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Abstract

In mechanism design theory, a designer would like to implement a desired social choice function which specifies her favorite outcome for each possible profile of all agents' types. Since the designer does not know each agent's private type, what she can do is to construct a mechanism and choose an outcome after observing a profile of agents' strategies. There is a dilemma in the sense that even if the designer is not satisfied with some outcome, she has to obey the mechanism designed by herself and announce this outcome. In this paper, we generalize the mechanism design theory to a case where the designer can take some action to actively adjust agents' private types, and yield a more favorite outcome. After defining a series of notions such as adjustable types, optimal adjustment cost and profitably Bayesian implementability, we propose that the traditional notion of Bayesian incentive compatibility does not hold in this generalized case. Finally, we construct a model to illustrate that the auctioneer can obtain an expected profit greater than what she obtains in the traditional optimal auction.

Key words: Mechanism design; Optimal auction; Bayesian Nash implementation.

1 Introduction

In the framework of mechanism design theory [1–4], there are one designer and some agents. [1] The designer would like to implement a desired social choice function which specifies her favorite outcome for each possible profile of agents' types. However, agents' types are modelled as their private properties and unknown to the designer. The designer only knows the distribution of agents'

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[1] The designer is denoted as "She", and the agent is denoted as "He".

types. In order to implement the social choice function, the designer constructs a mechanism which specifies each agent's strategy set (*i.e.*, the allowed actions of each agent) and an outcome function (*i.e.*, a rule for how agents' actions get turned into a social choice). During the process of an mechanism, the designer may be in a dilemma in the sense that even if some profile of agents' strategies leads to an outcome with low revenue, she has to announce it because she must obey the mechanism designed by herself.

The designer may improve her situation by holding a charity auction. Engers and McManus [5] proposed that agents' bids in a first-price charity auction are greater than those in a standard (non-charity) auction because of the charitable benefit that winners receive from their own payments. Besides the charity auction, there may exist another way for the designer to increase her revenue.

For example, let the designer be an auctioneer who sells a good, and each agent be a bidder whose initial valuation to the good (*i.e.*, private type) is low. In order to obtain as much profit as possible, suppose the designer is able to take some costly action (*e.g.*, advertisement) to adjust each agent's valuation to the good. Without loss of generality, we assume that each agent's valuation and bid both increase concavely with the growth of the cost spent by the designer, and the designer's utility is a linear function of the winner agent's bid. From the viewpoint of the designer, as long as her marginal utility is greater than her marginal cost, it is worthwhile for her to continue investing on this costly adjustment. Obviously, the designer will obtain the maximum profit when her marginal utility is equal to her marginal cost. Thus, if the designer can adjust agents' types, she may *actively* escape from the above-mentioned dilemma and yield an outcome better than what would happened without doing so.

In this paper, we generalize the mechanism design theory to a case where the designer can take some action to adjust agents' types. In Section 2, we give a series of notions and propositions. In Section 3, we construct a model to show that by adjusting agents' types, the designer can obtain an expected profit greater than what she can obtain in the traditional optimal auction model.

2 Theoretical analysis

Following Section 23.B of MWG's textbook [1], we consider a setting with one designer and I agents, indexed by $i = 1, \cdots, I$. These agents make a collective choice from a set X of possible alternatives. We make the following two assumptions:

Assumption 1: Prior to the choice, each agent i is assumed to observe a private parameter (*i.e.*, *type* θ_i) which determines his preference over alternatives in X. Let Θ_i be the set of agent i's all possible types, and $\theta_i^0 \in \Theta_i$ be agent i's *initial type*.

Assumption 2: The designer constructs a mechanism and announces an outcome after observing a profile of agents' strategies. She is assumed to be able to take some action to adjust all agents' private types before the mechanism works. This action is characterized by the relevant cost spent by the designer.

Definition 1: Given each agent i's initial type $\theta_i^0 \in \Theta_i$, suppose the designer spends non-negative cost $c \in R^+$ to adjust agents' types.[2] Each agent i's preference over the alternatives in X is determined by his *adjustable type* $\theta_i^c \in \Theta_i$. Let $\Theta = \Theta_1 \times \cdots \times \Theta_I$, $\theta = (\theta_1, \cdots, \theta_I) \in \Theta$. For each $i = 1, \cdots, I$, let

$$\begin{aligned}
&\theta^0 = (\theta_1^0, \cdots, \theta_I^0) \in \Theta, \\
&\theta_{-i}^0 = (\theta_1^0, \cdots, \theta_{i-1}^0, \theta_{i+1}^0, \cdots, \theta_I^0), \\
&\theta^c = (\theta_1^c, \cdots, \theta_I^c) \in \Theta, \\
&\theta_{-i}^c = (\theta_1^c, \cdots, \theta_{i-1}^c, \theta_{i+1}^c, \cdots, \theta_I^c).
\end{aligned}$$

A *type adjustment function* is denoted as $\mu(\theta, c) : \Theta \times R^+ \to \Theta$, in which $\mu(\theta, 0) = \theta$ for any $\theta \in \Theta$, *i.e.* zero cost means no type adjustment. Let $\theta^c = \mu(\theta^0, c)$, $\phi^0(\theta^0) = (\phi_1^0(\theta_1^0), \cdots, \phi_I^0(\theta_I^0))$ be the probability density function of initial type profile $\theta^0 \in \Theta$, and $\phi^c(\theta^c) = (\phi_1^c(\theta_1^c), \cdots, \phi_I^c(\theta_I^c))$ be the probability density function of adjustable type profile $\theta^c \in \Theta$. For each $i = 1, \cdots, I$, let

$$\begin{aligned}
&\phi_{-i}^0(\theta_{-i}^0) = (\phi_1^0(\theta_1^0), \cdots, \phi_{i-1}^0(\theta_{i-1}^0), \phi_{i+1}^0(\theta_{i+1}^0), \cdots, \phi_I^0(\theta_I^0)), \\
&\phi_{-i}^c(\theta_{-i}^c) = (\phi_1^c(\theta_1^c), \cdots, \phi_{i-1}^c(\theta_{i-1}^c), \phi_{i+1}^c(\theta_{i+1}^c), \cdots, \phi_I^c(\theta_I^c)).
\end{aligned}$$

Assumption 3: The designer is assumed to know $\phi^0(\theta^0)$ and type adjustment function $\mu(\theta, c)$ for any $c \geq 0$.

Definition 2: For any $x \in X$, each *agent i's utility* is denoted as $u_i(x, \theta_i) \in R$, where $\theta_i \in \Theta_i$. The *designer's utility* is denoted as $u_d(x) \in R$.

Definition 23.B.1 [1]: A *social choice function* (SCF) is a function $f : \Theta \to X$ that, for each possible profile of the agents' types $\theta \in \Theta$, assigns a collective choice $f(\theta) \in X$.

Definition 3: Given an SCF f and $\phi^0(\theta^0)$, for any $c \geq 0$, the *designer's expected utility* is denoted as $\bar{u}_d(c) = E_{\theta^c} u_d(f(\theta^c))$, and her *initial expected utility* is denoted as $\bar{u}_d(0) = E_{\theta^0} u_d(f(\theta^0))$ for the case of zero cost.

[2] The adjustment is observable to all agents, and the relevant cost c is common knowledge among all agents and the designer.

Definition 4: Given an SCF f and $\phi^0(\theta^0)$, for any $c \geq 0$, the *designer's expected profit* is denoted as $\bar{p}_d(c) = \bar{u}_d(c) - c$, and her *initial expected profit* is denoted as $\bar{p}_d(0) = \bar{u}_d(0)$ for the case of zero cost.

Assumption 4: $\bar{u}_d(c)$ is assumed to be a concave function that satisfies the following inequalities,

$$\frac{\partial \bar{u}_d(c)}{\partial c} > 0, \quad \frac{\partial^2 \bar{u}_d(c)}{\partial c^2} < 0, \quad \text{for any } c \geq 0.$$ [3]

Proposition 1: If there exists $c^* \geq 0$ such that

$$\left.\frac{\partial \bar{u}_d(c)}{\partial c}\right|_{c=c^*} = 1, \quad i.e. \quad \left.\frac{\partial \bar{p}_d(c)}{\partial c}\right|_{c=c^*} = 0,$$

then the designer will obtain the maximum expected profit $\bar{p}_d(c^*)$ at $c = c^*$. c^* is denoted as the *optimal adjustment cost.* By assumption 4, there holds

$$\left.\frac{\partial \bar{u}_d(c)}{\partial c}\right|_{c=0} \geq 1, \quad i.e. \quad \left.\frac{\partial \bar{p}_d(c)}{\partial c}\right|_{c=0} \geq 0.$$

Proposition 2: If the designer's expected utility $\bar{u}_d(c)$ and expected profit $\bar{p}_d(c)$ satisfy the following condition,

$$\left.\frac{\partial \bar{u}_d(c)}{\partial c}\right|_{c=0} < 1, \quad i.e. \quad \left.\frac{\partial \bar{p}_d(c)}{\partial c}\right|_{c=0} < 0, \tag{1}$$

then the designer will obtain the maximum expected profit at $c = 0$.

Definition 23.B.3 [1]: A *mechanism* Γ is a collection of I strategy sets $S_1, \cdots, S_I$, and an outcome function g, *i.e.*, $\Gamma = (S_1, \cdots, S_I, g(\cdot))$. A *strategy* of each agent i in Γ is a function $s_i(\cdot) : \Theta_i \to S_i$. Let $s(\cdot) = (s_1(\cdot), \cdots, s_I(\cdot))$. The outcome function is defined by $g(s(\cdot)) : \Theta \to X$.

Assumption 5: Assume that in a mechanism Γ, each agent i can play his strategy $s_i(\cdot)$ without any cost. Hence agent i's utility $u_i(x, \theta_i)$ is just his profit obtained from the outcome x. [4]

Definition 23.D.1 [1]: The strategy profile $s^*(\cdot) = (s_1^*(\cdot), \cdots, s_I^*(\cdot))$ is a *Bayesian Nash equilibrium* of mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ if, for all agent

[3] See the example given in Section 3, when each agent i's adjustable type is a square root function with the designer's cost as specified by Eq(5) and the social choice function is specified by Eq(6), then the inequalities in Formula (1) holds.

[4] For example, suppose that each agent is a bidder in an auction, then each agent can be considered to submit his bid to the auctioneer without any cost.

i and all $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i] \geq E_{\theta_{-i}}[u_i(g(\hat{s}_i, s_{-i}^*(\theta_{-i})), \theta_i)|\theta_i], \quad (2)$$

for all $\hat{s}_i \in S_i$.

Definition 23.D.2 [1]: The mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ *implements the social choice function* $f(\cdot)$ *in Bayesian Nash equilibrium* if there is a Bayesian Nash equilibrium of Γ, $s^*(\cdot) = (s_1^*(\cdot), \cdots, s_I^*(\cdot))$, such that $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$.

Definition 5: Given an SCF f and $\phi^0(\theta^0)$, f is *profitably Bayesian implementable* if the following conditions are satisfied:
1) The optimal adjustment cost $c^* > 0$, which means that the distribution of agents' private types are adjusted to $\phi^{c^*}(\theta^{c^*})$.
2) There are a mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ and a strategy profile $s^*(\cdot) = (s_1^*(\cdot), \cdots, s_I^*(\cdot))$ such that:
(i) For all agent i, and all $\theta_i^{c^*} \in \Theta_i$,

$$E_{\theta_{-i}^{c^*}}[u_i(g(s_i^*(\theta_i^{c^*}), s_{-i}^*(\theta_{-i}^{c^*})), \theta_i^{c^*})|\theta_i^{c^*}] \geq E_{\theta_{-i}^{c^*}}[u_i(g(\hat{s}_i, s_{-i}^*(\theta_{-i}^{c^*})), \theta_i^{c^*})|\theta_i^{c^*}] \quad (3)$$

for all $\hat{s}_i \in S_i$. [5]
(ii) $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$.

Proposition 3: Given an SCF f and $\phi^0(\theta^0)$, if f is profitably Bayesian implementable, then the designer's expected profit at the optimal adjustment cost is greater than her initial expected profit.
Proof: Given that f is profitably Bayesian implementable, then the optimal adjustment cost $c^* > 0$. By Proposition 1, $\bar{p}_d(c^*) > \bar{p}_d(0)$. $\square$

Definition 23.B.5 [1]: A *direct mechanism* is a mechanism $\Gamma' = (S_1', \cdots, S_I', g'(\cdot))$ in which $S_i' = \Theta_i$ for all i and $g'(\theta) = f(\theta)$ for all $\theta \in \Theta$.

Definition 23.D.3 [1]: The social choice function $f(\cdot)$ is *truthfully implementable in Bayesian Nash equilibrium* (or *Bayesian incentive compatible*) if $s_i'^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ $(i = 1, \cdots, I)$ is a Bayesian Nash equilibrium of the direct mechanism $\Gamma' = (S_1', \cdots, S_I', g'(\cdot))$, in which $S_i' = \Theta_i$, $g' = f$. That is, if for all $i = 1, \cdots, I$ and all $\theta_i \in \Theta_i$,

$$E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i] \geq E_{\theta_{-i}}[u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)|\theta_i], \quad (4)$$

[5] Note that after the designer spends the optimal adjustment cost c^*, each agent i's private type is adjusted from his initial type θ_i^0 to $\theta_i^{c^*}$. In Formula (3), the probability density function of type profile $\theta_{-i}^{c^*} = (\theta_1^{c^*}, \cdots, \theta_{i-1}^{c^*}, \theta_{i+1}^{c^*}, \cdots, \theta_I^{c^*})$ is $\phi_{-i}^{c^*}(\theta_{-i}^{c^*})$. As a comparison, in the traditional notion of Bayesian Nash equilibrium, there is no type adjustment. Thus, the probability density function of type profile $\theta_{-i} = (\theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_I)$ in Formula (4) is just $\phi_{-i}^0(\theta_{-i}^0)$.

for all $\hat{\theta}_i \in \Theta_i$.

Proposition 4: Given an SCF f and $\phi^0(\theta^0)$, if f is profitably Bayesian implementable, then the traditional notion of Bayesian incentive compatibility does not hold in this generalized case.

Proof: Given that f is profitably Bayesian implementable, then the optimal adjustment cost $c^* > 0$. Note that formula (3) is based on the distribution $\phi^{c^*}_{-i}(\theta^{c^*}_{-i})$ (see Footnote 5), and the designer's expected profit is $\bar{p}_d(c^*)$.
As a comparison, in the notion of Bayesian incentive compatibility (see Definition 23.D.3), there is a direct mechanism $\Gamma' = (S'_1, \cdots, S'_I, g'(\cdot))$, in which $S'_i = \Theta_i$, $g' = f$, and the strategy profile $s'^*(\theta) = \theta$ for all $\theta \in \Theta$ is a Bayesian Nash equilibrium of Γ'. By the definition of direct mechanism (see Definition 23.B.5), the designer does not take any type adjustment action during the process of mechanism. Thus, formula (4) is based on the distribution $\phi^0_{-i}(\theta^0_{-i})$ (see Footnote 5), and the designer's expected profit is just $\bar{p}_d(0)$.
Note that formula (4) cannot be inferred from formula (3) because the distribution $\phi^0_{-i}(\theta^0_{-i})$ is distinct from the distribution $\phi^{c^*}_{-i}(\theta^{c^*}_{-i})$. Thus, the traditional notion of Bayesian incentive compatibility does not hold in this generalized case. □

3 Example

Following the auction model in MWG's book (Page 863, [1]), we suppose there are one designer and two agents. The designer is an auctioneer who wants to sell a good, and each agent i's initial valuation to the good is $\theta_i \geq 0$, *i.e.*, $\Theta_i = R^+$. We consider a first-price-sealed-bid auction setting: Each agent i is allowed to submit a sealed bid $b_i \geq 0$. The bids are then opened, and the agent with the higher bid gets the good, and must pay money equal to his bid to the auctioneer.

Suppose each agent i's initial valuation θ^0_i is drawn independently from the uniform distribution on $[0, 1]$. The distribution is known by the designer but the exact value of each θ^0_i is agent i's private information. In order to obtain as much profit as possible, the designer spends some advertisement cost $c \geq 0$ to increase each agent i's valuation to the good. Let $\beta > 0$ be a coefficient, suppose each agent i's valuation (*i.e.*, his adjustable type) is a square root function of the cost c,

$$\theta^c_i = (1 + \beta\sqrt{c})\theta^0_i. \tag{5}$$

Thus,

$$\frac{\partial \theta^c_i}{\partial c} = \frac{\beta\theta^0_i}{2\sqrt{c}}, \qquad \frac{\partial^2 \theta^c_i}{\partial c^2} = -\frac{\beta\theta^0_i}{4}c^{-3/2}.$$

That is, for any $c \geq 0$, the following formulas hold:

$$\left.\frac{\partial \theta_i^c}{\partial c}\right|_{c=0} = +\infty, \ \frac{\partial \theta_i^c}{\partial c} > 0, \ \frac{\partial^2 \theta_i^c}{\partial c^2} < 0.$$

Let $\theta = (\theta_1, \theta_2)$, consider the social choice function

$$f(\theta) = (y_1(\theta), y_2(\theta), y_d(\theta), u_1(\theta), u_2(\theta), u_d(\theta)), \tag{6}$$

in which

$$\begin{aligned}
y_1(\theta) &= 1, \quad \text{if } \theta_1 \geq \theta_2; \quad = 0 \text{ if } \theta_1 < \theta_2 \\
y_2(\theta) &= 1, \quad \text{if } \theta_1 < \theta_2; \quad = 0 \text{ if } \theta_1 \geq \theta_2 \\
y_d(\theta) &= 0, \quad \text{for all } \theta \in \Theta \\
u_1(\theta) &= -\theta_1 y_1(\theta)/2 \\
u_2(\theta) &= -\theta_2 y_2(\theta)/2 \\
u_d(\theta) &= [\theta_1 y_1(\theta) + \theta_2 y_2(\theta)]/2.
\end{aligned}$$

The subscript d stands for the designer, and the subscript $1, 2$ stands for the agent 1 and agent 2 respectively. $y_i = 1$ means that agent i gets the good.

Now we will investigate whether this social choice function is Bayesian implementable. We will look for an equilibrium in which each agent i's strategy $b_i(\cdot)$ takes the form $b_i(\theta_i^c) = \alpha_i \theta_i^c = \alpha_i(1+\beta\sqrt{c})\theta_i^0$ for $\alpha_i \in [0, 1]$.

Suppose that agent 2's strategy has this form, and consider agent 1's problem. For each possible θ_1^c, agent 1 wants to solve the following problem:

$$\max_{b_1 \geq 0} (\theta_1^c - b_1) \text{Prob}(b_2(\theta_2^c) \leq b_1).$$

Because agent 2's highest possible bid is $\alpha_2(1+\beta\sqrt{c})$ when $\theta_2^0 = 1$, it is evident that agent 1's bid b_1 should never more than $\alpha_2(1+\beta\sqrt{c})$. Since θ_2^0 is uniformly distributed on $[0, 1]$, and $b_2(\theta_2^c) = \alpha_2(1+\beta\sqrt{c})\theta_2^0 \leq b_1$ if and only if $\theta_2^0 \leq b_1/[\alpha_2(1+\beta\sqrt{c})]$, hence we can write agent 1's problem as:

$$\max_{0 \leq b_1 \leq \alpha_2(1+\beta\sqrt{c})} \frac{(\theta_1^c - b_1)b_1}{\alpha_2(1+\beta\sqrt{c})}$$

The solution to this problem is

$$b_1^*(\theta_1^c) = \begin{cases} \theta_1^c/2, & \text{if } \theta_1^0/2 \leq \alpha_2 \\ \alpha_2(1+\beta\sqrt{c}), & \text{if } \theta_1^0/2 > \alpha_2 \end{cases}.$$

Similarly,

$$b_2^*(\theta_2^c) = \begin{cases} \theta_2^c/2, & \text{if } \theta_2^0/2 \leq \alpha_1 \\ \alpha_1(1+\beta\sqrt{c}), & \text{if } \theta_2^0/2 > \alpha_1 \end{cases}.$$

Letting $\alpha_1 = \alpha_2 = 1/2$, we see that the strategies $b_i^*(\theta_i^c) = \theta_i^c/2$ for $i = 1, 2$ constitute a Bayesian Nash equilibrium for this mechanism. Thus, there is a Bayesian Nash equilibrium of this mechanism that indirectly yields the outcomes specified by the social choice function $f(\theta)$, and hence $f(\theta)$ is Bayesian Nash implementable.

Let us consider the designer's expected profit:

$$\bar{p}_d(c) = (1 + \beta\sqrt{c})E[\theta_1^0 y_1(\theta^c) + \theta_2^0 y_2(\theta^c)]/2 - c.$$

The designer's problem is to choose an optimal cost $c \geq 0$ to maximize her expected profit, *i.e.*,

$$\max_{c \geq 0}(1 + \beta\sqrt{c})E[\theta_1^0 y_1(\theta^c) + \theta_2^0 y_2(\theta^c)]/2 - c$$

According to Appendix A, the designer's initial expected profit is $\bar{p}_d(0) = E[\theta_1^0 y_1(\theta^c) + \theta_2^0 y_2(\theta^c)]/2 = 1/3$. Thus, the designer's problem is reformulated as:

$$\max_{c \geq 0}(1 + \beta\sqrt{c})/3 - c$$

It can be easily derived that the optimal cost $c^* = \beta^2/36$. By Definition 5, $f(\theta)$ is profitably Bayesian implementable. The maximum expected profit of the designer is:

$$\bar{p}_d(c^*) = (1 + \beta\sqrt{c^*})/3 - c^* = \frac{1}{3}(1 + \frac{\beta^2}{12}).$$

Obviously, when $\beta > \sqrt{3}$, there exists $\bar{p}_d(c^*) > 5/12$. Note that the designer's maximum expected profit in the optimal auction with two bidders is 5/12 (see Page 23, the ninth line from the bottom, Ref [6]). Therefore, by adjusting agents' types, the seller can obtain an expected profit greater than the maximum expected profit given by the traditional optimal auction.

The profit of the winner agent i is:

$$u_i(f(\theta^{c^*}), \theta_i^{c^*}) = \theta_i^{c^*} - b_i^*(\theta_i^{c^*}) = \theta_i^0(1 + \beta\sqrt{c^*})/2 = \frac{\theta_i^0}{2}(1 + \frac{\beta^2}{6}) > \frac{\theta_i^0}{2}.$$

It can be seen that the winner's profit is also increased when agents' types are adjustable.

4 Conclusions

Traditionally, agents' types are considered as private and endogenous values, which means that the designer has no way to know and adjust each agent's type. Thus, although the designer constructs a mechanism in order to implement her favorite social choice function, she behaves just like *a passive observer* after receiving a profile of agents' strategies: *i.e.*, she must obey the mechanism and announce the outcome specified by the outcome function, no matter whether she is satisfied with the outcome or not.

This paper generalizes the traditional mechanism design theory to a case where agents' types can be adjusted by the designer. In the generalized case, by adjusting agents' types the designer behaves just like *an active modulator* who can choose an optimal adjustment cost and maximize her expected profit.

In Section 2, we define a series of notions such as adjustable types, optimal adjustment cost, profitably Bayesian implementability and so on. Then we propose that the traditional notion of Bayesian incentive compatibility does not hold in this generalized case. In Section 3, we construct a model to show that by adjusting agents' types, the designer can obtain an expected profit greater than the maximum expected profit yielded by the traditional optimal auction. At the same time, the winner agent's profit is also increased.

Appendix

As specified in Section 3, θ_1^0 and θ_2^0 are drawn independently from the uniform distribution on $[0,1]$. Let Z be a random variable defined as $Z=\theta_1^0 y_1(\theta^c)+\theta_2^0 y_2(\theta^c)$.

$$f_{\theta_1^0}(z)=\begin{cases}0, & z<0\\ 1, & z\in[0,1]\\ 0, & z>1\end{cases}.$$

$$F_{\theta_1^0}(z)=Prob\{\theta_1^0\leq z\}=\begin{cases}0, & z<0\\ z, & z\in[0,1]\\ 1, & z>1\end{cases}.$$

$$F_Z(z)=[F_{\theta_1^0}(z)]^2=\begin{cases}0, & z<0\\ z^2, & z\in[0,1]\\ 1, & z>1\end{cases}.$$

Therefore,

$$f_Z(z)=\begin{cases}0, & z<0\\ 2z, & z\in[0,1]\\ 0, & z>1\end{cases}.$$

As a result,

$$E(Z)=\int_0^1 z\cdot 2zdz=\int_0^1 2z^2dz=2/3.$$

Therefore, $E[\theta_1^0 y_1(\theta^c) + \theta_2^0 y_2(\theta^c)]/2 = 1/3$. According to Eq (6), the designer's initial expected profit and utility are $\bar{p}_d(0) = \bar{u}_d(0) = 1/3$.

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References

[1] A. Mas-Colell, M.D. Whinston and J.R. Green, Microeconomic Theory, Oxford University Press, 1995.

[2] Y. Narahari et al, Game Theoretic Problems in Network Economics and Mechanism Design Solutions, Springer, 2009.

[3] R. Serrano, The Theory of Implementation of Social Choice Function, SIAM Review, vol.46, No.3, 377-414, 2004.

[4] R. Myerson, Optimal Auction Design, Mathematics of Operations Research, vol.6, No.1, 58-73, 1981.

[5] M. Engers and B. McManus, Charity Auctions, International Economic Review, vol.48, No.3, 953-994, 2007.

[6] V. Krishna, Auction Theory (Second Edition), Academic Press, 2010.