On the Extension and Decomposition of a Preorder under Translation Invariance

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Abstract
We prove the existence, for a translation-invariant preorder on a divisible commutative group, of a complete preorder extending the preorder in question and satisfying translation invariance (theorem 1). We also prove that the extension may inherit a property of continuity (theorem 2). This property of continuity may lead to scalar invariance. By seeking to clarify the relationship between continuity and scalar invariance under translation invariance, we are led to formulate a theorem that asserts the existence of a continuous linear weak representation under a certain condition (theorem 3). The application of these results in a space of infinite real sequences shows that this condition is weaker than the axiom super weak Pareto, and that the latter is itself weaker than the axiom monotonicity for non-constant preorders. Thus, theorem 3 is a strengthening of theorem 4 of Mabrouk 2011. It also makes it possible to show the existence of a sequence of continuous linear preorders whose lexicographic combination constitutes the finest combination coarser than the preorder in question (theorem 4). This decomposition makes it possible to handle continuous functions instead of preorders when one looks for optima, which may be more practical. Finally we apply this decomposition to the preorder catching-up. Several examples are provided.

1- Introduction
The present paper establishes the existence, for any preorder on a divisible commutative group satisfying translation invariance, of a complete preorder extending the given preorder and satisfying translation invariance (section 3, theorem 1). In Demuynck-Lauwers 2009 the existence of an extension under the conditions translation invariance and scalar invariance is proven. However, the result proved here is stronger in the sense that it is freed from the scalar invariance assumption. The proof of theorem 1 follows the same diagram as the proof of Szpilrajn 1930 theorem which may be stated as follows. For any reflexive and transitive binary relation (i.e. a preorder) on a given set, there

1I am grateful to two anonymous referees who, when reviewing other papers, suggested me to study the issues of extending a preorder under translation-invariance and the relationship between translation-invariance and scalar-invariance.
exists a complete preorder which is an extension of the given preorder. Starting
from a preorder satisfying translation invariance, one adds comparisons on some
pairs of alternatives in such a way that translation invariance remains satisfied.
Then, an argument based on Zorn’s lemma makes it possible to extend the
procedure to the whole space. We give two examples, the first of which shows
the existence of a complete translation-invariant strict preorder on \( \mathbb{R} \) which
transgresses scalar invariance and the second shows the existence of a complete
translation-invariant preorder satisfying the social choice axioms strong Pareto
and fixed-step anonymity on a set \( X^{\mathbb{N}_0} \), where \( X \) is a divisible commutative
group.

Then, we prove a second extension theorem which asserts that the former
extension result (theorem 1) holds under an additional requirement of continu-
ity (section 4, theorem 2). The proof is an adaptation of the proof of Jaffray
1975 to the translation invariance case. It relies on the construction of a re-
lation that is used to "clean" the extended preorder given by theorem 1 from
undesirable rankings that transgress the continuity requirement. As applica-
tion, we prove the existence of a complete, translation-invariant, strong Pareto,
fixed-step anonymous and upper-semi-continuous preorder on \( \mathbb{R}^{\mathbb{N}_0} \) which is an
extension of a given preorder which satisfies the same axioms except complete-
ness. We also prove that the property of continuity under a norm topology leads
to scalar invariance.

Seeking to better understand the relationship between continuity and scalar
invariance under translation invariance, we offer an example of a complete pre-
order on \( \mathbb{R}^{\mathbb{N}_0} \), translation-invariant, continuous with respect to the \( l_1 \) topology
but not scalar invariant. We are then led to propose a continuity requirement,
linear continuity, equivalent to scalar invariance, and a theorem that asserts the
existence of a continuous linear weak representation under a certain condition
(section 5, theorem 3). The proof is similar to that of theorem 4 in Mabrouk
2011, except that the open convex cone used is different and the condition super
weak Pareto is replaced by a sufficient condition which can be formulated in
more general spaces.

Back in the context of infinite real sequences, it turns out that the sufficient
condition in weaker than monotonicity and super weak Pareto (section 6). Thus,
theorem 3 is a strengthening of theorem 4 of Mabrouk 2011. Moreover, using
theorem 3, we prove that, for a non-trivial preorder, monotonicity is stronger
than super weak Pareto.

A successive application of theorem 3 makes it possible to show the existence
of a sequence of continuous and linear preorders whose lexicographic combina-
tion constitutes the finest linear continuous combination coarser than a given
preorder (section 7, theorem 4). Although theorem 4 invokes several times theo-
rem 3, which is non-constructive, it may be used along with other specific informa-
tion to gain some insight on the preorder and to handle continuous functions
instead of preorders when one looks for optima. As an example, section 8 studies

\(^2\)See Alcantud-Diaz 2014 for an overview on the applications and extensions of Szpilrajn
theorem.
the decomposition of the catching-up preorder under two different norms.

2- Preliminaries

$\mathbb{N}_0$ is the set of positive integers. $n, i$ symbolize positive integers. $Q$ is the set of rational numbers. $(X, +)$ is a divisible commutative group. $B$ being a binary relation on $X$ and $x, y$ two elements of $X$, $xBy$ is denoted $x \succsim_B y$. The symbols $\leq, \geq, <, >$ are used for the natural order on $\mathbb{R}$. A reflexive and transitive binary relation on $X$ is a preorder on $X$. If, on top of that, for all $x, y$ either $x \succsim_B y$ or $x \precsim_B y$, it is a complete preorder. A binary relation $B_1$ is said to be a subrelation to a binary relation $B_2$, or $B_2$ an extension of $B_1$, if for all $x, y$ in $X$, 

\[ x \succsim_{B_1} y \implies x \succsim_{B_2} y \]

and

\[ x \precsim_{B_1} y \implies x \precsim_{B_2} y \]

**Axiom Translation Invariance (TI)** A preorder $R$ satisfies translation invariance if:

\[ \forall (x, y) \in X \times X, \forall u \in X, [x \succsim_R y \implies x + u \succsim_R y + u] \]

**Axiom Division Invariance (DI)** A preorder $R$ satisfies division invariance if:

\[ \forall x \in X, \forall n \in \mathbb{N}_0, \left[ x \succsim_R y \implies \frac{1}{n}x \succsim_R \frac{1}{n}y \right] \]

**Lemma 1** If a preorder $R$ on $X$ satisfies TI, then there exists a preorder $\hat{R}$ on $X$ of which $R$ is a subrelation and such that $\hat{R}$ satisfies TI and DI.

**Proof:** First, notice that under $R$, it is possible to sum inequalities. Indeed, by TI, if $a, b, u, v$ are such that $a \succsim_R b$ and $u \succsim_R v$, then $a + u \succsim_R b + u$ and $b + u \succsim_R b + v$. By transitivity, $a + u \succsim_R b + v$. For each $n$, consider the binary relation $R_n$ defined by

\[ x \succsim_{R_n} y \text{ iff } nx \succsim_R ny \]

If $x, y$ are such that $x \succsim_R y$, we can sum $n$ times this inequality. Thus, $x \succsim_{R_n} y$. Likewise, it is easily seen that $x \precsim_R y$ implies $x \precsim_{R_n} y$. As a result, $\hat{R}$ is a subrelation to $R_n$. Moreover, $R_n$ is reflexive and transitive. It is easily checked that $R_n$ satisfies TI.

Consider the binary relation

\[ \hat{R} = \cup_{n \in \mathbb{N}_0} R_n \]

defined on $X$ by $x \succsim_{\hat{R}} y$ iff there is $n$ such that $x \succsim_{R_n} y$.

$\hat{R}$ is a subrelation to $\hat{R}$. Moreover, $\hat{R}$ is reflexive and transitive. It is a preorder. Since for each $n$, $R_n$ satisfies TI, we deduce that $\hat{R}$ satisfies TI. The lemma is proved if we show that $\hat{R}$ satisfies DI. Let $x, y$ be such that $x \succsim_{\hat{R}} y$. There exists a positive integer $m$ such that $x \succsim_{R_m} y$. Thus $mx \succsim_R my$. We
can write that as $mn(\frac{1}{n} x) \succeq_R mn(\frac{1}{n} y)$. Thus $\frac{1}{n} x \succeq_R \frac{1}{n} y$. What implies $\frac{1}{n} x \succeq_R \frac{1}{n} y$. $R$ satisfies DI. \(\square\)

**Remark 1** (i) It is easily seen that $\hat{R}$ is the minimal preorder satisfying TI and DI, of which $R$ is a subrelation. (ii) If $R$ is complete, since $R$ is a subrelation to $\hat{R}$, we have necessarily $R = \hat{R}$. This shows that if the preorder is complete, TI implies DI. \(\Diamond\)

### 3- The Translation-Invariant Extension Theorem

**Theorem 1** Let $R$ be a preorder on $X$ satisfying TI. Then there exists a complete preorder on $X$ satisfying TI, of which $R$ is a subrelation.

**Proof:** If $R$ is a complete preorder, there is nothing to prove. Suppose that $R$ is not complete. Consider the preorder $\hat{R}$ built in the proof of lemma 1, and the set $\mathcal{R}$ of all preorders on $X$ satisfying TI and DI, and of which $R$ is a subrelation. $\mathcal{R}$ is not empty since $\hat{R} \in \mathcal{R}$. Let $(\mathcal{R}_\alpha)$ be a chain in $\mathcal{R}$, i.e. for any $\alpha, \alpha'$, $\mathcal{R}_\alpha$ is a subrelation to $\mathcal{R}_{\alpha'}$ or $\mathcal{R}_{\alpha'}$ is a subrelation to $\mathcal{R}_\alpha$. Notice that (i) the relation $\cup_\alpha (\mathcal{R}_\alpha)$ defined on $X$ by: $x \cup_\alpha (\mathcal{R}_\alpha) y$ iff there is $\alpha$ such that $xR_\alpha y$, is a preorder, (ii) it satisfies TI and DI, (iii) $R$ is a subrelation to $\cup_\alpha (\mathcal{R}_\alpha)$, (iv) for all $\alpha$, $\mathcal{R}_\alpha$ is a subrelation to $\cup_\alpha (\mathcal{R}_\alpha)$. Hence, in the set $\mathcal{R}$, every chain admits an upper bound. According to Zorn’s lemma, $\mathcal{R}$ admits at least a maximal element. Denote $M$ such a maximal element in $\mathcal{R}$. Suppose we can prove the following claim:

**Claim 1** For any non-complete $R'$ in $\mathcal{R}$ and any pair of $R'$-incomparable alternatives $(x_0, y_0)$, there exists a preorder $R'_1$ in $\mathcal{R}$ to which $R'$ is a subrelation and such that $x_0$ and $y_0$ are $R'_1$-comparable.

Then, if $M$ were not complete, there would exist a preorder in $\mathcal{R}$ to which $M$ is a strict subrelation. This would contradict that $M$ is maximal in $\mathcal{R}$. Therefore, if the claim holds, $M$ would be necessarily complete. $M$ would be the preorder we are looking for.

What remains of the proof is devoted to establish claim 1. This is done through the following 6 steps.

If there is a non-complete preorder in $\mathcal{R}$, the theorem is proved since $\mathcal{R}$ is not empty. Let $R'$ be a non-complete preorder in $\mathcal{R}$ and $x_0, y_0$ be two $R'$-incomparable elements of $X$.

Consider the binary relation $B$ on $X$: $x \gtrsim_B y$ iff either $x \gtrsim_R y$ or there is a positive rational $q$ such that $x - y = q(x_0 - y_0)$.

We prove successively that the two clauses of the definition of $B$ are exclusive (step 1), that the indifference relations are equal (step 2), that $R'$ is a subrelation to $B$ (step 3), that $B$ is weakly acyclic (this prepares for transitivity) (step 4), that $R'$ is a subrelation to the transitive closure of $B$ (step 5), that the transitive closure of $B$ satisfies TI and DI (step 6). The transitive closure of $B$ is then the required preorder.

**Step 1**: the two clauses are exclusive. If there is a positive rational $q$ such that $x - y = q(x_0 - y_0)$, then $x, y$ are $R'$-incomparable. Suppose not. For instance suppose $x \gtrsim_{R'} y$. By TI, $x - y \gtrsim_{R'} 0$. By DI, for all $n$, $\frac{1}{n} (x - y) \gtrsim_{R'} 0$. 

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Recall that it is possible to sum to inequalities (see the proof of lemma 1). We sum $m$ times the inequality $\frac{1}{2} (x - y) \leq R'$, $0$, $m$ being a positive integer. We obtain $\frac{m}{2} (x - y) \leq R'$. Take $\frac{m}{2} = q$. It gives $x_0 - y_0 \leq R'$, $0$, what contradicts $x_0, y_0$ being incomparable. The case $y \not\leq R' x$ is similar.

**Step 2:** equivalence of indifferences. Clearly, $x \sim_R y \Rightarrow x \sim_R y$. We show now that $x \sim_R y$ entails $x \sim_R y$. According to the definition of $B$, it is enough to prove that $x$ and $y$ are necessarily $R'$-comparable. Suppose not. Then $x \not\geq_R y$ implies that there is some positive rational $q$ such that $x - y = q(x_0 - y_0)$. We have also $y \not\leq_R x$. Thus, for some positive rational $q'$, $y - x = q'(x_0 - y_0)$. We see that this gives $q'(x_0 - y_0) = -q(x_0 - y_0)$, what implies $x_0 - y_0 = 0$ because $q,q'$ are both positive. But that contradicts $x_0, y_0$ being $R'$-incomparable.

**Step 3:** $R'$ is a subrelation to $B$. This is a direct consequence of $x \not\leq_R y \Rightarrow x \not\leq_B y$ (definition of $B$) and $x \sim_R y \Leftrightarrow x \sim_R y$ (step 2).

**Step 4:** $B$ is weakly-acyclic. We show that for all $x,y,z$ in $X : x \not\leq_B y$ and $y \not\leq_B z$ or non($z \not\leq_B x$).

One of the four following cases is implied by $x \not\leq_B y$ and $y \not\leq_B z$.

1. $x \not\leq_R y$ and $y \not\leq_R z$.
2. There are $q, q'$ such that $x - y = q(x_0 - y_0)$ and $y - z = q'(x_0 - y_0)$.
3. $x \not\leq_R y$ and there is a $q$ such that $y - z = q(x_0 - y_0)$.
4. There is $q$ such that $x - y = q(x_0 - y_0)$ and $y \not\leq_R z$.

Consider successively the four cases:

1. By transitivity of $R'$ : $x \not\leq_R x$. Thus, $x \not\leq_B z$.
2. $x - y = q(x_0 - y_0)$ and $y - z = q'(x_0 - y_0)$ entails $x - z = (q + q')(x_0 - y_0)$. Thus $x \not\leq_B z$.
3. Suppose we had $z \not\leq_B x$. Then we would have either $z \not\leq_R x$ or $z - x = q''(x_0 - y_0)$. Both possibilities contradict $x \not\leq_R y$ and $y - z = q'(x_0 - y_0)$. Indeed, with $x \not\leq_R y$, $z \not\leq_R x$ gives $z \not\leq_R y$ what contradicts $y - z = q'(x_0 - y_0)$ (step 1); whereas $y - z = q'(x_0 - y_0)$ implies $y - x = (q' + q'')(x_0 - y_0)$, what contradicts $x \not\leq_R y$. As a result, we have non($z \not\leq_B x$).
4. (This case is similar to (3)).

**Remark 2** Let $x,y,z$ be such that $x \not\leq_B y$ and $y \not\leq_B z$. Weak acyclicity entails that if one of the comparisons $x \not\leq_B y$ and $y \not\leq_B z$ is a strict preference, then either the comparison on $(x,z)$ is $x \not\leq_R z$ or $z$ and $x$ are $B$-incomparable.

**Step 5:** $R'$ is a subrelation to the transitive closure of $B$. Consider $\overline{B}$ the transitive closure of $B$ defined by: $x \not\leq_{\overline{B}} y$ if there is a sequence $(z_i)_{i=1}^n$ such that $x \not\leq_B z_1, z_1 \not\leq_B z_2, ..., z_n \not\leq_B y$. It is clear that $x \not\leq_R y$ implies $x \not\leq_{\overline{B}} y$ (step 3). $R'$ is a subrelation to $B$. It is enough to prove that $x \not\leq_{\overline{B}} y$ implies non($y \not\leq_R x$).

Consider the statement $Q_n$: "If there is a sequence $(z_i)_{i=1}^n$ such that $x \not\leq_B z_1 \not\leq_B z_2, ..., z_n \not\leq_B y$, then non($y \not\leq_R x$)." Let’s prove by induction that $Q_n$ is true for all positive integers. Notice that when the sequence $(z_i)$ has $n$ terms, there is $n+1$ successive comparisons.

$n = 1$: We have $x \not\leq_B z_1 \not\leq_B y$. By step 4, we have $x \not\leq_B y$ or non($y \not\leq_B x$). Both possibilities contradict $y \not\leq_R x$. So, we have non($y \not\leq_R x$).

Suppose that $Q_n$ is true and let’s show that $Q_{n+1}$ is true. Consider the sequence of $n+2$ comparisons: $x \not\leq_B z_1 \not\leq_B z_2, ..., z_n \not\leq_B y, z_{n+1} \not\leq_B y$. Each one of these comparisons comes either from the clause $x \not\leq_R y$ or the clause $x - y = q(x_0 - y_0)$ of the definition of $B$. If there is two successive
comparisons coming from the clause \( x \preceq_R y \), say \( z_p \preceq_R z_{p+1} \preceq_R z_{p+2} \) (with \( p = 0, \ldots, n + 2 \) and the convention: \( z_0 = x \) and \( z_{n+2} = y \)), by transitivity of \( R \) we have: \( x \preceq_B \ldots \preceq_B z_p \preceq_B \ldots \preceq_B y \) which constitutes a sequence of \( n + 1 \) comparisons. By \( Q_n \) we have \( \text{non}(y \succ_R x) \). If there is two successive comparisons coming from the clause \( x - y = q(x_0 - y_0) \), say \( z_p \preceq_B z_{p+1} \preceq_B z_{p+2} \), then \( z_p - z_{p+1} = q(x_0 - y_0) \) and \( z_{p+1} - z_{p+2} = q'(x_0 - y_0) \). Thus, \( z_p - z_{p+2} = (q + q') \implies \). We have again reduced the number of comparisons to \( n + 1 \). Thus, we have also \( \text{non}(y \succ_R x) \). It remains to consider the cases where the comparisons are alternate. Two cases must be considered: \( n + 2 \) even and \( n + 2 \) odd.

\( n + 2 \) even: The sequence of comparisons either begin or ends with a comparison from \( R' \). Suppose it begins with a comparison from \( R' \): \( x \preceq_{R'} z_1 \preceq_B z_2 \ldots \preceq_{R'} z_{n+1} \preceq_B y \). Apply \( Q_n \) to \( z_1 \preceq_B z_2 \ldots \preceq_{R'} z_{n+1} \preceq_B y \). It gives \( \text{non}(y \succ_{R'} z_1) \). Since \( x \preceq_{R'} z_1 \), we cannot have \( y \succ_{R'} x \). If the sequence of comparisons ends with a comparison from \( R' \), the proof is similar. So it is omitted.

\( n + 2 \) odd: If the sequence of comparisons begins with a comparison from \( R' \), the proof is also similar. So it is omitted. If the sequence of comparisons begins with a comparison from the clause \( x - y = q(x_0 - y_0) \), we have

\[
x \preceq_B \ldots \preceq_B z_{n+1} \preceq_B y
\]

Denote \((x, z_1)\) by \((\alpha_1, \beta_1)\), \((z_2, z_3)\) by \((\alpha_2, \beta_2)\), \(\ldots\) \((z_{2(p-1)}, z_{2p-1})\) by \((\alpha_p, \beta_p)\) with \( p = 1, \ldots, \frac{n+1}{2} \) and the convention \( z_0 = x \) and \( z_{n+2} = y \). Since comparisons \( x \preceq_B z_1 \preceq_B z_2 \ldots \preceq_B z_{n+1} \preceq_B y \) come from the clause \( x - y = q(x_0 - y_0) \), we have \( \alpha_p - \beta_p = q_p(x_0 - y_0) \) for \( p = 1, \ldots, \frac{n+1}{2} \). Moreover, according to \((1)\), \( \beta_p \preceq_{R'} \alpha_{p+1} \) for \( p = 1, \ldots, \frac{n+1}{2} \). Thus

\[
\begin{align*}
\alpha_1 - q_1(x_0 - y_0) & \preceq_{R'} \alpha_2 \\
\alpha_2 - q_2(x_0 - y_0) & \preceq_{R'} \alpha_3 \\
& \ldots \\
\alpha_{(n+1)/2} - q_{(n+1)/2}(x_0 - y_0) & \preceq_{R'} \alpha_{(n+3)/2}
\end{align*}
\]

We can sum these inequalities (this is established in the proof of lemma 1). We obtain

\[
\alpha_1 + \sum_{p = 2}^{(n+1)/2} \alpha_p - \sum_{p = 2}^{(n+1)/2} q_p(x_0 - y_0) \preceq_{R'} \alpha_{(n+3)/2}
\]

By \( TI \) we obtain

\[
\alpha_1 = \sum_{p = 1}^{(n+1)/2} q_p(x_0 - y_0) \preceq_{R'} \alpha_{(n+3)/2}
\]

But \( \alpha_1 = x_1 \) and \( \alpha_{(n+3)/2} = y \). Denote \( q = \sum_{p = 1}^{(n+1)/2} q_p \). Thus

\[
x - q(x_0 - y_0) \preceq_{R'} y
\]

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By **TI**, \( x - y \gtrsim_{R'} q(x_0 - y_0) \). If we had \( y \gtrsim_{R'} x \), it would give \( 0 \gtrsim_{R'} x - y \). By transitivity of \( R' \) and by **TI**, \( x_0 \) and \( y_0 \) would be \( R' \)-comparable, which is not the case. As a result, we have non(\( y \gtrsim_{R'} x \)). Step 5 is proved.

**Remark 3** \( R' \) is a subrelation to \( \overline{B} \), but \( B \) is not.\( \Box \)

**Step 6:** \( \overline{B} \) satisfies **TI**. As \( R' \) is translation-invariant, \( B \) is clearly translation-invariant. It is easily deduced that \( \overline{B} \) is also translation-invariant. Likewise, it is easily seen that \( \overline{B} \) satisfies **DI**. Thus, \( \overline{B} \) is the required preorder.\( \Box \)

**Corollary 1** Let \( B \) be a reflexive binary relation satisfying **TI**. Then there exists a complete preorder satisfying \( \overline{TI} \), of which \( B \) is a subrelation, if \( B \) is a subrelation to its transitive closure.

**Proof:** Necessity: the condition that \( B \) is a subrelation to its transitive closure is necessary and sufficient for the existence of a complete preorder of which \( B \) is a subrelation (Suzumura 1976, Bossert 2008). Sufficiency: denote \( \overline{B} \) the transitive closure of \( B \). It easily seen that \( \overline{B} \) is a preorder satisfying **TI**. Apply theorem 1 to \( \overline{B} \) to deduce that there exists a complete preorder satisfying \( \overline{TI} \), of which \( \overline{B} \) is a subrelation. Since \( B \) is a subrelation to \( \overline{B} \), it is also a subrelation to the complete preorder.\( \Box \)

**Example 1:** A translation-invariant and complete strict preorder on \( \mathbb{R} \) with \( \pi \) smaller than 0 and 0 smaller than 1.

Consider the following binary relation \( \gtrsim \) on \( \mathbb{R} \):

\[
x \gtrsim y \text{ if there is two nonnegative rationals } q, q' \text{ such that } x - y = -q + q' \pi
\]

\( \gtrsim \) is reflexive, transitive and satisfies **TI**. Moreover, \( \gtrsim \) is a strict preorder, which means that \( x \gtrsim y \) and \( y \gtrsim x \) implies \( x = y \). Indeed \( x - y = -q + q' \pi \) and \( y - x = -q_1 + q_1' \pi \) yields \( 0 = (x - y) + (y - x) = -(q + q_1) + (q' + q_1') \pi \). Thus \( (q + q_1) = (q' + q_1') \pi \). We must have \( q' + q_1' = 0 \) otherwise \( \pi \) would be rational. Thus we also have \( q + q_1 = 0 \). Since \( q, q_1, q', q_1' \) are nonnegative, we have \( q = q_1 = q' = q_1' = 0 \) and \( x = y \).

Theorem 1 asserts the existence of a translation-invariant and complete preorder, say \( \gtrsim_{\pi} \), of which \( \gtrsim \) is a subrelation. \( \gtrsim_{\pi} \) is strict like \( \gtrsim \) (i.e. \( x \sim_{\pi} y \Leftrightarrow x = y \)). Observe that \( \gtrsim_{\pi} \) respects the natural order of rationals. But it does not coincide with the natural order of reals. Moreover it does not satisfy invariance with respect to multiplication by a positive scalar (scalar invariance) since if you multiply 0 \( \gtrsim_{\pi} \) 1 by \( \pi \) the inequality is reversed. Finally, \( \gtrsim_{\pi} \) is not continuous. Consider a positive sequence of rational \( (q_n) \) such that \( \lim q_n = \frac{1}{\pi} \). **TI** allows to multiply an inequality by a positive rational. Multiplying \( \pi \) \( \gtrsim_{\pi} \) 0 by \( q_n \) yields \( q_n \pi \gtrsim_{\pi} q_n.0 = 0 \) for all \( n \). But \( \lim q_n \pi = 1 \gtrsim_{\pi} 0 \). A question then arises: can Scalar-Invariance still be transgressed under **TI** and continuity? An answer is provided in section 5 and 6.

**Example 2:** Existence of a translation-invariant, strong-Pareto, fixed-step-anonymous and complete preorder on \( X^{n_0} \), where \( X \) is a divisible commutative group equipped with a complete preorder \( R \) satisfying **TI**.
It is possible to prove the existence of such a preorder using the ultrafilter technique, as in Fleurbaey-Michel 2003, Lauwers 2009. We prove here this existence without using ultrafilters, which are highly nonconstructive objects. Although our theorem 1 also makes use of the axiom of choice, one may consider that our method is nevertheless more constructive in the sense that it indicates the concrete steps of adding comparisons.

Let \( Y = X^{\mathbb{N}_0} \), let \( R' \) be a preorder on \( Y \). We first give the following definitions:

**Fixed-Step Permutation:** (Fleurbaey-Michel 2003) \( \sigma \) is a fixed-step permutation if there exist \( k \in \mathbb{N}_0 \) such that for all \( n \in \mathbb{N}_0 \), \( \sigma([1, \ldots, kn]) = \{1, \ldots, kn\} \).

**Axiom Fixed-Step Anonymity:** Denote \( \sigma(x) \) the sequence obtained by permuting the components of \( x \in Y \) according to the permutation \( \sigma \). \( R' \) is fixed-step-anonymous if for all \( x \in Y \) and fixed-step permutation \( \sigma \), we have \( x \sim_{R'} \sigma(x) \).

**Axiom Strong Pareto:** \( R' \) is strong-Pareto if, for all \( x, y \in Y \) such that \( \forall i \in \mathbb{N}_0 \; x_i \geq_R y_i \) and \( x_j >_R y_j \) for some \( j \), we have \( x \succ_{R'} y \) (\( x_i, y_i \) denote the \( i^{th} \) component of resp. \( x, y \)).

Pareto axioms capture the idea that an increase of the components of a vector must increase the ranking of the vector. Anonymity axioms express a requirement of symmetry in the treatment of individuals or dates.

**The Fixed-Step Catching-up SC.** For all \( x, y \in Y \), \( x \succ_{\text{SC}} y \) iff there exist \( k, m \in \mathbb{N}_0 \) such that, for all \( n \in \mathbb{N}_0 \) with \( n > m \), we have

\[
\sum_{i=1}^{kn} x_i \geq \sum_{i=1}^{kn} y_i
\]

\( \text{SC} \) is a fixed–step-anonymous preorder (Fleurbaey-Michel 2003).

**Proposition 1:** There exists a translation-invariant, strong-Pareto, fixed-step-anonymous and complete preorder on \( Y \).

**Proof:** Apply theorem 1 to \( \text{SC} \). There exists a translation-invariant and complete preorder \( R' \) on \( Y \) of which \( \text{SC} \) is a subrelation. \( \text{SC} \) being a subrelation to \( R' \) entails that \( R' \) satisfies strong Pareto and fixed-step anonymity. \( R' \) is the required preorder.

4- **Continuity**

For a given nontrivial preorder \( R \) on a divisible commutative group \( X \), \( \tau_+(R) \) is the associated upper-order-topology, i.e. the topology generated by the base of open intervals: \( \beta_+(R) = \{\{x \in X : x \prec_R a\} : a \in X\} \).

**Theorem 2:** Let \( R \) be a preorder on \( X \) satisfying \( \text{TI} \). Then there exists a complete preorder \( R' \) on \( X \) satisfying \( \text{TI} \), of which \( R \) is a subrelation, and such that \( \tau_+(R') \subset \tau_+(R) \).

**Proof:** The following proof is an adaptation of the proof of Jaffray 1975 to a translation-invariant preorder. We start from a translation-invariant complete preorder which extends \( R \), whose existence is guaranteed by theorem 1. We
then apply a clause\(^3\) to "clean up" rankings that do not respect the upper-order-topology. It turns out that this clause is also translation-invariant, which makes it possible to build the desired preorder.

**Step 1:** Building the complete preorder. Let \(R_1\) be a complete preorder extending \(R\) and satisfying TI. Let \(x, y \in X\). Consider the following clause :
\[
C(x, y): \text{"There exists } B \in \beta_+(R) \text{ containing } x \text{ such that, for all } B' \in \beta_+(R) \text{ containing } y, \text{we can find } x' \in B' \text{ such that for all } z \in B, \text{ we have } z \not\prec \text{R}_1 x'."
\]
Because \(R_1\) satisfies TI, it is easily seen that if \(C(x, y)\) is true, \(C(x+h, y+h)\) is true for all \(h \in X\). Moreover, if \(C(x, y)\) is true, it is clear that we cannot have \(C(y, x)\) true. Thus, we can define a asymmetric relation \(R_2\) checking TI as follows: \(x \not\prec \text{R}_2 y\) iff \(C(x, y)\) is true.

We prove now that \(R_2\) is negatively transitive, i.e.
\[
\text{not}(x \not\prec \text{R}_2 y) \text{ and not}(y \not\prec \text{R}_2 z) \text{ implies not}(x \not\prec \text{R}_2 z)
\]
We have:
\[
\text{Not}(x \not\prec \text{R}_2 y) \iff \forall B_1 \in \beta_+(R) \text{ containing } x \text{ such that } [\text{for all } x'_1 \in B'_1 \text{ there exists } x''_1 \in B_1 \text{ such that } x''_1 \succ x'_1] .
\]
\[
\text{Not}(y \not\prec \text{R}_2 z) \iff \forall B_2 \in \beta_+(R) \text{ containing } y \text{ such that } [\text{for all } x'_2 \in B'_2 \text{ there exists } x''_2 \in B_2 \text{ such that } x''_2 \succ x'_2].
\]
Let \(B_1\) be in \(\beta_+(R)\) containing \(x\) and \(B'_1\) be the interval which existence is asserted by the clause "not\((x \not\prec \text{R}_2 y)\)". Take \(B'_1\) as the interval \(B_2\) of the clause "not\((y \not\prec \text{R}_2 z)\)". Thus, there exists \(B'_2 \in \beta_+(R)\) containing \(z\) such that \([\text{for all } x'_2 \in B'_2 \text{ there exists } x''_2 \in B'_2 \text{ such that } x''_2 \succ x'_2]\). Now apply the clause "not\((x \not\prec \text{R}_2 y)\)" for \(x'_2\) instead of \(x'_1\) and deduce that there exists \(x''_1 \in B_1\) such that \(x''_1 \succ R_1 x'_2\). By transitivity of \(R_2\), \(x''_1 \succ R_1 x''_2\) and \(x''_2 \succ R_1 x'_2\) gives \(x''_1 \succ R_1 x''_2\).

Summing up: for some \(B_1 \in \beta_+(R)\) containing \(x\), we have found \(B'_2 \in \beta_+(R)\) containing \(z\) such that \([\text{for all } x'_2 \in B'_2 \text{ there exists } x''_1 \in B_1 \text{ such that } x''_1 \succ R_1 x'_2]\). This is exactly the clause not\((x \not\prec \text{R}_2 z)\).

Since asymmetry and negative transitivity imply transitivity, \(R_2\) is transitive.

Now let \(R'\) be the following binary relation:
\[
x \preceq R' y \iff [(x \not\prec \text{R}_2 y) \text{ or not}(x \not\prec \text{R}_2 y)]
\]

The transitivity and negative transitivity of \(R_2\) implies the transitivity of \(R'\). Moreover, \(R'\) is complete and satisfies TI.

**Step 2:** \(R\) is a subrelation to \(R'\). Let \(x, y\) be such that \(x \not\prec R y\). In the clause \(C(x, y)\), take \(B = \{z \in X : z \not\prec R y\}\). We have \(x \in B\) and for all \(B'\) containing \(y\), we have \(z \not\prec R_1 y\) for all \(z \in B\). Hence the clause \(C(x, y)\) is true and \(x \not\prec \text{R}_2 y\).

\(^3\)This clause combines the two clauses proposed by Jaffray 1975 in the proof of his theorem 1, the first of which defines a preorder on \(\beta_+(R)\) and the second a preorder on \(X\).
Consequently, $x \prec_{R'} y$. If $x, y$ are such that $x \sim_R y$, the clause $C(x, y)$ cannot be satisfied. To see it, it suffices to notice that an interval containing $x$ necessarily contains $y$ and vice versa. If we take $B' = B$ in the clause $C(x, y)$, there is no $x'$ in $B$ such that for all $z \in B$, we have $z \prec_{R_2} x'$. Thus we have not $(x \prec_{R_2} y)$. In the same way, we have not $(y \prec_{R_2} x)$. Consequently, $x \sim_{R'} y$.

It remains to show that $\tau_+(R') \subset \tau_+(R)$. Let $y \in X$. We show that any subset in $\beta_+(R')$, the base of open intervals generating $\tau_+(R')$, is open with respect to $\tau_+(R)$. Let $x \in B = \{z \in X : z \prec_{R'} y\}$. By the definition of $R'$, there is $B_x$ in $\beta_+(R)$, containing $x$, such that for all $B_y \in \beta_+(R)$ containing $y$, we can find $x' \in B_y$ such that for all $z \in B_x$, we have $z \prec_{R_1} x'$. We can see that this implies that for all $z \in B_x$, we have $z \prec_{R'} y$. Hence $B_x \subset B$. Recap: for all $x$ in $B$, we found $B_x$ in $\beta_+(R)$ containing $x$ such that $B_x \subset B$. As a result, $B$ is a union of open sets of $\tau_+(R)$. It is thus an open set of $\tau_+(R)$.

Remark 4: Theorem 2 holds if we replace $\tau_+ (R)$ and $\tau_+ (R')$ respectively by $\tau_-(R)$ and $\tau_-(R')$ the lower-order-topologies.\)

Remark 5: The inclusion $\tau_+ (R') \subset \tau_+ (R)$ entails the upper semicontinuity of the extension with respect to any topology on $X$ stronger than $\tau_+ (R)$. Upper semicontinuity is used here in the sense that lower sections $\{x \in X : x \prec_R a\}$ are open. But it is not necessary for the topology on $X$ to be stronger than $\tau_+ (R)$ to have the upper semicontinuity of the extension. For more information on this issue, see Jaffray 1975, section 5.\)

Axiom Scalar Invariance (SI): For all nonnegative real $\alpha$ and vectors $x, y$ in a real vector space equipped with a preorder $R$, $x \succeq_R y \implies \alpha x \succeq_R \alpha y$.

Corollary 2: Let $Y$ be a real normed vector space. Denote $t$ the topology induced by the norm of $Y$ (i.e. the norm topology). Let $R$ be a preorder on $Y$ satisfying $\textbf{TI}$ and $\tau_+ (R) \subset t$. Let $R'$ be one of the complete preorders which existence is asserted by theorem 2, i.e. a complete preorder of which $R$ is a subrelation, satisfying $\textbf{TI}$ and such that $\tau_+ (R') \subset \tau_+ (R)$. Then $R'$ satisfies $\textbf{SI}$.

Proof: We have $\tau_+ (R') \subset t$. Let $\alpha$ be a nonnegative real and $x, y$ two vectors in $Y$ such that $x \succeq_R y$. Using $\textbf{TI}$ and $\textbf{DI}$ we get $q(x - y) \succeq_{R'} 0$ for any nonnegative rational number $q$. Let $(q_n)$ be a nonnegative sequence of rationals converging to $\alpha$. The sequence $q_n(x - y)$ converges to $\alpha (x - y)$. On the other hand, $q_n(x - y) \in P = \{z \in Y : z \succeq_{R'} 0\}$ and $P$ is closed since $\tau_+ (R') \subset t$. Thus, the limit of the sequence $(q_n(x - y))$, which is $\alpha (x - y)$, belongs to $P$. As a result $\alpha (x - y) \succeq_{R'} 0$. What yields, by $\textbf{TI}$, $\alpha x \succeq_{R'} \alpha y$.\)

An immediate consequence of corollary 2 is the following:

Corollary 3: Let $R$ be a complete preorder on $Y$, a real normed vector space, satisfying $\textbf{TI}$ and $\tau_+ (R) \subset t$, where $t$ is the norm topology of $Y$. Then $R$ satisfies $\textbf{SI}$.

Remark 6: $\tau_+ (R) \subset t$ is a continuity requirement. Under that continuity requirement and $\textbf{TI}$, $\textbf{SI}$ is, in a sense, satisfied since every complete preorder extending the original preorder and satisfying the same axiom of continuity and $\textbf{TI}$ must satisfy $\textbf{SI}$.\)

Remark 7: Demuynck-Lauwers 2009 showed that a given preorder satisfying $\textbf{TI}$ and $\textbf{SI}$ can be extended into a complete preorder satisfying $\textbf{TI}$ and $\textbf{SI}$. Corollary 2 shows that if, in addition, the initial preorder satisfies upper semi-
continuity, then it admits an extension which also satisfies upper semicontinuity in addition to the axioms TI and SI. ♦

Remark 8: On the relationship between SI and continuity, while corollary 3 presents SI as a consequence of TI and a condition of continuity, Weibull 1985 theorem A has shown that under conditions TI, SI, and two other conditions C4 and C5, a complete preorder verifies a strong condition of continuity that results in continuous representability, i.e. the existence of a real-valued order-preserving continuous function. On the other hand, Mitra-Ozbek 2013 introduce another continuity condition called scalar continuity which, under monotonicity (see definition in section 6), implies representability (Mitra-Ozbek 2013, proposition 2). If we add condition TI, it is not difficult to see that SI follows, as well as Weibull 1985 condition C4. Proposition 3 below gives a weaker continuity condition which turns out to be equivalent to SI. ♦

Example 3: Consider the following relation SC′ on \( \mathbb{R}^{N_0} \):

\[
x \succeq_{SC'} y \iff \exists k \in \mathbb{N}_0, \liminf_{n} \sum_{i=1}^{kn} (x_i - y_i) \geq 0
\]

SC′ is a preorder verifying strong Pareto and fixed-step anonymity. In their lemma 3, Fleurbaey-Michel 2003 show that SC′ is continuous with respect to the \( l_1 \)-norm.

Proposition 2: There exists a complete preorder SC′′ on \( \mathbb{R}^{N_0} \), of which SC′ is a subrelation, and which is translation-invariant, strong Pareto, fixed-step-anonymous and upper semi-continuous with respect to the \( l_1 \)-topology.

Proof: The \( l_1 \)-topology is induced by the \( l_1 \)-norm: \( \sum_{i=1}^{+\infty} |x_i| \). By theorem 2, there exists a translation-invariant, upper-semi-continuous and complete preorder SC″ on \( \mathbb{R}^{N_0} \) of which SC′ is a subrelation. SC′ being a subrelation to SC″ entails that SC″ satisfies strong Pareto and fixed-step anonymity. SC″ is the required preorder. □

Remark 9: In accordance with remark 4, proposition 2 holds if we replace "upper-semi-continuous" by "lower-semi-continuous". ♦

5- Scalar Invariance and the Continuous Linear Weak Representation

Let Y be a real vector space and R a preorder on Y satisfying TI. Remark 6 and Remark 8 show that, under TI, there is a relationship between SI and the concept of continuity. Notice that theorem 2 of Mitra-Ozbek 2010 also suggests that⁴. However, under TI, although continuity according to the norm topology implies SI (corollaries 2 and 3), this is not true for \( l_1 \)-topology as shown in the following example.

⁴In the finite-dimensional case, they show that TI, strong Pareto and minimal individual symmetry are not sufficient to warrant linear representability, hence continuity. To that aim, they construct an interesting example of a complete preorder on \( \mathbb{R}^n \), \( n \geq 2 \) satisfying these axioms but not SI (Mitra-Ozbek 2010, section 3).
Example 4: A translation-invariant preorder, $l_1$-continuous but not scalar-invariant. We build a preorder on $Y = \mathbb{R}^2$, in the manner of Svensson 1980. Consider the equivalence relation $r: x \Leftrightarrow y$ iff $\sum_{i=1}^{+\infty} |x_i - y_i| < +\infty$. Denote $x_\alpha = (\alpha, \alpha, \ldots)$ where $\alpha \in \mathbb{R}$. Each $x_\alpha$ can be considered as a representative of a distinct equivalence class $X_\alpha$. Denote $(X_3)$ the remaining equivalence classes and for each $X_\beta$, denote $x_\beta$ a representative. Consider the set $M = \{x_\alpha\} \cup \{x_\beta\}$. As the addition of equivalence classes for the relation $r$ is well defined, let $R$ be the preorder on $M$ defined by $x \preceq_R y$ if there is $\alpha \in \mathbb{R}$ such that $\alpha \preceq_R 0$ and $x - y = x_\alpha$ (see example 1 for the definition of $\preceq_R$). $R$ is TI. According to theorem 1 there is a complete preorder $R'$ on $M$ satisfying TI of which $R$ is a subrelation. Define a complete preorder $R''$. In each equivalence class $X_\gamma$ (where $\gamma = \alpha$ or $\beta$) exactly as in Svensson 1980 page 1255, (iv). $x, y$ being two elements of $Y$, denote $\bar{x}, \bar{y}$ the representatives in $M$ of their respective classes. Consider the preorder $R''$ defined as follows: (i) (if $\bar{x} \neq \bar{y}$ then $x \preceq_{R''} y$ if $\bar{x} \preceq_{R'} \bar{y}$), (ii) (if $\bar{x} = \bar{y} = x_\gamma$ then $x \preceq_{R''} y$ if $x \preceq_{R_\gamma} y$). $R''$ is TI. As in Svensson 1980, one shows that $R''$ is continuous with respect to the $l_1$-topology. However $R''$ is not SI.

It can thus be said that continuity with respect to the $l_1$-topology is not a sufficiently strong requirement to guarantee SI. Conversely, we may try to find out what level of continuity is verified if SI is verified. For example, the lexicographic order on $Y = \mathbb{R}^2$ is translation-invariant and scalar-invariant but it is not continuous. If $Y$ is the space of the real bounded sequences $\ell_\infty$, the preorder $SC'$ is translation-invariant and scalar-invariant but it is not continuous under the norm $\|x\| = \sup|x_i|$.

The following notion of continuity is proposed:

**Linear Continuity:** A preorder on $Y$ satisfies linear continuity if the preorder induced on every straight line of $Y$ equipped with the canonical topology of the real line, is continuous.

**Proposition 3:** Let $R'$ be a complete and translation-invariant preorder on $Y$. $R$ satisfies SI iff $R$ satisfies linear continuity.

**Proof:** If $R'$ satisfies linear continuity, take a nonzero vector $u$ in $P = \{v: v \preceq_{R'} 0\}$. Using TI and DI we show that for every nonnegative rational $q$ we have $qu \preceq_{R'} 0$. By continuity of the induced order on the straight line generated by $u$, we obtain that for every nonnegative real $\alpha$ we have $\alpha u \preceq_{R'} 0$. Thus, for any $x, y$ in $Y$ such that $x \preceq_{R'} y$, we can take $u = x - y$ and conclude that $\alpha x \preceq_{R'} \alpha y$ for every nonnegative real $\alpha$. This establishes that $R'$ satisfies SI. Conversely, if $R'$ satisfies SI, take a straight line $D$ in $Y$. There is two possible situations. Either $u \sim_{R'} 0$ for all $u \in D$ or there is $u \in D$ such that $u \succ_{R'} 0$. In the first situation, the induced order on $D$ is indifference. It is continuous. In the second situation, by SI, for all real $\delta$ we have $\delta u \preceq_{R'} 0 \Leftrightarrow \delta \geq 0$. Let $\lambda_\alpha \rightarrow \lambda$ in $\mathbb{R}$ and $\lambda_\alpha u \preceq_{R'} \lambda' u$ for all $n$. By TI, $(\lambda_\alpha - \lambda') u \preceq_{R'} 0$ and, consequently, by SI, $\lambda_\alpha - \lambda' \geq 0$. We deduce that $\lambda - \lambda' \geq 0$ and $\lambda u \preceq_{R'} \lambda' u$. The induced order on $D$ is thus upper-semi-continuous. We show in the same way that it is lower-semi-continuous.\[\square\]
Remark 10: Proposition 3 remains valid for a non-complete preorder verifying TI and DI.

Norm continuity (i.e., continuity with respect to the norm topology) implies scalar continuity as defined by Mitra-Ozbek 2013, which, in turn, implies linear continuity. A preorder may satisfy SI without norm continuity, as $SC'$ in $l_\infty$, or without scalar continuity, as the lexicographic order in $\mathbb{R}^2$. But of course both of them satisfy linear continuity since they satisfy SI.

We would now like to know to what extent we can approach a translation-invariant preorder by a coarser preorder and respecting TI and SI. A preorder $R_1$ is said to be finer than a preorder $R_2$, or $R_2$ coarser than $R_1$, if $x \succsim_{R_1} y \Rightarrow x \succsim_{R_2} y$. As a first step, the following theorem establishes the existence of a continuous linear functional on a real normed vector space which separates a convex set. First we show that $\partial Y$ is open and convex. It remains to show that for all positive real $\mu$, we have $\mu \in \partial Y$. Thus, it is a convex cone.

Theorem 3: Let $Y$ be a normed vector space and $R'$ be a complete translation-invariant preorder on $Y$. Suppose that the interior of $P_\mu^+(R')$, denoted $P_\mu^{o}(R')$, is not empty. Then there exists a non-zero, continuous linear functional $\varphi$ on $Y$ such that for all $x, y \in Y$, $\varphi(x) > \varphi(y) \Rightarrow x \succsim_{R'} y$.

Proof: The proof is similar to that of theorem 4 in Mabrouk 2011, except that the set $Q$ is here $\bigcup_{x \in R' \setminus 0} \left( x + P_\mu^{o}(R') \right)$ and the condition super weak Pareto is replaced by the condition $P_\mu^{o}(R') \neq \emptyset$. The argument is based on the geometrical form of Hahn-Banach theorem who asserts the existence of a continuous linear functional on a real normed vector space which separates a convex set having a non-empty interior from a point out of the interior of the convex set. First we show that $Q$ is open and convex. $x + P_\mu^{o}(R')$ is open as a translation of a non-empty open set. $Q$ is open as a union of open sets.

Since the sets $\{ x : x \succsim_{R'} 0 \}$ and $P_\mu^{o}(R')$ are closed under addition, $Q$ is closed under addition. It remains to show that for all positive real $\mu$ and $x + p$ in $Q$, we have $\mu(x + p) \in Q$. Let $k_n, m_n$ two sequences of positive integers such that $\lim \frac{k_n}{m_n} = \mu$. Let $p_n = \left( \mu - \frac{k_n}{m_n} \right) x + \mu p$. We have $\mu p \in P_\mu^{o}(R')$. $P_\mu^{o}(R')$ being open, there is an integer $N$ such that $p_N$ is in $P_\mu^{o}(R')$. Since $\mu(x + p) = \frac{k_n}{m_N} x + p_N$ and $\frac{k_n}{m_N} x \succsim_{R'} 0$ and $p_N \in P_\mu^{o}(R')$, we have $\mu(x + p) \in Q$. This proves that $Q$ is convex. By invoking Hahn-Banach theorem, there is a continuous linear functional $\varphi$ which separates $Q$ from 0, i.e., $\varphi(x) > 0$ for all $x \in Q$. Now take $x, y \in Y$ such that $x \succsim_{R'} y$.

\footnote{According to the terminology in Mitra-Ozbek 2013.}

\footnote{Clearly, $\mu p \in P_\mu^{o}(R')$. It is an interior point of $\lambda P_\mu^{o}(R')$ because if $B$ is an open sphere of center $\mu p$ and radius $\tau$ and $B \subset P_\mu^{o}(R')$, then $B' = \mu B$ is an open sphere of center $\mu p$ and radius $\mu \tau$ and $B' \subset P_\mu^{o}(R')$.}
and $p$ in $P^\circ_+(\mathbb{R})$. Let $(\alpha_n)$ be a sequence of positive reals decreasing to 0. We have $x - y + \alpha_n p \in Q$. Thus $\varphi(x - y + \alpha_n p) > 0$. The continuity of $\varphi$ yields $\lim_n \varphi(x - y + \alpha_n p) = \varphi(x - y) \geq 0$. We have shown that for all $x, y$ in $Y$, $x \succeq_R y$ implies $\varphi(x) \geq \varphi(y)$. Consequently, $\varphi(x) > \varphi(y)$ implies $x \succ_R y$. □

**Remark 11**: $\varphi$ is unique up to a positive multiplicative factor. Indeed, let $\varphi_1, \varphi_2$ be two continuous linear weak representations of $R'$. We have

$$\varphi_1(x) > \varphi_1(y) \Rightarrow x \succ_R y \Rightarrow \varphi_2(x) \geq \varphi_2(y) \quad (2)$$

It is known that the sets $\ker \varphi_1 = \{ x : \varphi_1(x) = 0 \}$ and $\ker \varphi_2 = \{ x : \varphi_2(x) = 0 \}$ are hyperplanes. Taking $y = 0$ in (2) yields $\varphi_1(x) > 0 \Rightarrow \varphi_2(x) \geq 0$. If $\ker \varphi_1$ were different from $\ker \varphi_2$, there would exist $h_1$ in $\ker \varphi_1$ such that $\varphi_2(h_1) > 0$ and $h_2$ in $\ker \varphi_2$ such that $\varphi_1(h_2) > 0$. Hence $\varphi_1(h_2 - h_1) = \varphi_1(h_2) > 0$ and $\varphi_2(h_2 - h_1) = -\varphi_2(h_1) < 0$. This contradicts the implication $\varphi_1(x) > 0 \Rightarrow \varphi_2(x) \geq 0$. Consequently, $\ker \varphi_1 = \ker \varphi_2$. Denote $H = \ker \varphi_1 = \ker \varphi_2$. Let $v \not\in H$. Every $x$ in $Y$ can be written in a unique manner $x = \alpha v + u$, where $u \in H$. Thus

$$\varphi_2(x) = \varphi_2(\alpha v + u) = \alpha \varphi_2(v) = \frac{\varphi_2(v)}{\varphi_1(v)} \varphi_1(\alpha v + u) = \frac{\varphi_2(v)}{\varphi_1(v)} \varphi_1(\alpha v) \varphi_1(v) \varphi_1(x)$$

The factor $\frac{\varphi_2(v)}{\varphi_1(v)}$ is positive since $\varphi_1, \varphi_2$ have always the same sign.

### 6- Pareto and Monotonicity Axioms

We are now in the space $l^\infty_\omega = \{ (x_1, x_2, ...) : x_i \in \mathbb{R} \text{ and } \sup |x_i| e^{-r_i} < +\infty \}$, where $r$ is a real. The norm is $\|x\| = \sup |x_i| e^{-r_i}$. This space is suitable for studying economic decisions in discrete time, infinite horizon and exponentially growing economy (if $r > 0$). Denote $l^\infty_{\omega+} = \{ x \in l^\infty_\omega : \text{ for all } i, x_i > 0 \}$. If $r = 0$, the economy remains bounded. Let $R'$ be a translation-invariant and complete preorder on $l^\infty_\omega$. Theorem 4 of Mabrouk 2011 and the present theorem 3 differ in that the axiom super weak Pareto used in theorem 4 of Mabrouk 2011, is replaced by the assumption $P^\circ_+(R') \neq \emptyset$ in theorem 3. Proposition 4 below shows that the former condition is stronger than the latter when the space is $l^\infty_\omega$. The present formulation is therefore more general. In addition, we are interested in the condition of monotonicity because, on the one hand, it is a minimum requirement of efficiency for any preorder intended to rank economic alternatives. On the other hand, it turns out that in $l^\infty_\omega$, monotonicity fulfills the sufficient condition of theorem 3 for the existence of a weak representation (proposition 5).

**Axiom Super Weak Pareto**: if $\inf(x_i - y_i) e^{-r_i} > 0$ then $x \succ_R y$ (or $\varphi(x) > \varphi(y)$ if the axiom is applied to a functional $\varphi$).

**Axiom Monotonicity**: If $x_i - y_i \geq 0$ for all $i$, then $x \succeq_R y$ (or $\varphi(x) \geq \varphi(y)$ if the axiom is applied to a functional $\varphi$).

**Proposition 4**: Let $R'$ be a translation-invariant, super weak Pareto and complete preorder on $l^\infty_\omega$. Then $P^\circ_+(R') \neq \emptyset$. 

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there is some
exists a non-zero, continuous linear functional
we have
thus \( x \in P_+^{\circ}(R') \). Consequently \( l_{\infty}^{\circ} \subset P_+^{\circ}(R') \), what yields
so \( P_+^{\circ}(R') \neq \emptyset \). □

**Proposition 5:** Let \( R' \) be a non-constant, translation-invariant, monotone and complete preorder on \( l_{\infty} \). Then \( P_+^{\circ}(R') \neq \emptyset \).

**Proof:** \( R' \) non-constant and translation-invariant entails that \( P_+^{\circ}(R') \neq \emptyset \). Let \( x \in P_+^{\circ}(R') \) and \( \varepsilon > 0 \). Monotonicity entails that the vector \( y \) defined by \( y_i = x_i + \|x\| e^{-r i} + \varepsilon \) is in the set \( A = P_+^{\circ}(R') \cap l_{\infty}^{\circ} \). Hence \( A \neq \emptyset \). For all \( x \) in \( A \), the set \( x + l_{\infty}^{\circ} \) is included in \( A \). Consequently, the interior of \( x + l_{\infty}^{\circ} \), which is \( x + l_{\infty}^{\circ} \), is included in \( A \) and we have \( A \neq \emptyset \). By monotonicity, for all \( x \) in \( A \) and all real \( \lambda > 1 \), we have \( \lambda x \succ_R^l x \succ_R^l 0 \). Let \( x \in A \). The set \( a = \{ \alpha > 0 : \forall \lambda \geq \alpha, \lambda x \succ_R^l 0 \} \) contains 1 and admits 0 as lower bound. Thus it admits an infimum, say \( \alpha_0 \). Suppose \( \alpha_0 > 0 \). Let \( \beta \in ]0, \alpha_0 [ \). We would have \( \beta \notin A \). In other words, there exists \( N \geq \beta \) such that \( \lambda^N x \not\succ_R^l 0 \). By TI, for all positive rational \( q \), we would have \( q \lambda^N x \not\succ_R^l 0 \). We may choose \( q \) such that \( q \lambda^N > 1 \). However the inequality \( q \lambda^N x \not\succ_R^l 0 \) contradicts the fact that for all real \( \lambda > 1 \), we have \( \lambda x \succ_R^l 0 \). Consequently, \( \alpha_0 = 0 \). In other words, for all \( \lambda > 0 \), we have \( \lambda x \succ_R^l 0 \). Thus \( A \subset P_+^{\circ}(R') \). What entails \( A \subset P_+^{\circ}(R') \). Since \( A \neq \emptyset \), we must have \( P_+^{\circ}(R') \neq \emptyset \). □

The following lemma is needed to prove corollary 4.

**Lemma 2:** Let \( \varphi \) be a nonzero linear functional on \( l_{\infty}^{\circ} \). Then \( \varphi \) is monotone iff \( \varphi \) is super weak Pareto.

**Proof:** (i) non super weak Pareto \( \Rightarrow \) non monotone: \( \varphi \) nonzero entails that there is some \( p \) in \( l_{\infty}^{\circ} \) such that \( \varphi(p) > 0 \). \( \varphi \) non-super-weak-Pareto \( \Rightarrow \) there is some \( u \) in \( l_{\infty}^{\circ} \) such that \( \inf u_i e^{-r i} > 0 \) and \( \varphi(u) \leq 0 \). Let \( \delta = \frac{1}{2} \inf u_i e^{-r i} \). We check easily that \( v_i = u_i - \delta p_i > 0 \) for all \( i \). But \( \varphi(v) = \varphi(u) - \delta \varphi(p) < 0 \). All components of \( v \) are positive and its image is negative. \( \varphi \) is not monotone.
(ii) non monotone \( \Rightarrow \) non super weak Pareto: There is some \( u \) in \( l_{\infty}^{\circ} \) such that \( u_i \geq 0 \) and \( \varphi(u) < 0 \). If there is no \( p \) in \( l_{\infty}^{\circ} \) such that \( \inf p_i e^{-r i} > 0 \) and \( \varphi(p) > 0 \), then \( \varphi \) is not super weak Pareto. If there is such a vector \( p \) in \( l_{\infty}^{\circ} \), let \( \delta = -\frac{\varphi(u)}{\varphi(p)} \) and \( v = u + \delta p \). We have \( \inf v_i e^{-r i} = \inf (u_i + \delta p_i) e^{-r i} > 0 \) and \( \varphi(v) = \varphi(u) + \delta \varphi(p) = 0 \). Thus \( \varphi \) is not super weak Pareto. □

**Corollary 4:** Let \( R' \) be a non-constant, translation-invariant and complete preorder on \( l_{\infty}^{\circ} \). If \( R' \) is monotone, \( R' \) is super weak Pareto.

**Proof:** According to proposition 5, \( R' \) non-constant, translation-invariant, monotone and complete implies \( P_+^{\circ}(R') \neq \emptyset \). According to theorem 3, there exists a non-zero, continuous linear functional \( \varphi \) on \( Y \) such that for all \( x, y \) in \( l_{\infty}^{\circ} \), \( \varphi(x) \succ_R^l \varphi(y) \Rightarrow x \succ_R^l y \), or, equivalently, \( x \succ_R^l y \Rightarrow \varphi(x) \geq \varphi(y) \). Consequently, if \( x, y \) are such that \( x_i - y_i \geq 0 \) for all \( i \), then \( x \succ_R^l y \) and \( \varphi(x) \geq \varphi(y) \). Consequently, \( \varphi \) is monotone. By lemma 2, \( \varphi \) is super-weak-
Pareto. If \( \inf(x_i - y_i)e^{-r_i} > 0 \) then \( \varphi(x) > \varphi(y) \) and \( x \succ_{R^\infty} y \).

In short, for a non-constant, translation-invariant and complete preorder \( R' \) on \( l_\infty^r \), we have the following implications:

\[
R' \text{ monotone } \Rightarrow R' \text{ super weak Pareto } \Rightarrow P_*^\varnothing(R') \neq \emptyset
\]

**Remark 12:** These implications and theorem 3 show that a monotone and non-constant preorder on \( l_\infty^r \) has a weak representation. So we find proposition 1 of Mitra-Ozbek 2013, in another context and following a different path.\footnote{Remark 16: Monotonicity implies super weak Pareto for a non-constant translation-invariant and complete preorder but the converse is not true. Concerning theorem 3, if the preorder is not complete, one can only assert the existence of a continuous linear functional \( \varphi \) such that \( \varphi(x) > \varphi(y) \) implies non-constant preorder on \( l_\infty^r \), where \( R \) is the preorder in question. Moreover, \( \varphi \) may not be unique as shown in the following preorder \( R \) on \( \mathbb{R}^2 \): \( x \succeq_R y \) iff \( x_1 \geq y_1 \) and \( x_2 \geq y_2 \). Every functional of the form \( \varphi(x) = ax_1 + bx_2 \), where \( a, b \) are two positive reals, satisfies \( x \succeq_R y \Rightarrow \varphi(x) \geq \varphi(y) \).\footnote{Remark 15: Proposition 4 and proposition 5 hold if the preorder is not complete. Concerning theorem 3, if the preorder is not complete, one can only assert the existence of a continuous linear functional \( \varphi \) such that \( \varphi(x) > \varphi(y) \) implies non-constant preorder on \( l_\infty^r \), where \( R \) is the preorder in question. Moreover, \( \varphi \) may not be unique as shown in the following preorder \( R \) on \( \mathbb{R}^2 \): \( x \succeq_R y \) iff \( x_1 \geq y_1 \) and \( x_2 \geq y_2 \). Every functional of the form \( \varphi(x) = ax_1 + bx_2 \), where \( a, b \) are two positive reals, satisfies \( x \succeq_R y \Rightarrow \varphi(x) \geq \varphi(y) \).}

\[
\text{Let } Y \text{ be a real normed vector space. If } R_1 \text{ and } R_2 \text{ are two preorders on } Y, \text{ denote } (R_1, R_2) \text{ their lexicographic combination defined as in Remark 14.}
\]

**7- Linear Continuous Lexicographic Decomposition:**

Let \( Y \) be a real normed vector space. If \( R_1 \) and \( R_2 \) are two preorders on \( Y \), denote \((R_1, R_2)\) their lexicographic combination defined as in Remark 14.

**Linear Continuous Lexicographic Combination (LCLC):** A preorder \( R \) on \( Y \) is said to be a LCLC if there exists a sequence, finite or infinite, of continuous linear functionals \( (\varphi_1, \varphi_2, \ldots) \) on \( Y \) such that \( R = (\preceq_{\varphi_1}, \preceq_{\varphi_2}, \ldots) \) where \( \preceq_{\varphi_n} \) is the preorder defined by \( x \preceq_{\varphi_n} y \) iff \( \varphi_n(x) \geq \varphi_n(y) \).

Notice that what is important for the preorder is the value of \( \varphi_n \) on the subspace \( H_{n-1} \) defined by \( H_0 = Y \) and \( H_n = \ker \varphi_n \cap H_{n-1} \) for \( n \geq 1 \). If \( R \) is non-constant and if \( \varphi_n \) is zero on \( H_{n-1} \), it plays no role in the definition of \( R \) and can be wiped out. Hence, if \( R \) is non-constant we will suppose that
every functional \( \varphi_n \) is non-zero on \( H_{n-1} \). By convention, if \( R \) is constant, the sequence \( (\varphi_n) \) amounts to \( \varphi_1 \) which is zero. A LCLC is complete, translation-invariant and scalar-invariant. By proposition 3, a LCLC is linear-continuous. But a linear-continuous translation-invariant and scalar-invariant preorder is not necessarily a LCLC even if it is complete (see the example of section 8 and particularly remark 19). It should be noted that a LCLC is linear-continuous but generally not continuous, as the standard lexicographic preorder on \( \mathbb{R}^2 \).

**Lemma 3:** If \( L \) is a LCLC on a real vector space \( Z \), then the preorder \( L' \) induced by \( L \) on a subspace \( Z' \) of \( Z \) is also a LCLC.

**Proof:** Obvious. However it must be noticed that the sequence of linear functionals may not be the same if one of the functionals is zero on its new corresponding subspace. In that case, that functional is wiped out.\( \square \)

**Lemma 4:** If a LCLC \( L \) on a real vector space \( Z \) is not constant. Then

\[
P_+^o(L) = \{ x \in Z : \varphi(x) > 0 \}
\]

where \( \varphi \) is the functional defining the first component of \( L \).

**Proof:** \( L \) not constant entails that \( \varphi \) is non zero. Clearly \( \{ x \in Z : \varphi(x) > 0 \} \subset P_+(L) = P_+^o(L) \). Since \( \{ x \in Y : \varphi(x) > 0 \} \) is open, \( \{ x \in Y : \varphi(x) > 0 \} \subset P_+^o(L) \).

Conversely, every point of \( P_+^o(L) \) satisfies \( \varphi(x) \geq 0 \). If \( \varphi(x) = 0 \), let \( u \) be such that \( \varphi(u) > 0 \) and \( \alpha \) a positive real. Denote \( y = x - \alpha u \). We have \( \varphi(y) = \varphi(x - \alpha u) = -\alpha \varphi(u) < 0 \). Thus \( y \prec_L 0 \). For a given neighborhood of \( x \), one can make \( \alpha \) as small as necessary for \( y \) to be in that neighborhood. This proves that if \( \varphi(x) = 0 \), \( x \) in not in the interior of \( P_+^o(L) \). Therefore

\[
P_+^o(L) = \{ x \in Y : \varphi(x) > 0 \} \]

**Lemma 5:** Let \( L_1, L_2 \) be two LCLCs on a real vector space \( Z \) such that \( L_2 \) is coarser than \( L_1 \). Then either \( L_2 \) is constant or it has the same first component than \( L_1 \).

**Proof:** If \( L_1 \) is constant, \( L_2 \) must obviously be constant. If \( L_1 \) is not constant and \( L_2 \) is constant, there is nothing to prove. The remaining case is \( L_1 \) and \( L_2 \) not constant. Let \( \varphi_1 \) be the first component of \( L_1 \) and \( \varphi_2 \) be the first component of \( L_2 \). According to lemma 4 \( P_+^o(L_j) = \{ x \in Z : \varphi_j(x) > 0 \} \), \( j = 1, 2 \). Moreover \( L_2 \) coarser than \( L_1 \) entails \( P_+^o(L_2) \subset P_+^o(L_1) \). Consequently

\[
P_+^o(L_2) \subset P_+^o(L_1).
\]

What yields \( \{ x \in Z : \varphi_2(x) > 0 \} \subset \{ x \in Z : \varphi_1(x) > 0 \} \). Taking the closure of these two spaces: \( \{ x \in Z : \varphi_2(x) \geq 0 \} \subset \{ x \in Z : \varphi_1(x) \geq 0 \} \).

We must also have \( \{ x \in Z : \varphi_1(x) \geq 0 \} \subset \{ x \in Z : \varphi_2(x) \geq 0 \} \) because if there was \( y \in Y \) such that \( \varphi_1(y) \geq 0 \) and \( \varphi_2(y) < 0 \), we would have \( \varphi_1(-y) \leq 0 \) and \( \varphi_2(-y) > 0 \), what would contradicts \( \{ x \in Z : \varphi_2(x) \geq 0 \} \subset \{ x \in Z : \varphi_1(x) \geq 0 \} \).

Consequently \( \{ x \in Z : \varphi_2(x) \geq 0 \} = \{ x \in Z : \varphi_1(x) \geq 0 \} \). With the same argument as in Remark 11, we conclude that \( \varphi_1 = \varphi_2 \) up to a positive multiplicative factor.\( \square \)

**Theorem 4:** Let \( R \) be a complete translation-invariant preorder on \( Y \). Consider the sequences of preorders \( (R_n)_{1 \leq n \leq n_{\text{max}}} \), functionals \( (\varphi_n)_{1 \leq n \leq n_{\text{max}}} \) and subspaces \( H_0 = Y \) and \( (H_n)_{1 \leq n \leq n_{\text{max}}} \) built by a successive application of
theorem 3 as follows:
- stage 1: $R_1 = R$. If $P^\omega_+(R) = \emptyset$, then $\varphi_1 = 0$, $n_{\text{max}} = 1$, $H_1 = H_0$ and the construction of the sequences $(R_n)$, $(\varphi_n)$ and $(H_n)$ stops. If $P^\omega_+(R) \neq \emptyset$, then $\varphi_1$ is the functional, unique up to a positive multiplicative factor, given by theorem 3 applied to the preorder $R_1$ on $H_0 = Y$, $H_1$ is defined as $H_1 = \ker \varphi_1 \cap H_0$ and $R_2$ is the preorder induced by $R$ on $H_1$.

- stage 2: In the subspace $H_1$ equipped with the relative topology, if $P^\omega_+(R_2) = \emptyset$, then $\varphi_2 = 0$, $n_{\text{max}} = 2$, $H_2 = H_1$ and the construction of the sequences $(R_n)$, $(\varphi_n)$ and $(H_n)$ stops. If $P^\omega_+(R_2) \neq \emptyset$, then $\varphi_2$ is the functional, unique up to a positive multiplicative factor, given by theorem 3 applied to the preorder $R_2$ on $H_1$, $H_2 = \ker \varphi_2 \cap H_1$ and $R_3$ is the preorder induced by $R$ on $H_2$.

- stage $n$: In the subspace $H_{n-1}$ equipped with the relative topology, if $P^\omega_+(R_n) = \emptyset$, then $\varphi_n = 0$, $n_{\text{max}} = n$, $H_n = H_{n-1}$ and the construction of the sequences $(R_n)$, $(\varphi_n)$ and $(H_n)$ stops. If $P^\omega_+(R_n) \neq \emptyset$, then $\varphi_n$ is the functional, unique up to a positive multiplicative factor, given by theorem 3 applied to the preorder $R_n$ on $H_{n-1}$, $H_n = \ker \varphi_n \cap H_{n-1}$ and $R_{n+1}$ is the preorder induced by $R$ on $H_n$.

... Then, the LCLC $L(R) = (\varphi_1, \varphi_2, \ldots, \varphi_{n_{\text{max}}})$ with $n_{\text{max}}$ possibly infinite, is the finest LCLC coarser than $R$ and every LCLC on $Y$ coarser than $R$ is either constant or of the form $(\varphi_1, \varphi_2, \ldots, \varphi_n)$, with $n \leq n_{\text{max}}$.

**Proof:** Let $L'$ be a LCLC on $Y$ coarser than $R$. By lemma 5, either $L'$ is constant, or it has the same first component than $L(R)$. If $\varphi_1 \neq 0$, apply again lemma 5 to the preorders $L'_2$ and $L(R)_2$ induced respectively by $L'$ and $L(R)$ on the space $H_1 = \ker \varphi_1 \cap H_0$ equipped with the relative topology. Again, either $L'_2$ is constant, or it has the same first component than $L(R)_2$, which is the second component of $L(R)$. Repeat this operation until the component of rank $n_{\text{max}}$. This shows that $L' = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, with $n \leq n_{\text{max}}$.

$L(R)$ may be referred to as the linear continuous lexicographic decomposition of $R$. Although theorem 4 involves several times theorem 3 which is non-constructive, it may be used along with other specific information to gain some insight on the preorder, as in section 8.

**Corollary 5:** Let $R^1, R^2$ be two non-constant LCLCs on $Y$. Denote $R^1 = (\varphi_1^1, \varphi_2^1, \ldots, \varphi_{n_1}^1)$ and $R^2 = (\varphi_1^2, \varphi_2^2, \ldots, \varphi_{n_2}^2)$. If $R^2$ is coarser than $R^1$, then $n_1 \geq n_2$ and for all $n \leq n_2$, $\varphi_n^1 = \varphi_n^2$ on the subspace $H_{n-1}^1$ (defined as in theorem 4 for the preorder $R^1$) up to a positive multiplicative factor. If $n_1 = n_2$ then $R^1 = R^2$.

**Proof:** It is a direct application of Theorem 4. □

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7In fact, $\varphi_n$ is well defined on $H_{n-1}$ not on $Y$. To be fully in line with the definition of a LCLC, one should consider extensions of $\varphi_n$ on $Y$. But this will play no role in the calculus of the preorder. In order not to overload the text unnecessarily, I keep $\varphi_n$ in the definition of $L(R)$. □
Remark 17: If the preorder is not complete, then the sequence
\[(\varphi_1, \varphi_2, \ldots, \varphi_n)\]
may not be unique. See Remark 15.

Example 5: Consider the following linear functionals on $\mathbb{R}^3$, $\varphi_1(x) = x_1 + x_2 + x_3$, $\varphi_2(x) = -x_1$. Denote $\preceq^3$ the preorder $\preceq_R$ (defined in example 1) applied to the third component. Consider the complete preorder on $\mathbb{R}^3$, $\mathcal{R} = (\varphi_1^3, \varphi_2^3, \varphi_3^3)$. The first step is to determine $P^o_+ (\mathcal{R})$. We have $x \succ_R 0$ iff (i) $\varphi_1(x) > 0$ or (ii) $\varphi_2(x) > 0$ or (iii) $\varphi_3(x) > 0$ and $x \succ_3 0$. Therefore $x \in P^o_+ (\mathcal{R})$ iff (i) $\varphi_1(x) > 0$ or (ii) $\varphi_2(x) = 0$ and $\varphi_3(x) > 0$. Thus $P^o_+ (\mathcal{R}) = \{ x \in \mathbb{R}^3 : \varphi_1(x) > 0 \}$. We can apply theorem 3.

$\varphi_1$ is a continuous linear functional that separates $P^o_+ (\mathcal{R})$ from 0. Thus $H_1 = \{ x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \}$. The second step is to apply again theorem 3 in the vector space $H_1$ equipped with the induced topology. The induced preorder on $H_1$ is $\preceq_R$ defined by $x \preceq_R y$ iff $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$ and (i) $x_1 < y_1$ or (ii) $x_1 = y_1$ and $x_3 \preceq_3 y_3$. Thus $P^o_+ (\mathcal{R}_2) = \{ x \in H_1 : x_1 < 0 \}$. This subspace being open in $H_1$, we have $P^o_+ (\mathcal{R}_2) = P^o_+ (\mathcal{R}_2)$. The continuous and linear functional $\varphi_2$ separates $P^o_+ (\mathcal{R}_2)$ from 0. Thus $H_2 = \{ x \in H_1 : x_1 = 0 \}$. The induced order in $H_2$ is $\preceq_R$ defined by $x \preceq_R y$ iff $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$ and $x_1 = y_1 = 0$ and $x_3 \preceq_3 y_3$. Therefore $P^o_+ (\mathcal{R}_3) = \emptyset$. The preorder $(\varphi_1^3, \varphi_2^3)$ is the finest LCLC coarser than $\mathcal{R}$.

8- Decomposition of the Catching-up Preorder
8-1- Decomposition under the Supnorm

Consider the catching-up preorder $\mathcal{C}$ defined on $l_\infty$ by $x \preceq_C y$ iff there is $n_0$ such that $n \geq n_0 \Rightarrow \sum_{i=1}^n x_i \geq \sum_{i=1}^n y_i$. The norm is the supnorm $\|x\| = \sup \{ |x_i| \}$. Because $\mathcal{C}$ is scalar-invariant, the set $Q$ used in the proof of theorem 3 is equal to $P_+^o (\mathcal{C}) = P_+^o (\mathcal{C})$, that is, the interior of $\{ x \in l_\infty : x \succ_C 0 \}$.  

Proposition 7: $P_+^o (\mathcal{C}) = \{ x \in l_\infty : \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^n x_i > 0 \}$

Proof: Condition $x \succ_C 0$ is equivalently written

$$\exists n_0 \text{ such that } n \geq n_0 \Rightarrow \sum_{i=1}^n x_i \geq 0$$

and for all $n$ there is $m \geq n$ such that $\sum_{i=1}^m x_i > 0$

where $n_0, n, m$ are positive integers. $x \in P_+^o (\mathcal{C})$ means that there is a positive real $r$ such that the sphere of center $x$ and radius $r$, $B(x, r)$, is in $P_+ (\mathcal{C})$, that is, for all $y$ in $l_\infty$ such that $\|y - x\| \leq r$, we have
\[ \exists n_0 \text{ such that } n \geq n_0 \Rightarrow \sum_{i=1}^{n} y_i \geq 0 \]

and for all \( n \) there is \( m \geq n \) such that \( \sum_{i=1}^{n} y_i > 0 \)

For \( x \) to be in \( P_+^a(C) \), it is enough that there is \( r > 0 \) such that

\[ \exists n_0 \text{ such that } n \geq n_0 \Rightarrow \sum_{i=1}^{n} (x_i - r) \geq 0 \]

and for all \( n \) there is \( m \geq n \) such that \( \sum_{i=1}^{n} (x_i - r) > 0 \)

This last condition is satisfied if \( \lim inf \frac{1}{n} \sum_{i=1}^{n} x_i > 0 \). Conversely, suppose \( \lim inf \frac{1}{n} \sum_{i=1}^{n} x_i = r' > 0 \). Let \( r = \frac{r'}{2} \). Then, clearly, \( B(x,r) \) is in \( P_+^a(C) \). As a result, \( P_+^a(C) = \left\{ x \in l_\infty : \lim inf \frac{1}{n} \sum_{i=1}^{n} x_i > 0 \right\} \).

Denote \( \pi_n = \frac{1}{n} \sum_{i=1}^{n} x_i \) and \( a = \{ x \in l_\infty \text{ such that the sequence } \pi_n \text{ converges} \} \).

\( a \) is a subspace of \( l_\infty \). Denote \( \pi_\infty = \lim \pi_n \) and \( \alpha_\infty \) the linear functional on \( a \) defined by \( \alpha_\infty(x) = \pi_\infty \). Let \( x \in P_+^a(C) \).

Let \( \varphi_1 \) be a non-zero continuous linear functional as given by theorem 3 and Remark 15. Denote \( c \) the set of converging sequences, \( l_1 \) the set of absolutely converging sequences and \( \delta_\infty \) the linear functional on \( c \) defined by \( \delta_\infty(x) = x_\infty = \lim x_n \).

**Proposition 8** \( \varphi_1 \) is equal to \( \alpha_\infty \) on \( a \), up to a positive multiplicative factor.

**Proof:** There is unique linear continuous functionals \( \varphi^1_1 \) and \( \varphi^2_1 \) on \( l_\infty \) such that \( \varphi_1 = \varphi^1_1 + \varphi^2_1 \), with \( \varphi^1_1 \in l_1 \) and the restriction of \( \varphi^2_1 \) to \( c \) is proportional to \( \delta_\infty \) (Yosida-Hewitt theorem\(^8\)). In the other hand, the preorder \( C \) satisfies the axiom finite anonymity which states that the ranking of a sequence does not change if one permutes two terms of the sequence. Since the preorder \( \preceq_C \) is coarser than \( C \), it inherits finite anonymity. Thus, the terms of the sequence \( \varphi\) are equal. Since their limit is 0, we have necessarily \( \varphi^1_1 = 0 \) and \( \varphi^2_1 = \varphi^1_2 \).

Consider the sequences \( y, z \) in \( a \) defined by \( y_n = \lim inf \pi_k \) and \( z_n = \lim sup \pi_k \) for all \( n \). For all positive real \( \varepsilon \), there is a positive integer \( n_0 \) such that \( n \geq n_0 \Rightarrow \lim inf \pi_k - \varepsilon \leq \pi_n \leq \lim sup \pi_k + \varepsilon \). Therefore \( y - \varepsilon.e \leq_C x \leq_C z + \varepsilon.e \) where \( \varepsilon = (1,1...) \), what entails \( \varphi_1(y - \varepsilon.e) \leq \varphi_1(x) \leq \varphi_1(z + \varepsilon.e) \). Since \( \varphi_1 \) is proportional to \( \delta_\infty \) on \( c \), there is a real \( \lambda \) such that

\[ \lambda(x - \varepsilon) = \lambda(\lim inf \pi_k - \varepsilon) \leq \varphi_1(x) \leq \lambda(\lim sup \pi_k + \varepsilon) \]

\(^8\)For the convenience of the reader, a statement and a proof of the theorem is given in the appendix with the objects and notations used in the present paper.
These inequalities being satisfied for all $\varepsilon$, we have $\lambda \liminf \pi_k \leq \varphi_1(x) \leq \lambda \limsup \pi_k$. We have $\lambda \geq 0$ because $\varphi_1$ inherits monotonicity from $C$ and $\lambda \neq 0$ because $\varphi_1$ is non-zero. Hence, on $a$, $\varphi_1$ is equal to $\alpha_\infty$ up to a positive multiplicative factor.\[\square\]

**Remark 18**: It is obvious that, on the set $a$, $\alpha_\infty(x) > \alpha_\infty(y) \Rightarrow x \succ_C y$. But it is less obvious that the restriction on $a$ of every continuous linear functional that separates $P_+(C)$ from 0, is equal to $\alpha_\infty$ up to a positive multiplicative factor.\[\lozenge\]

The second stage consist in studying the preorder $C$ in $H_1 = \ker \varphi_1$, denoted $C_2$. We have $P_+(C_2) = \{x \in H_1 : x \succ_{C_2} 0\}$. We prove by contradiction that $P_+(\sigma(C_2)) = \emptyset$. Thus, by theorem 4, the preorder $C$ has no second component.

**Proposition 9**: $P_+(\sigma(C_2)) = \emptyset$.

**Proof**: Suppose not. Let $\varphi_2$ be a non-zero continuous linear functional as given by theorem 3 and Remark 15 applied to $C_2$ in $H_1$. Denote $s$ the vector subspace of $H_1 : s = \{x \in H_1 : \lim_{n \to \infty} \sum_i x_i \text{ exists}\}$ and $\sigma$ the linear functional on $s$ defined by $\sigma(x) = \lim_{n \to \infty} \sum_i x_i$. Note that $\sigma$ is not continuous. For all $x, y$ in $s$, we have $\sigma(x) > \sigma(y) \Rightarrow x \succ_{C_2} y \Rightarrow \varphi_2(x) \geq \varphi_2(y)$. With the same reasoning than that of Remark 11 applied on $\sigma$ and $\varphi_2$ in the subspace $s$, we arrive to the result that $\sigma$ is equal to $\varphi_2$ on $s$, up to a positive multiplicative factor. But $\varphi_2$ is continuous and $\sigma$ is not. A contradiction.\[\square\]

8-2- Decomposition under a Stronger Norm

It is possible to propose a more precise decomposition of $C$ by using a stronger norm, hence a stronger topology. However, this will be at the cost of a narrower domain than $l_\infty$. For example let’s take the norm $\|x\| = \sup |x_n| + \lim \sum_i |x_i - \pi_i|$ defined on the vector space

\[c_1 = \left\{ x \in l_\infty : \sup |x_n| + \lim \sum_i |x_i - \pi_i| < +\infty \right\}\]

For $n \geq 2$, we have $x_n = n\pi_n - (n-1)\pi_{n-1}$. Thus

\[x_n - \pi_n = (n-1)(\pi_n - \pi_{n-1})\]

**Lemma 6**: The space $c_1$ is included in the space $e$ of converging sequences.

**Proof**: Denote $u_n = \pi_n - \pi_{n-1}$ for $n \geq 2$ and $u_1 = \pi_1 = x_1$. We have $\sum_i |x_i - \pi_i| = \sum_i (i-1) |u_i|$. Thus, $\sum_i |u_i| \leq \sum_i (i-1) |u_i| < +\infty$ and $\pi_n = \sum_i u_i$ converges. Moreover, $\sum_i |x_i - \pi_i| < +\infty$ entails $\lim(x_n - \pi_n) = 0$. As a result, $x_n$ converges.\[\square\]
In exactly the same way as for the space $l_\infty$ equipped with the supnorm, we prove that in $c_1$ equipped with the norm $\|x\| = \sup \{x_i\} + \lim \sum_{i=1}^{n} |x_i - \pi_i|$, we have $P^\circ (C) = \{ x \in c_1 : \lim x_n > 0 \}$. The linear functional $\delta_\infty (x) = \lim x_n$, or strictly speaking $\delta_\infty$, is clearly the first component of the preorder $C$ on the space $c_1$.

To determine the second component, $\varphi_2$, let’s consider the preorder $C_2$, restriction of $C$ on $H_1 = \ker \delta_\infty$, and calculate $P^\circ (C_2)$ in $H_1$.

**Proposition 10:** The second component of $C$ is the functional $\sigma$ defined on $H_1$ by $\sigma(x) = \sum_{i=1}^{+\infty} x_n$, up to a positive multiplicative factor.

**Proof:** In $H_1$ we have $\lim \pi_n = \lim x_n = 0$. Together with (3), the equality $\pi_n = \sum_{i=1}^{n} u_i$ yields:

$$\sum_{i=1}^{n} \pi_i + \sum_{i=1}^{n} (i - 1) u_i = n \sum_{i=1}^{n} u_i$$

On the other hand $\lim \sum_{i=1}^{n} u_i = \lim \pi_n = 0$. Thus $\sum_{i=1}^{n} u_i + \sum_{i=n+1}^{+\infty} u_i = 0$. Replacing $\sum_{i=1}^{n} u_i$ by $-n \sum_{i=n+1}^{+\infty} u_i$ in (4), we get $\sum_{i=1}^{n} \pi_i + \sum_{i=1}^{n} (i - 1) u_i = -n \sum_{i=n+1}^{+\infty} u_i$.

Furthermore $n \sum_{i=1}^{+\infty} u_i \leq n \sum_{i=1}^{+\infty} u_i \leq \sum_{i=1}^{+\infty} (i - 1) |u_i| \to 0$ when $n \to +\infty$. Since $\sum_{i=1}^{+\infty} (i - 1) |u_i| < +\infty$, equation (4) yields

$$\sum_{i=1}^{+\infty} \pi_n + \sum_{i=1}^{+\infty} (n - 1) u_n = 0$$

(5)

It follows that $\sum_{i=1}^{+\infty} \pi_n$ converges (i.e. it has a finite limit). Since $\sum_{i=1}^{+\infty} |x_n - \pi_n| < +\infty$, $\sum_{i=1}^{+\infty} (x_n - \pi_n)$ converges. Therefore the sum $\sigma(x) = \sum_{i=1}^{+\infty} x_n$ converges on $H_1$.

We prove now that $\sigma$ is continuous on $H_1$. Let $x, x'$ be in $H_1$ such that $x' \to x$. Using (5) and (3) we get

$$\left| \sum_{i=1}^{+\infty} \pi'_{n} - \pi_n \right| = \left| \sum_{i=1}^{+\infty} (n - 1) (u'_{n} - u_n) \right| \leq \sum_{n=1}^{+\infty} (n - 1) \left| (u'_{n} - u_n) \right|$$

$$= \sum_{i=1}^{+\infty} \left| (x'_n - \pi'_n) - (x_n - \pi_n) \right| = \sum_{i=1}^{+\infty} \left| (x'_n - x_n) - (\pi'_n - \pi_n) \right|$$

By definition, $\sum_{i=1}^{+\infty} \left| (x'_n - x_n) - (\pi'_n - \pi_n) \right| \to 0$ when $x' \to x$. 

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Thus $\left\| \sum_{k=1}^{\infty} (x'_n - x_n) \right\| - \left\| \sum_{k=1}^{\infty} (x''_n - x_n) \right\| \leq \left( \sum_{k=1}^{\infty} |x'_n - x_n| \right) - \left( \sum_{k=1}^{\infty} |x''_n - x_n| \right) \to 0$

As a result, $\sum_{k=1}^{\infty} x'_n \to \sum_{k=1}^{\infty} x_n$ and $\sigma$ is continuous on $H_1$. The set $\sigma^{-1}([0, +\infty[)$ is clearly not empty, open and included in $P_+ (C_2)$. Hence $P_+ (C_2) \neq \emptyset$.

To prove the unicity of $\sigma$, notice that $\sigma (x) > \sigma (y) \Rightarrow x \succ y$. Let $\varphi_2$ be a non-zero continuous linear functional as given by theorem 3 and Remark 15. We thus have $\sigma (x) > \sigma (y) \Rightarrow x \succ y \Rightarrow \varphi_2 (x) \geq \varphi_2 (y)$.

Hence, the implication (2) holds and we can apply Remark 11. $\varphi_2$ is equal to $\sigma$ on $H_1$ up to a positive multiplicative factor $\Box$

The third stage consists in calculating $P_+ (C_3)$ in $H_2 = \ker \sigma$, where $C_3$ is the restriction of $C$ to $H_2$. Note that $P_+ (C_3)$ is not empty. For example the sequence $x_n = - \frac{1}{1 + \frac{1}{n-1}}$ for $n \geq 2$ and $x_1 = 1$ is in $P_+ (C_3)$.

**Proposition 11:** $P_+ (C_3) = \emptyset$.

**Proof:** Let $x$ be in $P_+ (C_3)$. We prove now that in every neighborhood of $x$ in $H_2$ one can find $y$ such that $\not{x} \succ C 0$. To that end, we build $y$ by adding a small term such that the sum becomes episodically negative while remaining in the neighborhood of $x$. Since $x$ is in $P_+ (C_3)$, for all $n$ there is an even integer $k > n$ such that $\sum_{i=1}^{k} x_i > 0$. Denote $k(n) = k$. Let $\delta$ be a positive real and $m$ and $p$ be two positive integers such that $p > m$. There is an integer $p' > m$ such that $\left\| \sum_{i=1}^{k(p')} x_i \right\| < \frac{1}{\delta}$. Denote $k_m(p) = p'$. It is always possible to choose $k$ and $p'$ such that functions $k$ and $k_m$ are increasing. Define $y^n_m$ as follows: if there is $p$ such that $n = k(k_m(p))$ then $y^n_m = x_n - (1 + \delta) \sum_{i=1}^{n} x_i$; if there is $p$ such that $n = 1 + k(k_m(p))$ then $y^n_m = x_n + (1 + \delta) \sum_{i=1}^{n-1} x_i$; else $y^n_m = x_n$. For $n = k(k_m(p))$ we have $\sum_{i=1}^{n} y^n_m = \sum_{i=1}^{n} x_i - (1 + \delta) \sum_{i=1}^{n} x_i = -\delta \sum_{i=1}^{n} x_i < 0$.

Therefore, $\sum_{i=1}^{n} y^n_m$ is episodically negative, which yields $\not{x} \succ C 0$. We check easily that $y^n_m$ is bounded and that $\sum_{i=1}^{n} y^n_m = 0$. Moreover, if there is $p$ such that $n = k(k_m(p))$ then $y^n_m = x_n - (1 - \frac{1}{n})(1 + \delta) \sum_{i=1}^{n} x_i$; if there is $p$
such that \( n = 1 + k(k_m(p)) \) then \( y^m_n - \overline{y}^m_n = x_n - \overline{x}_n - (1 + \delta) \left( \sum_{i=1}^{n-1} x_i \right) \); else \( y^m_n - \overline{y}^m_n = x_n - \overline{x}_n \). Therefore, for \( N \) sufficiently large:

\[
\sum_{1}^{N} |y^m_n - \overline{y}^m_n| \leq \sum_{1}^{N} |x_n - \overline{x}_n| + 2(1 + \delta) \sum_{k(k_m(p)) \leq N} \sum_{1}^{k(k_m(p))} x_i \leq \sum_{1}^{N} |x_n - \overline{x}_n| + 2(1 + \delta) \sum_{k(k_m(p)) \leq N} \frac{1}{2p}
\]

Hence, \( \sum_{1}^{+\infty} |y_n - \overline{y}_n| \) converges, what shows that \( y \) is in \( H_2 \). Using again the expression of \( y_n - \overline{y}_n \), we get:

\[
\sum_{1}^{N} |(y^m_n - x_n) - (\overline{y}^m_n - \overline{x}_n)| \leq 2(1 + \delta) \sum_{k(k_m(p)) \leq N} \sum_{1}^{k(k_m(p))} x_i \leq 2(1 + \delta) \sum_{m < p} \frac{1}{2p} \text{ Thus}
\]

\[
\sum_{1}^{+\infty} |(y^m_n - x_n) - (\overline{y}^m_n - \overline{x}_n)| \leq 2(1 + \delta) \sum_{m < p} \frac{1}{2p} \to 0 \text{ when } m \to +\infty
\]

and

\[
\sup_{n} |y^m_n - x_n| = \sup_{n \geq m} |y^m_n - x_n| \leq \sup_{n \geq m} |y^m_n| + \sup_{n \geq m} |x_n| \to 0 \text{ when } m \to +\infty
\]

\( \Box \)

Hence, by theorem 4, \( C \) has not a third component in \( c_1 \) equipped with the norm \( \|x\| = \sup \{x_n\} + \lim \sum_{1}^{n} |x_i - \overline{x}_i| \).

**Remark 19:** Since either under the supnorm or \( c_1 \)-norm, the decomposition of the catching-up preorder \( C \) is strictly coarser than \( C \), so is the decomposition of every complete preorder extending \( C \). Such a complete preorder provides an example of a complete, translation-invariant and scalar-invariant preorder which is not reducible to a \( LCLC \).
Appendix: Decomposition of an element $y \in l^*_\infty$

The following is a statement and proof of the Yosida-Hewitt theorem without resorting to concepts related to measures.

Recall that $\delta_\infty$ is the linear functional defined on $c$, the space of real converging sequences, by $\delta_\infty(x) = \lim x_n$, and $l^*_\infty$ is the dual of $l_\infty$, the space of real bounded sequences.

**Theorem:** Let $y \in l^*_\infty$. Then we can write in a unique manner:

$$y = y_1 + y_2$$

where $y_1$ verifies:

$$\sum_{i=1}^{+\infty} |y_{1i}| < +\infty$$

and $y_2$ is such as its restriction to $c$ is proportional to $\delta_\infty$.

**Proof:**

**Step 1:** Projection from $l^*_\infty$ on $l_1$. For $i \geq 1$, let $e_i$ be the element of $l_\infty$ such that all its components are zero except the $i^{th}$ which is 1. Let $y \in l^*_\infty$. Let $x \in c_0$. We have $\sum_1^n x_i e_i \rightarrow x$, so $y | (\sum_1^n x_i e_i) \rightarrow y | x$, then $\sum_1^{+\infty} x_i (y | e_i) = y | x$. One the other hand, $y$ continuous $\Rightarrow \|y| e_i\| \leq \|f\|$. Since $\|e_i\| = 1$, we get $|(y| e_i)| \leq \|y\|$ for all $i \geq 1$. Let $\alpha \in [0, 1]$. Take $x_n = \text{sign}(y | e_n) \cdot \frac{1}{n}$. We have $x = (x_n)_{n \geq 1} \in c_0$ and

$$\sum_{i=1}^{+\infty} \frac{|y| e_n}{n^{\alpha}} = \|f(x)\| \leq \|x\| \cdot \|f\| = \|f\|$$

Let $\varphi(\alpha) = \sum_{i=1}^{+\infty} \frac{|y| e_n}{n^{\alpha}}$. Then $\varphi$ is bounded and decreasing on $[0, 1]$. Hence, it has a finite limit as $\alpha \rightarrow 0$. We can show easily that this limit is $\sum_{i=1}^{+\infty} |(y| e_n)|$. Thus the sequence $(y| e_n)_{n \geq 1}$ is in $l_1$. Denote $\Phi$ the mapping from $l^*_\infty$ to $l_1$ which associates to $y$ the sequence $(y| e_i)_{i \geq 1}$. $\Phi$ is a projection from $l^*_\infty$ to $l_1$. Indeed, it is a linear transformation and, considering $l_1$ as a subset of $l^*_\infty$, if $y \in l_1$ then $\Phi(y) = y$.

**Step 2:** Decomposition of an element $y \in l^*_\infty$ by $\Phi$. Consider the mapping Identity $I$ from $c_0$ to $l_\infty$

$$I : c_0 \longrightarrow l_\infty$$

with $\Phi = I^*$. Given that the adjoint of a continuous linear operator is continuous and that $I$ is linear and continuous, $\Phi$ is a continuous linear operator. Furthermore, we have (Luenberger p155) $R(I)^\perp = N(I^*)$ where $R(I) = \{y \in l_\infty \ | \ \exists x \in c_0 : I(x) = y\} = c_0$ and $N(I^*) = \{x \in c_0 \ | \ I^*(x) = 0\}$ which means $N(\Phi) = c_0^\perp$. For $y \in l^*_\infty$, define $k = \Phi(y) - y$. We can write $y = \Phi(y) + k$, with $\Phi(y) \in l_1$ and $k \in c_0^\perp$. We have decomposed an element $y$ of $l^*_\infty$ as a sum of an element of $l_1$ and an element of $c_0^\perp$. We easily show that this decomposition is unique.
Step 3: Study of $c_0^\perp$. We have:

$$\|\delta_\infty\| = \sup_{x \in c} \frac{\lim x_n}{\|x\|} = \sup_{x \in c} \frac{\lim x_n}{\sup |x_n|} = 1$$

and

$$\forall \alpha \in \mathbb{R} : \|\alpha \delta_\infty\| = |\alpha| \|\delta_\infty\| = |\alpha|$$

so we can apply Hahn-Banach theorem, and extend $\alpha \delta_\infty$ with an element of $l_\infty^*$, say $\beta$. Denote $B$ the set of such linear functionals. We now show that $c_0^\perp = B$.

We see easily that $B$ is a vector subspace of $l_\infty^*$ included in $c_0^\perp$. Reciprocally, let $\beta \in c_0^\perp$ and $x \in c$. Denote $e = (1, 1, \ldots)$. We have $x - (\delta_\infty \mid x)e \in c_0$, so $\langle \beta \mid (x - (\delta_\infty \mid x)e) \rangle = 0$. Thus $\beta \mid x = \langle \beta \mid e \rangle (\delta_\infty \mid x)$. This proves that the restriction of $\beta$ to $c$ is proportional to $\delta_\infty$. Then $\beta \in B$ and $c_0^\perp \subset B$.

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