1 \leq 1$.
(ii) If $\kappa_3 < \min\{\mu, \kappa_2\}$ *then firms 1 and 2 choose prices in the interval* $[p_0, 1]$ *and obtain profit* $\pi_i = p_0\kappa_i$, *where* p_0 *is given in (21), while firm 3 obtains profit* $\pi_3 > p_0\kappa_3$ *and chooses its price in an interior range* $[\hat{p}, p_1]$, *where* $p_0 < \hat{p} < p_1 < 1$.

In contrast to the consideration set framework, where segmented price competition of the "overlapping duopoly" form was sometimes seen, here firms always compete head-to-head in prices. All firms offer prices in a lower price range when part (i) applies, while all firms offer prices in an intermediate range under part (ii). Likewise, here firms each offer prices from an interval, while with consideration sets a firm might choose high and low prices but not intermediate prices.

Unlike the model with consideration sets, here it is not possible that entry into a duopoly market can harm consumers. To see this, consider two incumbents, 1 and 2, with respective capacities κ_1 and $\kappa_2 \leq \kappa_1$. If $\kappa_1 + \kappa_2 \leq 1$ then there is no competition between these firms, consumers have zero surplus, and entry can only improve consumer surplus. Suppose then that $\kappa_1 + \kappa_2 > 1$ (and that $\kappa_1 < 1$), in which case industry profit without entry is

$$(\kappa_1 + \kappa_2)\frac{1 - \kappa_2}{\kappa_1} .$$

Suppose a third firm enters, with capacity κ_3. Since demand was already met, entry leaves welfare unchanged and consumers are harmed if and only industry profit rises. If $\kappa_3 \geq 1 - \kappa_2$ then no firm has any captive customers after entry, equilibrium price is $p \equiv 0$ and consumers benefit from entry. Otherwise, if $\kappa_3 < 1 - \kappa_2$ firm 1 has captive demand but firm 3 does not, so that part (ii) of Proposition 3 applies, with minimum price $p_0 = \frac{1-\kappa_2-\kappa_3}{\kappa_1}$. If π_3 denotes the entrant's profit, the change in profit due to entry is

$$(\kappa_1 + \kappa_2)p_0 + \pi_3 - (\kappa_1 + \kappa_2)\frac{1 - \kappa_2}{\kappa_1} = \pi_3 - \kappa_3\frac{\kappa_1 + \kappa_2}{\kappa_1} .$$

However, the entrant cannot make profit greater than κ_3 (which is the profit if it supplies its capacity at price $p = 1$), and so the change in profit is negative and consumers benefit from entry.

Unlike our main model with consideration sets, in the capacity framework our assumption of unit demand makes a significant difference to—and simplifies—the analysis. De Francesco and Salvadori (2015) have studied triopoly in the richer and more complex situation where aggregate demand is downward sloping (such that revenue is concave), under the assumption of an efficient rationing rule, and show that additional possibilities can then arise in equilibrium. For example, the smallest firm might have an atom at its maximum price, with the result that the two larger firm do not choose prices immediately above this maximum price and there are gaps in the set of prices offered in the market.[16]

6 Conclusions

The aim of this paper has been to explore, in a parsimonious framework with price-setting firms and homogeneous products, how patterns of consumer consideration matter for competitive outcomes. Different patterns of consideration not only yield different levels of equilibrium profits and consumer surplus, they can also determine pricing patterns in terms of which firms are direct price competitors. In this regard a distinction has emerged between

[16] They also find a configuration where the middle firm only chooses high prices, which is not possible with unit demand.

settings in which all firms are direct price competitors, and more segmented settings in which some firms always price high and others price low.

The former pricing pattern was a feature of the cases of duopoly, symmetry and independent reach discussed in Section 2. The equilibrium profit of each firm was its reach multiplied by the highest captive-to-reach ratio, and firms were direct competitors in the sense that there was a range of (low) prices that all firms might choose in equilibrium. With independent reach, however, only larger firms would ever choose prices higher than that range. The same is true in triopoly with parameters that accord with cases (i) or (ii) of Proposition 2. Case (ii), though, had the interesting feature of a middle range of prices that one of the firms would never offer.

A quite different pattern of profits and prices was found with nested reach (under the conditions of Proposition 1) and in case (iii) of the triopoly analysis. Some firms then make less profit than their reach multiplied by the highest captive-to-reach ratio, and price segmentation arises. In particular, there is no range of prices that all firms might charge: some only ever price high and some only ever price low. Nevertheless, the presence of the firms that price high bears down on the prices that lower-pricing firms charge in equilibrium.

The analysis of capacity-constrained price competition showed no such distinction, at least with three suppliers and unit demand, when there was always a range of prices offered by all firms. However, when one firm is small in the sense of part (ii) of Proposition 3, that firm prices only in a middle range, and its profit per unit of capacity exceeds that of the other firms, in contrast to any situation that arose in triopoly in the consideration set model. More generally, the consideration set model allows for richer patterns of competitive interaction than the capacity model. For instance, in the former scenario one firm's reach can lie inside another's, entry can leave the number of captives customers unaffected, it is possible for small firms to have a high proportion of captive customers, and a firm can have different overlaps with similar-sized rivals, while none of these features can happen in the capacity model.

Our analysis has taken as given the structure of consideration sets (and capacities), though we have considered some effects of market entry. This provides a basis for more general analysis that endogenizes patterns of competition by way of, for example, advertising and other marketing efforts by firms, and search by consumers.[17] A theme of such analysis will be that the profitability of advertising, say, depends on how it affects patterns of consumer awareness. For example, greater awareness of a firm's product might mean less profit for the firm, not more.

References

Acemoglu, D., K. Bimpikis, and A. Ozdaglar (2009): "Price and Capacity Competition," *Games and Economic Behavior*, 66, 1–26.

Armstrong, M., and J. Vickers (2018): "Discriminating Against Captive Customers," Oxford Department of Economics Discussion Paper 858.

Baye, M., D. Kovenock, and C. De Vries (1992): "It Takes Two to Tango: Equilibria in a Model of Sales," *Games and Economic Behavior*, 4, 493–510.

Baye, M., and J. Morgan (2001): "Information Gatekeepers on the Internet and the Competitiveness of Homogeneous Product Markets," *American Economic Review*, 91(3), 454–474.

Bulow, J., and J. Levin (2006): "Matching and Price Competition," *American Economic Review*, 96(3), 652–668.

Burdett, K., and K. Judd (1983): "Equilibrium Price Dispersion," *Econometrica*, 51(4), 955–969.

Butters, G. (1977): "Equilibrium Distributions of Sales and Advertising Prices," *Review of Economic Studies*, 44(3), 465–491.

Chen, Y., and M. Riordan (2007): "Price and Variety in the Spokes Model," *Economic Journal*, 117, 897–921.

——— (2008): "Price-Increasing Competition," *Rand Journal of Economics*, 39(4), 1042–1058.

[17]For instance, Ireland (1993) and McAfee (1994) study a model where firms first invest in reach and then compete in price, while Butters (1977) studies the situation where firms choose their reach and price simultaneously. In all of these papers the reach of each firm is assumed to be independent.

Chioveanu, I., and J. Zhou (2013): "Price Competition with Consumer Confusion," *Management Science*, 59(11), 2450–2469.

Dasgupta, P., and E. Maskin (1986): "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *Review of Economic Studies*, 53(1), 1–26.

De Francesco, M., and N. Salvadori (2013): "Bertrand-Edgeworth Competition in an Almost Symmetric Oligopoly," *Studies in Microeconomics*, 1(2), 213–219.

——— (2015): "Bertrand-Edgeworth Games Under Triopoly: The Payoffs," MPRA discussion paper 64638.

Hirata, D. (2009): "Asymmetric Bertrand-Edgeworth Oligopoly and Mergers," *The B.E. Journal of Theoretical Economics (Topics)*, 9(1).

Honka, E., A. Hortacsu, and M. A. Vitorino (2017): "Advertising, Consumer Awareness, and Choice: Evidence From the U.S. Banking Industry," *Rand Journal of Economics*, 48(3), 611–646.

Ireland, N. (1993): "The Provision of Information in a Bertrand Oligopoly," *Journal of Industrial Economics*, 41(1), 61–76.

Janssen, M., and E. Rasmusen (2002): "Bertrand Competition Under Uncertainty," *Journal of Industrial Economics*, 50(1), 11–21.

Manzini, P., and M. Mariotti (2014): "Stochastic Choice and Consideration Sets," *Econometrica*, 82(3), 1153–1176.

McAfee, R. P. (1994): "Endogneous Availability, Cartels, and Merger in an Equilibirium Price Dispersion," *Journal of Economic Theory*, 62, 24–47.

Narasimhan, C. (1988): "Competitive Promotional Strategies," *Journal of Business*, 61, 427–449.

Piccione, M., and R. Spiegler (2012): "Price Competition Under Limited Comparability," *Quarterly Journal of Economics*, 127(1), 97–135.

Rosenthal, R. (1980): "A Model in which an Increase in the Number of Sellers Leads to a Higher Price," *Econometrica*, 48(6), 1575–1579.

Sovinsky Goeree, M. (2008): "Limited Information and Advertising in the U.S. Personal Computer Industry," *Econometrica*, 76(5), 1017–1074.

Spiegler, R. (2006): "The Market for Quacks," *Review of Economic Studies*, 73(4), 1113–1131.

——— (2011): *Bounded Rationality and Industrial Organization*. Oxford University Press, Oxford, UK.

Szech, N. (2011): "Welfare in Markets where Consumers Rely on Word of Mouth," mimeo.

Varian, H. (1980): "A Model of Sales," *American Economic Review*, 70(4), 651–659.

Vives, X. (1999): *Oligopoly Theory: Old Ideas and New Tools*. MIT Press, Cambridge.

Technical Appendix

Proof of Proposition 1:

We construct an equilibrium of the stated form. The profit of the largest firm n is $\pi_n = \beta_n$, its number of captive customers, and denote the profit of smaller firms by π_i. In the highest interval $[p_{n-1}, 1]$ used by the two largest firms, these firms are sure to be undercut by all smaller rivals, and so in this price range their CDFs must satisfy

$$\beta_n + \beta_{n-1}(1 - F_{n-1}(p)) = \frac{\beta_n}{p} \ ; \ \beta_{n-1}(1 - F_n(p)) = \frac{\pi_{n-1}}{p} \ .$$

Since $F_n(p_{n-1}) = 0$ it follows that p_{n-1} and π_{n-1} are related as

$$\pi_{n-1} = \beta_{n-1} p_{n-1} \ .$$

We have $F_{n-1}(1) = 1$, while the largest firm has an atom at $p = 1$ with probability $1 - F_n(1) = \pi_{n-1}/\beta_{n-1} = p_{n-1}$.

In the lowest interval $[p_1, p_2]$ used by the two smallest firms, these firms are sure to undercut all larger rivals, and so in this range their CDFs must satisfy

$$\beta_2 + \sigma_1(1 - F_1(p)) = \frac{\pi_2}{p} \ ; \ \sigma_1(1 - F_2(p)) = \frac{\pi_1}{p}$$

and since $F_1(p_1) = F_2(p_1) = 0$ it follows that

$$\pi_1 = \sigma_1 p_1 \ ; \ \pi_2 = (\sigma_1 + \beta_2) p_1 \ .$$

Since $F_1(p_2) = 1$ we have $\pi_2 = \beta_2 p_2$, which combined with the previous expression for π_2 implies that

$$p_2 = \frac{\sigma_1 + \beta_2}{\beta_2} p_1 \ . \tag{22}$$

If there are just three firms, these are the two price intervals in the equilibrium. With more than three firms there are intermediate intervals, and in the interval $[p_i, p_{i+1}]$, where $1 < i < n - 1$, firms i and $i + 1$ are active and will be undercut by smaller rivals and undercut their larger rivals. Therefore, in this range their CDFs must satisfy

$$\beta_{i+1} + \beta_i(1 - F_i(p)) = \frac{\pi_{i+1}}{p} \ ; \ \beta_i(1 - F_{i+1}(p)) = \frac{\pi_i}{p} \ . \tag{23}$$

Since $F_{i+1}(p_i) = 0$ it follows that

$$\pi_i = \beta_i p_i \ .$$

An intermediate firm i, where $2 \leq i \leq n-1$, is active in both the intervals $[p_{i-1}, p_i]$ and $[p_i, p_{i+1}]$, and its CDF F_i needs to be continuous across the threshold price p_i. At the price p_i we therefore require that

$$\frac{\pi_{i-1}}{\beta_{i-1} p_i} = 1 - F_i(p_i) = \frac{1}{\beta_i}\left(\frac{\pi_{i+1}}{p_i} - \beta_{i+1}\right) , \tag{24}$$

where in the case of $i = 2$ we have written $\beta_1 = \sigma_1$. If we write $p_n = 1$ then we have $\pi_i = \beta_i p_i$ for all firms $1 \leq i \leq n$, and so for $2 \leq i \leq n-1$ expression (24) entails expression (9). This is a second-order difference equation in p_i where p_1 is free, p_2 is given in (22), and the terminal condition $p_n = 1$ serves to pin down p_1. It is clear from (22) and (9) that the sequence $p_1, p_2, p_3, ...$ is an increasing sequence of price thresholds. This completes the description of the candidate equilibrium.

We next show that no firm has an incentive to deviate from its described strategy. By construction, firm i is indifferent between choosing any price in the interval $[p_{i-1}, p_{i+1}]$, assuming its rivals follow the stated strategies. We need to check that a firm's profit is no higher if it chooses a price outside this interval. Consider first an upward price deviation, which is only relevant if $i < n-1$. If $i < n-2$ and firm i chooses a price above p_{i+2} is has no demand since firm $i+1$ is sure to set a lower price and all firm i's potential customers also consider firm $(i+1)$'s price. Suppose then that $i < n-1$ and firm i chooses a price $p \in [p_{i+1}, p_{i+2}]$, in which case it has demand β_i if its price is below the prices of both rivals $i+1$ and $i+2$. Therefore, from (23) its profit with such a price is

$$p\beta_i[1 - F_{i+1}(p)][1 - F_{i+2}(p)] = \frac{\beta_i \pi_{i+1}}{\beta_{i+1}^2}\left(\frac{\pi_{i+2}}{p} - \beta_{i+2}\right) = p_{i+1}\frac{\beta_i \beta_{i+2}}{\beta_{i+1}}\left(\frac{p_{i+2}}{p} - 1\right) .$$

This profit decreases from $\pi_i = \beta_i p_i$ at $p = p_{i+1}$ to zero at $p = p_{i+2}$. We deduce that firm i cannot increase its profit by choosing a price above p_{i+1}.

Next consider a downward price deviation, so that firm i chooses a price below p_{i-1} (which is only relevant when $i > 2$). Suppose that this firm chooses a price in the interval $[p_j, p_{j+1}]$, where $j \leq i-2$. The firm will undercut all firms larger than firm $j+1$, and so obtain demand at least $\beta_{j+2} + ... + \beta_i$. It will also serve the segment β_{j+1} if it undercuts firm $j+1$ and it will additionally serve the segment β_j if it undercuts both firms j and $j+1$. Putting this together implies that the firm's profit with price $p \in [p_j, p_{j+1}]$ is

$$p\left\{\beta_{j+2} + ... + \beta_i + (1 - F_{j+1}(p))(\beta_{j+1} + \beta_j(1 - F_j(p))\right\} . \tag{25}$$

Given the CDFs in (23), this profit is a convex function of p and so must be maximized in this range either at p_j or at p_{j+1}. Therefore, we can restrict our attention to deviations by firm $i > 2$ to the threshold prices $\{p_1, p_2, ..., p_{i-2}\}$. If it chooses price p_j where $2 \le j \le i-2$, expression (25) implies its profit is

$$p_j \left\{\beta_{j+1} + ... + \beta_i + \beta_j(1 - F_j(p_j))\right\} \ .$$

Expression (24) implies that $\beta_j(1 - F_j(p_j))$ is equal to $\beta_{j+1}(\frac{p_{j+1}}{p_j} - 1)$, in which case the above deviation profit with price p_j is

$$p_j \left(\beta_{j+1} + ... + \beta_i + \beta_{j+1}(\frac{p_{j+1}}{p_j} - 1)\right) = \beta_{j+1}p_{j+1} + (\beta_{j+2} + ... + \beta_i)p_j \ . \tag{26}$$

One can check that expression (26) holds also for $j = 1$. We need to show that (26) is no higher than firm i's equilibrium profit, which is $\pi_i = \beta_i p_i$. We do this in two steps: (i) we show that (26) is increasing in j given i, so that $j = i - 2$ is the most tempting of these deviations for firm i, and (ii) we show (26) is below $\beta_i p_i$ when $j = i - 2$.

To show (i), suppose that $i \ge 4$, which is the only relevant case, and suppose that $1 \le j \le i - 3$. Then firm i's deviation profit with price p_{j+1} from (26) is

$$\begin{aligned}
\beta_{j+2}p_{j+2} + (\beta_{j+3} + ... + \beta_i)p_{j+1} &= \beta_{j+1}p_j + \beta_{j+2}p_{j+1} + (\beta_{j+3} + ... + \beta_i)p_{j+1} \\
&\ge \beta_{j+1}p_j + \beta_{j+2}p_{j+1} + (\beta_{j+3} + ... + \beta_i)p_{j+1} - (\beta_{j+2} - \beta_{j+1})(p_{j+1} - p_j) \\
&= \beta_{j+1}p_{j+1} + \beta_{j+2}p_j + (\beta_{j+3} + ... + \beta_i)p_{j+1} \ge \beta_{j+1}p_{j+1} + (\beta_{j+2} + ... + \beta_i)p_j
\end{aligned}$$

where the final expression is the firm's deviation profit with price p_j, which proves claim (i). (Here, the first equality follows from (9), the first inequality follows from (8) and the fact that $\{p_j\}$ is an increasing sequence, while the final inequality follows from $\{p_j\}$ being an increasing sequence.)

To show claim (ii), suppose that $i \ge 3$ which is the only relevant case, and observe that

$$\begin{aligned}
\beta_i p_i &= \beta_{i-1}p_{i-2} + \beta_i p_{i-1} \\
&\ge \beta_{i-1}p_{i-2} + \beta_i p_{i-1} - (\beta_i - \beta_{i-1})(p_{i-1} - p_{i-2}) \\
&= \beta_{i-1}p_{i-1} + \beta_i p_{i-2}
\end{aligned}$$

where the final expression is (26) when $j = i - 2$. (Here, the first equality follows from (9) and the inequality follows from $\{\beta_i\}$ being an increasing sequence.) This completes the proof that the stated strategies constitute an equilibrium.

Proof of Proposition 2:

First consider the situation where $\alpha_{12}\sigma_3 > |t_1 - t_2|$, so that either part (i) or (ii) applies, in which case we will construct an equilibrium where each firm chooses the lowest price $p_0 = \alpha_1/\sigma_1$ and firm i's profit is $\pi_i = \sigma_i p_0$.

Note that demand (10) can be written in the form

$$q_i(p) = \alpha_i + \frac{1}{\alpha}[\alpha(1 - F_j(p)) + \alpha_{ik}][\alpha(1 - F_k(p)) + \alpha_{ij}] - \frac{\alpha_{ij}\alpha_{ik}}{\alpha} ,$$

and so the equilibrium condition $pq_i(p) \equiv \pi_i = \sigma_i p_0$ for prices in firm i's support can be expressed in the factorized form

$$z_i(p) = [\alpha(1 - F_j(p)) + \alpha_{ik}][\alpha(1 - F_k(p)) + \alpha_{ij}] , \tag{27}$$

where we have written

$$z_i(p) \equiv \alpha\left[\sigma_i \frac{p_0}{p} - \alpha_i\right] + \alpha_{ij}\alpha_{ik} = (\alpha + \alpha_{ij})(\alpha + \alpha_{ik}) - \alpha\sigma_i\left[1 - \frac{p_0}{p}\right] . \tag{28}$$

Here, $z_i(p)$ is a decreasing function of p and $z_i(1) > 0$, so $z_i(p)$ is positive throughout the range $[p_0, 1]$. Note that the firm with the smallest z_i/σ_i is the firm with the largest t_i in (11). In particular, (27) implies that for a price in the support of all three firms the function $[\alpha(1 - F_i(p)) + \alpha_{jk}]z_i(p)$ is the same for each firm, and firm i's CDF satisfies

$$\alpha[1 - F_i(p)] + \alpha_{jk} = \sqrt{\frac{z_j(p)z_k(p)}{z_i(p)}} . \tag{29}$$

(As required, $F_i(p_0) = 0$ for each firm.) Define the functions from (29):

$$\phi_i(p) = 1 - \frac{1}{\alpha}\left[\sqrt{\frac{z_j(p)z_k(p)}{z_i(p)}} - \alpha_{jk}\right] . \tag{30}$$

When these functions are increasing, they will be the CDFs $F_i(p)$ used in the range where all firms are active. However, there are circumstances—covered by part (ii) of this proposition—where one of these ϕ_i can decrease, when we will need to "iron" ϕ_i to make it a valid CDF.

Differentiating ϕ_i in (30) yields

$$\begin{aligned} \phi_i'(p) &= -\frac{1}{2\alpha}\sqrt{\frac{z_j(p)z_k(p)}{z_i(p)}}\left[\frac{z_j'}{z_j} + \frac{z_k'}{z_k} - \frac{z_i'}{z_i}\right] \\ &= \frac{p_0}{2p^2}\sqrt{\frac{z_j(p)z_k(p)}{z_i(p)}}\left[\frac{\sigma_j}{z_j} + \frac{\sigma_k}{z_k} - \frac{\sigma_i}{z_i}\right] , \end{aligned} \tag{31}$$

where the second equality follows from (28). It follows that if $\phi_i'(p)$ is zero or negative it must be the firm with the largest σ_i/z_i, i.e. the largest t_i. At a price p such that $\phi_i'(p) = 0$ we have

$$\frac{\sigma_i}{z_i} = \frac{\sigma_j}{z_j} + \frac{\sigma_k}{z_k} \ ,$$

in which case differentiating (31) shows that ϕ_i'' at this price has the sign of

$$-\frac{\sigma_j}{z_j^2}z_j' - \frac{\sigma_k}{z_k^2}z_k' + \frac{\sigma_i}{z_i^2}z_i' = \frac{\alpha p_0}{p^2}\left[\frac{\sigma_j^2}{z_j^2} + \frac{\sigma_k^2}{z_k^2} - \frac{\sigma_i^2}{z_i^2}\right]$$

$$= \frac{\alpha p_0}{p^2}\left[\frac{\sigma_j^2}{z_j^2} + \frac{\sigma_k^2}{z_k^2} - \left(\frac{\sigma_j}{z_j} + \frac{\sigma_k}{z_k}\right)^2\right] < 0$$

so that ϕ_i is single-peaked in price, while ϕ_j and ϕ_k are everywhere increasing in price.

Note that condition (15) is equivalent to

$$\alpha_{12}z_3(1) \geq \alpha_{13}z_2(1) \ . \tag{32}$$

We claim that there exists a unique price $p_1 \in (p_0, 1]$ such that $\phi_3(p_1) = 1$. To see this, consider the continuous function

$$\zeta(p) \equiv z_1(p)z_2(p) - \alpha_{12}^2 z_3(p) \ , \tag{33}$$

which satisfies $\zeta(p_0) > 0 \geq \zeta(1)$. (Since $z_1(1) = \alpha_{12}\alpha_{13}$, the latter inequality follows from (32) and is strict if that inequality is strict.) Therefore, there exists a price $p_1 \in (p_0, 1]$ that satisfies $\zeta(p_1) = 0$, which by (30) is equivalent to $\phi_3(p_1) = 1$. There can be no other root of ζ in $(p_0, 1]$ since $p^2\zeta(p)$ is quadratic in p and so has no more than two roots, and having two roots in $(p_0, 1]$ is not compatible with $\zeta(p_0) > 0 \geq \zeta(1)$. Therefore there is a unique p_1 in $(p_0, 1]$ where $\zeta(p_1) = 0$ and for p in that range $\zeta(p)$ has the sign of $(p_1 - p)$.

(i) Now suppose that (16) holds, in which case we construct an equilibrium of the stated form where firms 1 and 2 choose prices in $[p_0, 1]$ and firm 3 chooses price in $[p_0, p_1]$. At price p_1 where $\zeta = 0$, expression (30) implies for $i, j \in \{1, 2\}$ with $i \neq j$ that

$$\begin{aligned} 1 - \phi_i(p_1) &= \frac{1}{\alpha}\left(\frac{z_j(p_1)}{\alpha_{12}} - \alpha_{j3}\right) \\ &= \frac{1}{\alpha_{12}}\left(\frac{\sigma_j p_0}{p_1} - \alpha_j\right) \\ &\geq \frac{a_j}{\alpha_{12}}\left(\frac{1}{p_1} - 1\right) \geq 0 \ . \end{aligned} \tag{34}$$

We have $\phi_i'(p_1) \geq 0$ in (31), in which case ϕ_i is increasing in the range $[p_0, p_1]$, if and only if

$$0 \leq \left[\frac{\sigma_j}{z_j} + \frac{\sigma_3}{z_3} - \frac{\sigma_i}{z_i} \right] z_1 z_2 = \sigma_j z_i + \sigma_3 \alpha_{12}^2 - \sigma_i z_j \ ,$$

where the equality follows from $\zeta(p_1) = 0$. Since $\sigma_i z_j - \sigma_j z_i = (\alpha + \alpha_{12})(t_i - t_j)$, this is equivalent to the condition

$$\alpha_{12}\sigma_3 \geq \frac{\alpha + \alpha_{12}}{\alpha_{12}}(t_i - t_j) \ . \tag{35}$$

Therefore if (16) holds the three CDFs $F_i = \phi_i$ in (30) increase from zero in the range $[p_0, p_1]$, where $F_3(p_1) = 1$ and $F_1(p_1) \leq 1$ and $F_2(p_1) \leq 1$.

For prices above p_1 firms 1 and 2 compete alone, always being undercut by firm 3. Therefore, in the range $[p_1, 1]$ their two CDFs satisfy

$$\alpha_1 + \alpha_{12}(1 - F_2(p)) = \sigma_1 \frac{p_0}{p} \ ; \ \alpha_2 + \alpha_{12}(1 - F_1(p)) = \sigma_2 \frac{p_0}{p} \ . \tag{36}$$

From (34) these CDFs join continuously at $p = p_1$ with F_1 and F_2 in the lower range $[p_0, p_1]$, and are increasing in the range $[p_1, 1]$ such that $F_2(1) = 1$ and firm 1 has an atom at $p = 1$ with probability $1 - F_1(1) = \frac{1}{\alpha_{12}}(\sigma_2 \frac{\alpha_1}{\sigma_1} - \alpha_2) \geq 0$.

The final requirement for this to constitute an equilibrium is that firm 3 does not wish to set a price above p_1 (if $p_1 < 1$). If it did so its profit would be

$$p \times \{\alpha_3 + \alpha_{13}(1 - F_1(p)) + \alpha_{23}(1 - F_2(p)) + \alpha(1 - F_1(p))(1 - F_2(p))\} \ ,$$

where F_1 and F_2 are given in (36), which is a convex function of p since the coefficient on $1/p$ is positive. Therefore, if firm 3 has no incentive to choose the price $p = 1$ it also has no incentive to choose any price in $(p_1, 1)$. Firm 1's atom at $p = 1$ is $\frac{1}{\alpha_{12}}(\sigma_2 \frac{\alpha_1}{\sigma_1} - \alpha_2)$, which from (32) is no greater than $\frac{1}{\alpha_{13}}(\sigma_3 \frac{\alpha_1}{\sigma_1} - \alpha_3)$. Therefore, firm 3's profit if it chooses $p = 1$ does not exceed

$$\alpha_3 + \alpha_{13} \left[\frac{1}{\alpha_{13}}(\sigma_3 \frac{\alpha_1}{\sigma_1} - \alpha_3) \right] = \sigma_3 p_0$$

which is the firm's profit in the candidate equilibrium.

(ii) Suppose next that (17) holds. Since (16) no longer holds, expression (35) shows that $\phi_i'(p_1) < 0$ for the firm $i = 1, 2$ which has the higher t_i. We claim that $\phi_i(p_1) > 0$ if and only if $\frac{t_i - t_j}{\alpha_{12}} < \sigma_3$. To see this, note from (34) that $\phi_i(p_1) > 0$ if and only if

$$\frac{p_0}{p_1} < \frac{\alpha_{12} + \alpha_j}{\sigma_j} \iff \zeta(\frac{\sigma_j p_0}{\alpha_{12} + \alpha_j}) > 0 \ ,$$

where the equivalence comes from $\zeta(p)$ in (33) having the sign of $(p_1 - p)$ for $p \in (p_0, 1]$. By using the right-hand expression for z in (28), and noting that when $p = \frac{\sigma_j p_0}{\alpha_{12}+\alpha_j}$ we have $1 - \frac{p_0}{p} = \frac{t_i}{\sigma_1\sigma_2}$, we obtain

$$\zeta(\frac{\sigma_j p_0}{\alpha_{12} + \alpha_j}) = \alpha_{12}\alpha\frac{\alpha + \alpha_{j3}}{\sigma_j}[t_j - t_i + \alpha_{12}\sigma_3] \ ,$$

which yields the claim.

Since $\phi_i(p_1) > 0$, let $\hat{p}$ be the unique price in (p_0, p_1) such that $\phi_i(\hat{p}) = \phi_i(p_1)$. This price is unique since ϕ_i is a single-peaked function, and ϕ_i is increasing in the range $[p_0, \hat{p}]$. Expression (34) implies that $\phi_i(p_1)$ and $\phi_j(p_1)$ are both below 1 while by construction $\phi_3(p_1) = 1$. Since only ϕ_i can be non-monotonic, all three ϕ functions in (30) are increasing and below 1 in the range $[p_0, \hat{p}]$, and these functions comprise the equilibrium CDFs in this price range.

In the intermediate price range $[\hat{p}, p_1]$ only firms j and 3 are active, and are undercut by firm i with probability $\phi_i(\hat{p}) = \phi_i(p_1)$. Therefore, from (10) the CDFs F_j and F_3 in this range satisfy

$$\alpha_j + \alpha_{ij}(1 - \phi_i(\hat{p})) + \alpha_{j3}(1 - F_3(p)) + \alpha(1 - \phi_i(\hat{p}))(1 - F_3(p)) = \sigma_j\frac{p_0}{p} \qquad (37)$$

$$\alpha_3 + \alpha_{i3}(1 - \phi_i(\hat{p})) + \alpha_{j3}(1 - F_j(p)) + \alpha(1 - \phi_i(\hat{p}))(1 - F_j(p)) = \sigma_3\frac{p_0}{p} \ . \qquad (38)$$

These functions are increasing in p, and at $p = p_1$ they coincide with ϕ_j and ϕ_3 in (30), which we know satisfy $\phi_j(p_1) \leq 1$ and $\phi_3(p_1) = 1$.

The remainder of the proof closely follows that for part (i). For prices in the range $[p_1, 1]$, only firms 1 and 2 are active, and their CDFs are given by (36), which again join up correctly across the threshold $p = p_1$. It remains to show that firm 3 does not wish to set a price above p_1 (if $p_1 < 1$) and that firm i does not wish to set a price in the intermediate range $[\hat{p}, p_1]$. The first of these requirements is satisfied as in the proof of part (i). For the second, firm i's profit with a price in $[\hat{p}, p_1]$ would be

$$p \times \{\alpha_i + \alpha_{i3}(1 - F_3(p)) + \alpha_{ij}(1 - F_j(p)) + \alpha(1 - F_j(p))(1 - F_3(p))\} \ ,$$

where F_j and F_3 are given in (38)–(37). This deviation profit is again convex in p, and by construction it is equal at the endpoints $p = \hat{p}$ and $p = p_1$. Therefore, this profit cannot be higher in the interior of $[\hat{p}, p_1]$, which completes the proof for part (ii).

(iii) The principal part of the proof of this final part of the proposition consists of demonstrating the following lemma.

Lemma 2 *Suppose that firm i has the highest t in (11) and that the two remaining firms satisfy $\sigma_j \geq \sigma_k$. Then if*

$$\alpha_{ij}\sigma_k \leq t_i - t_j \ , \tag{39}$$

there are prices p_0 and p_1, with $p_0 < p_1 \leq 1$, such that it is an equilibrium for firms j and k to choose prices in the lower range $[p_0, p_1]$ and for firms i and j to choose prices in the upper range $[p_1, 1]$.

Proof of Lemma 2: There are two subcases to prove, depending on which of firms i and j has an atom at $p = 1$. Suppose first that it is firm i which has the atom at $p = 1$, and so has profit $\pi_i = \alpha_i$. Since firms j and k have p_0 as their minimum price in the proposed equilibrium, their profits are $\pi_j = \sigma_j p_0$ and $\pi_k = \sigma_k p_0$. In the candidate equilibrium firms j and k will undercut firm i in the range $[p_0, p_1]$, and from (10) their CDFs satisfy

$$\alpha_j + \alpha_{ij} + (\alpha_{jk} + \alpha)(1 - F_k(p)) = \sigma_j \frac{p_0}{p} \ ; \tag{40}$$

$$\alpha_k + \alpha_{ik} + (\alpha_{jk} + \alpha)(1 - F_j(p)) = \sigma_k \frac{p_0}{p} \ . \tag{41}$$

Here, since $\sigma_k \leq \sigma_j$ we see that F_k reaches 1 before F_j does. Since firm k's maximum price in this candidate equilibrium is p_1, (40) implies that p_0 and p_1 are related by

$$\alpha_j + \alpha_{ij} = \sigma_j \frac{p_0}{p_1} \tag{42}$$

(so that $p_1 > p_0$).

For prices in the upper range $[p_1, 1]$ firms i and j compete and are sure to be undercut by firm k, so their CDFs satisfy

$$\alpha_i + \alpha_{ij}(1 - F_j(p)) = \frac{\alpha_i}{p} \ ; \ \alpha_j + \alpha_{ij}(1 - F_i(p)) = \sigma_j \frac{p_0}{p} \ . \tag{43}$$

Here, (42) implies that $F_i(p_1) = 0$ as required, and we also have $F_j(1) = 1$. In order for $F_j(\cdot)$ to be continuous at the threshold price p_1, we require that

$$\frac{1}{\alpha_{jk} + \alpha}\left(\sigma_k \frac{p_0}{p_1} - \alpha_k - \alpha_{ik}\right) = \frac{1}{\alpha_{ij}}\left(\frac{\alpha_i}{p_1} - \alpha_i\right) \ .$$

From (42), this implies that the minimum price is $p_0 = P$ where

$$P = \frac{\alpha_i(\alpha_j + \alpha_{ij})}{\alpha_i\sigma_j + \alpha_{ij}(\sigma_j - \sigma_k)} \ , \tag{44}$$

which in turn implies that

$$p_1 = \frac{\alpha_i\sigma_j}{\alpha_i\sigma_j + \alpha_{ij}(\sigma_j - \sigma_k)} \ .$$

(Note that the denominator in these expressions is positive given $\sigma_k \leq \sigma_j$, and we have $p_0 < p_1 \leq 1$ where $p_1 = 1$ if $\sigma_j = \sigma_k$.) In this candidate equilibrium we require that firm i is willing to price all the way up to $p = 1$, i.e., that $F_i(1) \leq 1$, which from (43) requires that

$$P \geq \frac{\alpha_j}{\sigma_j} \ .$$

There are two possible deviations that we need to check are unprofitable to make this an equilibrium. First, firm i cannot have an incentive to choose a low price in $[p_0, p_1)$. Using the convexity argument at the end of the proof for part (i), it suffices to check it is not profitable for it to choose the lowest price $p_0 = P$. Since its profit at price P is $\sigma_i P$, we require that $P \leq \alpha_i/\sigma_i$, which is equivalent to condition (39).

Second, if $p_1 < 1$ we need to check that firm k does not have an incentive to choose a high price in $(p_1, 1]$. Again, its profit is convex in p in this range, and so it has no such incentive provided that its profit at $p = 1$ is no higher than $\sigma_k P$. But its profit at $p = 1$ is $\alpha_k + \alpha_{ik}(1 - F_i(1))$, which is no greater than $\alpha_k + \alpha_{ik}$, and so a sufficient condition for this deviation to be unprofitable is that $\sigma_k P \geq \alpha_k + \alpha_{ik}$. But

$$\begin{aligned}
\sigma_k P - (\alpha_k + \alpha_{ik}) &= \sigma_j - \sigma_k(1 - P) - (\alpha_j + \alpha_{ij}) \\
&= \sigma_j - \sigma_k(1 - P) - (\sigma_j + \frac{\alpha_{ij}}{\alpha_i}(\sigma_j - \sigma_k))P \\
&= (\sigma_j - \sigma_k)(1 - \frac{\alpha_i + \alpha_{ij}}{\alpha_i}P) \\
&\geq (\sigma_j - \sigma_k)(1 - \frac{\alpha_i + \alpha_{ij}}{\sigma_i}) \geq 0 \ .
\end{aligned}$$

Here, the second equality follows from the definition of P in (44), while the first inequality follows from $P \leq \alpha_i/\sigma_i$ which is equivalent to (39).

The second sub-case in the lemma has firm j with the atom at 1, so that $\pi_j = \alpha_j$, which implies the minimum price is $p_0 = \alpha_j/\sigma_j$, in which case (40) implies that the threshold price where firm k drops out is $p_1 = \frac{\alpha_j}{\alpha_j + \alpha_{ij}}$. For firm k to drop out before firm j, we again require that $\sigma_k \le \sigma_j$. If π_i denotes firm i's profit, in the range $[p_1, 1]$ the CDFs need to satisfy

$$\alpha_i + \alpha_{ij}(1 - F_j(p)) = \frac{\pi_i}{p} \ ; \ \alpha_j + \alpha_{ij}(1 - F_i(p)) = \frac{\alpha_j}{p} \ . \tag{45}$$

(Note that $F_i = 0$ at p_1 and $F_i = 1$ at $p = 1$, as required.) For F_j to be continuous at the threshold price $p_1 = \frac{\alpha_j}{\alpha_j + \alpha_{ij}}$ we require

$$\frac{1}{\alpha_{jk} + \alpha}\left(\frac{\sigma_k}{\sigma_j}(\alpha_j + \alpha_{ij}) - \alpha_k - \alpha_{ik}\right) = \frac{1}{\alpha_{ij}}\left(\frac{\alpha_j + \alpha_{ij}}{\alpha_j}\pi_i - \alpha_i\right) \ .$$

Noting that the left-hand side above is simply $(\sigma_j - \sigma_k)/\sigma_j$, this condition requires that

$$\pi_i = \frac{\alpha_i \alpha_j}{\sigma_j P} \ , \tag{46}$$

where P was given in (44). For firm i not to want to choose the minimum price $p_0 = \alpha_j/\sigma_j$ we require

$$\frac{\alpha_i \alpha_j}{\sigma_j P} \ge \sigma_i \frac{\alpha_j}{\sigma_j}$$

so that $P \le \alpha_i/\sigma_i$ which is equivalent to (39). For firm j to be willing to price up to 1 in (45) we require that $\pi_i \ge \alpha_i$, which from (46) requires that

$$P \le \frac{\alpha_j}{\sigma_j} \ .$$

The final condition to check is that firm k has no incentive to set a high price. Firm k's profit in this candidate equilibrium is $\sigma_k \alpha_j/\sigma_j$, and if it chooses $p = 1$ it obtains profit $\alpha_k + \alpha_{jk}(1 - F_j(1))$. Therefore, we require that

$$\sigma_k \frac{\alpha_j}{\sigma_j} - \alpha_k \ge \alpha_{jk}(1 - F_j(1)) = \frac{\alpha_i \alpha_{jk}}{\alpha_{ij}}\left(\frac{\alpha_j}{\sigma_j P} - 1\right) = \frac{\alpha_{jk}}{\alpha_j + \alpha_{ij}}(\alpha_j - \alpha_i - \sigma_k \frac{\alpha_j}{\sigma_j}) \tag{47}$$

where the first equality follows from (45). This inequality holds since

$$\begin{aligned}
&(\alpha_j + \alpha_{ij})(\sigma_k \frac{\alpha_j}{\sigma_j} - \alpha_k) - \alpha_{jk}(\alpha_j - \alpha_i - \sigma_k \frac{\alpha_j}{\sigma_j}) \\
&\quad = (\sigma_j - \alpha)\sigma_k \frac{\alpha_j}{\sigma_j} - \alpha_{jk}(\alpha_j - \alpha_i) - \alpha_k(\alpha_j + \alpha_{ij}) \\
&\quad \ge \alpha_j \sigma_k - (\alpha_{jk} + \alpha)(\alpha_j - \alpha_i) - \alpha_k(\alpha_j + \alpha_{ij}) \\
&\quad = \alpha_j \alpha_{ik} + \alpha_i(\alpha_{jk} + \alpha) - \alpha_k \alpha_{ij} \ge \alpha_j \alpha_{ik} \ge 0 \ .
\end{aligned}$$

Here, the first inequality uses the implication of $P < \alpha_j/\sigma_j$ that $\alpha_j - \alpha_i \ge \sigma_k \frac{\alpha_j}{\sigma_j}$ (which can be seen from the final equality in (47).) The second inequality uses assumption (39).

In sum, provided that condition (39) holds, we have found the desired equilibrium. If P in (44) is above α_j/σ_j then firm i has the atom at $p = 1$ and minimum price offered by firms j and k is equal to P. If P is below α_j/σ_j then it is firm j which has the atom at $p = 1$, and the minimum price offered by firms j and k is equal to α_j/σ_j. This completes the proof of the lemma.

Now return to the proof of part (iii), so that firms 1, 2 and 3 are as described in the statement of the Proposition and (18) holds. First suppose that $t_2 \geq t_1$, so that (18) entails $t_2 > t_1 + \alpha_{12}\sigma_3$. Then we claim that $\sigma_3 \leq \sigma_1$. For if not we would have $t_2 - t_1 \geq \alpha_{12}\sigma_3$ and $t_1 - t_2 \geq \alpha_{12}(\sigma_2 - \sigma_1)$, where the second inequality follows since

$$t_1 - t_2 = (\sigma_2 - \alpha_2 - \alpha_{12})\sigma_1 - (\sigma_1 - \alpha_1 - \alpha_{12})\sigma_2 = (\alpha_1 + \alpha_{12})\sigma_2 - (\alpha_2 + \alpha_{12})\sigma_1 \geq \alpha_{12}(\sigma_2 - \sigma_1) ,$$

where the final inequality is due to the assumption that firm 1 has the highest captive-to-reach ratio. These two inequalities imply

$$\sigma_1 \geq \sigma_2 + \sigma_3 \geq \sigma_2 + \sigma_1$$

if $\sigma_3 \geq \sigma_1$, which is a contradiction. Therefore, since $\sigma_3 \leq \sigma_1$ we have

$$t_2 > t_1 + \alpha_{12}\sigma_3 = t_1 + t_3 - \alpha\sigma_3 \geq t_1 + t_3 - \alpha\sigma_1 = \alpha_{23}\sigma_1 + t_3 \geq t_3 \ .$$

Therefore, firm 2 has the largest t of all three firms and $\sigma_3 \leq \sigma_1$, and so we can apply Lemma 1 (with $i = 2$, $j = 1$ and $k = 3$ in the statement of Lemma 2).

Second suppose instead that $t_1 \geq t_2$, so that (18) entails $t_1 \geq t_2 + \alpha_{12}\sigma_3$. Again, if $\sigma_3 \leq \sigma_2$ the same argument as above shows that $t_1 \geq t_3$, and Lemma 1 applies (with $i = 1$, $j = 2$ and $k = 3$ in the statement of the lemma). The remaining case has $\sigma_3 \geq \sigma_2$. Note that for any i and j we have

$$t_i - t_j + \alpha_{ij}(\sigma_i - \sigma_j) = \sigma_i\sigma_j(\rho_i - \rho_j) \ , \tag{48}$$

and using this implies that

$$t_1 - t_3 = \alpha_{13}\left(\frac{\sigma_1\sigma_3}{\alpha_{13}}(\rho_1 - \rho_3) + \sigma_3 - \sigma_1\right) \geq \alpha_{13}\left(\frac{\sigma_1\sigma_2}{\alpha_{12}}(\rho_1 - \rho_2) + \sigma_3 - \sigma_1\right)$$
$$= \alpha_{13}\left(\sigma_1 - \sigma_2 + \frac{t_1 - t_2}{\alpha_{12}} + \sigma_3 - \sigma_1\right) \geq \alpha_{13}\left(2\sigma_3 - \sigma_2\right) \geq \alpha_{13}\sigma_2 \ .$$

(Here, the first inequality follows from (15), the second equality follows from (48), the second inequality follows from assumption $t_1 \geq t_2 + \alpha_{12}\sigma_3$, and the final inequality follows since $\sigma_3 \geq \sigma_2$.) Therefore $t_1 - t_3 \geq \alpha_{13}\sigma_2$, and in particular t_1 is the largest t. Thus the conditions of Lemma 2 are satisfied (with $i = 1$, $j = 3$ and $k = 2$ in the statement of the lemma), and firm 2 operates only in the lower range while firm 3 uses the full range. Regardless of whether firm 1 or firm 2 has the larger t_i, one can check the the minimum price p_0 in the lemma is weakly below firm 1's captive-to-reach ratio. This completes the proof of part (iii).

Proof of Proposition 3:

(i) First suppose that $\kappa_3 \geq \mu$, in which case the discussion in the text shows that each firm's supply is (19) and each firm has the same minimum price p_0 in (21) and firm i makes profit $\pi_i = \kappa_i p_0$. The CDFs which support this equilibrium are derived as follows. With supply given by (19) and profit $\pi_i = \kappa_i p_0$, a price p in firm i's support satisfies

$$\mu F_j(p) F_k(p) = \kappa_i \left(1 - \frac{p_0}{p}\right) . \tag{49}$$

For a price p in all three supports, $\kappa_i F_i(p)$ must be the same for all firms, and equal to

$$\kappa_i F_i(p) = \sqrt{\frac{\kappa_1\kappa_2\kappa_3}{\mu}\left(1 - \frac{p_0}{p}\right)} . \tag{50}$$

Here, all CDFs are zero at $p = p_0$ and are increasing functions of p above p_0. Firm 3 has the largest F in (50) and $F_3(p)$ reaches 1 at the price p_1 given by

$$p_1 = \frac{p_0}{1 - \mu\frac{\kappa_3}{\kappa_1\kappa_2}} = \frac{1 - \mu\frac{1}{\kappa_1}}{1 - \mu\frac{\kappa_3}{\kappa_1\kappa_2}} .$$

Note that $p_1 \leq 1$ above, with strict inequality if $\kappa_2 > \kappa_3$.

If $\kappa_2 = \kappa_3$ (but still assuming $\kappa_3 \geq \mu$) then $p_1 = 1$, and the equilibrium involves each firm's CDF being given by (50) throughout the range $[p_0, 1]$. Here, $F_2(p) \equiv F_3(p)$ and $F_2(1) = F_3(1) = 1$, while firm 1 has an atom at $p = 1$ with probability $\frac{\kappa_1 - \kappa_2}{\kappa_1}$.

If $\kappa_3 < \kappa_2$ (but still assuming $\kappa_3 \geq \mu$) then $p_1 < 1$ and above p_1 the two largest firms compete as duopolists, sure to be undercut by firm 3. From (49) their CDFs satisfy

$$\kappa_1 F_1(p) = \kappa_2 F_2(p) = \frac{\kappa_1\kappa_2}{\mu}\left(1 - \frac{p_0}{p}\right) . \tag{51}$$

Therefore, $F_2(1) = 1$ while firm 1 has an atom with probability $\frac{\kappa_1-\kappa_2}{\kappa_1}$ at $p = 1$. The final condition to check is that firm 3 does not wish to deviate to a high price $p > p_1$. If it did so, (51) implies that its change in profit would be

$$p[\kappa_3 - \mu F_1(p)F_2(p)] - p_0\kappa_3 = \left(1 - \frac{p_0}{p}\right)\left[\kappa_3 p - \frac{\kappa_1\kappa_2}{\mu}(p - p_0)\right] .$$

However, the term $[\cdot]$ is zero at $p = p_1$ and decreasing in p since

$$\kappa_1\kappa_2 - \mu\kappa_3 = (\kappa_1 - \kappa_3)(\kappa_2 - \kappa_3) + \kappa_3(1 - 2\kappa_3) > 0 .$$

(Here, $1 - 2\kappa_3 > 0$ since we have $\kappa_2 + \kappa_3 < 1$.) This completes the proof of part (i) except for the case where $\kappa_3 = \kappa_2$ and $\kappa_3 < \mu$, which we deal with later.

(ii) Now suppose that $\kappa_3 < \mu$, so that the smallest firm has no captive customers. We divide the proof into two subcases, depending on whether or not firm 2 has captive customers. Suppose first that firms 1 and 2 have captive customers but firm 3 does not, so $\kappa_3 < \mu \le \kappa_2 \le \kappa_1$. (In particular, we cannot have $\kappa_3 = \kappa_2$ in this case.) Then (20) entails

$$s_i(p) = \kappa_i(1 - F_j(p)) + (1 - \kappa_j - \kappa_3F_3(p))F_j(p) = \kappa_i - F_j(p)(\mu - \kappa_3(1 - F_3(p))) \quad (52)$$

for $i, j = 1, 2$ and

$$s_3(p) = \kappa_3(1 - F_1(p)F_2(p)) . \quad (53)$$

We claim in this case that there is an equilibrium where firms 1 and 2 choose prices in the range $[p_0, 1]$ and have profit $\pi_i = \kappa_i p_0$, where p_0 is given by (21), and where firm 3 only chooses price in an interior interval $[\hat{p}, p_1]$, where $p_0 < \hat{p} < p_1 < 1$, and obtains profit $\kappa_3 > \pi_3 > \kappa_3 p_0$.

Since $ps_i(p) = \pi_i = \kappa_i p_0$ for $i = 1, 2$, where s_i is given in (52), we have $\kappa_1 F_1(p) = \kappa_2 F_2(p)$. For $p < \hat{p}$, where firms 1 and 2 are sure to undercut firm 3, (52) implies that

$$\kappa_1 F_1(p) = \kappa_2 F_2(p) = \frac{\kappa_1\kappa_2}{\mu - \kappa_3}\left(1 - \frac{p_0}{p}\right) . \quad (54)$$

Clearly these CDFs equal zero at $p = p_0$ and increase with p. If firm 3 did choose a price $p < \hat{p}$, with the supply function in (53) its profit $ps_3(p)$ would be

$$p\kappa_3(1 - F_1F_2) = p\kappa_3\left(1 - \frac{\kappa_1\kappa_2}{(\mu - \kappa_3)^2}\left(1 - \frac{p_0}{p}\right)^2\right) , \quad (55)$$

where the equality follows from (54), and the derivative of this profit is proportional to

$$(\mu - \kappa_3)^2 - \kappa_1\kappa_2\left(1 - \left(\frac{p_0}{p}\right)^2\right) \quad (56)$$

which is decreasing in p and positive at $p = p_0$. Since $p_0 < \frac{1-\kappa_2}{\kappa_1}$, a sufficient condition for (56) to be strictly negative at $p = 1$ is

$$\kappa_1\kappa_2\left(1-\left(\frac{1-\kappa_2}{\kappa_1}\right)^2\right)-(\mu-\kappa_3)^2>0\ .$$

But the left-hand side of this can be written

$$\begin{aligned}\kappa_1\kappa_2\left(1-\left(\frac{1-\kappa_2}{\kappa_1}\right)^2\right)-(\mu-\kappa_3)^2 &= \frac{\mu-\kappa_3}{\kappa_1}\left(\kappa_2(1+\kappa_1-\kappa_2)-\kappa_1(\kappa_1+\kappa_2-1)\right)\\ &= \frac{\mu-\kappa_3}{\kappa_1}(\kappa_2(1-\kappa_2)+\kappa_1(1-\kappa_1))>0\ .\end{aligned}$$

We deduce that there is a unique $\hat{p} \in (p_0, 1)$ such that (56) is equal to zero, which is therefore given by

$$\hat{p} = \frac{p_0}{\sqrt{1-\frac{(\mu-\kappa_3)^2}{\kappa_1\kappa_2}}}\ . \tag{57}$$

(The previous inequality shows that the root in the denominator is real.) Since $\hat{p}$ will be the smallest price in firm 3's support, this firm's profit will be equal to its profit at $p = \hat{p}$, which from (55)–(57) is

$$\begin{aligned}\pi_3 &= \hat{p}\kappa_3\left(1-\frac{\kappa_1\kappa_2}{(\mu-\kappa_3)^3}\left(1-\frac{p_0}{\hat{p}}\right)^2\right)\\ &= \hat{p}\kappa_3\left(1-\frac{\kappa_1\kappa_2}{(\mu-\kappa_3)^3}\left(1-\left(\frac{p_0}{\hat{p}}\right)^2\right)\frac{1-\frac{p_0}{\hat{p}}}{1+\frac{p_0}{\hat{p}}}\right)\\ &= \hat{p}\kappa_3\left(1-\frac{1-\frac{p_0}{\hat{p}}}{1+\frac{p_0}{\hat{p}}}\right)\\ &= \frac{2\kappa_3}{\frac{1}{p_0}+\frac{1}{\hat{p}}} = \frac{2p_0\kappa_3}{1+\sqrt{1-\frac{(\mu-\kappa_3)^2}{\kappa_1\kappa_2}}}\ .\end{aligned} \tag{58}$$

By construction, firm 3's profit would be lower than π_3 if it chose a price below $\hat{p}$.

Turn next to the range for prices where all three firms are active, $p \in [\hat{p}, p_1]$. Using the observation that $\kappa_1 F_1 = \kappa_2 F_2$, with supply functions (52)-(53) the solution to $ps_i(p) = \pi_i$ for $i = 1, 2, 3$ is

$$\kappa_1F_1(p) = \kappa_2F_2(p) = \sqrt{\kappa_1\kappa_2\left(1-\frac{\pi_3}{p\kappa_3}\right)} \tag{59}$$

and

$$\kappa_3(1-F_3(p)) = \mu-\left(1-\frac{p_0}{p}\right)\sqrt{\frac{\kappa_1\kappa_2}{1-\frac{\pi_3}{p\kappa_3}}}\ . \tag{60}$$

Using the first line in (58), we see that $F_3(\hat{p}) = 0$ and that F_1 and F_2 join up continuously on either side of the threshold price $\hat{p}$. Expression (59) implies that F_1 and F_2 are increasing in p in $[\hat{p}, p_1]$. From (60), $F_3(p)$ is increasing if $(1 - \frac{p_0}{p})/\sqrt{1 - \frac{\pi_3}{p\kappa_3}}$ is increasing, which is the case if

$$\frac{\pi_3}{\kappa_3}\left(\frac{1}{p_0} + \frac{1}{p}\right) \leq 2 \ ,$$

which holds for all $p \geq \hat{p}$ if it holds at $\hat{p}$. However, from (58) this inequality holds with equality at $\hat{p}$, and so F_3 in (60) is an increasing function for $p \geq p_0$. (In particular, the density for firm 3's price is zero at $p = \hat{p}$.)

Evaluating F_i at $p = 1$ in (59) and (60) reveals that $F_1(1) < 1$, $F_2(1) < 1$ and $F_3(1) > 1$. That this is so for F_1 is clear, while to show this for F_2 note that $1 - F_2(1)$ has the sign

$$\kappa_2 - \kappa_1\left(1 - \frac{\pi_3}{\kappa_3}\right) > \kappa_2 - \kappa_1(1 - p_0) \geq \kappa_2 - \kappa_1\left(1 - \left(1 - \frac{\kappa_2}{\kappa_1}\right)\right) = 0 \ .$$

(Here, the first inequality follows since $\pi_3 > \kappa_3 p_0$ and the second follows since $p_0 = 1 - \mu/\kappa_1 \geq 1 - \kappa_2/\kappa_1$.) Finally, observe that $1 - F_3(1)$ in (60) has the sign of $\left(1 - \frac{\pi_3}{\kappa_3}\right)\kappa_1 - \kappa_2$, which by the above argument is negative as required. Firm 3's maximum price, $p_1 < 1$, therefore satisfies $F_3(p_1) = 1$, or

$$\left(1 - \frac{p_0}{p_1}\right)\sqrt{\frac{\kappa_1\kappa_2}{1 - \frac{\pi_3}{p_1\kappa_3}}} = \mu \ . \tag{61}$$

The remaining region is where $p > p_1$, when firms 1 and 2 are sure to be undercut by firm 3, when we again have F_1 and F_2 given by (51). By construction, these two CDFs join up continuously on either side of the price p_1 and they are increasing in p for $p \geq p_1$. As in the regime where all firms had captive demand, firm 1 has an atom at $p = 1$ with probability $\frac{\kappa_1 - \kappa_2}{\kappa_1}$.

The final condition to check is that firm 3 has no incentive to choose a price above p_1. Similarly to (55), firm 3's profit with price $p \geq p_1$ is given by

$$p\kappa_3(1 - F_1F_2) = p\kappa_3\left(1 - \frac{\kappa_1\kappa_2}{\mu^2}\left(1 - \frac{p_0}{p}\right)^2\right) \ ,$$

where the equality follows from (51). As in (56), the derivative of firm 3's profit is proportional to

$$\mu^2 - \kappa_1\kappa_2\left(1 - \left(\frac{p_0}{p}\right)^2\right)$$

which decreases with p. Therefore, if the above expression is negative at $p = p_1$ firm 3 has no incentive to choose a price above p_1. However, this expression at $p = p_1$ has the sign of

$$\begin{aligned}1 - \frac{\kappa_1\kappa_2}{\mu^2}\left(1 - \left(\frac{p_0}{p_1}\right)^2\right) &= 1 - \frac{\kappa_1\kappa_2}{\mu^2}\left(1 - \frac{p_0}{p_1}\right)^2 \frac{p_1+p_0}{p_1-p_0} \\ &= 1 - \frac{p_1+p_0}{p_1-p_0}\left(1 - \frac{\pi_3}{\kappa_3 p_1}\right) \\ &= \frac{p_0}{p_1-p_0}\left[\frac{\pi_3}{\kappa_3}\left(\frac{1}{p_0}+\frac{1}{p_1}\right) - 2\right] \\ &< \frac{p_0}{p_1-p_0}\left[\frac{\pi_3}{\kappa_3}\left(\frac{1}{p_0}+\frac{1}{\hat{p}}\right) - 2\right] = 0 \ .\end{aligned}$$

Here, the second equality follows since firm 3's profit is equal to π_3 at $p = p_1$, while the final equality follows from (58).

The second subcase for part (ii) is when only firm 1 has captive consumers, so that $\kappa_3 \leq \kappa_2 < \mu < \kappa_1$. In this case (20) implies

$$s_1(p) = \kappa_1 - (\mu - \kappa_2)F_3(p) - (\mu - \kappa_3)F_2(p) - (1-\kappa_1)F_2(p)F_3(p) \ ,$$

while for $i, j = 2, 3$ we have

$$s_i(p) = \kappa_i - [\mu - \kappa_j + (1-\kappa_1)F_j(p)]F_1(p) \ .$$

We will construct an equilibrium where firms 1 and 2 choose prices in the range $[p_0, 1]$ and have profit $\pi_i = \kappa_i p_0$, where p_0 is given by (21). With profits $\pi_i = \kappa_i p_0$, the equilibrium condition $ps_i(p) = \pi_i$ with the above supply functions implies that for p in 1's price support

$$\kappa_1\left(1 - \frac{p_0}{p}\right) = (\mu - \kappa_3)F_2 + (\mu - \kappa_2)F_3 + (1-\kappa_1)F_2F_3 \ ,$$

which can be written in the factorized form

$$\kappa_1(1-\kappa_1)\left(1 - \frac{p_0}{p}\right) + (\mu-\kappa_2)(\mu-\kappa_3) = [\mu - \kappa_3 + (1-\kappa_1)F_3][\mu - \kappa_2 + (1-\kappa_1)F_2)] \ . \quad (62)$$

For firm 2 we have

$$\kappa_2\left(1 - \frac{p_0}{p}\right) = [\mu - \kappa_3 + (1-\kappa_1)F_3]F_1 \ , \quad (63)$$

while for firm 3 we have

$$\kappa_3 - \frac{\pi_3}{p} = [\mu - \kappa_2 + (1-\kappa_1)F_2)]F_1 \quad (64)$$

where π_3 is to be determined.

First consider the loose end from part (i) where $\kappa_2 = \kappa_3 = \kappa$ say and $\kappa < \mu$. Then we claim that the equilibrium has $\pi_1 = \kappa_1 p_0$, $\pi_2 = \pi_3 = \kappa p_0$ and $F_2 = F_3 = F$ say. From (62), F satisfies

$$[\mu - \kappa + (1-\kappa_1)F(p)]^2 = \kappa_1(1-\kappa_1)\left(1 - \frac{p_0}{p}\right) + (\mu-\kappa)^2 ,$$

which implies that $F(p)$ is an increasing function of p with $F(p_0) = 0$ and $F(1) = 1$. Expression (63) then implies

$$F_1(p) = \kappa \frac{1 - \frac{p_0}{p}}{\sqrt{\kappa_1(1-\kappa_1)\left(1-\frac{p_0}{p}\right) + (\mu-\kappa)^2}} = \kappa \sqrt{\frac{1-\frac{p_0}{p}}{\kappa_1(1-\kappa_1) + \frac{(\mu-\kappa)^2}{1-\frac{p_0}{p}}}}$$

which is increasing in p and satisfies $F_1(0) = 0$ and $F_1(1) = \mu/\kappa_1$ (where the latter point is most easily seen from (63) using the fact $F(1) = 1$). This completes the loose end from part (i).

Now suppose that $\kappa_3 < \kappa_2$. We claim there is again an equilibrium of the stated form. For $p < \hat{p}$, where firms 1 and 2 are sure to undercut firm 3, the two CDFs again are given by (54). If firm 3 deviated to a price $p < \hat{p}$, from (64) its profit would be

$$y(p) \equiv p(\kappa_3 - [\mu - \kappa_2 + (1-\kappa_1)F_2]F_1) ,$$

where the CDFs are as in (54). We can express this profit $y(p)$ compactly as

$$y(p) = bp_0 - ap - \frac{c}{p}p_0^2 ,$$

with the positive constants given by

$$c = \frac{(1-\kappa_1)\kappa_1\kappa_2}{(\mu-\kappa_3)^2} \ ; \ a = \left(1 - \frac{(\mu-\kappa_3)(\kappa_2-\kappa_3)}{\kappa_1\kappa_2}\right)c \ ; \ b = \kappa_3 + a + c \ .$$

Then

$$y'(p) = c\left(\frac{p_0}{p}\right)^2 - a \tag{65}$$

which is positive at $p = p_0$ and decreasing in p. Since $p_0 < \frac{1-\kappa_2}{\kappa_1}$, a sufficient condition for $y'(1) < 0$ is

$$c\left(\frac{1-\kappa_2}{\kappa_1}\right)^2 - a < 0 \Leftrightarrow 1 - \left(\frac{1-\kappa_2}{\kappa_1}\right)^2 - \frac{(\mu-\kappa_3)(\kappa_2-\kappa_3)}{\kappa_1\kappa_2} > 0 \ .$$

However, the right-hand side can be written as

$$1 - \left(\frac{1-\kappa_2}{\kappa_1}\right)^2 - \frac{(\mu-\kappa_3)(\kappa_2-\kappa_3)}{\kappa_1\kappa_2} = \frac{\mu-\kappa_3}{\kappa_1^2\kappa_2}\left(\kappa_2(1-\kappa_2) + \kappa_1\kappa_3\right) > 0 \ .$$

Therefore, there is a unique price $\hat{p} \in (p_0, 1)$ which makes the derivative zero in (65), and this is given by

$$\hat{p} = p_0 \sqrt{\frac{c}{a}} = \frac{p_0}{\sqrt{1 - \frac{(\mu-\kappa_3)(\kappa_2-\kappa_3)}{\kappa_1\kappa_2}}} \ . \tag{66}$$

Since $\hat{p}$ will be the smallest price in firm 3's support, this firm's profit will be equal to its profit at $p = \hat{p}$, which is

$$\pi_3 = y(\hat{p}) = \left[\kappa_3 + c\left(1 - \sqrt{\frac{a}{c}}\right)^2\right] p_0 \tag{67}$$

so that

$$\pi_3 = \left[\kappa_3 + \frac{1-\kappa_1}{(\mu-\kappa_3)^2}\left(\sqrt{\kappa_1\kappa_2} - \sqrt{\kappa_1\kappa_2 - (\mu-\kappa_3)(\kappa_2-\kappa_3)}\right)^2\right] p_0 \ . \tag{68}$$

Consider next the region $\hat{p} \le p \le p_1$ where all three firms are active. From (62) we see that (63) and (64) imply that

$$F_1(p) = \sqrt{\frac{k_2\left(\kappa_3 - \frac{\pi_3}{p}\right)}{\kappa_1(1-\kappa_1) + \frac{(\mu-\kappa_2)(\mu-\kappa_3)}{1-\frac{p_0}{p}}}} \ , \tag{69}$$

which increases with p. Expressed in terms of F_1 in (69) the other two CDFs in (63)-(64) are given by

$$F_2(p) = 1 - \frac{\kappa_3}{1-\kappa_1}\left[1 - \frac{\left(1 - \frac{\pi_3}{p\kappa_3}\right)}{F_1(p)}\right] \tag{70}$$

and

$$F_3(p) = 1 - \frac{\kappa_2}{1-\kappa_1}\left[1 - \frac{\left(1 - \frac{p_0}{p}\right)}{F_1(p)}\right] \ . \tag{71}$$

Since $\frac{\pi_3}{\kappa_3} \ge p_0$, it follows that if $F_3'(p) \ge 0$ then $F_2'(p) \ge 0$. Expression (71) implies that F_3 is increasing if and only if $(1 - p_0/p)/F_1(p)$ increases with p. From (69), which can be written in the modified form

$$F_1(p) = \sqrt{\frac{(1-\frac{p_0}{p})\left(1 - \frac{\pi_3}{\kappa_3 p}\right)}{1 - \frac{\kappa_1(1-\kappa_1)}{\kappa_2\kappa_3}p_0(\frac{1}{p}-1)}} \ , \tag{72}$$

and taking logs, one sees that the derivative of $(1 - p_0/p)/F_1(p)$ has the sign of

$$\frac{1}{1-\frac{p_0}{p}} + \frac{\frac{(1-\kappa_1)\kappa_1}{\kappa_2\kappa_3}}{1 - \frac{(1-\kappa_1)\kappa_1}{\kappa_2\kappa_3}p_0\left(\frac{1}{p}-1\right)} - \frac{\frac{\pi_3}{p_0\kappa_3}}{1-\frac{\pi_3}{p\kappa_3}} = \frac{\frac{(1-\kappa_1)\kappa_1}{\kappa_2\kappa_3}}{1 - \frac{(1-\kappa_1)\kappa_1}{\kappa_2\kappa_3}p_0\left(\frac{1}{p}-1\right)} - \frac{\frac{\pi_3}{\kappa_3 p_0} - 1}{\left(1-\frac{\pi_3}{p\kappa_3}\right)\left(1-\frac{p_0}{p}\right)} \ .$$

From (72) this has the sign of

$$\frac{(1-\kappa_1)\kappa_1}{\kappa_2}F_1^2(p) - \left(\frac{\pi_3}{p_0} - \kappa_3\right) \ . \tag{73}$$

This increases with p, and so $F_3(p)$ is increasing for $p \geq \hat{p}$ provided that (73) is non-negative at $p = \hat{p}$. Since F_1 is continuous across the threshold $p = \hat{p}$, $F_1(p_1)$ is given by (54), and substituting this into (73) yields

$$\begin{aligned} \frac{(1-\kappa_1)\kappa_1}{\kappa_2}F_1^2(p) - \left(\frac{\pi_3}{p_0} - \kappa_3\right) &= \kappa_3 + \frac{(1-\kappa_1)\kappa_1}{\kappa_2}\left(\frac{\kappa_2}{\mu_3 - \kappa_3}\left(1 - \frac{p_0}{p}\right)\right)^2 - \frac{\pi_3}{p_0} \\ &= \kappa_3 + c\left(1 - \frac{p_0}{p}\right)^2 - \frac{\pi_3}{p_0} \\ &= \kappa_3 + c\left(1 - \sqrt{\frac{a}{c}}\right)^2 - \frac{\pi_3}{p_0} = 0 \ . \end{aligned}$$

Here, the final expression is equal to zero from (67). Therefore (73) is zero at $p = \hat{p}$. (In particular, we again have $F_3'(\hat{p}) = 0$.)

Thus, all three CDFs are increasing for $p \geq \hat{p}$. Evaluating the CDFs in (69)–(71) at $p = 1$, and noting that $\kappa_3 p_0 < \pi_3 < \kappa_3$, reveals that $F_1(1) < 1$, $F_2(1) < 1$ and $F_3(1) > 1$. Therefore, there exists a unique price $p_1 \in (\hat{p}, 1)$ such that $F_3(p_1) = 1$. Above p_1 only firms 1 and 2 compete, and are sure to be undercut by firm 3. From (62) and (63), the two CDFs are given by

$$F_1(p) = 1 - \frac{p_0}{p} \ ; \ 1 - F_2(p) = \frac{\kappa_1}{\kappa_2}p_0\left(\frac{1}{p} - 1\right) \ . \tag{74}$$

Here, these CDFs increase with p, $F_2(1) = 1$, while firm 1 has at atom at $p = 1$ with probability p_0. The final point to check is that firm 3 has no incentive to choose a price above p_1. If it did, from (64) its profit would be

$$p\left[\kappa_3(1 - F_1) - (1-\kappa_1)F_1(1-F_2)\right] = \kappa_3 p_0 - p(1-\kappa_1)\left(1 - \frac{p_0}{p}\right)\frac{\kappa_1}{\kappa_2}p_0\left(\frac{1}{p} - 1\right) \ ,$$

where the equality follows after substituting F_1 and F_2 from (74). However, since the coefficient on $1/p$ in this expression is now positive this is a *convex* function of p, and so the firm has no incentive to choose price $p < p_1$ provided that its profit at $p = 1$ is no greater than π_3. However, setting $p = 1$ in the profit expression yields $\kappa_3 p_0$, which is indeed below π_3. This completes the proof for part (ii).