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2008

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ABSTRACT

The Shewhart, Bonferroni-adjustment and analysis of means (ANOM) control chart are typically applied to monitor the mean of a quality characteristic. The Shewhart and Bonferroni procedure are utilized to recognize special causes in production process, where the control limits are constructed by assuming normal distribution for known parameters (mean and standard deviation), and approximately normal distribution regarding to unknown parameters. The ANOM method is an alternative to the analysis of variance method. It can be used to establish the mean control charts by applying equicorrelated multivariate non-central t distribution. In this paper, we establish new control charts, in phases I and II monitoring, based on normal and t distributions having as a cause a known (or unknown) parameter (standard deviation). Our proposed methods are at least as effective as the classical Shewhart methods and have some advantages.

Mathematics Subject Classification: 62P30, 62E15

KEY WORDS: Shewhart, Bonferroni-adjustment, Analysis of means, Average run length, False alarm probability

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1. Introduction

The Shewhart, the Bonferroni-adjustment and the analysis of means control charts are common techniques for monitoring the process mean. Shewhart (1931) proposed a scheme for detecting out-of-control signals and shifts in the mean from its target value μ_0. Ott (1975), Rocke (1989), Ryan (1989), Chen (1997), Quesenberry (1997), Smith (1998), Maravelakis, et al., (2002), Woodall et al. (2004), Montgomery (2005), and several other authors modified and extended the Shewhart control charts. The Shewhart procedure usually is based on at least 20 to 25 sample group sizes (k) and at least 4 to 6 sample subgroup sizes (n). This procedure with known mean and standard deviation parameters is based on a random variable that follows the normal distribution. When the mean and standard deviation are unknown the procedure is based on a statistic that follows approximately the normal distribution. The values of the subgroup averages ($\bar{X}_{i.} = \sum_{j=1}^{n} X_{ij} / n$) are plotted on the Shewhart control chart that includes the center line $E(\bar{X}_{i.})$ and the control limits $E(\bar{X}_{i.}) \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\bar{X}_{i.})}$, where the quality characteristics X_{ij} for $i = 1,2,...,k$ and $j = 1, 2,...,n$ (j^{th} observation in i^{th} subgroup) are assumed to be independent identically normally distributed with mean μ and variance σ^2.

Ryan (1989) introduced the Bonferroni-adjustment control limits as an alternative to the Shewhart approach. The control limits are given by $E(\bar{X}_{i.}) \pm Z_{\alpha/2k}\sqrt{\operatorname{var}(\bar{X}_{i.})}$. In other words, to construct the Bonferroni control limits the value α of the Shewhart control limits is replaced by the value α / k.

Ott (1967) introduced the ANOM control chart for comparing a group of means in order to see if any one of them differs significantly from the overall mean. Schilling (1973) extended this scheme to what he called the ANOM for treatment effects or ANOME.

Ott's procedure is carried out by comparing the sample mean values to the overall grand mean, about which decision lines have been constructed. If a sample mean lies outside these decision lines, it is declared to be significantly different from the grand mean. The main difference between the Bonferroni and ANOM charts is that in the first the sample group and subgroup sizes (k,n) are usually as large as 20 or more ($k \geq 20$), and 4 or more ($n \geq 4$), respectively to compute the control limits, whereas in the second $k \geq 2$ and $n \geq 2$ is sufficient to compute the decision lines.

Ott's method is based on the multiple significance test proposed by Halperin et al. (1955). Later, Nelson (1982) obtained the exact critical points of $h_{(\alpha/2,k,v)}$, and used the decision lines $\bar{\bar{X}}_{..} \pm h_{(\alpha/2,k,v)} S_b \sqrt{(k-1)/(kn)}$, where the critical point $h_{(\alpha/2,k,v)}$ depends on k, $v = k(n-1)$ (degrees of freedom in S_b), and the significance level α, with,

$$S_b = \left(\sum_{i=1}^{k}\sum_{j=1}^{n}(X_{ij} - \bar{X}_{i.})^2 / (k(n-1))\right)^{1/2} \quad ; \quad \bar{\bar{X}}_{..} = \sum_{i=1}^{k}\sum_{j=1}^{n} X_{ij} / (kn).$$

Some other applications of the ANOM for testing the interaction effects were investigated by Ramig (1983), Nelson (1988), Wludyka and Nelson (1997), and Budsaba et al. (2000). A full review of the ANOM technique is given by Rao (2005).

According to equicorrelated multivariate non-central t distribution for constructing the ANOM scheme, Tsai et al. (2005) introduced a control chart for a random variable $W_i = (\bar{X}_{i.} - \bar{\bar{X}}_{..})$, with the center line 0, and the control limits $0 \pm t_{\alpha/2,v}\sqrt{\bar{V}\frac{k+1}{kn}}$, where

$$\bar{V} = \sum_{i=1}^{k}\sum_{j=1}^{n}(X_{ij} - \bar{X}_{i})^2 / (k(n-1)) \text{ and } v = k(n-1).$$

In this paper, we introduce new control charts, in phase I and II, to monitor the mean of a quality characteristic when the standard deviation (σ) is known or unknown. The purpose in phase I is to perceive the stability and variation in a process over time. We are concerned with ongoing monitoring to detect assignable causes in the process in phase II controlling. Useful recognitions of phase I and phase II applications have been studied already, for example, by Kang and Albin (2000), Woodall (2000), Hawkins et al. (2003), Woodall et al. (2004), Montgomery (2005), and Jensen et al. (2006).

The proposed control limits with known or unknown σ are established for random variables that follow the normal distribution and t distribution, exactly. Another property of the proposed methods is that the values of both sample group and subgroup sizes (k and n) for computing the control limits, need to be grater than 1.

The paper is organized as follows. In sections 2, 3, 4, we set out the Shewhart, Bonferroni, ANOM, and new control charts, respectively. The probability of a false alarm for the Shewhart and the strategy proposed here is compared in section 5. The in-control average run lengths are described in section 6 for the Shewhart and the proposed charts. In section 7, the results and some recommendations for constructing the control limits are presented.

2. The Shewhart and Bonferroni Control Chart

Assume that the random variables X_{ij}, for $i=1,2,...,k$ and $j=1,2,...,n$, which measures the quality of process, are independent normally distributed with mean μ and variance σ^2. The Shewhart control limits for this quality characteristic with known parameters and confidence $1-\alpha$ are $\mu \pm Z_{\alpha/2}\sigma/\sqrt{n}$, where the center line of control chart is μ. If the mean and standard deviation of the quality characteristic are unknown, they are estimated by the unbiased statistics $\bar{\bar{X}}_{..}$ and $\bar{S}/c_4$ where,

$$\bar{S}=\sum_{i=1}^{k} S_i/k \ ; \ S_i=(\sum\nolimits_{j=1}^{n}(X_{ij}-\bar{X}_{i.})^2/(n-1))^{1/2} \ ; \ c_4=(\frac{2}{n-1})^{1/2}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}.$$

The random variable $S_i\sqrt{(n-1)}/\sigma$ is chi distributed with $n-1$ degrees of freedom. The mean and the standard deviation of the statistic S_i are $c_4\sigma$ and $\sigma\sqrt{1-c_4^2}$, respectively. (The constant value c_4 depends only on the sample subgroup size (n)).

The Shewhart chart with unknown parameters is constructed on the statistic $(\bar{X}_{i.}-\bar{\bar{X}}_{..})/(\bar{S}/(c_4\sqrt{n}))$ in phase I and $(\bar{X}_{i.}^{f}-\bar{\bar{X}}_{..})/(\bar{S}/(c_4\sqrt{n}))$ in phase II, where $\bar{X}_{i.}^{f}$ indicates a subgroup average for future observations. These statistics follow approximately the normal distribution for large sample sizes. As a consequence, the center line and the control limits for the Shewhart chart with unknown parameters are,

$$U\hat{C}L=\bar{\bar{X}}_{..}+Z_{\alpha/2}\bar{S}/(c_4\sqrt{n}) \;;\; \hat{C}L=\bar{\bar{X}}_{..} \;;\; L\hat{C}L=\bar{\bar{X}}_{..}-Z_{\alpha/2}\bar{S}/(c_4\sqrt{n})\ . \tag{1}$$

The unknown standard deviation σ can be also estimated by the unbiased statistic $\bar{R}/d_2$, where the statistic $\bar{R}$ is the average range and the constant value d_2 is the mean range of the standard normal variables. This statistic gives the Shewhart control limits as

$$\bar{\bar{X}}_{..}\pm Z_{\alpha/2}\bar{R}/(d_2\sqrt{n})\ . \tag{2}$$

Equation (2) is also based approximately on the normal distribution with large sample sizes.

The Bonferroni-adjustment control chart to improve the probability of one or more false alarms of the Shewhart chart was suggested by Ryan (1989). The Bonferroni-adjustment control limits with known and unknown parameters for retrospective monitoring in phase I are, respectively,

$$\mu\pm Z_{\alpha/2k}\,\sigma\big/\sqrt{n}\ , \tag{3}$$

$$\bar{\bar{X}}_{..}+Z_{\alpha/2k}\bar{S}/(c_4\sqrt{n})\ . \tag{4}$$

For constructing equations (3) and (4), the value α of Shewhart control limits is replaced by the value α / k.

3. The Analysis of Means Control Chart

The analysis of means can be thought of as an alternative to the Bonferroni method, since it also considers a group of sample averages instead of one average at a time in order to determine whether any of the sample averages differ much from the overall mean. The construction of ours and the ANOM strategies are based on the t distribution, hence a brief description of the ANOM technique is presented here.

The random variables X_{ij} are iid normal variables with mean μ and variance σ^2. Therefore, in phase I, the correlated random variables $\bar{X}_{i.} - \bar{\bar{X}}_{..}$ and $\bar{X}_{i'.} - \bar{\bar{X}}_{..}$ for $i \neq i' = 1,2,...,k$, follow the normal distribution with mean 0 and variance $\sigma^2(k-1)/(kn)$. Let $T_i = (\bar{X}_{i.} - \bar{\bar{X}}_{..}) / S_{\bar{X}_{i.} - \bar{\bar{X}}_{..}}$. The ANOM control chart is based on the joint statistic $(T_1, T_2, ..., T_k)$ that is equicorrelated multivariate non-central t distributed with equicorrelations $\rho = -1/(k-1)$. The statistic T_i follows the t distribution with $k(n-1)$ degrees of freedom. Here,

$$S_{\bar{X}_{i.} - \bar{\bar{X}}_{..}} = \hat{\sigma}_{\bar{X}_{i.} - \bar{\bar{X}}_{..}} = S_b \sqrt{(k-1)/(kn)} \quad ; \quad S_b^2 = \hat{\sigma}^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} (X_{ij} - \bar{X}_{i.})^2 / (k(n-1)).$$

Nelson (1982) defined the joint probability of T_i for $i = 1, 2, ..., k$ as $P\left[\bigcap_{i=1}^{k} |T_i| \leq h_{(\alpha/2,k,\nu)}\right] = 1 - \alpha$. Thus, $P[|T_i| \leq h_{(\alpha/2,k,\nu)}] = 1 - \alpha' \geq 1 - \alpha$, such that α' is unknown and $\alpha' \leq \alpha$. This probability results the ANOM chart with center line $\bar{\bar{X}}_{..}$ and approximately the following limits,

$$UCL = \bar{\bar{X}}_{..} + h_{(\alpha/2,k,\nu)} S_b \sqrt{(k-1)/(kn)} \quad ; \quad LCL = \bar{\bar{X}}_{..} - h_{(\alpha/2,k,\nu)} S_b \sqrt{(k-1)/(kn)}. \qquad (5)$$

Here, the exact critical values $h_{(\alpha/2,k,v)}$ depend on the desired level of significance (α), the sample sizes k, and the degrees of freedom $v = k(n-1)$.

Nelson (1982) and (1993) calculated the critical values $h_{(\alpha/2,k,v)}$ to satisfy

$$P[|T_1| \le h_{(\alpha/2,k,v)}, |T_2| \le h_{(\alpha/2,k,v)}, \dots, |T_k| \le h_{(\alpha/2,k,v)}] = 1-\alpha .$$

The left side of this equation is,

$$K\int_0^\infty \int_0^\infty \left[g(sh,y,\rho)\right]^k s^{v-1} \exp\left[-(y^2+vs^2)/2\right] d_y d_s , \tag{6}$$

where,

$$g(sh,y,\rho) = 2\mathrm{Re}\left[\Phi\left\{\frac{sh-y\sqrt{\rho}}{\sqrt{1-\rho}}\right\}\right]-1 ; \Phi(x+iy) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left[-(u+iy)^2/2\right]du ; K = 2\sqrt{\frac{2}{\pi}}\left[\frac{v}{2}\right]^{v/2} / \Gamma\left[v/2\right].$$

The function $\mathrm{Re}\left[\Phi(.)\right]$ is the real part of $\Phi(.)$, and $i=\sqrt{-1}$. Nelson (1993) numerically evaluated the double integral (6). The values $h_{\alpha/2,k,v}$ can be computed by replacing different values of the desired level α and the constants k and v. Tables of the critical values $h_{\alpha/2,k,v}$ are given by Nelson (1993) for various values of k, v and α.

4. A New Control Chart

As previously suggested, the new charts to monitor the mean quality characteristic with known or unknown parameter σ are exactly based on the normal and the t distribution, respectively. The proposed charts are dependent only on the parameter σ.

In phase I, we have $\bar{X}_{i.} - \bar{\bar{X}}_{..} \sim N(0, \sigma^2(k-1)/(kn))$. Therefore, with known variance σ^2, the new control limits are

$$\bar{\bar{X}}_{..} \pm Z_{\alpha/2}\sigma\sqrt{(k-1)/(kn)} , \tag{7}$$

since, $P(|\bar{X}_{i.} - \bar{\bar{X}}_{..}| / [\sigma\sqrt{(k-1)/(kn)}] \le Z_{\alpha/2}) = 1-\alpha$. In this case, the center line is $\bar{\bar{X}}_{..}$. For the construction of new control chart with unknown parameter σ^2, it is known that the random variables $(\bar{X}_{i.}-\bar{\bar{X}}_{..})/\sqrt{\sigma^2(k-1)/(kn)}$ and $k(n-1)S_b^2/\sigma^2=\sum_i^k\sum_j^n(X_{ij}-\bar{X}_{i.})^2/\sigma^2$ follow the standard normal distribution and the chi-square distribution, respectively. According to *Cochran's Theorem*, these random variables are independent. Therefore, the following statistic is t distributed with $k(n-1)$ degrees of freedom ($k>1$ and $n>1$),

$$T_i=\frac{(\bar{X}_{i.}-\bar{\bar{X}}_{..})/\sqrt{\sigma^2(k-1)/(kn)}}{\sqrt{\dfrac{k(n-1)S_b^2/\sigma^2}{k(n-1)}}}=\frac{\bar{X}_{i.}-\bar{\bar{X}}_{..}}{S_b\sqrt{(k-1)/(kn)}}\sim t\left(k(n-1)\right).$$

As a result, the new control chart with unknown variance is given by:

$$\begin{aligned} UCL &= \bar{\bar{X}}_{..}+t_{\alpha/2,k(n-1)}S_b\sqrt{(k-1)/(kn)} \\ CL &= \bar{\bar{X}}_{..} \\ LCL &= \bar{\bar{X}}_{..}-t_{\alpha/2,k(n-1)}S_b\sqrt{(k-1)/(kn)}\ , \end{aligned} \tag{8}$$

where, $P(|\bar{X}_{i.}-\bar{\bar{X}}_{..}|/[S_b\sqrt{(k-1)/(kn)}]\le t_{\alpha/2,k(n-1)})=1-\alpha$.

For controlling future subgroups, $\bar{X}_{i.}^f$, the variance of $\bar{X}_{i.}^f-\bar{\bar{X}}_{..}$ is evaluated to be $\sigma^2(k+1)/kn$. In phase II, the random variables $(\bar{X}_{i.}^f-\bar{\bar{X}}_{..})/(\sigma\sqrt{(k+1)/(kn)})$ and $(\bar{X}_{i.}^f-\bar{\bar{X}}_{..})/(S_b\sqrt{(k+1)/(kn)})$ follow the standard normal distribution and the t distribution, respectively. As a result, the proposed control limits, in phase II, with known and unknown σ are,

$$\bar{\bar{X}}_{..}\pm Z_{\alpha/2}\sigma\sqrt{(k+1)/(kn)}\ , \tag{9}$$

$$\bar{\bar{X}}_{..}\pm t_{\alpha/2,k(n-1)}S_b\sqrt{(k+1)/(kn)}\ . \tag{10}$$

Here, the sample group and subgroup sizes required to construct our proposed charts, with known and unknown parameter σ, are greater than 1, i.e. $k>1$ and $n>1$.

5. The Performance of Retrospective Charts

Let the individual events G_i denote that the subgroup averages $\bar{X}_{i.}$ exceed the control limits of in control process. If these events are independent, then the sequence of trials comparing $\bar{X}_{i.}$ with UCL will be a sequence of Bernoulli trials and the overall occurrences of G_i will be a Binomial random variable with parameters k and $P(G_i)$. However, in the case of unknown parameters, these events for the Bonferroni and ANOM charts are not independent. Hence a performance comparison between these charts for historical data in phase I is given based on a simulation study. We also use simulation to study the probability $P(G_i)$ for the estimated Shewhart chart, since the control limits are the approximations of true limits. For our proposed charts, in the case of the known and unknown parameter σ, the $P(G_i)$ can be easily evaluated theoretically.

For the Shewhart chart with known parameters the probability of at least a false signal is $1-(1-\alpha)^k$, since the events G_i follow the Binomial distribution. Ryan (1989) showed that this probability is approximately equal to $k\alpha$. Hence, Ryan suggested the Bonferroni-adjustment scheme for the mean control limits, where the probability of one or more false alarms is improved to the desired value $\alpha \simeq 1-(1-\alpha/k)^k$, which is less than $1-(1-\alpha)^k$ for the Shewhart scheme. As already mentioned, the ANOM method is an alternative to the Bonferroni method, maintaining approximately the overall false alarm probability at the desired α. Nedumaran and Pignatiello (2005) compared this probability for the Bonferroni and ANOM procedures. The performance measure for these charts is the overall probability of a false signal. Based on their study, the actual probability of having at least one false alarm, using Monte Carlo simulation experiments (20,000 times), for the ANOM approach is slightly less than the one of the Bonferroni approach, and very close to the desired value α.

To compare the Shewhart scheme to our scheme, we use a performance measure the probability of a false alarm. In this case, the k subgroups of size n are generated (20,000 times) from a stable in-control iid normal process. The estimated control limits are obtained according to (1) for the Shewhart strategy with unknown parameters and according to (7) and (8) for our strategies with known and unknown parameter. Table 1 shows the results of the estimated probability of a false alarm.

Table 1 about here

It can be concluded that the proposed new schemes, for small and large sizes k and n, perform better than the Shewhart scheme, in the sense that, the estimated false alarm probability of the proposed schemes is very close to the intended α. Indeed, in theory the desired α can be exactly attained applying the proposed schemes (7) and (8). However, because of the small errors in simulation experiments and the fact that the random sample sizes are not large enough this cannot be achieved.

6. Average Run Length

The average run length (ARL) is the average number of subgroups that are plotted before a subgroup average indicates an out-of-control condition. The ARL can be calculated as $ARL = 1/p$, under the condition that the process observations are uncorrelated. Here, p is the probability that a point exceeds the control limits.

The average run length is considered for future subgroups, when the process is in control i.e. $\mu = \mu_0$, by plotting each subgroup on the control chart immediately after each sample is collected. Let the individual events G_i^f denote that the subgroup averages $\bar{X}_{i.}^f$ exceeds the control limits of the in control process.

In the case where the events G_i^f are independent, the sequence of trials, to compare $\bar{X}_{i.}^f$ with UCL, will be a sequence of Bernoulli trials and the run length between occurrences of G_i^f will be a Geometric random variable with probability $P(G_i^f)$. The in-control average run length will be $1/\alpha = 1/P(G_i^f)$,

$$P(G_i^f) = P(\bar{X}_{i.}^f \leq LCL \ \ or \ \ \bar{X}_{i.}^f \geq UCL \,\big|\, \mu = \mu_0).$$

Quesenberry (1993) suggested that the $P(G_i^f)$ for a classical Shewhart 3σ control chart in case of the known parameters is equal to $\alpha = 0.0027$, and with unknown parameters is approximately,

$$P(G_i^f) = 2[1 - \Phi(3\{1 + \frac{1}{k}[1 + \frac{9(1-c_4^2)}{c_4^2}]\}^{-1/2})], \tag{11}$$

where $\Phi(.)$ indicates the standard normal distribution function. Using equation (11), the $P(G_i^f)$ for the often recommended values $k = 20$ and $n = 4$ is 0.0048, which is greater than the intended $\alpha = 0.0027$. Quesenberry (1993) recommended sample sizes of about $400/(n-1)$ to construct the classical Shewhart chart. Following this recommendation, for $m = 133$ and $n = 4$, the intended probability of a false alarm, i.e. 0.0027, will be obtained. As a result, the usual recommendations on the sample sizes are not sufficient to ensure that the Shewhart estimated control limits are close enough to the true limits. The $P(G_i^f)$ for the proposed methods (9) and (10) with known and unknown σ is equal to the desired value α for both small and large sample sizes.

The events G_i^f and $G_{i'}^f$, $i \neq i'$, for the Shewhart chart with known parameters are uncorrelated, since the control limits are the constant values and the subgroup averages $\bar{X}_{i.}^f$ and $\bar{X}_{i'.}^f$ are independent. Thus, the run length between occurrences of G_i^f is a Geometric random variable with probability $P(G_i^f) = \alpha$ and the ARL equals to $1/\alpha$. But, these events for the Shewhart chart with unknown parameters and the proposed chart with known and unknown parameter are not independent, since the random variables $\bar{X}_{i.}^{f} - U\hat{C}L$ and $\bar{X}_{i'.}^{f} - U\hat{C}L$ are not independent. The correlation between these random variables for the Shewhart method is,

$$corr(\bar{X}_{i.}^{f} - U\hat{C}L, \bar{X}_{i'.}^{f} - U\hat{C}L) = [1 + k\{1 + \frac{z_{\alpha/2}^{2}(1-c_4^2)}{c_4^2}\}^{-1}]^{-1},$$

while, for the proposed method with known parameter is $1/(k+1)$, and for unknown parameter is $[1+k\{1+(k+1)t_{\alpha/2,k(n-1)}^{2}(1-\psi^2)\}^{-1}]^{-1}$. In this case, ψ is an unbiasing factor to estimate σ, where $E(S_b/\psi)=\sigma$, $\mathrm{var}(S_b/\psi)=\sigma^2(1-\psi^2)/\psi^2$. The statistic S_b is chi distributed with $k(n-1)$ degrees of freedom. Based on the raw moment function of chi distribution, ψ is,

$$\psi = \sqrt{\frac{2}{k(n-1)}}\Gamma\left(\frac{k(n-1)+1}{2}\right) / \Gamma\left(\frac{k(n-1)}{2}\right).$$

The correlations evaluated for the Shewhart and the proposed methods rely on k and n, where these are always positive. These correlations decrease when we use larger sample sizes k and n. As a consequence, for the Shewhart method with unknown parameters (1) and the proposed method with known and unknown parameter (9) and (10), the distribution of run length between occurrences of the events G_i^f is not a Geometric distribution. Hence, when the parameters are unknown, the ARL cannot be evaluated based on the mean of a Geometric distribution. To overcome this problem, the $ARL = 1/\alpha$ is estimated by the simulation experiments. The existence of correlation between the events G_i^f increases the ARL, making it greater than the intended ARL. Under these circumstances, the control limits (10) are not suitable for accomplishing the desired ARL. Hence, we propose the following approximate control limits as an alternative for (10),

$$\bar{\bar{X}}_{..} \pm t_{\alpha/2,k(n-1)} S_b \psi / \sqrt{n} \,. \tag{12}$$

Table 2 shows the results of simulation experiments for equations (1), (9), (10) and (12). For each entry in Table 2, the mean control limits are computed corresponding to k samples of size n, and future samples are generated from an in control process until a subgroup average is found outside the control limits. The number of samples is one observation from the run length distribution. This procedure is replicated 20,000 times. Each table entry is the average of observations from the run length distribution.

Table 2 about here

As already mentioned, the probability $P(G_i^f)$, corresponding to (1), is approximated to be greater than the intended α. This indicates a reason to decrease the in control ARL for the Shewhart scheme. On the other hand, the correlation between the events G_i^f causes an increase of the ARL. Based on Table 2, it can be concluded that the ARL for the classical Shewhart scheme is less than the desired ARL. For the proposed limits (10) the ARL is greater than $1/\alpha$, although the $P(G_i)$ is exactly equal α. This is due to the correlation between the events G_i^f. According to simulation experiments, the performance of the proposed schemes (9) and (12), to achieve the intended in control ARL is more satisfactory than the one of the schemes (1) and (10). The probability of a false alarm for the scheme (9) is equal to α, and for the scheme (12) is relatively greater than α.

7. Conclusion

It has been shown that the procedures suggested in this paper, in both phases I and II, have three advantages over the classical Shewhart method: first the proposed scheme is established using small sample sizes; second the in-control ARL of the new procedure is very close to the desired ARL; third the false alarm probability corresponding to the proposed methods equals the intended α.

It has been suggested in the literature to use the ANOM and the Bonferroni procedures to monitor historical data in phase I controlling. These methods maintain the overall false alarm probability approximately at a desired level α. The ANOM scheme performs better than the Bonferroni technique in achieving an overall probability of a false signal at the desired α.

We recommend using the proposed strategies if the individual occurrence of events G_i and G_i^f is required, and the ANOM strategy if the overall occurrence of events G_i is considered. The ANOM and the proposed methods are constructed on the statistic $\bar{X}_{i.} - \bar{\bar{X}}_{..}$ that includes more information than $\bar{X}_{i.}$ used for the Shewhart and Bonferroni methods. Moreover, the distribution function of $\bar{X}_{i.} - \bar{\bar{X}}_{..}$ relies only on the parameter σ, whereas, that of $\bar{X}_{i.}$ depends on both parameters μ and σ.

Acknowledgments

The authors are grateful to Professor William H. Woodall and the referee for helpful suggestions and comments that led to substantial improvements of the paper.

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Table 1 Estimated probability of a false alarm, for intended $\alpha = 0.1$

k	Approach/n	5	10	20
	(1)	0.0721	0.0783	0.0823
5	(7)	0.1101	0.0996	0.0947
	(8)	0.1091	0.0989	0.0982
	(1)	0.0801	0.0831	0.0868
15	(7)	0.0994	0.1023	0.0987
	(8)	0.0978	0.1039	0.0980
	(1)	0.0861	0.0918	0.0901
25	(7)	0.1026	0.0997	0.1031
	(8)	0.1062	0.0941	0.1063
	(1)	0.0939	0.0924	0.0966
35	(7)	0.1006	0.0989	0.1011
	(8)	0.1018	0.0972	0.1028

Table 2 Estimated in-control ARL, for intended $\alpha = 0.1$

k	Approach/n	5	10	20
5	(1)	7.27	7.81	8.84
	(9)	11.54	11.41	11.02
	(10)	16.8	16.21	15.41
	(12)	9.09	9.21	9.12
15	(1)	7.43	8.25	8.92
	(9)	10.31	11.02	10.06
	(10)	15.4	14.96	13.83
	(12)	9.46	9.49	10.01
25	(1)	8.31	9.19	8.83
	(9)	10.12	9.98	10.04
	(10)	13.76	14.03	13.89
	(12)	9.72	10.08	9.17
35	(1)	9.39	9.42	9.31
	(9)	10.64	10.14	10.11
	(10)	12.62	13.72	12.31
	(12)	10.19	9.90	9.94