Modeling Returns of Stock Indexes through Fractional Brownian Motion Combined with Jump Processes and Modulated by Markov Chains

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(Modelado de rendimientos de índices bursátiles mediante movimiento fraccional browniano combinado con procesos de saltos y modulado por cadenas de Markov)

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Abstract
We develop a mathematical model useful to describe the stochastic dynamics and return distribution of the stock indexes of world's main economies (USA, Eurozone, UK and Japan) and of the main emerging markets (China, Brazil, and México) incorporating risk factors as: idiosyncratic volatility, market volatility, and regime-switching volatility. It is assumed that the returns of the stock indexes are driven by fractional Brownian motions combined with Poisson processes and modulated by Markov chains. To do that, we calibrate Jump-GARCH models and estimate Markov regime-switching stochastic volatility models. The proposed models properly describe the stochastic dynamics of the returns of the stock indexes under study during 1994-2017. The main empirical finding is that the USA stock market stays in high volatility most of the time and presents more jumps than other indexes, and that Brazil stock market has the biggest intensity of jumps during 1994-2017. The outcome supports the hypothesis of long-term memory of stock markets.

JEL Classification: N20, B23, C02.
Keywords: Stock index return, fractional Brownian motion, Markov regime-switching, jump processes.

Resumen
Esta investigación desarrolla un modelo matemático útil para describir la dinámica estocástica y la distribución de los rendimientos de los índices bursátiles de las principales economías del mundo (EE.UU. Zona Euro, Reino Unido y Japón) y de los mayores mercados emergentes (China, Brasil y México) mediante la incorporación de factores de riesgo como: volatilidad idiosincrática, volatilidad del mercado y volatilidad de cambio de régimen. Se supone que los rendimientos de los índices bursátiles son conducidos por movimientos fraccionales brownianos combinados con procesos de Poisson y modulados por cadenas de Markov. Para lograr este objetivo se calibran modelos Jump-GARCH y se estiman modelos de volatilidad estocástica de cambio de régimen markoviano. Los modelos propuestos describen adecuadamente la dinámica estocástica de los rendimientos de los índices bursátiles bajo estudio durante 1994-2017. Los principales hallazgos empíricos reflejan que el mercado de capitales de EE.UU. se mantiene en alta volatilidad la mayor parte del tiempo y presenta más saltos que los otros índices y el mercado accionario de Brasil presenta grandes saltos con mayor intensidad. El resultado sostiene la hipótesis de memoria de largo pazo en el mercado de capitales.

Clasificación JEL: N20, B23, C02.
Palabras clave: rendimiento de índices de acciones, movimiento fraccional browniano, cambio de régimen markoviano, procesos de salto.

1. Introduction

Over a long period of time, numerous studies on the stochastic dynamics and return distribution of stock market have emerged; however, there are still irregularities and stylized facts that need to be elucidated and explained. At present, the evolution of the stock market indexes follow complex dynamics coming from intricate global investment strategies, being necessary the generation of models with more sophisticated tools. Most of the available models in the specialized literature can be broadly classified in two large groups: models seeking to explain the fundamental value of stocks and models describing stock prices (Krause, 2001). In the latter, recent investigations have been focused on volatility of aggregate stock markets through cross-section analysis (Ang, 2004). Some other studies have included stochastic calculus to model stock returns and time-varying volatility. For example, Christoffersen *et al*. (2009) build up a two-factor stochastic volatility model useful to generate time-varying correlation. In the same line, Johnson (2002) develops a stochastic volatility model with time-varying correlation between returns and volatility. Also, An *et al*. (2014) use cross-section analysis of option volatilities to forecast stock returns. Finally, López-Herrera *et al*. (2009) study the long-term dependence on returns and volatilities.

Some other studies focus on return distributions with time-varying moments. In this regard, Carr and Wu (2007) propose a stochastic skew model for foreign exchange rates; Pham and Touzi (1996) explore the stochastic volatility on equilibrium state prices; Durham and Park (2012) stress on stochastic volatility in stock returns and found that return distributions have time-varying skewness and kurtosis; finally, Harvey and Siddique (1999) examine time-varying skewness through a GARCH model, and suggest that the relation between skewness and variance of stock returns is linked to the seasonal variations in the conditional moments.

Another factor that has been relevant in examining returns dynamics is the volatility of volatility.[1] Some studies have demonstrated that the variance risk premium depends on volatility of volatility. For instance, Das and Sundaram (1999) examine the volatility of volatility and the correlation between the innovations in asset pricing. Also, Durham and Park (2012) develop a mixed jump-diffusion process on options with volatility of volatility (*cf*. Ang *et al*., 2006).

An important characteristic of stock markets is the presence of unexpected and sudden jumps. In this regard, Martijn *et al*. (2015) suggest, by using cross-section analysis, that stock returns have high sensitivity to jumps and volatility risks. Moreover, Du and Kapadia (2011) argue that the index VIX has a critical degree of bias related to jumps. Also, Branger *et al*. (2007) propose an equilibrium model with jumps and stochastic volatility to describe the dynamic behaviour of stock returns. In the same line, Heston (1993) deals with Poisson jumps through a stochastic volatility model with returns affected by three factors: diffusive volatility shocks, diffusive price shocks, and price jumps. Finally, Bates (2008) examines investors' behaviour about how they treat extreme events and common events related to jumps of stock returns. These authors treat jumps and diffusion risks separately (*cf*. Johnson, 2002).

[1] Volatility of volatility is a measure of volatility expect of the n-day forward price of the volatility and this drive nearby volatility options price.

There are also other studies addressing options and futures markets as Garland *et al*. (2012). These authors develop a stochastic volatility model to assess several characteristics that are consistent with variation in the shape of return distributions by including regime-switching to feature random changes in the volatility of volatility, leverage effect, and jump intensity. Santa-Clara and Yan (2010) present a model of option prices when the volatility of the diffusion shocks and the intensity of the jumps change over time and show that diffusive volatility and jump intensity capture the ex-ante risk assessed by investors of the S&P500 index options. Moreover, Vallejo and Venegas-Martínez (2017) model the dynamics of asset prices with time-inhomogeneous Markov chains and applying fractional Brownian motion with multiple Poisson jumps (*cf*. Venegas-Martínez, 2001 and 2008).

The above investigations have highlighted the importance of including the effect of the volatility, volatility of volatility, and jumps and regime-switching on stock returns. This paper contributes to the current literature by extending previous investigations in a model that includes risk factors as idiosyncratic volatility, market volatility, and volatility of volatility. To do that, we use fractional Brownian motion combined with jump processes and modulate volatility with Markov chains. In order to calibrate the proposed model, we estimate a Jump-Garch model, a Markov regime-switching model, and the corresponding Hurst coefficient.

This paper is organized as follows: section 2 presents the proposed stochastic model of stock index returns; section 3 describes the data and defines the endogenous and exogenous variables; section 4 calibrates the proposed model; finally, section 4 provides the conclusions.

2. Modelling Stock Index Returns

This section presents the theoretical background of fractional Brownian motion combined with Poisson process modulated with Markov switching-regime stochastic volatility. Most of the empirical studies suggest that market volatility varies over time and stocks with high sensitivities to both jump and volatility risks have low expected returns (Cremers *et al*. 2015). Durham and Park, (2012) propose a Markov regime-switching model of both volatility of volatility and jump intensity to determine the skewness and kurtosis of stock returns. Also, Vallejo-Jiménez and Venegas-Martínez (2017) develop a model that explains the dynamics of asset prices that are driven by multiple jumps, fractional Brownian motion, and Markov regime switching.[2]

In our multifactor risk model returns are driven by a fractional Brownian motion combined with Poisson jumps and modulated by Markov regime switching aligned (as a particular case) with Durham and Park (2012):

$$\mathrm{d}y_t = \left(\mu - \tfrac{1}{2} v_t^2\right) \mathrm{d}t + v_t \mathrm{d}B_{1t}^H \tag{1}$$

$$\mathrm{d}v_t = a(b - v_t)\mathrm{d}t + \sigma_i \mathrm{d}B_{2t}^H + \gamma \mathrm{d}N_t \tag{2}$$

[2] See also Christoffersen *et al*. (2009) and Ang *et al*. (2006).

$$E = \{\sigma_1, \sigma_2\} \tag{3}$$

where $\mathrm{d}y_t$ is a dependent variable determining the dynamics of the stock index return, $\mathrm{d}B_{1t}^H$ and $\mathrm{d}Bt_{2t}^H$ are independent fractional Brownian motions, H is the Hurst parameter, μ is the annual mean of returns, $\mathrm{d}N_t$ is a Poisson jumps, v_t is the idiosyncratic volatility, E is the regime state (low volatility an high volatility), σ_i is the volatility state, dv_t is the volatility of volatility, a is the speed adjustment parameter, b in the long run mean, and γ_i, i =1, 2, are mean jump sizes.[3]

A Markov regime-switching process (Hamilton, 2005) is a nonlinear time series model that integrates multiple structures to shows the behaviour of a state variable in different regimes. The probabilities of switching from state σ_i to state σ_j are given by the following transition matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

The fractional Brownian motion B_t^H is defined on a fixed probability space with its augmented filtration $(\Omega, F, (F_t)_{t\in[0,T]}, P)$ and $H \in (0,1)$ is the Hurst coefficient (Taqqu, 2013). The Hurst coefficient H is used as a measure of long-term memory of time series. It describes the irregularity of the motion, predict the stock return and reflect the autocorrelation on returns. It is worth mentioning that if $H \neq \frac{1}{2}$, then B_t^H is not a semimartingale (Mandelbrot and Van Ness, 1968). In this case:[4]

H = ½, the process is Brownian motion or Wiener process

H > ½, the increments are positively correlate (long memory)

H < ½, the increments are negatively correlated (mean reverting).

Regarding our proposal, it is important to point out that Cajueiro and Tabak (2005) find that the Hurst coefficient on Brazilian stock market is time-varying; Jamdee and Los (2005) show that European options have long memory and are dependent on volatility; and Bender (2000) suggest that the law of one price holds in a market where the stock is driven by fractional Brownian motion.

We review briefly the ARCH model, which is useful to explain the trend of large residuals to cluster together (Engle, 1982). The ARCH model is given by:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \, \varepsilon_{t-i}^2 \tag{3}$$

where σ_t^2 is the conditional variance, ω and α are unknown parameters, and ε_{t-i}^2 is the lag of the random error term. In the GARCH model the variance term depends of the lagged variance as well as the lagged square residuals. It allows evaluate different type of persistence in volatility (Bollerslev, 1986). The GARCH model is represented by:

[3] Other papers dealing with jump-diffusion processes are Venegas-Martínez (2000) and (2001) and Venegas-Martínez and González-Aréchiga (2000).

[4] Duncan and Pasik-Duncan (1991) introduce an integration theory named Wick-Itô-Skorohod integral for fractional Brownian motion.

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \, \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i \, \sigma_{t-j}^2 \qquad \omega, \alpha, \beta > 0 \tag{4}$$

where σ_t^2 is the conditional variance, ω and α and β are unknown parameters, ε_{t-i}^2 is the lag of the random deviation term, σ_{t-j}^2 is the lag of the variance, α is the component of the influence of random deviation in the previous period, β is the component of the variance in the previous period, and $\alpha + \beta$ is the level of persistence.

On the other hand, the Jump-GARCH model is an alternative for modelling the dynamics of stock indexes when sudden and unexpected jumps occur (Chen, Lin and Lin, 2013). In this case, two stochastic innovations, $\varepsilon_{t,1}$ and $\varepsilon_{t,2}$, capture the dynamic of the return with no jump and jump, respectively. The innovations $\varepsilon_{t,1}$ and $\varepsilon_{t,2}$ are independent and satisfies:

$$\varepsilon_t = . \, \varepsilon_{t,1} + . \, \varepsilon_{t,2} \tag{5}$$

The first innovation means that market stay normal (no jump), thus

$$\varepsilon_{t,1} = \sigma_t u_t, \qquad u_t \sim N(0,1) \tag{6}$$

and

$$\mathrm{E}\big(\varepsilon_{t,1} | \, y_{t-1} \big) = 0. \tag{7}$$

The second innovation describes the unexpected jump, hence:

$$\varepsilon_{t,2} = N_t - \mathrm{E}(N_t | y_{t-1}) = \sum_{k=1}^{n_t} \gamma_{t,k} - \theta \lambda_t, \qquad n_t | y_{t-1} \sim P(\lambda_t) \tag{8}$$

and

$$\varepsilon_{t,2} = \mathrm{E}\big(\varepsilon_{t,2} | \, y_{t-1} \big) = 0 \tag{9}$$

where y_t stands for the dynamics of the return, N_t is the jump component, $\gamma_{t,k}$ is the jump size, λ_t is the jump intensity, θ is the component of jump intensity, and n_t denotes the number of jumps. The Poisson process N_t with intensity parameter λ satisfies:

$$\mathrm{P}\{\text{One jump on d}t\} = \mathrm{P}\{\mathrm{d}N_t = 1\} = \lambda dt + o(\mathrm{d}t) \tag{10}$$

$$\mathrm{P}\{\text{None jump on } dt\} = \mathrm{P}\{\mathrm{d}N_t = 0\} = 1 - \lambda \mathrm{d}t + o(\mathrm{d}t). \tag{11}$$

Hence,

$$\mathrm{P}\{\text{More that one jump on d}t\} = \mathrm{P}\{\mathrm{d}N_t \geq 1\} = o(\mathrm{d}t). \tag{12}$$

Then,

$$\mathrm{E}[\mathrm{d}N_t] = \mathrm{Var}[\mathrm{d}N_t] = \lambda \mathrm{d}t, \tag{13}$$

$$\mathrm{Cov}(\mathrm{d}W_t, \mathrm{d}N_t) = 0. \tag{14}$$

3. Data Description

Our proposal captures and describes the dynamics of the stock index returns under study. The data for USA (S&P 500), Eurozone (EuroStoxx50), United of Kingdom (FTSE100), Japan (Nikkei), China (Hang Seng), México (IPC) and Brazil (Bovespa) were obtained from Bloomberg and includes daily returns of each stock index. The USA is considered as a benchmark since it is the world largest economy and it has the biggest financial market.[5]

The sample period of our analysis begins on January 1994 and ends on December 2017. The purpose of this study is to capture in our proposal the dynamics of stock market indexes in and out crises periods. The most relevant events are the bubble dot com in 2001, the subprime mortgage recession in 2008, the Eurozone debt crisis in 2011, the Brexit in June 2016, and the power takeover of president Trump in December 2016. The idiosyncratic volatility is represented by the deviation standard. The market volatility is calculated through the VIX index, which is a measure of 30 days expected volatility of the U.S. stock market, calculated from real-time mid quote prices of S&P500 call and put options index (CBOE). Finally, the volatility of volatility is the square return of measure by the VIX index. The parameters for Markov regime switching, Jump-GARCH, and the Hurst coefficient are calculated by different econometric techniques.

4. Empirical Analysis

The estimation of the Markov regime-switching models describing the grade of volatility of the previous period of the returns of S&P500, Eurostoxx50, FTSE100, Nikkei, Hang Seng, IPC and Bovespa indexes are shown in Table 1.

Table 1. Transition probabilities of the Markov Regime-Switching Model

Probabilities	S&P500	EuroStoxx50	FTSE100	IPC	Bovespa	Nikkei	Hang Seng
p_{11}	0.41424	0.52744	0.54349	0.58354	0.56479	0.46037	0.48052
p_{12}	0.58576	0.47255	0.45650	0.41646	0.43521	0.53963	0.51949
p_{21}	0.29279	0.40675	0.50486	0.38694	0.44869	0.54721	0.37474
p_{22}	0.70721	0.59324	0.49513	0.61306	0.55131	0.45279	0.62526

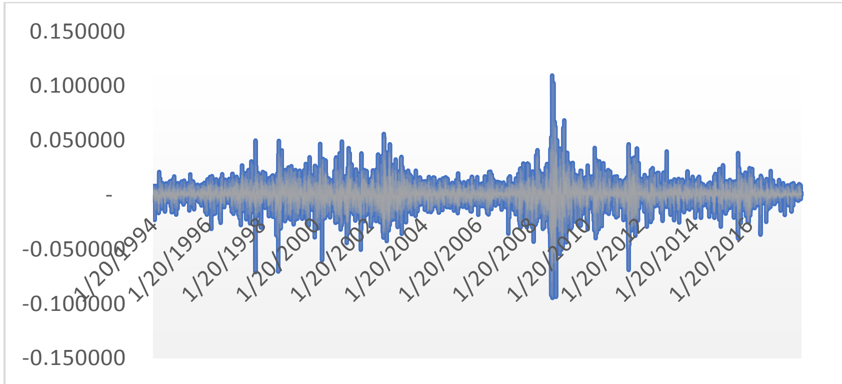
Source: own elaboration with Bloomberg data and E-views software

Table 1 shows that the S&P500 has 70% of probability to stay in high volatility from one period to another, followed by the Hang Seng with 62%. While, IPC, Bovespa and FTSE100 has more probability to stay in low volatility more time than S&P500, EuroStoxx50, Nikkei and Hang Seng. For S&P500 is easy to transit to high volatility with 58%, and just 29% of the time changed from high to low volatility.

The following Figures show the returns of S&P 500, Eurostoxx50, FTSE100, Nikkei, Hang Seng, IPC and Bovespa indexes from 1994 to 2017, these reflect higher jumps on the most relevant economic event as bubbles, crises and politiques decisions around the world; see Figures 1 to 7.

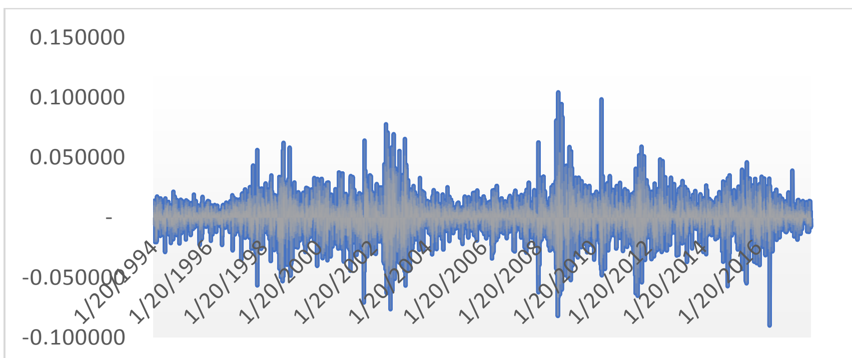
[5] Investors of financial markets take often decisions on basis of Eurozone and UK economic data.

Figure 1. Returns of S&P 500 (1994-2017)



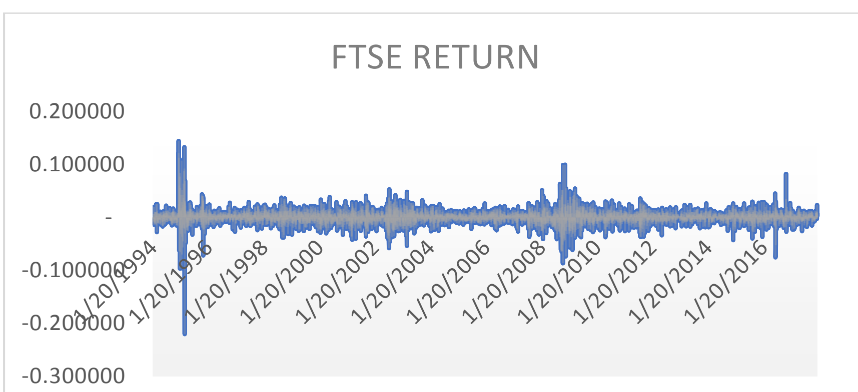
Source: own elaboration with Bloomberg data

Figure 2. Returns of EuroStoxx50 (1994-2017)



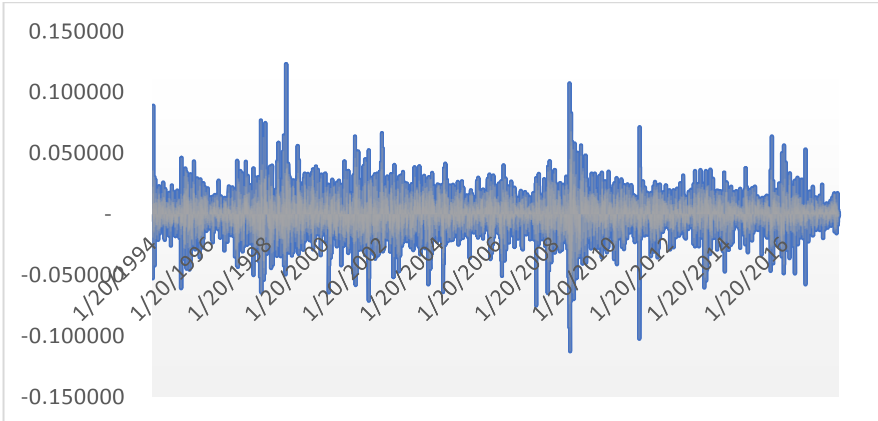
Source: own elaboration with Bloomberg data

Figure 3. Returns of FTSE100 (1994-2017)



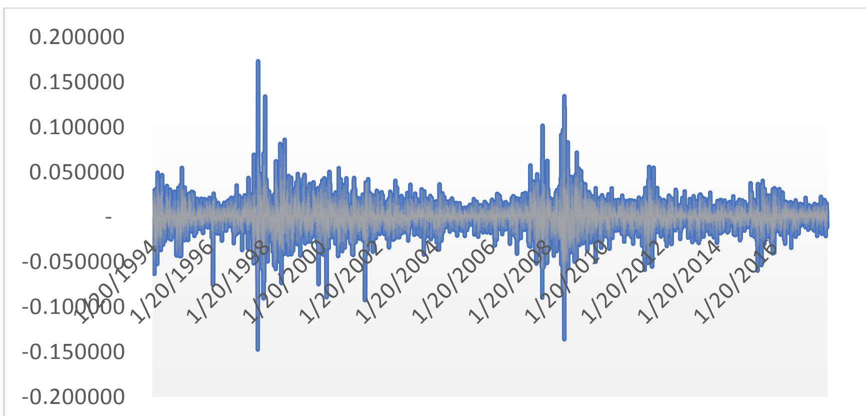
Source: own elaboration with Bloomberg data

Figure 4. Returns of Nikkei (1994-2017)



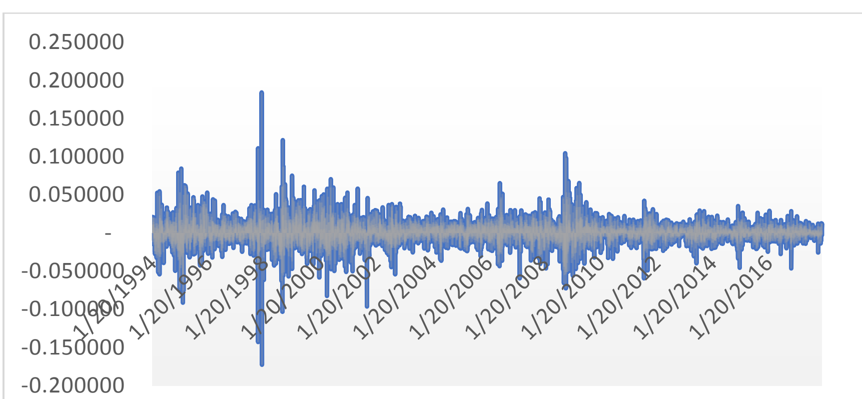
Source: own elaboration with Bloomberg data

Figure 5. Returns of Hang Seng (1994-2017)



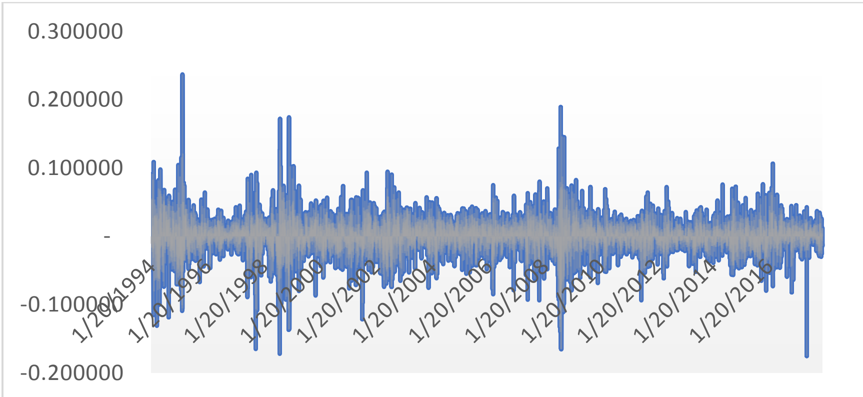
Source: own elaboration with Bloomberg data

Figure 6. Returns of IPC (1994-2017)



Source: own elaboration with Bloomberg data

Figure 7. Returns of Bovespa (1994-2017)



Source: own elaboration with Bloomberg data

Table 2 shows the summary economic events that have had impact on the stock index returns.

Table 2. Summary of Economic and Geopolitical Shocks (1994-2017)

Event	S&P500	EuroStoxx50	FTSE100	Nikkei	Hang Seng	IPC	Bovespa
Dot Com Bubble	End of 2000 and 2001	2001 and beginning of 2002	Slightly	From 2000 to 2002	First semester of 2000	End of 2000 and 2001	Year 2000
Subprime Mortgage Recession	End of 2008 and beginning of 2009	End of 2008 and beginning of 2009	Second half of 2008	Year 2009	From 2008 to 2009	End of 2008 and beginning of 2009	Year 2009
Eurozone Debt Crises	Slightly	4T2011	Slightly	Beginning of 2011	Beginning of 2012	Slightly	Slightly
Brexit	Slightly	June 2016	June 2016	June of 2016	Slightly	Slightly	June 2016
Power takeover President Trump	Slightly	Slightly	Slightly	End of 2016	Slightly	End of 2016	End of 2016

Source: own elaboration.

In order to estimate the parameters of the Jump-GARCH model, we compute the log likelihood, evaluate the log likelihood of a GARCH model on the residuals, examine the jumps, and transform the accumulated first and second moments. Table 3 shows the estimates of the parameters of the Jump-GARCH model by using maximum likelihood for the returns of the stock indexes S&P500, EuroStoxx50, FTSE100, Nikkei, Hang Seng, IPC and Bovespa. The calibration of parameters is as follows: α is the random deviation in the previous period is close to 0.10; β is the lag of the variance is close to 0.90; $\alpha + \beta$ means the persistence, it is more than 0.95, thus there is evidence of a GARCH effect; θ indicates that jumps are related with negative movements on the price for developed economies; and γ the size of a jump and Bovespa index has the highest jump intensity. The percentage of jumps ($\%J$) is between and 47% and 52%, for S&P500 is 52%, and for Bovespa is 47%.

Finally, The mean (μ) of the return is the highest for IPC and the lowest for Nikkei during the period of study.

Table 3. Estimation of fractional Brownian motion combined with Jump-GARCH

PARAMETERS	S&P500	EuroStoxx50	FTSE100	IPC	Bovespa	Nikkei	Hang Seng
μ	0.0002940	0.0001540	0.0004290	0.0004958	0.0002748	0.0000286	0.0001576
ν^2	0.0001365	0.0002031	0.0001737	0.0002323	0.0006977	0.0002410	0.0002626
ν	0.011684	0.014252	0.013178	0.015241	0.026414	0.015523	0.016204
$\mathcal{H}$	0.556061	0.533003	0.542380	0.516423	0.560925	0.539031	0.535857
%J	52.59522	49.21371	50.42269	49.55913	47.35933	50.06818	50.05000
λ	0.000638	0.000569	0.000498	0.000664	0.000982	0.000365	0.000533
θ	-0.037814	-0.013797	-0.036261	0.087828	0.065490	-0.109516	0.042823
γ	0.000001	0.000002	0.000003	0.000001	0.000016	0.000005	0.000002
α	0.094715	0.081803	0.117276	0.081836	0.117298	0.101665	0.071503
β	0.894829	0.911664	0.864243	0.917049	0.860306	0.880997	0.921362

Statistically significant at 95% confidence level
Source: own elaboration with Bloomberg data

5. Conclusions

We have studied Jump-GARCH model and modulated Markov chains to describe the stochastic dynamics and return distribution of the stock indexes S&P500, EuroStoxx50, FTSE100, Nikkei, Hang Seng, IPC and Bovespa. The outcome supports the hypothesis of long-term memory of stock indexes, the irregularity of the motion, and the evidence of high volatility on stock market from 1994 to 2017.

After calibrating our model, it can be seen that the stock indexes that have major probability to stay in high volatility are S&P500 with 70% and Hang Seng with 62%; while, IPC, Bovespa and FTSE100 have high probability to stay in low volatility, 58%, 56% and 54%, respectively. The percentage of changes from high to low volatility from one period to another is just 29% for S&P500. Nikkei has the bigger change to move from high to low volatility but it does not stay for much time in low volatility. We found that S&P500 and Hang Seng are more volatile than other indexes. Moreover, from the GARCH estimation is observed that Bovespa and FTSE100 have the highest lag random deviation (0.1172); Hang Seng has the highest lag variance, 0.9213; Bovespa has the biggest size and intensity of jumps; however, S&P has more percentage of jumps in that period. Finally, the behaviour of the return of all indexes follows a fractional Brownian motion since the Hurst coefficient ($\mathcal{H}$) is greater than 0.5. It means that the increments are positively correlated, and the series have long-term memory.

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