Innovation and Inequality in a Monetary Schumpeterian Model with Heterogeneous Households and Firms

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Abstract

This study develops a Schumpeterian growth model with heterogeneous households and heterogeneous firms to explore the effects of monetary policy on innovation and income inequality. Household heterogeneity arises from an unequal distribution of wealth. Firm heterogeneity arises from random quality improvements and a cost of entry. We find that under endogenous firm entry, inflation has inverted-U effects on economic growth and income inequality. We also calibrate the model for a quantitative analysis and find that the model is able to match the growth-maximizing inflation rate and the inequality-maximizing inflation rate that we estimate using cross-country panel data. Finally, we simulate the utility-maximizing level of inflation and explore how it is affected by the wealth holdings of households.

JEL classification: O30, O40, D30, E41
Keywords: inflation, income inequality, economic growth, heterogeneity

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1 Introduction

The seminal study by Tobin (1965) initiated an influential literature in macroeconomics that explores the relationship between inflation and economic growth. Studies in this literature have focused on how inflation affects economic growth via the accumulation of physical capital and/or human capital.[1] However, an important insight from the seminal study by Solow (1956) is that economic growth is ultimately driven by technological progress. Therefore, it is important to also understand the effects of inflation in a growth model with endogenous technological progress. Marquis and Reffett (1994) explore the effects of inflation in the R&D-based growth model developed by Romer (1990). However, this early study by Marquis and Reffett (1994) and many subsequent studies in this branch of the literature have mostly focused on a representative-household setting with homogeneous firms. In this study, we find that the interdependence between heterogeneous households and heterogeneous firms leads to novel results.

Specifically, we develop a monetary Schumpeterian growth model with heterogeneous firms and heterogenous households. We model firm heterogeneity in the Schumpeterian quality-ladder model by assuming that the step size of quality improvements is randomly drawn from a Pareto distribution. Then, to allow for endogenous firm entry, we assume that R&D entrepreneurs need to pay an entry cost to enter the market after observing the step size of their quality improvements. As a result, an entrepreneur would enter the market if and only if her quality improvement is sufficiently large, which in turn generates an endogenous distribution of quality improvements that are implemented. Motivated by the empirical evidence in Piketty (2014), we consider an unequal distribution of wealth as an important source of income inequality. Therefore, we model household heterogeneity in the Schumpeterian model by assuming that households have different levels of wealth in order to generate an endogenous income distribution. Within this growth-theoretic framework, we explore the effects of monetary policy on innovation and income inequality. In summary, we find that inflation has inverted-U effects on economic growth and income inequality under endogenous firm entry.

The inverted-U effect of inflation on economic growth under endogenous entry of heterogeneous firms can be explained as follows. Inflation increases the cost of R&D via the cash-in-advance (CIA) constraint on R&D and decreases the arrival rate of innovation, which is a negative effect of inflation on economic growth.[2] The lower rate of creative destruction however increases the expected value of future profits and the market value of inventions, which in turn lowers the entry threshold for quality improvements. With more inventions being implemented, inflation also has a positive effect on economic growth. These positive and negative effects together generate an inverted-U effect of inflation on economic growth so long as the entry cost is sufficiently large.

Interestingly, this inverted-U effect of inflation on economic growth also leads to an inverted-U effect of inflation on income inequality in the Schumpeterian model. In our model, income inequality is increasing in the ratio of wealth income to wage income. Therefore, either an increase in the real interest rate or an increase in the value of financial assets would increase income inequality. Given the Euler equation under which the real interest rate is increasing in the growth rate of consumption, the abovementioned inverted-U effect of inflation on economic

[1] See for example Stockman (1981), Abel (1985), Dotsey and Ireland (1996), Gillman and Kejak (2005), Ho *et al.* (2007) and Wong (2016).

[2] Marquis and Reffett (1994) also find a negative effect of inflation on R&D, which is supported by empirical evidence based on cross-country panel regressions in Chu *et al.* (2015).

growth causes an inverted-U effect on the real interest rate and hence also an inverted-U effect on income inequality. Furthermore, inflation has both positive and negative effects on the value of financial assets. On the one hand, by slowing down the rate of creative destruction, inflation increases the market value of monopolistic firms, which in turn increases the value of financial assets. On the other hand, by lowering the entry threshold for quality improvements, inflation reduces the average step size of quality improvements implemented in the market and decreases the average markup ratio, which in turn decreases the market values of monopolistic firms and financial assets. Combining all these effects yields an overall inverted-U effect of inflation on income inequality, which exists only under endogenous entry of heterogeneous firms. We also calibrate the model to perform a quantitative analysis and find that our model is able to match a growth-maximizing inflation rate of 5% and an inequality-maximizing inflation rate of 12% that are estimated using cross-country panel data. Finally, we simulate the utility-maximizing level of inflation and explore how it is affected by the wealth holdings of households.

This study relates to the literature on innovation and economic growth. Romer (1990) develops the seminal R&D-based growth model in which economic growth is driven by the invention of new products. Segerstrom *et al.* (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992) develop the Schumpeterian quality-ladder model in which economic growth is driven by the innovation of higher-quality products. For tractability, these seminal studies and many subsequent studies assume a constant step size of quality improvement. Important exceptions include Klette and Kortum (2004) and Minniti *et al.* (2013). Minniti *et al.* (2013) develop a Schumpeterian growth model with random step sizes of quality improvements drawn from a Pareto distribution.[3] This study extends the elegant model in Minniti *et al.* (2013) by allowing for a Hopenhayn-Melitz-type entry cost to generate endogenous entry of heterogeneous firms[4] and introducing heterogenous households with different asset holdings. In other words, this study contributes to the literature by developing a Schumpeterian growth model with two dimensions of heterogeneity among households and firms.

This study also relates to the literature on innovation and inflation. In this literature, the seminal study by Marquis and Reffett (1994) analyzes the effects of inflation on innovation in a variant of the Romer variety-expanding model. Subsequent studies analyze the effects of inflation in the Schumpeterian quality-ladder model; see for example Chu and Lai (2013), Chu and Cozzi (2014), Chu *et al.* (2015), He and Zou (2016), Huang *et al.* (2017), Neto *et al.* (2017), He (2018) and Lin *et al.* (2018).[5] However, all these studies feature a constant step size of quality improvement. As a result, these studies predict a monotonic relationship between inflation and economic growth, which is different from the inverted-U relationship between inflation and economic growth often found in empirical studies.[6] As a result, Chu *et al.* (2017) develop a monetary Schumpeterian growth model with endogenous entry of heterogeneous firms,[7] and

[3] A recent study by Iwaisako and Ohki (2018) develops a Schumpeterian growth model with random quality improvements drawn from a uniform distribution.

[4] See also Baldwin and Robert-Nicoud (2008), Haruyama and Zhao (2008) and Gustafsson and Segerstrom (2010) who adapt a similar entry cost into the R&D-based growth model with heterogeneous firms, but they do not consider random increments on the quality ladder.

[5] See also Funk and Kromen (2010), Benigno and Fornaro (2018) and Oikawa and Ueda (2018), who consider sticky prices in the Schumpeterian growth model. Our study assumes flexible prices in order to focus on the effects of inflation on long-run growth.

[6] See for example Bick (2010) and Lopez-Villavicencio and Mignon (2011) for recent studies.

[7] See also Arawatari *et al.* (2018) and Hori (2018) who consider monetary policy in the presence of heterogeneity in the productivity of R&D entrepreneurs.

they show that their model can generate an inverted-U relationship between inflation and economic growth and match empirical estimates of the growth-maximizing inflation rate under plausible parameter values. However, all the abovementioned studies feature a representative household; therefore, they cannot be used to analyze the implications of monetary policy on the income distribution. Therefore, this study introduces heterogeneous households into this literature in order to analyze the effects of monetary policy on income inequality in addition to innovation and economic growth.

This study also relates to the literature on innovation and income inequality. Representative studies include Chou and Talmain (1996), Li (1998), Zweimuller (2000), Foellmi and Zweimuller (2006), Kiedaisch (2017), Aghion *et al.* (2018), Grossman and Helpman (2018) and Jones and Kim (2018). These studies focus on the relationship between income inequality and innovation. Our study complements these interesting studies by exploring the effects of monetary policy on innovation and income inequality. Chu and Cozzi (2018) explore the effects of R&D subsidies and patent policy on income inequality, but not monetary policy. More importantly, Chu and Cozzi (2018) focus on a Schumpeterian growth model with a constant step size of quality improvement. We show that endogenous entry of heterogeneous firms is necessary for the emergence of an inverted-U effect of inflation on income inequality that is consistent with our empirical finding.[8]

In the New Keynesian literature, recent studies such as McKay and Reis (2016) and Kaplan *et al.* (2018) introduce heterogeneous agents into the standard New Keynesian model to explore the effects of government policies on inequality via nominal rigidity. In the New Monetarist literature, studies have also introduced heterogeneous agents into the search-theoretic monetary model to explore the effects of inflation on inequality via search and matching frictions; see Rocheteau *et al.* (2018) for a recent study and a discussion of earlier studies. Our study differs from these interesting studies by exploring the effects of inflation on inequality in a monetary Schumpeterian growth model with heterogeneous agents in which inflation affects inequality via R&D and innovation. In other words, we focus on the long-run effects of monetary policy on the macroeconomy, which complement the interesting effects, emphasized by the New Keynesian model and the search-theoretic monetary model, at different time horizons.

The rest of this study is organized as follows. Section 2 presents the model and solves the market equilibrium of the aggregate economy. Section 3 explores the distributions of wealth and income. Section 4 analyzes the effects of monetary policy. Section 5 provides a quantitative analysis. Section 6 concludes. Proofs are relegated to the appendix.

2 A Schumpeterian model with heterogeneous firms and heterogeneous households

The Schumpeterian quality-ladder model is based on Aghion and Howitt (1992) and Grossman and Helpman (1991). We consider the monetary Schumpeterian growth model in Chu *et al.* (2017) featuring (a) a CIA constraint on R&D,[9] (b) lab-equipment specifications for innovation

[8] See also Natob (2015) who finds an inverted-U effect of inflation on income inequality using dynamic panel regressions with cross-country data.

[9] See Chu *et al.* (2015) for a discussion of empirical evidence for the presence of cash requirements on R&D and also Berentsen *et al.* (2012) for a discussion of theoretical justifications and microfoundations.

and entry that use final good as the input, (c) random quality improvements as in Minniti *et al.* (2013) and (d) a fixed entry cost that generates endogenous entry of heterogeneous firms as in Hopenhayn (1992) and Melitz (2003). Furthermore, we follow Chu and Cozzi (2018) to introduce heterogeneous households with different asset holdings into the monetary Schumpeterian model.

2.1 Households

There is a unit continuum of households, which are indexed by $h \in [0, 1]$. They have identical homothetic preferences over consumption $c_t(h)$ but own different levels of wealth.[10] Household h has the following utility function:

$$u(h) = \int_0^\infty e^{-\rho t} \ln c_t(h) dt, \tag{1}$$

where the parameter $\rho > 0$ is the subjective discount rate.

Household h supplies one unit of labor to earn wage income w_t and maximizes utility $u(h)$ subject to

$$\dot{a}_t(h) + \dot{m}_t(h) = r_t a_t(h) - \pi_t m_t(h) + i_t b_t(h) + w_t + \tau_t - c_t(h). \tag{2}$$

$a_t(h)$ is the real value of financial assets (i.e., equity of monopolistic firms) owned by household h, and r_t is the real interest rate. $m_t(h)$ is the real value of cash holdings of household h, and π_t is the inflation rate. $b_t(h)$ is the amount of cash borrowed from household h by entrepreneurs for R&D, and i_t is the nominal interest rate.[11] The CIA constraint is given by $b_t(h) \leq m_t(h)$. Finally, the government provides a lump-sum transfer τ_t to each household.[12]

From standard dynamic optimization, household h's consumption path is given by

$$\frac{\dot{c}_t(h)}{c_t(h)} = r_t - \rho, \tag{3}$$

which shows that the growth rate of consumption is the same across households such that $\dot{c}_t(h)/c_t(h) = \dot{c}_t/c_t$ for all $h \in [0, 1]$, where $c_t \equiv \int_0^1 c_t(h) dh$ denotes aggregate consumption. Therefore, the growth rate of aggregate consumption is also given by

$$\frac{\dot{c}_t}{c_t} = r_t - \rho. \tag{4}$$

[10] Due to the households' homothetic preferences, the aggregate economy behaves as if there is a representative household; see for example Caselli and Ventura (2000). Nevertheless, the heterogeneity in wealth holdings still enables us to explore the endogenous distribution of income.

[11] It can be shown as a no-arbitrage condition that the rate of return on borrowing $b_t(h)$ must equal $r_t + \pi_t$.

[12] The transfer is financed by seigniorage. Alternatively, we can assume that seigniorage is used to finance a public good, in which case all our analytical results would be the same so long as the (potentially utility-enhancing) public good is non-productive.

2.2 Final good

Final good y_t is produced by a unit continuum of competitive firms using the following Cobb-Douglas production function:

$$y_t = L_t^{\theta} K_t^{1-\theta}, \tag{5}$$

where L_t is labor input and $\theta \in (0, 1)$ measures labor intensity in production. K_t is a composite of a unit continuum of differentiated intermediate goods $k_t(j)$ given by

$$K_t = \exp\left(\int_0^1 \ln k_t(j) dj\right). \tag{6}$$

From profit maximization using (5), the conditional demand function for L_t is

$$w_t L_t = \theta y_t. \tag{7}$$

From profit maximization using (5) and (6), the conditional demand function for $k_t(j)$ is

$$p_t(j) k_t(j) = (1-\theta)\, y_t, \tag{8}$$

where $p_t(j)$ is the price of $k_t(j)$.[13]

2.3 Intermediate goods

There is a unit continuum of industries indexed by $j \in [0, 1]$. In each industry j, there is a monopolistic industry leader, who holds a patent on the latest technology and dominates the market until the arrival of the next innovation.[14] The production function of the leader in industry j is

$$k_t(j) = q_t(j, \omega_j) x_t(j), \tag{9}$$

where $q_t(j, \omega_j)$ is the quality-level of the leader in industry j and ω_j is an integer denoting the quality vintage of the intermediate goods produced by the leader in industry j. $x_t(j)$ is the quantity of input j produced using final good with an one-to-one technology (i.e., $x_t(j)$ units of final good produce $x_t(j)$ units of input j). From Bertrand competition, the equilibrium price of $k_t(j)$ is a markup over the marginal cost $1/q_t(j, \omega_j)$ given by

$$p_t(j) = \frac{\lambda_t(j)}{q_t(j, \omega_j)}, \tag{10}$$

where the markup ratio $\lambda_t(j) \equiv q_t(j, \omega_j)/q_t(j, \omega_j - 1)$ is determined by the size of the quality improvement by the leader in industry j. The equilibrium level of monopolistic profit is

$$\Pi_t(j) = \left[\frac{\lambda_t(j) - 1}{\lambda_t(j)}\right] p_t(j) k_t(j) = \left[\frac{\lambda_t(j) - 1}{\lambda_t(j)}\right] (1-\theta) y_t \equiv \Pi_t(\lambda), \tag{11}$$

where the second equality uses (8).

[13] Here $p_t(j)$ is denominated in units of the final good.
[14] See Cozzi (2007) for a discussion of this Arrow replacement effect.

2.4 R&D and entry

In this section, we present the three steps of innovation. First, an entrepreneur invents a higher quality product. Then, the size of the quality improvement is randomly drawn from a Pareto distribution. Finally, if and only if the quality improvement is sufficiently large, then the entrepreneur would pay a fixed entry cost to enter the market.

2.4.1 Invention

R&D is performed by competitive entrepreneurs. If an entrepreneur employs $R_t(j)$ units of final good to engage in innovation in industry j, then she would succeed in inventing the next higher-quality product in the industry with an instantaneous probability $\phi_t(j)$ given by

$$\phi_t(j) = R_t(j)/\alpha_t. \tag{12}$$

To ensure balanced growth, $\alpha_t \equiv \alpha Q_t^{(1-\theta)/\theta}$ measuring the difficulty of R&D is increasing the aggregate technology level Q_t,[15] which is defined as

$$Q_t \equiv \exp\left(\int_0^1 \ln q_t(j,\omega_j)dj\right). \tag{13}$$

To facilitate the payment of $R_t(j)$, the entrepreneur needs to borrow the amount $\zeta R_t(j)$ of cash from households, where $\zeta \in (0,1]$ is the CIA parameter. The borrowing cost is determined by the nominal interest rate i_t. Therefore, the total cost of R&D is $(1+\zeta i_t)R_t(j)$. Let's use $v_t^e(j,\omega_j+1)$ to denote the expected value of an invention before the realization of the size of its quality improvement. The R&D condition is given by

$$\phi_t(j)v_t^e(j,\omega_j+1) = (1+\zeta i_t)\,R_t(j) \Leftrightarrow v_t^e(j,\omega_j+1) = (1+\zeta i_t)\,\alpha Q_t^{(1-\theta)/\theta}. \tag{14}$$

2.4.2 Random quality improvements

We follow Minniti *et al.* (2013) to assume that when an R&D entrepreneur invents a higher-quality product in industry j, the quality step size $\lambda_t(j) > 1$ is randomly drawn from a stationary Pareto distribution with the following probability density function:

$$f(\lambda) = \frac{1}{\kappa}\lambda^{-\frac{1+\kappa}{\kappa}}, \tag{15}$$

where the parameter $\kappa \in (0,1)$ determines the shape of the Pareto distribution. Given that the expected value of $\lambda_t(j)$ is equal across industries, (11) implies that the expected value $\Pi_t^e(j)$ of monopolistic profit $\Pi_t(j)$ is also the same across industries such that $\Pi_t^e(j) = \Pi_t^e$ for $j \in [0,1]$. Therefore, we follow the standard treatment to focus on the symmetric equilibrium in which the arrival rate of innovation is equal across industries, such that $\phi_t(j) = \phi_t$ for $j \in [0,1]$.[16] As a result, the expected value of an invention does not depend on j such that $v_t^e(j,\omega_j+1) = v_t^e$ for $j \in [0,1]$.

[15] Venturini (2012) provides empirical evidence for the presence of increasing R&D difficulty.

[16] Cozzi *et al.* (2007) provide a theoretical justification for the symmetric equilibrium to be the unique rational-expectation equilibrium in the Schumpeterian model.

2.4.3 Endogenous firm entry

Following Hopenhayn (1992) and Melitz (2003), we consider a fixed entry cost to generate an endogenous entry of heterogeneous firms. Let's denote $v_t(\lambda)$ as the *ex post* value of an invention (i.e., after the realization of the quality step size λ). In this case, the entry condition is

$$v_t(\lambda) \geq \beta_t, \tag{16}$$

where the entry cost $\beta_t = \beta Q_t^{(1-\theta)/\theta}$ is proportional to $Q_t^{(1-\theta)/\theta}$ to ensure balanced growth. Given that $\Pi_t(\lambda)$ is increasing in λ, there exists a threshold quality level $\tilde{\lambda}_t$ above which $v_t(\lambda) \geq \beta_t$ for all $\lambda \geq \tilde{\lambda}_t$. Also, it can be shown that $v_t(\lambda)/Q_t^{(1-\theta)/\theta}$ is stationary in equilibrium. Then, Lemma 1 shows that the threshold quality level $\tilde{\lambda}$ in $v_t(\tilde{\lambda}) = \beta_t$ is stationary.

Lemma 1 *There exists a unique and stationary threshold quality level* $\tilde{\lambda}_t = \tilde{\lambda}$ *for all* t.

Proof. See Appendix A. ■

Given the stationary threshold $\tilde{\lambda}$, Lemma 2 derives the no-arbitrage condition for the expected value v_t^e of an invention. In (17), $\Pr(\lambda \geq \tilde{\lambda}) = \tilde{\lambda}^{-1/\kappa}$ is the probability that a randomly drawn quality step size is larger than the threshold $\tilde{\lambda}$.

Lemma 2 *The no-arbitrage condition for the expected value* v_t^e *of an invention is*

$$r_t = \frac{\Pi_t^e + \dot{v}_t^e + \Pr(\lambda \geq \tilde{\lambda})\dot{\beta}_t - \Pr(\lambda \geq \tilde{\lambda})\phi_t[v_t^e + \Pr(\lambda \geq \tilde{\lambda})\beta_t]}{v_t^e + \Pr(\lambda \geq \tilde{\lambda})\beta_t}. \tag{17}$$

Proof. See Appendix A. ■

2.5 Monetary authority

We consider the nominal interest rate i_t as the policy instrument, which is exogenously set by the monetary authority. The Fisher equation is given by $i_t = \pi_t + r_t$, where $\pi_t \equiv \dot{P}_t/P_t$ is the inflation rate and P_t is the price level of final good. Given the aggregate nominal money balance $M_t \equiv P_t m_t$, the growth rate of the aggregate nominal money balance is

$$\mu_t \equiv \frac{\dot{M}_t}{M_t} = \pi_t + \frac{\dot{m}_t}{m_t} = i_t - r_t + \frac{\dot{m}_t}{m_t} = i_t - \rho - \frac{\dot{c}_t}{c_t} + \frac{\dot{m}_t}{m_t}, \tag{18}$$

where the last equality uses the aggregate consumption path in (4). It can be shown that given a stationary nominal interest rate i, aggregate consumption c_t and aggregate real money balance m_t grow at the same rate on the balanced growth path. Therefore, on the balanced growth path, the growth rate of the nominal money balance is determined by the nominal interest rate such that $\mu = i - \rho$. The government uses the seigniorage revenue $\dot{M}_t$ to finance a lump-sum transfer τ_t that has a real value given by

$$\tau_t = \frac{\dot{M}_t}{P_t} = \mu_t \frac{M_t}{P_t}, \tag{19}$$

which yields $\tau_t = (i - \rho)\, m_t$ on the balanced growth path.

2.6 Decentralized equilibrium

The equilibrium is a time path of allocations $\{c_t(h), a_t(h), m_t(h), b_t(h), y_t, L_t, k_t(j), x_t(j), R_t(j)\}$, a time path of prices $\{w_t, r_t, p_t(j), v_t(\lambda)\}$ and a time path of policies $\{i_t, \tau_t\}$. Also, at each instance of time, the following conditions hold:

- household $h \in [0,1]$ maximizes utility taking $\{w_t, r_t, i_t, \tau_t\}$ as given;
- competitive firms produce final good y_t to maximize profit taking prices as given;
- monopolistic firm $j \in [0,1]$ produces intermediate good $k_t(j)$ and chooses $\{x_t(i), p_t(j)\}$ to maximize profit;
- competitive R&D entrepreneurs choose $R_t(j)$ to maximize expected profit taking $v_t(\lambda)$ as given;
- the market-clearing condition for labor holds such that $L_t = 1$;
- the market-clearing condition for final good holds such that $\int_0^1 c_t(h)dh + \int_0^1 x_t(j)dj + \int_0^1 R_t(j)dj + \tilde{\lambda}^{-1/\kappa}\phi_t\beta_t = y_t$;
- the total amount of cash owned by households equals the amount of cash borrowed by entrepreneurs such that $\int_0^1 m_t(h)dh = \int_0^1 b_t(h)dh = \zeta \int_0^1 R_t(j)dj$;
- the total value of assets owned by households equals the value of all monopolistic firms such that $\int_0^1 a_t(h)dh = \int_0^1 v_t(j)dj \equiv v_t$;
- the monetary authority uses seigniorage to finance a lump-sum transfer $\tau_t = \dot{M}_t/P_t$.

2.7 Aggregate economy

First, we derive the growth rate of aggregate technology Q_t by differentiating the log of (13) with respect to time and using the law of large numbers:

$$\frac{\dot{Q}_t}{Q_t} = \left[\int_0^1 \ln \lambda_t(j)dj\right] \Pr(\lambda \geq \tilde{\lambda})\phi_t = \left[\int_{\tilde{\lambda}}^{\infty} (\ln \lambda)\,\tilde{f}(\lambda)d\lambda\right]\tilde{\lambda}^{-1/\kappa}\phi_t = (\ln\tilde{\lambda} + \kappa)\tilde{\lambda}^{-1/\kappa}\phi_t, \quad (20)$$

where the truncated density function $\tilde{f}(\lambda)$ as a result of the threshold $\tilde{\lambda}$ is defined as

$$\tilde{f}(\lambda) \equiv \frac{f(\lambda)}{\int_{\tilde{\lambda}}^{\infty} f(\lambda)d\lambda} = \tilde{\lambda}^{\frac{1}{\kappa}} f(\lambda). \quad (21)$$

In (20), $\tilde{\lambda}^{-1/\kappa}\phi_t$ is the *composite* arrival rate of *implementable* quality improvements and $\kappa + \ln\tilde{\lambda}$ is the average step size of implemented quality improvements. Then, we derive the aggregate production function for y_t in the following lemma:

Lemma 3 *The aggregate production function for y_t is given by*

$$y_t = \left(\frac{1-\theta}{\tilde{\lambda} e^{\kappa}} Q_t \right)^{\frac{1-\theta}{\theta}}. \tag{22}$$

Proof. See Appendix A. ■

The aggregate production function in (22) implies that the growth rate of aggregate output y_t is given by

$$g_t \equiv \frac{\dot{y}_t}{y_t} = \frac{1-\theta}{\theta} \frac{\dot{Q}_t}{Q_t} = \frac{1-\theta}{\theta} (\ln \tilde{\lambda} + \kappa) \tilde{\lambda}^{-1/\kappa} \phi_t, \tag{23}$$

where the last equality uses (20). Lemma 4 shows that given a stationary nominal interest rate i, the aggregate economy jumps to a unique and stable balanced growth path along which ϕ and g are also stationary.

Lemma 4 *The aggregate economy jumps to a unique and stable balanced growth path.*

Proof. See Appendix A. ■

The no-arbitrage condition for the *ex-post* value of an invention with $\lambda \geq \tilde{\lambda}$ is given by

$$\frac{\Pi_t(\lambda)}{v_t(\lambda)} = r + \tilde{\lambda}^{-1/\kappa} \phi - \frac{\dot{v}_t(\lambda)}{v_t(\lambda)} = \rho + \tilde{\lambda}^{-1/\kappa} \phi, \tag{24}$$

where the last equality uses the aggregate consumption path in (4) and the property that c_t and v_t both grow at the steady-state equilibrium growth rate in (23). Then, substituting (11) and (24) into the entry condition $v_t(\tilde{\lambda}) = \beta Q_t^{(1-\theta)/\theta}$, we obtain

$$\left(\frac{\tilde{\lambda} - 1}{\tilde{\lambda}} \right) \frac{1}{\beta} = \frac{(\rho + \tilde{\lambda}^{-1/\kappa} \phi) Q_t^{(1-\theta)/\theta}}{(1-\theta) y_t}. \tag{25}$$

From (17), the equilibrium value of v_t^e on the balanced growth path is determined by

$$\frac{\Pi_t^e}{v_t^e + \tilde{\lambda}^{-1/\kappa} \beta_t} = r_t + \tilde{\lambda}^{-1/\kappa} \phi - \frac{\dot{v}_t^e + \tilde{\lambda}^{-1/\kappa} \dot{\beta}_t}{v_t^e + \tilde{\lambda}^{-1/\kappa} \beta_t} = \rho + \tilde{\lambda}^{-1/\kappa} \phi, \tag{26}$$

where the last equality uses the aggregate consumption path in (4) and the property that c_t, v_t^e and β_t all grow at the steady-state equilibrium growth rate in (23). The expected value of monopolistic profit is given by

$$\Pi_t^e = \left[\int_{\tilde{\lambda}}^{\infty} \left(\frac{\lambda - 1}{\lambda} \right) f(\lambda) d\lambda \right] (1-\theta) y_t = \left[\frac{\tilde{\lambda} - 1/(1+\kappa)}{\tilde{\lambda}^{\frac{1+\kappa}{\kappa}}} \right] (1-\theta) y_t. \tag{27}$$

Substituting (26) and (27) into the R&D condition in (14) yields

$$\left[\frac{\tilde{\lambda} - 1/(1+\kappa)}{\tilde{\lambda}^{\frac{1+\kappa}{\kappa}}} \right] \frac{1}{(1+\zeta i)\, \alpha + \tilde{\lambda}^{-1/\kappa} \beta} = \frac{(\rho + \tilde{\lambda}^{-1/\kappa} \phi) Q_t^{(1-\theta)/\theta}}{(1-\theta) y_t}. \tag{28}$$

Combining (25) and (28), we obtain the following condition:

$$(\tilde{\lambda}-1)\tilde{\lambda}^{1/\kappa}=\frac{\beta}{\alpha}\frac{\kappa}{1+\kappa}\frac{1}{1+\zeta i}, \tag{29}$$

where the left-hand side is monotonically increasing in $\tilde{\lambda}$. Therefore, (29) implicitly determines the unique equilibrium value of $\tilde{\lambda}$ as a decreasing function in the nominal interest rate i. Using (22), (25) and (29), we obtain the following condition:

$$\tilde{\lambda}^{-1/\kappa}\phi=\frac{\tilde{\lambda}^{-(1/\kappa+1/\theta)}}{1+\zeta i}\frac{\kappa}{1+\kappa}\frac{(1-\theta)^{1/\theta}}{\alpha e^{\kappa(1-\theta)/\theta}}-\rho, \tag{30}$$

which determines the unique equilibrium value of the composite innovation rate $\tilde{\lambda}^{-1/\kappa}\phi$. The right-hand side of (30) is decreasing in the nominal interest rate i for a given value of $\tilde{\lambda}$; however, $\tilde{\lambda}$ is also decreasing in i. Therefore, the overall effect of i on $\tilde{\lambda}^{-1/\kappa}\phi$ is ambiguous. The following proposition summarizes the overall effects of i on $\tilde{\lambda}^{-1/\kappa}\phi$ and g.

Proposition 1 *If the entry cost parameter β is sufficiently large (small), then the nominal interest rate i has an inverted-U effect (a monotonically negative) on the composite innovation rate $\tilde{\lambda}^{-1/\kappa}\phi$ and the equilibrium growth rate g.*

Proof. See Appendix A. ■

Intuitively, when the entry cost β is zero, the nominal interest rate i has no effect on the distribution of quality improvements that are implemented because all firms enter the market. In this case, the entry threshold becomes $\tilde{\lambda}=1$, and the equilibrium growth rate $g=\frac{1-\theta}{\theta}\kappa\phi$ is monotonically decreasing in the nominal interest rate i via the innovation arrival rate ϕ. However, when the entry cost β is positive, the nominal interest rate i affects the entry threshold $\tilde{\lambda}$ in addition to the innovation arrival rate ϕ. In this case, $\Pr(\lambda\geq\tilde{\lambda})=\tilde{\lambda}^{-1/\kappa}$ is increasing in the nominal interest rate i because an increase in the nominal interest rate i reduces the entry threshold $\tilde{\lambda}$ and leads to more quality improvements being implemented. Intuitively, the nominal interest rate i has a negative effect on the arrival rate ϕ of future innovations, which in turn increases the expected value of the profit stream generated by an implemented quality improvement and decreases the minimum quality step size that makes incurring the entry cost profitable. Overall, the effects of the nominal interest rate i on the composite innovation rate $\tilde{\lambda}^{-1/\kappa}\phi$ and the equilibrium growth rate $g=\frac{1-\theta}{\theta}(\ln\tilde{\lambda}+\kappa)\tilde{\lambda}^{-1/\kappa}\phi$ become ambiguous and follow an inverted-U pattern when the entry cost β is sufficiently large.

3 Wealth and income distributions

In this section, we show that the wealth distribution is stationary and exogenously determined by its initial distribution. Then, we show that the income distribution is also stationary but endogenously affected by the nominal interest rate.

3.1 Wealth distribution

In equilibrium, household $h \in [0,1]$ lends all its cash to entrepreneurs such that $m_t(h) = b_t(h)$. Substituting this condition into (2) yields

$$\dot{a}_t(h) + \dot{b}_t(h) = r_t[a_t(h) + b_t(h)] + w_t + \tau_t - c_t(h), \tag{31}$$

where we have also used the Fisher equation $r_t = i_t - \pi_t$. Aggregating (31) for all h, we have

$$\dot{a}_t + \dot{b}_t = r_t(a_t + b_t) + w_t + \tau_t - c_t. \tag{32}$$

Let's denote $z_t(h) \equiv a_t(h) + b_t(h)$ as household h's wealth, which consists of financial assets and bond holdings. Then, we define $s_{z,0}(h) \equiv z_0(h)/z_0$ as the initial share of wealth owned by household h, and $s_{z,0}(h)$ is exogenously given at time 0. We consider a general distribution function of initial wealth share with a mean of one and a standard deviation of $\sigma_z > 0$.

Taking the log of wealth share $s_{z,t}(h) \equiv z_t(h)/z_t$ at time t and differentiating the resulting expression with respect to time yield

$$\frac{\dot{s}_{z,t}(h)}{s_{z,t}(h)} = \frac{\dot{z}_t(h)}{z_t(h)} - \frac{\dot{z}_t}{z_t} = \frac{c_t - w_t - \tau_t}{z_t} - \frac{c_t(h) - w_t - \tau_t}{z_t(h)}. \tag{33}$$

Then, (33) can be re-expressed as

$$\dot{s}_{z,t}(h) = \frac{c_t - w_t - \tau_t}{z_t} s_{z,t}(h) - \frac{s_{c,t}(h)c_t - w_t - \tau_t}{z_t}, \tag{34}$$

where $s_{c,t}(h) \equiv c_t(h)/c_t$ is the share of consumption by household h at time t. Taking the log of $s_{c,t}(h)$ and differentiating the resulting expression with respect to time yield

$$\frac{\dot{s}_{c,t}(h)}{s_{c,t}(h)} = \frac{\dot{c}_t(h)}{c_t(h)} - \frac{\dot{c}_t}{c_t}. \tag{35}$$

Given that $\dot{c}_t(h)/c_t(h) = \dot{c}_t/c_t$ from (3) and (4), (35) becomes $\dot{s}_{c,t}(h) = 0$ for all t, which in turn implies $s_{c,t}(h) = s_{c,0}(h)$ for all t.[17] Given that $\{a_t, b_t, z_t, c_t, w_t, \tau_t\}$ all grow at the same rate g in equilibrium, (34) represents a one-dimensional differential equation, which describes the potential evolution of $s_{z,t}(h)$ given an initial $s_{z,0}(h)$. In Appendix A, we show that the coefficient on $s_{z,t}(h)$ in (34) is positive and equal to ρ. Together with the fact that $s_{z,t}(h)$ is a state variable, the only solution consistent with long-run stability is $\dot{s}_{z,t}(h) = 0$ for all t, which is achieved by consumption share $s_{c,t}(h)$ jumping to its steady-state value shown in Appendix A. Lemma 5 shows that as an equilibrium outcome, the wealth distribution is stationary and remains the same as the initial distribution, which is exogenously given at time 0.[18]

Lemma 5 *The wealth share of household $h \in [0,1]$ is given by $s_{z,t}(h) = s_{z,0}(h)$ for all t.*

Proof. See Appendix A. ■

[17] $s_{c,0}(h)$ is an endogenous variable to be determined in Appendix A.

[18] See also Garcia-Penalosa and Turnovsky (2006) who show the stationarity of the wealth distribution in an AK growth model.

3.2 Income distribution

From (31), income earned by household h is given by

$$I_t(h) = r_t z_t(h) + w_t. \tag{36}$$

Aggregating (36) yields total income earned by all households given by

$$I_t = r_t z_t + w_t. \tag{37}$$

Combining (36) and (37) yields the share of income earned by household h given by

$$s_{I,t}(h) \equiv \frac{I_t(h)}{I_t} = \frac{s_{z,0}(h)\ r_t z_t + w_t}{r_t z_t + w_t}, \tag{38}$$

which also uses $z_t(h) = s_{z,t}(h) z_t = s_{z,0}(h) z_t$ from Lemma 5. The distribution function of income share $s_{I,t}(h)$ has a mean of one and the following standard deviation:

$$\sigma_{I,t} \equiv \sqrt{\int_0^1 [s_{I,t}(h) - 1]^2 dh} = \frac{r_t z_t}{r_t z_t + w_t} \sqrt{\int_0^1 [s_{z,0}(h) - 1]^2 dh} = \frac{r_t z_t / w_t}{1 + r_t z_t / w_t} \sigma_z, \tag{39}$$

which is also the coefficient of variation of income and is increasing in $r_t z_t / w_t$. As discussed in Chu and Cozzi (2018), income inequality $\sigma_{I,t}$ is increasing in $r_t z_t / w_t$ because an unequal distribution of wealth is the source of income inequality in the model. Therefore, whenever interest income $r_t z_t$ increases relative to wage income w_t, the degree of income inequality increases.

Lemma 6 *Income inequality is increasing in the ratio of interest income to wage income.*

Proof. Equation (39) shows that $\sigma_{I,t}$ is increasing in $r_t z_t / w_t$. ■

Recall that total wealth is given by $z_t = a_t + b_t$. The amount of financial assets a_t in the economy is given by

$$a_t = v_t = \int_{\tilde{\lambda}}^{\infty} v_t(\lambda) \tilde{f}(\lambda) d\lambda = \left[\tilde{\lambda}^{1/\kappa} (1 + \zeta i)\, \alpha + \beta\right] Q_t^{(1-\theta)/\theta}, \tag{40}$$

which uses $\int_{\tilde{\lambda}_t}^{\infty} v_t(\lambda) f(\lambda) d\lambda = (1 + \zeta i)\, \alpha_t + \tilde{\lambda}_t^{-1/\kappa} \beta_t$. Using (7), (22) and (40), we derive

$$\frac{a}{w} = \left[\tilde{\lambda}^{1/\kappa} (1 + \zeta i)\, \alpha + \beta\right] \frac{(\tilde{\lambda} e^{\kappa})^{(1-\theta)/\theta}}{\theta (1-\theta)^{(1-\theta)/\theta}}. \tag{41}$$

The amount of borrowing b_t in the economy is given by

$$b_t = \zeta R_t = \zeta \alpha Q_t^{(1-\theta)/\theta} \phi_t, \tag{42}$$

where the last equality uses (12). Using (7), (22) and (42), we derive

$$\frac{b}{w} = \zeta \alpha \phi \frac{(\tilde{\lambda} e^{\kappa})^{(1-\theta)/\theta}}{\theta (1-\theta)^{(1-\theta)/\theta}}. \tag{43}$$

Using (4), (41) and (43), we derive the ratio of total interest income to wage income as

$$\frac{rz}{w} = \frac{r(a+b)}{w} = (\rho + g) \left[\tilde{\lambda}^{1/\kappa} (1 + \zeta i)\, \alpha + \beta + \zeta \alpha \phi\right] \frac{(\tilde{\lambda} e^{\kappa})^{(1-\theta)/\theta}}{\theta (1-\theta)^{(1-\theta)/\theta}}, \tag{44}$$

where the growth rate g, the quality threshold $\tilde{\lambda}$ and the innovation arrival rate ϕ are determined by (23), (29) and (30), respectively.

4 Monetary policy on growth and inequality

In this section, we explore the effects of monetary policy on economic growth and income inequality. We begin by exploring the relationship between the inflation rate and the nominal interest rate. From the Fisher equation, the inflation rate is given by

$$\pi = i - r = i - g(i) - \rho, \tag{45}$$

where the last equality uses (4). Differentiating the steady-state equilibrium inflation rate π in (45) with respect to the nominal interest rate i yields

$$\frac{\partial \pi}{\partial i} = 1 - \frac{\partial g(i)}{\partial i}. \tag{46}$$

Therefore, so long as $\partial g(i)/\partial i < 1$, the relationship between the steady-state equilibrium inflation rate and the nominal interest rate is positive.[19] This positive *long-run* relationship between the inflation rate and the nominal interest rate is supported by empirical studies such as Mishkin (1992) and Booth and Ciner (2001). In the following sections, we explore the effects of the nominal interest rate on economic growth and income inequality. It is useful to note that any relationship between the nominal interest rate and growth/inequality would also apply to inflation and growth/inequality given the positive relationship between inflation and the nominal interest rate.

4.1 Monetary policy under a zero entry cost

We first consider the case of a zero entry cost $\beta = 0$. In this case, the threshold quality level becomes $\tilde{\lambda} = 1$. Then, the equilibrium growth rate in (23) becomes $g = \frac{1-\theta}{\theta}\kappa\phi$, where the innovation arrival rate ϕ in (30) simplifies to

$$\phi = \frac{1}{1+\zeta i}\frac{\kappa}{1+\kappa}\frac{(1-\theta)^{1/\theta}}{\alpha e^{\kappa(1-\theta)/\theta}} - \rho, \tag{47}$$

which is decreasing in the nominal interest rate i. As for the effect of the nominal interest rate i on income inequality, we know from Lemma 6 that we simply have to examine how i affects the ratio of total interest income to wage income in (44). We begin by examining separately the effects of i on ra/w and rb/w.

Under a zero entry cost, the ratio of asset interest income to wage income simplifies to

$$\frac{ra}{w} = \left(\rho + \frac{1-\theta}{\theta}\kappa\phi\right)\frac{1}{\rho+\phi}\frac{\kappa}{1+\kappa}\frac{1-\theta}{\theta}, \tag{48}$$

which uses (4), (41) and (47). Recall that the innovation arrival rate ϕ is decreasing in the nominal interest rate i. Therefore, (48) shows that the nominal interest rate i has two opposing effects on the ratio ra/w of asset interest income to wage income. First, an increase in i reduces the real interest rate $r = \rho + \frac{1-\theta}{\theta}\kappa\phi$ by decreasing innovation and the equilibrium growth

[19] Under our calibrated parameter values, the equilibrium inflation rate is indeed increasing in the nominal interest rate.

rate. This corresponds to the *interest-rate* effect of innovation on income inequality identified in Chu and Cozzi (2018), who consider R&D subsidies instead of monetary policy. Second, an increase in i reduces the rate of creative destruction and raises the asset-wage ratio a/w. This corresponds to the *asset-value* effect of innovation on income inequality in Chu and Cozzi (2018). Equation (48) shows that as $\rho \to 0$, the two effects cancel each other. For the more general case with $\rho > 0$, differentiating ra/w in (48) with respect to i yields the following result:

$$\frac{\partial ra/w}{\partial i} > 0 \Leftrightarrow \kappa < \frac{\theta}{1-\theta}. \tag{49}$$

Therefore, the positive *asset-value* effect of i on income inequality dominates the negative *interest-rate* effect of i on income inequality if and only if $\kappa < \theta/(1-\theta)$. This result generalizes the one in Chu and Cozzi (2018), who consider a symmetric quality step size and find that the asset-value effect of R&D subsidies dominates the interest-rate effect of R&D subsidies if and only if the quality step size is sufficiently small. In the case of asymmetric quality step sizes, the average quality step size is increasing in κ. Therefore, a small value of κ implies a small average quality step size, under which the asset-value effect dominates the interest-rate effect of monetary policy on income inequality.

Under a zero entry cost, the ratio of bond interest income to wage income simplifies to

$$\frac{rb}{w} = \left(\rho + \frac{1-\theta}{\theta}\kappa\phi\right)\zeta\phi\frac{\alpha(e^{\kappa})^{(1-\theta)/\theta}}{\theta(1-\theta)^{(1-\theta)/\theta}}, \tag{50}$$

which uses (4) and (43). Equation (50) shows that the ratio rb/w of bond interest income to wage income is increasing in the innovation arrival rate ϕ, which in turn is decreasing in the nominal interest rate i. The first negative effect is that an increase in i reduces the real interest rate $r = \rho + \frac{1-\theta}{\theta}\kappa\phi$ by decreasing innovation and the equilibrium growth rate. The second negative effect is that an increase in i decreases R&D and the amount of borrowing, which in turn decreases the bond-wage ratio b/w.

Combining (48) and (50) yields the ratio of total interest income to wage income given by

$$\frac{rz}{w} = \left(\rho + \frac{1-\theta}{\theta}\kappa\phi\right)\left[\frac{1}{\rho+\phi}\frac{\kappa}{1+\kappa}\frac{1-\theta}{\theta} + \zeta\phi\frac{\alpha(e^{\kappa})^{(1-\theta)/\theta}}{\theta(1-\theta)^{(1-\theta)/\theta}}\right]. \tag{51}$$

As $\rho \to 0$, the two effects of the nominal interest rate i on the ratio ra/w of asset interest income to wage income cancel each other. In this case, we are left with the negative effects of i on the ratio rb/w of bond interest income to wage income. For the more general case in which $\rho > 0$, the overall effect of i on rz/w depends on the relative value of κ and $\theta/(1-\theta)$. If $\kappa > \theta/(1-\theta)$, then the effects of i on ra/w and rb/w are both negative. In this case, the overall effect of the nominal interest rate on income inequality is negative. If $\kappa < \theta/(1-\theta)$, then the effect of i on ra/w is positive whereas the effect of i on rb/w is negative. In this case, the overall effect of the nominal interest rate on income inequality can be positive, negative or U-shaped. Proposition 2 summarizes these results.

Proposition 2 *Given a zero entry cost parameter β, an increase in the nominal interest rate has the following effects: (a) it has a negative effect on income inequality if $\kappa > \theta/(1-\theta)$ and (b) it may have a positive, negative or U-shaped effect on income inequality if $\kappa < \theta/(1-\theta)$.*

Proof. See Appendix A. ■

4.2 Monetary policy under a positive entry cost

We now consider the general case of a positive entry cost $\beta > 0$. Recall that the effects of the nominal interest rate on income inequality depend on how it affects the ratio of total interest income to wage income, which depends on r, a/w and b/w. Proposition 1 shows that the nominal interest rate has an inverted-U effect on the equilibrium growth rate under a sufficiently large entry cost β. Therefore, the nominal interest rate also has an inverted-U effect on the real interest rate $r = \rho + g$ under a sufficiently large entry cost β. It is useful to note that this *interest-rate* effect works through the quality threshold $\tilde{\lambda}$ in addition to the innovation arrival rate ϕ in the previous section and in Chu and Cozzi (2018).

We now consider how the nominal interest rate affects the asset-wage ratio a/w. Substituting (29) and (30) into (41) yields

$$\frac{a}{w} = \frac{1}{\rho + \tilde{\lambda}^{-1/\kappa}\phi} \frac{\tilde{\lambda} - 1/(1+\kappa)}{\tilde{\lambda}} \frac{1-\theta}{\theta}, \tag{52}$$

where the quality threshold $\tilde{\lambda} > 1$ is determined by (29) and decreasing in the nominal interest rate i. In the previous section with $\beta = 0$, the quality threshold is simply $\tilde{\lambda} = 1$. In this special case, the positive *asset-value* effect works through the innovation arrival rate ϕ, which in turn is decreasing in i. However, in the more general case with $\beta > 0$, the asset-value effect also works through the quality threshold $\tilde{\lambda}$ via two channels. First, as explained in the discussion of Proposition 1, an increase in the nominal interest rate i reduces the entry quality threshold $\tilde{\lambda}$, which in turn leads to more innovations being implemented and increases $\tilde{\lambda}^{-1/\kappa}$ in the composite creative destruction rate $\tilde{\lambda}^{-1/\kappa}\phi$. This effect works to decrease a/w as shown in (52). Second, the lower average quality step size also reduces the average markup ratio and the average value of monopolistic firms, which in turn decreases a/w. It is useful to note that these negative *asset-value* effects have the opposite sign as the one in the previous section by working through a different channel that is the quality threshold $\tilde{\lambda}$, which is absent in Chu and Cozzi (2018).

We now consider how the nominal interest rate affects the bond-wage ratio b/w. Substituting (29) and (30) into (43) yields

$$\frac{b}{w} = \zeta\alpha\tilde{\lambda}^{1/\kappa}\left[\frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\theta}} \frac{(1-\theta)^{1/\theta}}{\beta e^{\kappa(1-\theta)/\theta}} - \rho\right] \frac{(\tilde{\lambda}e^{\kappa})^{(1-\theta)/\theta}}{\theta(1-\theta)^{(1-\theta)/\theta}}. \tag{53}$$

Equation (53) shows that the nominal interest rate i affects b/w through $\tilde{\lambda}$ via multiple channels. The main effect is similar to and complements the one in the previous section but once again works through a different channel that the nominal interest rate reduces the quality threshold and the average quality step size, which in turn decreases the average markup ratio and the expected value of monopolistic profits. This general-equilibrium effect in turn reinforces the direct negative direct of i on R&D and the amount of borrowing as well as the bond-wage ratio b/w.

In Section 4.1, we find that in the case of a zero entry cost $\beta = 0$ and a positive discount rate $\rho > 0$, an increase in the nominal interest rate has both positive and negative effects on income inequality. In this section, we find that in the case of a positive entry cost $\beta > 0$, an increase in the nominal interest rate has additional effects on income inequality via endogenous firm entry. Therefore, when the entry cost β and the discount rate ρ are both positive and the CIA parameter ζ, which determines the effects of the nominal interest rate, is sufficiently large, we find that an increase in the nominal interest rate has a potentially inverted-U effect on income inequality. Specifically, Proposition 3 shows that the effect of the nominal interest rate on income inequality is firstly increasing and eventually decreasing.

Proposition 3 *If the product of the discount rate and the entry cost (i.e., $\rho\beta$) is positive and the CIA parameter ζ is sufficiently large, then income inequality is firstly increasing and eventually decreasing in the nominal interest rate i.*

Proof. See Appendix A. ■

It is important to note that this *inverted-U* effect of the nominal interest rate on income inequality is different from the *U-shaped* effect under a zero entry cost $\beta = 0$ in Proposition 2. Therefore, without the entry cost, it is impossible for the model to generate an inverted-U effect on income inequality. The reason is that as Proposition 1 shows, the nominal interest rate has an inverted-U effect on the equilibrium growth rate if and only if the entry cost β is sufficiently large. In other words, endogenous firm entry is necessary for the emergence of an inverted-U effect of the nominal interest rate on economic growth, which in turn generates an inverted-U effect on income inequality that is otherwise absent without endogenous firm entry.

The main mechanisms behind this inverted-U effect on income inequality can be summarized as follows. Given that the real interest rate is increasing in the growth rate of consumption, the inverted-U effect of the nominal interest rate on economic growth causes an inverted-U effect on the real interest rate. Furthermore, the nominal interest rate has both positive and negative effects on the value of assets. On the one hand, by slowing down the innovation arrival rate, the nominal interest rate increases the market value of monopolistic firms, which in turn increases the value of assets. On the other hand, by lowering the entry threshold for quality improvements, the nominal interest rate reduces the average step size of implemented quality improvements and decreases the average markup ratio, which in turn decreases the market values of monopolistic firms and assets. Combining all these effects yields an overall inverted-U effect of the nominal interest rate on income inequality, which exists only under endogenous entry of heterogeneous firms.

5 Quantitative analysis

In this section, we provide a quantitative analysis. In Section 5.1, we use cross-country data to estimate the empirical effects of inflation on economic growth and income inequality. In Section 5.2, we calibrate the model to data and our regression estimates before simulating the effects of inflation on growth and inequality. Section 5.3 explores how the wealth holdings of households affect inflation and growth.

5.1 Empirical estimation

To facilitate the subsequent calibration, we first provide an empirical estimation of the effects of inflation on economic growth and income inequality. Here we use cross-country panel data to estimate the following regressions:

$$\begin{aligned} g_{it} &= \gamma_1 \pi_{it} + \gamma_2 \pi_{it}^2 + \Gamma X_{it} + \delta_i + \epsilon_{it}, \\ \sigma_{it} &= \omega_1 \pi_{it} + \omega_2 \pi_{it}^2 + \Omega X_{it} + \overline{\delta}_i + \overline{\epsilon}_{it}, \end{aligned}$$

where g_{it} denotes the growth rate of real GDP in country i at time t, π_{it} denotes the inflation rate from the Consumer Price Index in country i at time t, and σ_{it} denotes income inequality in country i at time t. The Gini index, which is a conventional measure of income inequality, is collected from the World Income Inequality Database (WIID) version 3.4. This database also provides information on the income share of each decile group; e.g., the first decile group includes the poorest 10% of the population, whereas the tenth decile group includes the richest 10%. We calculate the ratio of income between the top group and the bottom group as an alternative measure of income inequality. X_{it} is a vector of the following control variables: a constant, the degree of openness, the unemployment rate, and investment risks. We follow Fan and Gao (2017) to use the investment profile index and the corruption index from the International Country Risk Guide to measure investment risks.[20] δ_i and $\overline{\delta}_i$ are the country fixed effects.[21] Summary statistics for all variables are reported in Appendix B.

To be consistent with our innovation-driven growth model, we focus on high-income countries and consider the data from 1995 to 2014. Since the difference between the maximum and the minimum values of the inflation rate in a few countries are too high (which are close to the value of hyperinflation rate and much higher than the value of galloping inflation rate), we delete the top 15% outliers.[22] In the first three columns, we define high-income countries according to the definition given by the WIID. In the last three columns, we define high-income countries according to the classification given by the World Bank (WB). Table 1 shows that the overall effects of inflation on economic growth and income inequality follow an inverted-U pattern. The growth-maximizing inflation rate is about 5%,[23] whereas the inequality-maximizing inflation rate is about 12%.[24]

One concern about our benchmark (reduced-form) regression results is that the nominal interest rate could enter both in the error term and inflation, which may lead to an endogeneity problem and bias our results. To address this issue, we use the lending interest rate collected from CEIC database as an instrumental variable for the inflation rate. As reported in Appendix B, the results are still robust and the threshold values are similar to the ones in Table 1.

[20] We rescale these two indexes into a number between zero and ten.

[21] If we control year fixed effects instead, our results (available upon request) still hold.

[22] If we delete the top 5% or 10% or 20% outliers instead, the results are still significant.

[23] This value is between the estimates reported in Bick (2010) and Lopez-Villavicencio and Mignon (2011).

[24] This estimate is much lower than the estimate in Natob (2015), who however focuses on developing countries.

Table 1: Effects of inflation on economic growth and income inequality						
	WIID			WB		
	growth	income inequality		growth	income inequality	
	(1)	(2)	(3)	(4)	(5)	(6)
π_{it}	2.368***	0.179***	0.159***	1.742***	0.184***	0.162***
	(0.506)	(0.043)	(0.054)	(0.513)	(0.043)	(0.054)
π_{it}^2	-0.242***	-0.008***	-0.007***	-0.143**	-0.008***	-0.007***
	(0.066)	(0.002)	(0.002)	(0.062)	(0.002)	(0.002)
country fixed effect	YES	YES	YES	YES	YES	YES
control variables	YES	YES	YES	YES	YES	YES
observations	612	448	448	740	452	452
R-squared	0.139	0.898	0.923	0.114	0.903	0.929

Note: *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Standard errors are in parentheses. Columns 1 and 4 correspond to the GDP growth rate. Columns 2-3 and 5-6 use different measures of income inequality. Specifically, columns 2 and 5 correspond to the income difference between the top 10% and the bottom 10% of the population. Columns 3 and 6 correspond to the Gini coefficient. In the first (last) three columns, high-income countries follow the classification by the WIID (WB).

5.2 Calibration and simulation

To perform a more realistic quantitative analysis, we generalize the model by introducing elastic labor supply $L_t(h) = 1 - l_t(h)$, where $l_t(h)$ denotes leisure. The generalized utility function is

$$u(h) = \int_0^\infty e^{-\rho t} \left\{ \ln c_t(h) + \eta \ln \left[\frac{l_t(h)}{(l_t)^\varepsilon} \right] \right\} dt, \tag{54}$$

where the parameter $\eta \geq 0$ determines the importance of leisure $l_t(h)$ in utility. We also allow for external habit, measured by the parameter $\varepsilon \in [0, 1]$, on leisure $l_t \equiv \int_0^1 l_t(h)dh$.[25] Furthermore, we impose a CIA constraint on consumption as in Chu and Cozzi (2014). With the additional cash requirement on consumption, the CIA constraint becomes $b_t(h) + \varphi c_t(h) \leq m_t(h)$, where the parameter $\varphi \in [0, 1]$ determines the fraction of consumption spending that is subject to the CIA constraint. We provide detailed derivations of this generalized model in Appendix C.

We now calibrate the model to perform a quantitative analysis on the relationship between inflation and economic growth/income inequality. The model features the following structural parameters $\{\zeta, \rho, \theta, \eta, \varphi, \kappa, \alpha, \beta, \varepsilon\}$ and policy instrument i. We consider a range of values for the CIA-R&D parameter $\zeta \in \{0.65, 1\}$.[26] We set the discount rate ρ to a conventional value of 0.05. We set the degree of labor intensity θ to a value of 0.56 in the US; see for example Karabarbounis and Neiman (2014). We calibrate the leisure parameter η by matching the

[25] It is useful to note that the presence of external habit on leisure does not affect the equilibrium allocations but only affects welfare. Therefore, the parameter ε enables us to explore its implications on the utility-maximizing inflation rate without affecting the equilibrium allocations. We do not include external habit on consumption as it is well known that it leads to a counterfactual comovement between hours worked and productivity shocks. For example, Khorunzhina (2015) finds that the counterfactual "response of hours worked [...] disappears once habit in consumption is ruled out but habit in leisure remains".

[26] We find that when ζ is too small, the calibrated value of κ becomes greater than unity.

average fraction of time devoted to labor supply L as 0.3. We calibrate the CIA-consumption parameter φ using the M1-consumption ratio from 1990 to 2016 in the US. As for the Pareto distribution parameter κ, we calibrate its value by matching an average TFP growth rate g of 0.5% from 1990 to 2016 in the US. As for the R&D cost parameter α and the entry cost parameter β, we calibrate their values by matching the growth-maximizing inflation rate and the inequality-maximizing inflation rate estimated in the previous section. We calibrate the monetary policy instrument i by matching the average inflation rate in the US, which is about 2.5% in the past two decades. Finally, we will consider the full range of values for the external habit parameter $\varepsilon \in [0, 1]$ when evaluating the welfare effects of inflation. The calibrated parameter values are summarized in Table 2.

Table 2: Calibration

ζ	ρ	θ	η	φ	κ	α	β	i
1.00	0.05	0.56	1.381	0.20	0.34	$2.57*10^{-5}$	0.19	0.08
0.65	0.05	0.56	1.379	0.20	0.90	$3.17*10^{-3}$	0.15	0.08

Figure 1 simulates the relationship between inflation and economic growth. We find that the relationship between inflation and economic growth follows an inverted-U pattern. The equilibrium growth rate is maximized at an inflation rate of 5.0%. After that, any further increase in inflation is associated with a decline in economic growth. For example, increasing the inflation rate from 5% to 15% leads to a decrease in the equilibrium growth rate that ranges from 0.006% (in the case of $\zeta = 0.65$) to 0.007% (in the case of $\zeta = 1$).

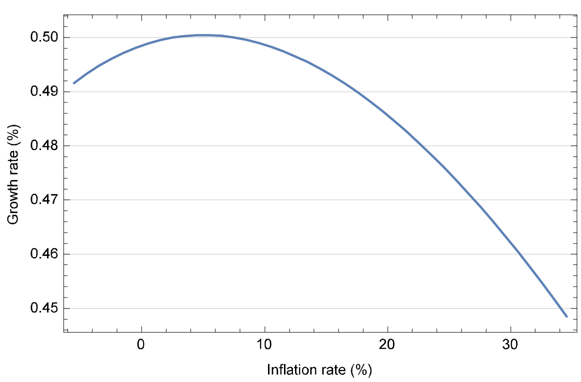

Figure 1a: Inflation and growth ($\zeta = 1$)

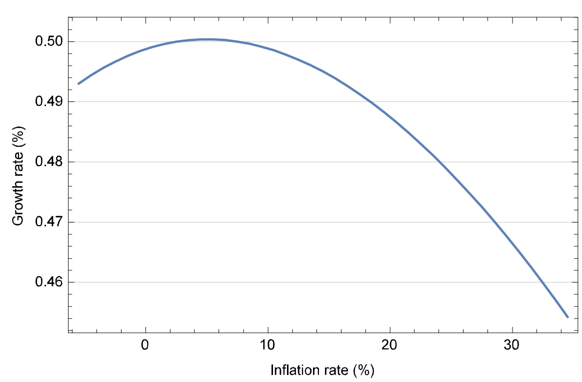

Figure 1b: Inflation and growth ($\zeta = 0.65$)

Figure 2 simulates the relationship between inflation and income inequality. We find that the relationship between inflation and income inequality also follows an inverted-U pattern. The coefficient of variation of income is maximized at an inflation rate of 12%. When the inflation rate increases from the benchmark value of 2.5% to 12%, the percent change in the coefficient of variation of income ranges from 0.10% (in the case of $\zeta = 0.65$) to 0.12% (in the case of $\zeta = 1$). When the inflation rate is above 12%, any further increase in inflation is associated with a decline in income inequality.

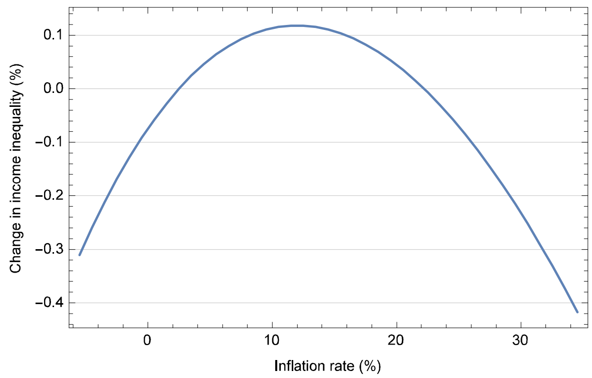

Figure 2a: Inflation and inequality ($\zeta = 1$)

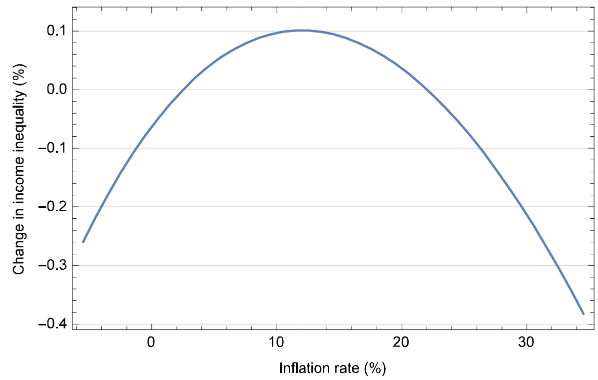

Figure 2b: Inflation and inequality ($\zeta = 0.65$)

Finally, we consider the relationship between inflation and the utility of the households that have a wealth share of unity (i.e., $s_{z,0}(h) = 1$).[27] This benchmark case corresponds to social welfare in a model with a representative household. Figure 3 plots the utility-maximizing inflation rates for $\varepsilon \in [0, 1]$. Unlike many previous studies that do not feature innovation and creative destruction,[28] the Friedman rule does not hold in our Schumpeterian model, and the welfare-maximizing nominal interest rate and inflation rate are both positive. As shown in Chu and Cozzi (2014), the Schumpeterian growth model features a negative externality of R&D in the form of a business-stealing effect. Therefore, an increase in the inflation rate that reduces R&D may improve welfare. Furthermore, in this Schumpeterian model with endogenous entry, an increase in the inflation rate has an additional positive effect on welfare by increasing the frequency of entries. Khorunzhina (2015) estimates that the degree of external habit ε on leisure is about 0.95, which corresponds to a welfare-maximizing inflation rate of about 14% in the case of $\zeta = 1$.

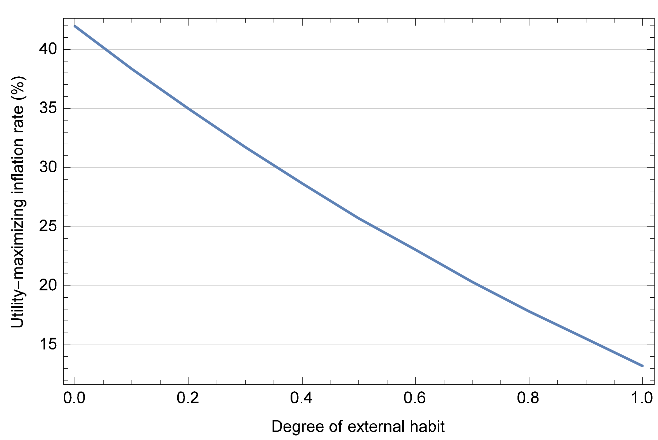

Figure 3a: Utility-maximizing inflation ($\zeta = 1$)

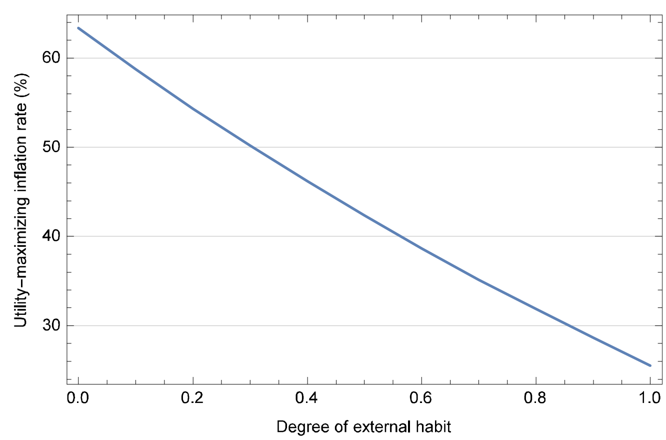

Figure 3b: Utility-maximizing inflation ($\zeta = 0.65$)

[27] In Section 6, we will explore how the wealth share $s_{z,0}(h)$ affects the utility-maximizing inflation rate.

[28] See for example Wong (2016) for a recent study that considers a new monetarist model with heterogeneous agents and the accumulation of human capital as an endogenous growth engine.

5.3 How wealth inequality affects inflation and growth

In this section, we explore how the wealth share $s_{z,0}(h)$ of a household determines its utility-maximizing inflation rate $\pi^*(h)$. We find that an increase in $s_{z,0}(h)$ leads to a decrease in $\pi^*(h)$. Intuitively, given that a household's consumption is increasing in its wealth, the negative asset-value effects of inflation are stronger for wealthier households, which in turn prefer a lower inflation rate. Figure 4 plots the negative relationship between $s_{z,0}(h)$ and $\pi^*(h)$ for the case of $\varepsilon = 0.95$. Given that the households' preferences on the inflation rate are single-peaked, the median voter theorem applies. Suppose the wealth share owned by the median household is $s_{z,0}(m)$. Then, an increase in $s_{z,0}(m)$ would affect the equilibrium growth rate in the economy by decreasing the inflation rate that is preferred by the median voter and selected in a majority-rule voting system. Figure 5 plots the relationship between $s_{z,0}(m)$ and the equilibrium growth rate $g^*(m)$ that corresponds to the utility-maximizing inflation rate of the median household. As $s_{z,0}(m)$ increases, the inflation rate preferred by the median household decreases, which in turn causes the equilibrium growth rate to increase initially until reaching the growth-maximizing inflation rate of 5% after which the equilibrium growth rate decreases.

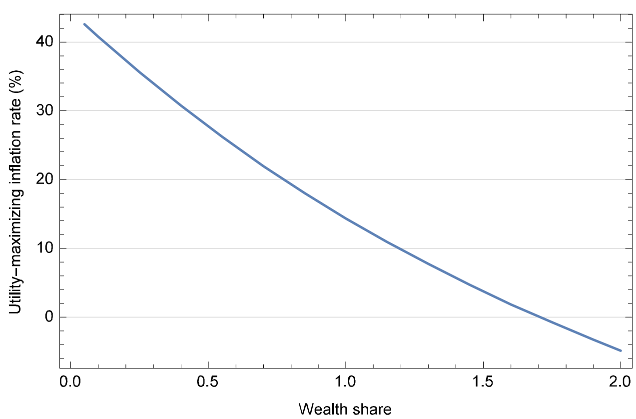

Figure 4a: Inflation and wealth share ($\zeta = 1$)

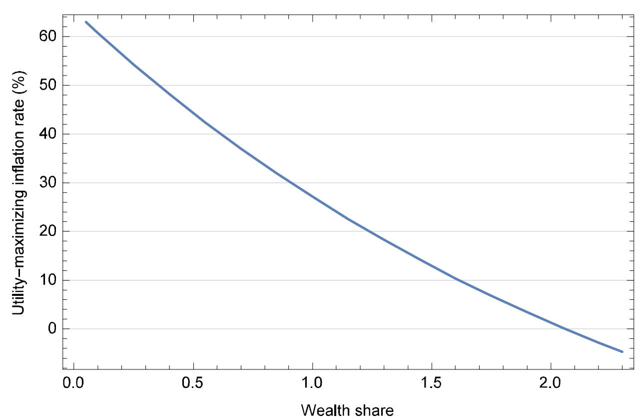

Figure 4b: Inflation and wealth share ($\zeta = 0.65$)

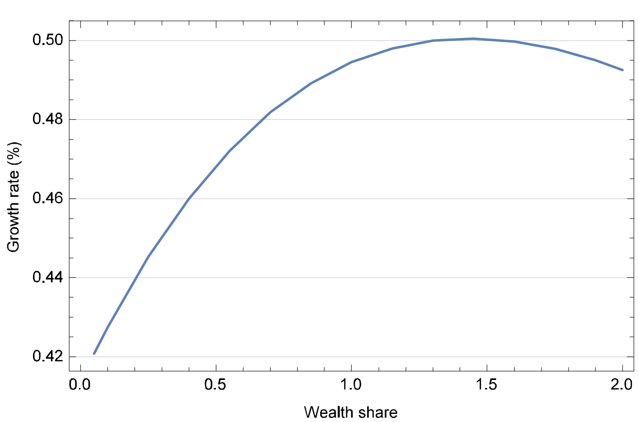

Figure 5a: Growth and wealth share ($\zeta = 1$)

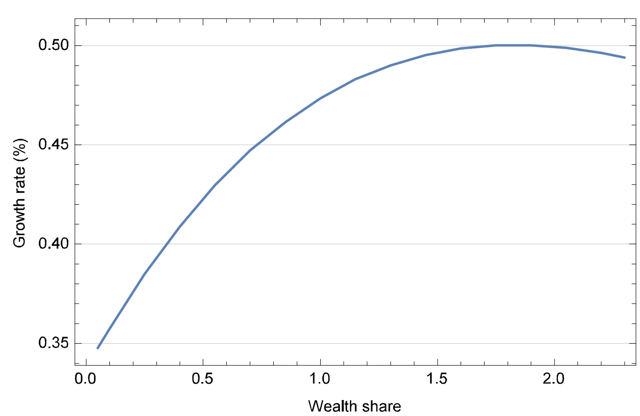

Figure 5b: Growth and wealth share ($\zeta = 0.65$)

6 Conclusion

In this study, we have developed a Schumpeterian growth model with two dimensions of heterogeneity among households and firms. We model household heterogeneity by assuming that households own different levels of wealth, which in turn generate an endogenous distribution of income. We model firm heterogeneity by assuming random quality improvements and a cost of entering a market, which together generate an endogenous distribution of implemented quality improvements. Both the income distribution and the implemented quality distribution are affected by monetary policy. Within this monetary growth-theoretic framework, we find that inflation has inverted-U effects on both economic growth and income inequality. Furthermore, we calibrate our model to match the growth-maximizing and inequality-maximizing inflation rates that are estimated using cross-country panel data. We also simulate the utility-maximizing level of inflation and explore how it is affected by the wealth holdings of households.

Finally, our model could feature scale effects as in the first-generation R&D-based growth model in Romer (1990), Segerstrom *et al.* (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992).[29] We sidestep this issue by normalizing the supply of labor to unity. Alternatively, one can remove scale effects in the Schumpeterian growth model by considering the semi-endogenous-growth approach in Segerstrom (1998) or the second-generation approach in Peretto (1998, 2007). We leave this potentially interesting extension to future research.

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Appendix A: Proofs

Proof of Lemma 1. It follows from (14) that $v_t^e + \tilde{\lambda}_t^{-1/\kappa}\beta_t = (1+\zeta i)\,\alpha_t + \tilde{\lambda}_t^{-1/\kappa}\beta_t$. Differentiating both sides of this equation with respect to time t yields

$$\dot{v}_t^e + \tilde{\lambda}_t^{-1/\kappa}\dot{\beta}_t = (1+\zeta i)\,\alpha_t \frac{\dot{\alpha}_t}{\alpha_t} + \tilde{\lambda}_t^{-1/\kappa}\beta_t \frac{\dot{\beta}_t}{\beta_t} \Leftrightarrow \frac{\dot{v}_t^e + \tilde{\lambda}_t^{-1/\kappa}\dot{\beta}_t}{v_t^e + \tilde{\lambda}_t^{-1/\kappa}\beta_t} = \frac{1-\theta}{\theta}\frac{\dot{Q}_t}{Q_t}, \tag{A1}$$

where the first equality cancels $\beta_t d(\tilde{\lambda}_t^{-1/\kappa})/dt$ from both sides and the second equality uses $\alpha_t = \alpha Q_t^{(1-\theta)/\theta}$ and $\beta_t = \beta Q_t^{(1-\theta)/\theta}$. Using (A1) and $\Pr(\lambda \geq \tilde{\lambda}_t) = \tilde{\lambda}_t^{-1/\kappa}$, we modify (17) as

$$r_t = \frac{\Pi_t^e}{v_t^e + \tilde{\lambda}_t^{-1/\kappa}\beta_t} + \frac{1-\theta}{\theta}\frac{\dot{Q}_t}{Q_t} - \tilde{\lambda}_t^{-1/\kappa}\phi_t. \tag{A2}$$

Similarly, we modify (24) for $\lambda = \tilde{\lambda}_t$ as

$$r_t = \frac{\Pi(\tilde{\lambda}_t)}{v_t(\tilde{\lambda}_t)} + \frac{1-\theta}{\theta}\frac{\dot{Q}_t}{Q_t} - \tilde{\lambda}_t^{-1/\kappa}\phi_t, \tag{A3}$$

which uses the entry condition $v_t(\tilde{\lambda}_t) = \beta_t = \beta Q_t^{(1-\theta)/\theta}$ and $\Pr(\lambda \geq \tilde{\lambda}_t) = \tilde{\lambda}_t^{-1/\kappa}$. From (A2) and (A3), we have

$$\frac{\Pi_t^e}{(1+\zeta i)\,\alpha + \tilde{\lambda}_t^{-1/\kappa}\beta} = \frac{\Pi_t(\tilde{\lambda}_t)}{\beta}, \tag{A4}$$

where

$$\Pi_t^e \equiv \int_{\tilde{\lambda}_t}^{\infty} \Pi_t(\lambda) f(\lambda) d\lambda = \left(\frac{\tilde{\lambda}_t - 1/(1+\kappa)}{\tilde{\lambda}_t^{\frac{1+\kappa}{\kappa}}}\right)(1-\theta)y_t \tag{A5}$$

and

$$\Pi_t(\tilde{\lambda}_t) = \frac{\tilde{\lambda}_t - 1}{\tilde{\lambda}_t}(1-\theta)y_t \tag{A6}$$

from (11). Using (A4)-(A6), we also have

$$\tilde{\lambda}_t^{\frac{1}{\kappa}}(\tilde{\lambda}_t - 1) = \frac{\kappa}{1+\kappa}\frac{1}{1+\zeta i}\frac{\beta}{\alpha}, \tag{A7}$$

which uniquely determines $\tilde{\lambda} > 1$ independent of t because the left-hand side of (A7) is increasing in $\tilde{\lambda}_t > 1$ and the right-hand side is independent of t. ■

Proof of Lemma 2. In the symmetric equilibrium, we have $v_t^e(i, \omega_i + 1) = v_t^e$, which can be expressed as

$$v_t^e \equiv \int_1^{\tilde{\lambda}} 0\; f(\lambda) d\lambda + \int_{\tilde{\lambda}}^{\infty} [v_t(\lambda) - \beta_t]\; f(\lambda) d\lambda = \int_{\tilde{\lambda}}^{\infty} v_t(\lambda) f(\lambda) d\lambda - \Pr(\lambda \geq \tilde{\lambda})\beta_t. \tag{A8}$$

Substituting the no-arbitrage condition for the value of an implemented innovation $v_t(\lambda) = [\Pi_t(\lambda) + \dot{v}_t(\lambda) - \Pr(\lambda \geq \tilde{\lambda})\phi_t v_t(\lambda)]/r_t$ into (A8) yields

$$r_t[v_t^e + \Pr(\lambda \geq \tilde{\lambda})\beta_t] = \Pi_t^e + \int_{\tilde{\lambda}}^{\infty} \dot{v}_t(\lambda) f(\lambda) d\lambda - \Pr(\lambda \geq \tilde{\lambda})\phi_t \int_{\tilde{\lambda}}^{\infty} v_t(\lambda) f(\lambda) d\lambda, \tag{A9}$$

which uses (A8) and $\Pi_t^e \equiv \int_{\tilde{\lambda}}^{\infty} \Pi_t(\lambda) f(\lambda) d\lambda$. Then, we use the R&D condition $v_t^e = (1+\zeta i)\,\alpha_t$ to derive

$$\int_{\tilde{\lambda}}^{\infty} v_t(\lambda) f(\lambda) d\lambda = (1+\zeta i)\,\alpha_t + \Pr(\lambda \geq \tilde{\lambda})\beta_t. \tag{A10}$$

Differentiating both sides in (A10) with respect to t yields

$$\int_{\tilde{\lambda}}^{\infty} \dot{v}_t(\lambda) f(\lambda) d\lambda = (1+\zeta i)\,\dot{\alpha}_t + \Pr(\lambda \geq \tilde{\lambda})\dot{\beta}_t. \tag{A11}$$

By substituting (A11) into (A9), with $v_t^e = (1+\zeta i_t)\,\alpha_t$, we can obtain

$$r_t[v_t^e + \Pr(\lambda \geq \tilde{\lambda})\beta_t] = \Pi_t^e + \dot{v}_t^e + \Pr(\lambda \geq \tilde{\lambda})\dot{\beta}_t - \Pr(\lambda \geq \tilde{\lambda})\phi_t[v_t^e + \Pr(\lambda \geq \tilde{\lambda})\beta_t], \tag{A12}$$

which is equivalent to (17). ■

Proof of Lemma 3. Substituting (8) and (10) into (6) yields

$$K_t = (1-\theta)\, y_t Q_t \exp\left(-\int_0^1 \ln \lambda_t(j) dj\right), \tag{A13}$$

which uses (13) for Q_t. Given that $\lambda_t(j) > \tilde{\lambda}$ for implemented innovations, the truncated distribution function for implemented innovations is as follows:

$$\tilde{f}(\lambda) \equiv \frac{f(\lambda)}{\int_{\tilde{\lambda}}^{\infty} f(\lambda) d\lambda} = \tilde{\lambda}^{\frac{1}{\kappa}} f(\lambda). \tag{A14}$$

By this,

$$\exp\left(-\int_0^1 \ln \lambda_t(j) dj\right) = \frac{1}{\tilde{\lambda} e^{\kappa}} \tag{A15}$$

holds. Substituting (A13)-(A15) into $y_t = K_t^{1-\theta}$ from (5) yields (22). ■

Proof of Lemma 4. Define $\tilde{c}_t \equiv c_t / Q_t^{(1-\theta)/\theta}$. Then, from (4), it holds that

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = r_t - \rho - \frac{1-\theta}{\theta}\frac{\dot{Q}_t}{Q_t}. \tag{A16}$$

From (A3) and (A6), with $v_t(\tilde{\lambda}) = \beta_t = Q_t^{(1-\theta)/\theta}\beta$, the real interest rate can be expressed as

$$r_t = \frac{\tilde{\lambda}-1}{\tilde{\lambda}}\frac{1-\theta}{\beta}\frac{y_t}{Q_t^{(1-\theta)/\theta}} + \frac{1-\theta}{\theta}\frac{\dot{Q}_t}{Q_t} - \tilde{\lambda}^{-1/\kappa}\phi_t. \tag{A17}$$

Substituting (22) and (A17) into (A16) yields

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{(1-\theta)^{1/\theta}}{\beta(e^{\kappa})^{(1-\theta)/\theta}} \frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\theta}} - \tilde{\lambda}^{-1/\kappa}\phi_t - \rho. \tag{A18}$$

Given (29), the right-hand side of (A18) is always decreasing in ϕ_t. To obtain the equilibrium expression of ϕ_t off the balanced growth path, we will derive the total demand for final goods. First, we use (8)-(10), and to have

$$\int_0^1 x_t(j)dj = \int_0^1 \frac{(1-\theta)\, y_t}{\lambda_t(j)} dj = (1-\theta)\, y_t \int_{\tilde{\lambda}}^{\infty} \frac{\tilde{f}(\lambda)}{\lambda} d\lambda = \frac{(1-\theta) y_t}{(1+\kappa)\,\tilde{\lambda}}. \tag{A19}$$

Then from (12), we have

$$\int_0^1 R_t(j)dj = \alpha Q_t^{(1-\theta)/\theta}\phi_t. \tag{A20}$$

Combining (A19) and (A20) with the final good market condition yields

$$\phi_t = \frac{1}{\alpha + \beta\tilde{\lambda}^{-1/\kappa}} \left[\left(1 - \frac{1-\theta}{(1+\kappa)\,\tilde{\lambda}}\right) \left(\frac{(1-\theta)}{\tilde{\lambda}e^{\kappa}}\right)^{(1-\theta)/\theta} - \tilde{c}_t \right] \equiv \underline{\phi}(\tilde{c}_t), \tag{A21}$$

which also uses (22) for y_t. Finally, by substituting (A21) into (A18), we have a one-dimensional differential equation in $\tilde{c}_t$. Given that ϕ_t decreases with $\tilde{c}_t$, the right-hand side of (A18) is increasing in $\tilde{c}_t$. The dynamics of $\tilde{c}_t$ is saddle-point stable; i.e., $\tilde{c}_t$ jumps to the unique steady-state $\tilde{c}$ at $t = 0$. Accordingly, (A18) determines the stationary equilibrium value of $\tilde{\lambda}^{-1/\kappa}\phi_t$ as in (30). Then, (A21) determines the steady-state value of $\tilde{c}_t$ as

$$\tilde{c} = \left(1 - \frac{1-\theta}{(1+\kappa)\,\tilde{\lambda}}\right) \left(\frac{1-\theta}{\tilde{\lambda}e^{\kappa}}\right)^{(1-\theta)/\theta} - \phi\left(\alpha + \beta\tilde{\lambda}^{-1/\kappa}\right). \tag{A22}$$

■

Proof of Proposition 1. By (29) and (30), we have

$$\tilde{\lambda}^{-1/\kappa}\phi = \frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\theta}} \frac{(1-\theta)^{1/\theta}}{\beta e^{\kappa(1-\theta)/\theta}} - \rho. \tag{A23}$$

We naturally focus on a non-trivial case where $\tilde{\lambda}^{-1/\kappa}\phi > 0$. There are, thus, lower and upper bounds of $\tilde{\lambda}$, say λ_- and λ_+, such that $\tilde{\lambda}^{-1/\kappa}\phi > 0$ holds if and only if $\tilde{\lambda} \in (\lambda_-, \lambda_+)$. Specifically, since

$$(\tilde{\lambda}^{-1/\kappa}\phi)' = \frac{1}{\theta} \frac{(1-\theta)^{1/\theta}}{\beta e^{\kappa(1-\theta)/\theta}} \frac{1-(1-\theta)\tilde{\lambda}}{\tilde{\lambda}^{1+1/\theta}},$$

$\tilde{\lambda}^{-1/\kappa}\phi$ is an inverted-U shaped function in $\tilde{\lambda}$ and maximized at $\tilde{\lambda} = 1/(1-\theta)$. Then, $\lambda_- < 1/(1-\theta) < \lambda_+$ holds. In addition, by (29), $\tilde{\lambda}$ is decreasing in i, thereby having an upper bound, denoted as $\overline{\lambda}$, due to $i \le 0$. It is easy to verify that $\overline{\lambda}$ increases from 1 to ∞ as β increases from 0. When β is such large that $\overline{\lambda} > 1/(1-\theta)$ holds, there is an inverted-U shaped relationship between $i \geq 0$ and $\tilde{\lambda}^{-1/\kappa}\phi$, noting $\tilde{\lambda}$ monotonically decreases with $i \geq 0$. Then, when β is small such that $\overline{\lambda} < 1/(1-\theta)$, the relationship is monotonically negative for any $i \geq 0$.

Differentiating (23) with respect to $\tilde{\lambda}$ yields

$$\left(\frac{dg}{d\tilde{\lambda}}\right)\frac{\beta\theta\tilde{\lambda}^{1+1/\theta}}{1-\theta} \tag{A24}$$
$$= \underbrace{\left(\frac{(1-\theta)^{1/\theta}}{e^{\kappa(1-\theta)/\theta}}\left(\tilde{\lambda}-1\right)-\beta\rho\tilde{\lambda}^{1/\theta}\right)}_{\equiv\Lambda_1(\tilde{\lambda})} - \underbrace{\frac{(1-\theta)^{1/\theta}}{\theta e^{\kappa(1-\theta)/\theta}}\left(\ln\tilde{\lambda}+\kappa\right)\left((1-\theta)\tilde{\lambda}-1\right)}_{\equiv\Lambda_2(\tilde{\lambda})},$$

where we have used (A23). Note the following properties: (a) $\Lambda_1((1/(1-\theta))) > 0$ and $\Lambda_2(1/(1-\theta)) = 0$; (b) $\Lambda_1(\tilde{\lambda})$ is an uni-modal function (maximized at some $\tilde{\lambda}$ that is higher than $1/(1-\theta)$) and $\Lambda_2(\tilde{\lambda})$ is a strictly increasing function; (c) $\Lambda_1(\lambda_-) = \Lambda_1(\lambda_+) = 0$; and (d) $\Lambda_1(\tilde{\lambda})$ is strictly concave and $\Lambda_2(\tilde{\lambda})$ is strictly convex. Taking into account these facts, with a usual graphical analysis, there must uniquely exist a threshold level of $\tilde{\lambda} \in (1/(1-\theta), \lambda_+)$, denoted as Λ^* in the figure, under (above) which $\Lambda_1(\tilde{\lambda}) > (<)\Lambda_2(\tilde{\lambda})$, that is, $dg/d\tilde{\lambda} > (<)0$. Recalling that $\overline{\lambda}$ increases with β and then $\tilde{\lambda}$ decreases with i, we can show that the relationship between i and g is also inverted-U shaped (negative) if β is large (small). ■

Proof of Lemma 5. From (3), (4), and (35), we can show that $s_{c,t}(h) = s_{c,0}(h)$ holds for all t. Substituting this condition into (34) yields

$$\dot{s}_{z,t}(h) = \frac{c_t - w_t - \tau_t}{z_t}s_{z,t}(h) - \frac{s_{c,0}(h)c_t - w_t - \tau_t}{z_t}. \tag{A25}$$

According to Lemma 4, $\{c_t, w_t, \tau_t, z_t, m_t\}$ all grow at the same rate g in equilibrium. Using (4) and (32), it is easy to obtain

$$\frac{c_t - w_t - \tau_t}{z_t} = r_t - \frac{\dot{z}_t}{z_t} = \rho > 0. \tag{A26}$$

Therefore, the coefficient on $s_{z,t}(h)$ in (A25) is always positive, which in turn implies that $\dot{s}_{z,t}(h) = 0$ for all t is the only solution of (A25) consistent with long-run stability. Finally, imposing $\dot{s}_{z,t}(h) = 0$ on (A25) yields the steady-state value of $s_{c,t}(h)$ given by

$$s_{c,0}(h) = 1 - \frac{\rho\left[1 - s_{z,0}(h)\right]}{c/z}, \tag{A27}$$

where we can make use of (40) and (42) to derive

$$\frac{c}{z} = \frac{\tilde{c}}{\tilde{\lambda}^{1/\kappa}(1+\zeta i)\alpha + \beta + \zeta\alpha\phi}. \tag{A28}$$

Note that $\tilde{c}$ is given by (A22). ■

Proof of Proposition 2. Differentiating rz/w in (51) with respect to ϕ yields

$$\frac{d(rz/w)}{d\phi} = \kappa\frac{\alpha(e^{\kappa})^{(1-\theta)/\theta}}{\theta(1-\theta)^{(1-\theta)/\theta}}\left[\tfrac{\rho}{(\rho+\phi)^2}\left(\tfrac{1-\theta}{\theta}\kappa - 1\right)\tfrac{1}{1+\kappa}\tfrac{(1-\theta)^{1/\theta}}{\alpha(e^{\kappa})^{(1-\theta)/\theta}} + \zeta\left(\tfrac{\rho}{\kappa} + 2\tfrac{1-\theta}{\theta}\phi\right)\right]. \tag{A29}$$

Given $\kappa > \theta/(1-\theta)$, (A29) shows that $d(rz/w)/d\phi > 0$. From (47), we know $d\phi/di < 0$. As a result, there is a negative effect of i and rz/w.

As for $\kappa < \theta/(1-\theta)$, we will show that there are three possibilities: for a feasible range of ϕ, (a) $d(rz/w)/d\phi < 0$, (b) $d(rz/w)/d\phi > 0$, or (c) $d(rz/w)/d\phi < (>)0$ if ϕ is smaller (larger). Before proceeding, it is useful to note that there is an upper bound of ϕ since $i \geq 0$ with (47), given by

$$\phi_+ \equiv \underbrace{\frac{\kappa}{1+\kappa}\frac{(1-\theta)^{1/\theta}}{\alpha e^{\kappa(1-\theta)/\theta}}}_{\equiv\vartheta} - \rho.$$

We will derive a sufficient conditions for each case, by focusing on both ends of $\phi \in (0, \phi_+]$.

First, by substituting $\phi \to 0$ (i.e., the lower bound) into (A29), we can show that $d(rz/w)/d\phi > 0$ holds at $\phi \to 0$ if

$$\left(1 - \frac{\kappa}{\theta/(1-\theta)}\right) < \frac{\zeta\rho^2}{\vartheta}. \tag{A30}$$

Moreover, it is easy to derive $d^2(rz_t/w_t)/d\phi^2 > 0$ when $\kappa < \theta/(1-\theta)$. As a result, $d(rz/w)/d\phi > 0$ holds for any $\phi \in (0, \phi_+]$. Given $d\phi/di < 0$, in this case, there is a negative effect of i on rz/w.

Second, it is straightforward to verify that $d(rz/w)/d\phi < 0$ holds at $\phi \to 0$ if (A30) is violated. In this case, by substituting $\phi = \phi_+$ into (A29), we can show that $d(rz/w)/d\phi < 0$ also holds at the upper bound, $\phi = \phi_+$, if and only if

$$\left(1 - \frac{\kappa}{\theta/(1-\theta)}\right) > \zeta\vartheta\left[\frac{\vartheta}{\rho}\frac{2\kappa}{\theta/(1-\theta)} + \left(1 - \frac{2\kappa}{\theta/(1-\theta)}\right)\right]. \tag{A31}$$

We know $d^2(rz_t/w_t)/d\phi^2 > 0$ when $\kappa < \theta/(1-\theta)$. As a result, $d(rz/w)/d\phi < 0$ holds for any $\phi \in (0, \phi_+]$. Given $d\phi/di < 0$, in this case, there is a positive effect of i on rz/w.

Finally, if (A31) does not hold, there is a threshold value of ϕ below (above) which $d(rz/w)/d\phi < (>)0$; i.e., there is a U-shaped relationship between i and rz/w. Therefore, the effect of i on rz/w can be negative, positive, or U-shaped. ■

Proof of Proposition 3. As in the proof of Proposition 1, we focus on the non-trivial case where $\tilde{\lambda}^{-1/\kappa}\phi > 0$, implying $\tilde{\lambda} \in (\lambda_-, \lambda_+)$ unless $\beta\rho = 0$. Recall that $1 < \lambda_- < 1/(1-\theta) < \lambda_+$. By (29) and (44), we have

$$\frac{rz}{w} = (\rho+g)\tilde{\lambda}^{(1-\theta)/\theta}\left[\frac{\tilde{\lambda} - 1/(1+\kappa)}{\tilde{\lambda}-1} + \frac{\zeta\alpha}{\beta}\phi\right]\frac{\beta(e^{\kappa})^{(1-\theta)/\theta}}{\theta(1-\theta)^{(1-\theta)/\theta}}. \tag{A32}$$

By differentiating this with respect to $\tilde{\lambda}$,

$$\Xi\frac{d(rz/w)}{d\tilde{\lambda}} = \frac{\tilde{\lambda}(\rho+g)'}{(\rho+g)} + \frac{1-\theta}{\theta} + \tilde{\lambda}\frac{\left(\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}-1}\right)' + \frac{\zeta\alpha}{\beta}\phi'}{\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}-1} + \frac{\zeta\alpha}{\beta}\phi} \equiv \Psi(\tilde{\lambda}), \tag{A33}$$

where $\Xi > 0$ is a composite variable that is strictly positive.[30] Here, we can derive from (23) and (A23) some components of $\Psi(\tilde{\lambda})$ as

$$\frac{\tilde{\lambda}(\rho+g)'}{(\rho+g)} = \frac{\tilde{\lambda}}{\rho+g}\frac{1-\theta}{\theta}\left[\frac{\tilde{\lambda}^{-1/\kappa}\phi}{\tilde{\lambda}} + \left(\ln\tilde{\lambda} + \kappa\right)\left(\tilde{\lambda}^{-1/\kappa}\phi\right)'\right], \tag{A33a}$$

[30] Here $\Xi \equiv \frac{\tilde{\lambda}w}{rz}$.

$$\left(\tilde{\lambda}^{-1/\kappa}\phi\right)' = \frac{(1-\theta)^{1/\theta}}{\beta e^{\kappa(1-\theta)/\theta}}\left(\frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\theta}}\right)', \tag{A33b}$$

$$\left(\frac{\tilde{\lambda}-1}{\tilde{\lambda}^{1/\theta}}\right)' = \frac{1}{\theta}\frac{1-(1-\theta)\tilde{\lambda}}{\tilde{\lambda}^{1+1/\theta}}, \tag{A33c}$$

$$\left(\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}-1}\right)' = -\frac{\kappa}{1+\kappa}\frac{1}{(\tilde{\lambda}-1)^2}, \tag{A33d}$$

and

$$\phi' = \left(\tilde{\lambda}^{1/\kappa}\right)'\left(\tilde{\lambda}^{-1/\kappa}\phi\right) + \left(\tilde{\lambda}^{1/\kappa}\right)\left(\tilde{\lambda}^{-1/\kappa}\phi\right)'. \tag{A33e}$$

By evaluating these at $\tilde{\lambda} \in \{\lambda_-, \lambda_+\}$ and substituting them into (A33), we can obtain[31]

$$\Psi(\tilde{\lambda}) = \underbrace{\tfrac{1}{\theta}\tfrac{1-(1-\theta)\tilde{\lambda}}{\tilde{\lambda}-1}\left[\tfrac{1-\theta}{\theta}\left(\ln\tilde{\lambda}+\kappa\right) + (\alpha\zeta)\tfrac{(1+\kappa)\rho}{\beta}\tfrac{\tilde{\lambda}^{1/\kappa}(\tilde{\lambda}-1)}{(1+\kappa)\tilde{\lambda}-1}\right]}_{\equiv\Psi_1(\tilde{\lambda})} + \underbrace{\tfrac{1-\theta}{\theta} - \tfrac{\kappa}{(1+\kappa)\tilde{\lambda}-1}\tfrac{\tilde{\lambda}}{\tilde{\lambda}-1}}_{\equiv\Psi_2(\tilde{\lambda})} \tag{A34}$$

which reflects $\tilde{\lambda}^{-1/\kappa}\phi = 0$ for $\tilde{\lambda} \in \{\lambda_-, \lambda_+\}$ with (30). For $\tilde{\lambda} = \lambda_-$, $\Psi_1(\tilde{\lambda}) > 0$ always holds due to $\lambda_- < 1/(1-\theta)$, but $\Psi_2(\tilde{\lambda}) \lesseqgtr 0$. For $\tilde{\lambda} = \lambda_+$, both $\Psi_1(\tilde{\lambda}) < 0$ and $\Psi_2(\tilde{\lambda}) > 0$ hold due to $\lambda_+ > 1/(1-\theta)$.

Given that $\tilde{\lambda}^{-1/\kappa}\phi$ is independent of ζ, λ_- and λ_+ are also independent of ζ. Thus, changes in ζ affect (A34) only through the second term of Ψ_1. Keeping $\tilde{\lambda} = \{\lambda_-, \lambda_+\}$ unchanged, it is possible to make $\Psi(\tilde{\lambda})$ larger (smaller) as one needs by increasing ζ, since the coefficient of Ψ_1, $\frac{1-(1-\theta)\tilde{\lambda}}{\tilde{\lambda}-1}$, is positive (negative) for $\tilde{\lambda} = \lambda_-$ ($\tilde{\lambda} = \lambda_+$). Therefore, for a sufficiently large ζ,[32] $\Psi(\lambda_-) > 0$ and $\Psi(\lambda_+) < 0$ hold; rz/w is first increasing and eventually decreasing in $\tilde{\lambda}$ on the feasible domain of $\tilde{\lambda}$. As we already mentioned, by (29), $\tilde{\lambda}$ has another upper bound, $\overline{\lambda}$, due to $i \geq 0$. Since, by (29), $\overline{\lambda}$ is decreasing in α and satisfies $\lim_{\alpha\to 0}\overline{\lambda} = \infty$, we can also prove that rz/w first increases and eventually decreases with i on the feasible domain of i, by taking an appropriately small value of α so that $\overline{\lambda} > 1/(1-\theta)$. ■

[31] We provide the detailed derivations of (A34) in an unpublished appendix; see Appendix D.

[32] It is worth noting that there exists a sufficient condition for the lower bound of ζ to be less than 1.

Appendix B: Data and regression results

Table B1: Summary statistics					
	Obs	Mean	Std.Dev.	Min	Max
Panel A	World Income Inequality Database				
Economic Growth	612	5.670	9.752	-26.946	32.453
Income Inequality	448	8.143	3.927	4.280	35.240
GINI Coefficient	448	29.693	4.967	22.500	55.433
Inflation Rate	612	2.315	1.839	-4.480	12.678
Unemployment	612	7.309	3.803	1.805	27.466
Openness	612	1.007	0.777	0.167	4.416
Investment Profile	612	8.432	1.568	4.236	10.000
Corruption	612	7.249	1.718	3.333	10.000
Interest Rate	501	6.388	3.576	0.200	20.679
Panel B	World Bank				
Economic Growth	740	6.366	10.582	-32.872	37.465
Income Inequality	452	8.300	4.082	4.280	35.240
GINI Coefficient	452	29.943	5.293	22.500	55.433
Inflation Rate	740	2.420	2.068	-4.480	12.678
Unemployment	740	6.937	4.060	0.700	27.466
Openness	740	1.014	0.718	0.167	4.416
Investment Profile	740	8.410	1.590	3.333	10.000
Corruption	740	6.801	1.921	3.333	10.000
Interest Rate	601	6.637	3.641	0.200	21.167

Notes: Income Inequality and GINI Coefficient are from the WIID. The former corresponds to the income difference between the top 10% and the bottom 10% of the population. Investment Profile and Corruption are two measures of investment risks, which are obtained from the International Country Risk Guide.

Table B2: Effects of inflation on economic growth and income inequality (IV regression)						
	WIID			WB		
	growth	income inequality		growth	income inequality	
	(1)	(2)	(3)	(4)	(5)	(6)
π_{it}	8.063**	0.490***	0.281**	10.056**	0.502***	0.286**
	(3.213)	(0.130)	(0.131)	(4.110)	(0.133)	(0.134)
π_{it}^2	-0.787**	-0.016***	-0.011**	-0.994**	-0.016***	-0.011**
	(0.324)	(0.004)	(0.005)	(0.417)	(0.005)	(0.005)
Kleibergen-Paap rk LM statistic	19.475	36.428	36.428	12.716	36.438	36.438
weak instrument (F-statistic)	12.897	47.083	47.083	7.400	47.375	47.375
country fixed effect	YES	YES	YES	YES	YES	YES
control variables	YES	YES	YES	YES	YES	YES
observations	501	326	326	601	330	330

Note: *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Standard errors are in parentheses. Columns 1 and 4 correspond to the GDP growth rate. Columns 2-3 and 5-6 use different measures of income inequality. Specifically, columns 2 and 5 correspond to the income difference between the top 10% and the bottom 10% of the population. Columns 3 and 6 correspond to the Gini coefficient. In the first (last) three columns, high-income countries follow the classification by the WIID (WB). The Kleibergen and Paap (2006) rk Lagrange Multiplier (LM) statistic reveals that our strategy passes the under-identification test. Also, the F-statistic of the weak-instrument test is above the critical value in Stock and Yogo (2005), which suggests that we can reject the weak-instrument hypothesis.

Appendix C: The generalized model

This appendix presents the key equilibrium conditions for the model with elastic labor supply under the utility function in (54). Equation (3) is the same, and labor supply is

$$w_t\left(1-L_t\left(h\right)\right)=\eta c_t\left(h\right)\left(1+\varphi i_t\right). \tag{C1}$$

The conditional demand functions for labor and intermediate goods are respectively

$$w_t=\theta y_t/L_t, \tag{C2}$$

$$K_t=L_t\left[\left(1-\theta\right)Q_t/\left(\tilde{\lambda}e^{\kappa}\right)\right]^{1/\theta}, \tag{C3}$$

where $L_t\equiv\int_0^1 L_t\left(h\right)dh$. From (C3), the aggregate production function can be derived as

$$y_t=L_t\left(\frac{1-\theta}{\tilde{\lambda}e^{\kappa}}\right)^{(1-\theta)/\theta}Q_t^{(1-\theta)/\theta}. \tag{C4}$$

Substituting (C4) into (25), (30) can be revised as follows:

$$\tilde{\lambda}^{-1/\kappa}\phi=\frac{\tilde{\lambda}^{-(1/\kappa+1/\theta)}}{1+\zeta i}\frac{\kappa}{1+\kappa}\frac{\left(1-\theta\right)^{1/\theta}}{\alpha e^{\kappa(1-\theta)/\theta}}L-\rho, \tag{C5}$$

where the condition for $\tilde{\lambda}$ in (29) remains unchanged. We substitute (C2) into (C1) to derive $\theta y_t\left(1-L\right)/L=\eta\left(1+\varphi i\right)c_t$. Combining this condition with the resource constraint $\int_0^1 c_t(h)dh+\int_0^1 x_t(j)dj+\int_0^1 R_t(j)dj+\tilde{\lambda}^{-1/\kappa}\phi\beta_t=y_t$ and (C4), we obtain

$$\theta\left(\frac{1-\theta}{\tilde{\lambda}e^{\kappa}}\right)^{(1-\theta)/\theta}\left(1-L\right)=\eta\left(1+\varphi i\right)\tilde{c}, \tag{C6}$$

where

$$\tilde{c}=\left[1-\frac{1-\theta}{\left(1+\kappa\right)\tilde{\lambda}}\right]\left(\frac{1-\theta}{\tilde{\lambda}e^{\kappa}}\right)^{(1-\theta)/\theta}L-\left(\alpha+\beta\tilde{\lambda}^{-1/\kappa}\right)\phi. \tag{C7}$$

Based on the CIA constraint $b_t(h)+\varphi c_t(h)=m_t(h)$, (31) can be revised as

$$\dot{z}_t(h)=r_tz_t(h)-i_t[m_t(h)-b_t(h)]+w_tL_t(h)+\tau_t-c_t(h)=r_tz_t(h)+w_tL_t(h)+\tau_t-\left(1+\varphi i_t\right)c_t(h), \tag{C8}$$

where $z_t(h)=a_t(h)+m_t(h)$. Aggregating (C8) for all h, we have

$$z_t=r_tz_t+w_tL_t+\tau_t-\left(1+\varphi i_t\right)c_t. \tag{C9}$$

In this generalized model, we know that $s_{c,t}(h)=s_{c,0}(h)$ still holds for all t. Given this condition, we combine (C8) and (C9) and use (C1) to derive

$$\dot{s}_{z,t}(h)=\frac{\left(1+\eta\right)\left(1+\varphi i\right)c_t-w_t-\tau_t}{z_t}s_{z,t}(h)-\frac{\left(1+\eta\right)\left(1+\varphi i\right)s_{c,0}(h)c_t-w_t-\tau_t}{z_t}. \tag{C10}$$

Using (4), (C1) and (C9) , it is easy to obtain $\left[(1+\eta)(1+\varphi i)c_t - w_t - \tau_t\right]/z_t = \rho > 0$. As a result, we know $\dot{s}_{z,t}(h) = 0$ for all t with long-run stability. Imposing $\dot{s}_{z,t}(h) = 0$ on (C10) yields the steady-state value of $s_{c,t}(h)$ given by

$$s_{c,0}(h) = 1 - \frac{\rho\left[1 - s_{z,0}(h)\right]}{(1+\eta)(1+\varphi i)c/z}, \tag{C11}$$

where we can make use of (40), (42) and $m_t(h) = b_t(h) + \varphi c_t(h)$ to derive

$$\frac{c}{z} = \frac{\tilde{c}}{\tilde{\lambda}^{1/\kappa}(1+\zeta i)\alpha + \beta + \zeta\alpha\phi + \varphi\tilde{c}}. \tag{C12}$$

From (C8), income earned by household h is given by

$$I_t(h) = r_t a_t(h) - \pi_t m_t(h) + i_t b_t(h) + w_t L_t(h) = r_t z_t(h) + w_t L_t(h) - \varphi i_t c_t(h). \tag{C13}$$

Aggregating (C13) yields total income earned by all households given by

$$I_t = r_t z_t + w_t L_t - \varphi i_t c_t. \tag{C14}$$

Combining (C13) and (C14) and using (C1), (38) can be revised as follows:

$$s_{I,t}(h) = \frac{s_{z,0}(h)\ r_t z_t + w_t - \left[(1+\eta)\varphi i_t + \eta\right] s_{c,0}(h) c_t}{r_t z_t + w_t - \left[(1+\eta)\varphi i_t + \eta\right] c_t}. \tag{C15}$$

The distribution of income share $s_{I,t}(h)$ has a mean of one and the following standard deviation:

$$\sigma_{I,t} = \frac{\frac{r_t z_t}{w_t} - \frac{\rho\left[(1+\eta)\varphi i_t + \eta\right]}{(1+\eta)(1+\varphi i_t)}\frac{z_t}{w_t}}{1 + \frac{r_t z_t}{w_t} - \left[(1+\eta)\varphi i_t + \eta\right]\frac{c_t}{w_t}}\sigma_z, \tag{C16}$$

where

$$r_t\frac{z_t}{w_t} = (\rho + g)\left[\tilde{\lambda}^{1/\kappa}(1+\zeta i)\alpha + \beta + \zeta\alpha\phi + \varphi\tilde{c}\right]\frac{(\tilde{\lambda}e^{\kappa})^{(1-\theta)/\theta}}{\theta(1-\theta)^{(1-\theta)/\theta}},$$

$$\frac{c_t}{w_t} = \frac{\tilde{c}(\tilde{\lambda}e^{\kappa})^{(1-\theta)/\theta}}{\theta(1-\theta)^{(1-\theta)/\theta}}.$$

Therefore, we can solve the six endogenous variables $\left\{\tilde{\lambda},\ \phi,\ L,\ \tilde{c},\ s_{c,0}(h),\ \sigma_I\right\}$ using (29), (C5), (C6), (C7), (C11) and (C16).

We impose balanced growth on (54) to derive the steady-state utility function as

$$u(h) = \frac{1}{\rho}\left\{\ln c_0(h) + \frac{g}{\rho} + \eta\ln\left[1 - L(h)\right] - \eta\varepsilon\ln(1-L)\right\}, \tag{C17}$$

where $c_0(h)$ is the balanced-growth level of consumption at time 0. Substituting $c_0(h) = c_0 s_{c,0}(h)$ into (C17) and then normalizing the initial Q_0 to unity yield

$$u(h) = \frac{1}{\rho}\left\{\ln\tilde{c} + \ln s_{c,0}(h) + \frac{g}{\rho} + \eta\ln\left[1 - L(h)\right] - \eta\varepsilon\ln(1-L)\right\}, \tag{C18}$$

where we make use of (C1), (C2) and (C4) to derive

$$1 - L(h) = \frac{\eta(1+\varphi i)}{\theta(1-\theta)^{(1-\theta)/\theta}}\left[\tilde{c}s_{c,0}(h)(\tilde{\lambda}e^{\kappa})^{(1-\theta)/\theta}\right].$$

Appendix D (not for publication)

Derivations of (A24). Differentiating (23) with respect to $\tilde{\lambda}$ yields

$$\frac{dg}{d\tilde{\lambda}} = \frac{1-\theta}{\theta}\left[(\ln\tilde{\lambda}+\kappa)'\tilde{\lambda}^{-1/\kappa}\phi_t + (\ln\tilde{\lambda}+\kappa)\left(\tilde{\lambda}^{-1/\kappa}\phi_t\right)'\right], \tag{D1}$$

which naturally yields (A24). ■

Derivations of (A34). Applying $\tilde{\lambda}^{-1/\kappa}\phi = 0$, $\phi = 0$, or $g = 0$ to (A33a)–(A33e) yields

$$\frac{\tilde{\lambda}(\rho+g)'}{(\rho+g)} = \frac{\tilde{\lambda}}{\rho+g}\frac{1-\theta}{\theta}\left(\ln\tilde{\lambda}+\kappa\right)\left(\tilde{\lambda}^{-1/\kappa}\phi\right)', \tag{D2}$$

$$\left(\tilde{\lambda}^{-1/\kappa}\phi\right)' = \frac{\rho}{\theta\tilde{\lambda}}\frac{1-(1-\theta)\tilde{\lambda}}{\tilde{\lambda}-1}, \tag{D3}$$

$$\left(\frac{\tilde{\lambda}-1/(1+\kappa)}{\tilde{\lambda}-1}\right)' = -\frac{\kappa}{1+\kappa}\frac{1}{(\tilde{\lambda}-1)^2}, \tag{D4}$$

and

$$\phi' = \left(\tilde{\lambda}^{1/\kappa}\right)\left(\tilde{\lambda}^{-1/\kappa}\phi\right)', \tag{D5}$$

in which we have used for $\left(\tilde{\lambda}^{-1/\kappa}\phi\right)'$

$$\frac{(1-\theta)^{1/\theta}}{\beta e^{\kappa(1-\theta)/\theta}} = \frac{\tilde{\lambda}^{-1/\kappa}\phi+\rho}{(\tilde{\lambda}-1)/\tilde{\lambda}^{1/\theta}} = \frac{\rho\tilde{\lambda}^{1/\theta}}{\tilde{\lambda}-1} \tag{D6}$$

from (A23) and $\tilde{\lambda}^{-1/\kappa}\phi = 0$. Substituting these into (A33) yields

$$\Psi(\tilde{\lambda}) = \frac{\tilde{\lambda}}{\rho}\left[\frac{1-\theta}{\theta}\left(\ln\tilde{\lambda}+\kappa\right) + (\alpha\zeta)\frac{(1+\kappa)\rho}{\beta}\frac{\tilde{\lambda}^{1/\kappa}(\tilde{\lambda}-1)}{(1+\kappa)\tilde{\lambda}-1}\right]\left(\tilde{\lambda}^{-1/\kappa}\phi\right)' + \frac{1-\theta}{\theta} - \frac{\kappa}{(1+\kappa)\tilde{\lambda}-1}\frac{\tilde{\lambda}}{\tilde{\lambda}-1}. \tag{D7}$$

Combining (D3) and (D7) yields (A34). ■