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Bayesian $MCMC$ analysis of periodic asymmetric power $GARCH$ models

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Abstract

A Bayesian $MCMC$ estimate of a periodic asymmetric power $GARCH$ (PAP-$GARCH$) model whose coefficients, power, and innovation distribution are periodic over time is proposed. The properties of the PAP-$GARCH$ model such as periodic ergodicity, finiteness of moments and tail behaviors of the marginal distributions are first examined. Then, a Bayesian $MCMC$ estimate based on Griddy-Gibbs sampling is proposed when the distribution of the innovationn of the model is standard Gaussian or standardized Student with a periodic degree of freedom. Selecting the orders and the period of the PAP-$GARCH$ model is carried out via the Deviance Information Criterion (DIC). The performance of the proposed Griddy-Gibbs estimate is evaluated through simulated and real data. In particular, applications to Bayesian volatility forecasting and Value-at-Risk estimation for daily returns on the S&P500 index are considered.

Keywords: Periodic Asymmetric Power $GARCH$ model, probability properties, Griddy-Gibbs estimate, Deviance Information Criterion, Bayesian forecasting, Value at Risk.

Mathematics Subject Classification: AMS 2000 Primary 62M10; Secondary 60F99

Proposed running head: Bayesian inference for PAP-GARCH models.

1. Introduction

Interest in $GARCH$ (generalized autoregressive conditionally heteroskedastic) models introduced by Engle (1982) and Bollerslev (1986) has increased drastically in the last three decades. These models, which aim to represent time series with a stochastic conditional volatility, have undergone an explosive evolution and seem nowadays to reach a full maturity stage. Numerous variants of the original $GARCH$ formulation

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have been proposed to reflect several well-known features of time series volatility such as (e.g. Francq and Zakoïan, 2010) volatility clustering, dependence without correlation, persistence in volatility, excess kurtosis, asymmetry in volatility (which means that negative and positive returns have different impact on the future volatility), and Taylor effect (which implies that absolute returns are more correlated than the squares). One of the most important formulations that has been found suitable to accommodate many of the mentioned stylized facts is the asymmetric power $GARCH$ model (AP-$GARCH$) proposed by Ding et *al.* (1993) and revisited by Pan et *al.* (2008) in an equivalent form (see also Hwang and Basawa, 2004; Hamadeh and Zakoïan, 2011; Francq and Zakoïan, 2013; Aknouche and Touche, 2015; Xia et *al.*, 2017). Compared to other competitive volatility models such as Markov Switching $GARCH$ models (e.g. Haas et *al.*, 2004; Francq and Zakoïan, 2008; Bauwens et *al.*, 2014), the AP-$GARCH$ specification has a simpler probability structure and is easier to estimate by maximum likelihood-type methods, a fact that makes it quite popular.

Despite its generality and usefulness, the AP-$GARCH$ model with rather time-invariant parameters seems unable to represent time series volatility whose distribution varies over time, such as volatility with seasonal or periodic behavior. In fact, it is often argued that time series volatilities, in particular the financial ones, exhibit a typical periodic pattern that cannot be adequately modeled by time-invariant parameter models. The day-of-the-week effect, the month-of-the-year effect and intraday high frequency return series are typical examples (e.g. Franses and Paap, 2000; Tsiakas, 2006; Osborn et *al.*, 2008; Aknouche, 2017; Bollerslev et *al.*, 2000; Smith, 2010; Rossi and Fantazani, 2015). Other non-financial applications of periodic volatility models include hourly electricity demand, intraday wind power and wind speed series (Ambach and Croonenbroeck, 2015; Ambach and Schmid, 2015; Ziel et *al.*, 2016).

Bollerslev and Ghysels (1996) introduced a periodic $GARCH$ (P-$GARCH$) model in which the volatility coefficients are periodic over time with period S. It is well recognized that the P-$GARCH$ model is sufficiently flexible and rich enough to represent periodicity in volatility and other useful characteristics (Franses and Paap, 2000, Osborn et *al.*, 2008; Rossi and Fantazani, 2015). However, it seems unable to capture some pathological features such as asymmetry in volatility (also called the leverage effect), Taylor's property and tail heaviness of the marginal distribution. In a mainly theoretical perspective, Aknouche et *al.* (2018) generalized the AP-$GARCH\,(1,1)$ model to the case where the coefficients, the power and the distribution of the model innovation are periodic over time. The proposed model, denoted by the acronym PAP-$GARCH_S\,(1,1)$ (periodic asymmetric power $GARCH$) is potentially advocated for the representation of many well-known features of return time series volatility, such as those captured by the P-$GARCH$ model and also the leverage effect, the Taylor effect and excess of kurtosis.

The existing literature on P-$GARCH$ models generally assumes the stationarity of the innovation, so the periodicity of the model is driven solely by the volatility coefficients (Bollerslev and Ghysels, 1996; Franses and Paap, 2000, Osborn et *al.*, 2008; Aknouche and Bibi, 2009; Aknouche and Al-Eid, 2012; Rossi and Fantazani, 2015; Ziel, 2015-2016). In many applications, this might be insufficient to represent certain seasonal return series that are characterized by time-varying shape marginal distributions. In the PAP-$GARCH_S\,(1,1)$ model proposed by Aknouche et *al.* (2018), the periodicity is rather manifested through both the volatility parameters and the distribution of the innovation sequence. This makes the model more flexible in representing periodic volatility, at just a minor cost of a few additional parameters.

For a general class of periodic conditionally heteroskedastic time series models that encompasses the PAP-$GARCH$ model, Aknouche et *al.* (2018) established the strong consistency and asymptotic normality of the generalized quasi-maximum likelihood estimate ($GQMLE$) under general weak and tractable assumptions. Despite many advantages of the $GQMLE$, an estimate based on the Bayesian approach might be an attractive addition. Indeed, such an estimate may have better finite properties, especially for small powers, and allows a reproducible Bayesian inference in a flexible manner (ex. Bauwens and Lubrano, 1998; Ardia, 2008; Xia et *al.*, 2017). In particular, Bayesian order and period selection via the DIC (Deviance Information Criterion, Spiegelhalter et *al.*, 2002), Bayesian forecasting of the volatility and Bayesian estimation of the conditional value at risk (VaR) are interesting applications (e.g. Aknouche, 2017, Xia et *al.*, 2017).

This paper deals with Bayesian $MCMC$ (Monte Carlo Markov Chains) inference for the general order PAP-$GARCH_S\,(p,q)$ model. First, some stability properties of this model such as periodic ergodicity, existence of moments and tail behavior of the S marginal distributions are examined. Next, an estimate based on the Bayesian Griddy-Gibbs sampling (cf. Ritter and Tanner 1992) is proposed. This technique is an extension of the Gibbs sampler to the case where the posterior density of some parameters has a complex or non-standard form, as in the PAP-$GARCH_S\,(p,q)$ case. Non-parametric diagnostic tools corresponding to the Griddy-Gibbs procedure, such as the Relative Numerical Inefficiency (RNI), the Numerical Standard Error (NSE) and some $MCMC$ correlation measurements are utilized while model selection is carried out using the DIC. The PAP-$GARCH$ model is then compared, through simulated and real series, to some of its subclasses, namely the P-$GARCH$ model, the periodic threshold $GARCH$ (PT-$GARCH$) model and the non-periodic AP-$GARCH$ model. Furthermore, applications to Bayesian forecasting of the volatility and the conditional VaR for the return of the $S\&P500$ index are presented. For this, two distributions of the innovation sequence are assumed. The first one is the standard Gaussian distribution making the innovation independent and identically distributed (iid). In the second case the innovation is rather independent and S-periodically distributed (ipd_S) having a standardized Student distribution with a S-periodic degree of freedom.

The rest of this paper proceeds as follows. Section 2 studies the structure of the PAP-$GARCH_S(p,q)$ model, namely periodic ergodicity, existence of moments and tail behavior of the S marginal distributions. Section 3 develops a Bayesian Griddy Gibbs estimate ($BGGE$) for the model. Section 4 assesses the performance of the $BGGE$ in finite samples through simulation experiments. Real applications to the $S\&P500$

returns under both the Gaussian and Student assumptions on the innovation sequence are considered in Section 5. Section 6 concludes while proofs of the main results are postponed to Section 7.

2. Structure of the $PAP\text{-}GARCH_S\,(p,q)$ model

Let $\{\epsilon_t, t\in\mathbb{Z}\}$ be a $PAP\text{-}GARCH_S\,(p,q)$ process with period S and orders p and q, given by the following equation

$$\begin{cases} \epsilon_t = \sigma_t\eta_t \\ \sigma_t^{\delta_t} = \omega_t + \sum_{i=1}^{q} \alpha_{ti+}(\epsilon_{t-i}^{+})^{\delta_{t-i}} + \alpha_{ti-}(\epsilon_{t-i}^{-})^{\delta_{t-i}} + \sum_{j=1}^{p} \beta_{tj}\sigma_{t-j}^{\delta_{t-j}} \end{cases}, \quad t\in\mathbb{Z}, \tag{2.1}$$

where $x^+ = \max(x,0)$, $x^- = -\min(x,0)$ and $\{\eta_t, t\in\mathbb{Z}\}$, called the model innovation, is a sequence of independent and S-periodically distributed (ipd_S) unobservable random variables ($S\geq 1$) such that η_t is independent of $\{\epsilon_i, i<t\}$. The volatility parameters $\omega_t>0$, $\alpha_{ti+}\geq 0$, $\alpha_{ti-}\geq 0$, $\beta_{tj}\geq 0$, and $\delta_t>0$ ($1\leq i\leq q$, $1\leq j\leq p$) are S-periodic over t. To emphasize the periodicity of model (2.1) we rewrite it in the following form

$$\begin{cases} \epsilon_{Sn+v} = \sigma_{Sn+v}\eta_{Sn+v} \\ \sigma_{Sn+v}^{\delta_v} = \omega_v + \sum_{i=1}^{q} \alpha_{vi+}(\epsilon_{Sn+v-i}^{+})^{\delta_{v-i}} + \alpha_{vi-}(\epsilon_{Sn+v-i}^{-})^{\delta_{v-i}} + \sum_{j=1}^{p} \beta_{vj}\sigma_{Sn+v-j}^{\delta_{v-j}} \end{cases} \quad \begin{matrix} 1\leq v\leq S, \\ n\in\mathbb{Z}, \end{matrix} \tag{2.2}$$

where for all $1\leq v\leq S$, the vth season (or channel) stands for the set $\{..., v-S, v, v+S, ...\}$. Model (2.1) proposed by Aknouche et *al.* (2018) for the case $p=q=1$ is quite general and covers a wide range of well-known $GARCH$-type models. For $S=1$, it is just the asymmetric power $GARCH$ ($AP\text{-}GARCH\,(p,q)$) model proposed by Ding et *al.* (1993) and revisited by Pan et *al.* (2008) (see also Francq and Zakoïan , 2013). It reduces to the periodic $GARCH\,(p,q)$ when $\delta_v=2$ and $\alpha_{v+}=\alpha_{v-}$ ($1\leq v\leq S$), to the periodic power $GARCH\,(p,q)$ when $\alpha_{v+}=\alpha_{v-}$ ($1\leq v\leq S$) and to the periodic threshold $GARCH\,(p,q)$ when $\delta_v=1$ for all $1\leq v\leq S$. Beside the stylized facts captured by the $AP\text{-}GARCH\,(p,q)$ model such as the leverage effect and the Taylor property (e.g. Granger, 2005; Haas, 2009; Aknouche and Touche, 2015), model (2.1) might also account for periodicity in volatility (see also Aknouche et *al.*, 2018 in the case $p=q=1$).

To study the probabilistic structure of the $PAP\text{-}GARCH_S\,(p,q)$ model (2.1), it is customary to put the model in a stochastic recurrence equation with rather ipd_S coefficients. Let $r=p+2q-2$,

$$\begin{aligned} \zeta_t &= \beta_{t+1,1} + \alpha_{t+1,1+}(\eta_t^{+})^{\delta_t} + \alpha_{t+1,1-}(\eta_t^{-})^{\delta_t}, \\ Y_t &= \left(\sigma_{t+1}^{\delta_{t+1}}, ..., \sigma_{t-p+2}^{\delta_{t-p+2}}, (\epsilon_t^{+})^{\delta_t}, (\epsilon_t^{-})^{\delta_t}, ..., (\epsilon_{t-q+2}^{+})^{\delta_{t-q+2}}, (\epsilon_{t-q+2}^{-})^{\delta_{t-q+2}}\right)' \in\mathbb{R}^r, \\ B_t &= (\omega_{t+1}, 0, ..., 0)\in\mathbb{R}^r, \end{aligned}$$

and

$$A_{t-1} = \begin{pmatrix} \zeta_{t-1} & \beta_{t,2} & \cdots & \beta_{t,p-1} & \beta_{t,p} & \alpha_{t,2+} & \alpha_{t,2-} & \cdots & \alpha_{t,q-1+} & \alpha_{t,q-1-} & \alpha_{t,q+} & \alpha_{t,q-} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ (\eta_{t-1}^{+})^{\delta_{t-1}} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ (\eta_{t-1}^{-})^{\delta_{t-1}} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \end{pmatrix}$$

Equation (2.1) may be cast in the following stochastic recurrence equation

$$Y_t = A_t Y_{t-1} + B_t, \qquad t \in \mathbb{Z}, \tag{2.3}$$

where $\{(A_t, B_t), t \in \mathbb{Z}\}$ is an ipd_S sequence valued in $\mathcal{M}_r(\mathbb{R}) \times \mathbb{R}^r$, $\mathcal{M}_r(\mathbb{R})$ being the set of square matrices with dimension r. Let $\gamma^{(S)}$ be the top Lyapunov exponent associated with the recurrence equation (2.3) which is given by (cf. Aknouche and Bibi, 2009)

$$\gamma^{(S)} = \inf \left\{ \frac{1}{n} E \log \| A_{nS} ... A_2 A_1 \|, \; n \geq 1 \right\}. \tag{2.4}$$

The following result gives necessary and/or sufficient conditions for equation (2.1) to have a unique strictly periodically stationary and periodically ergodic solution (see e.g. Aknouche et *al.*, 2018 for the definition of periodic ergodicity).

Theorem 2.1 i) *Assume that* $E \log |\eta_v|^{\delta_v} < \infty$ *for all* $1 \leq v \leq S$. *A necessary and sufficient condition for model* (2.3) *to have a unique nonanticipative strictly periodically stationary and periodically ergodic solution is that*

$$\gamma^{(S)} < 0. \tag{2.5}$$

This solution is given by

$$X_t = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} A_{t-i} B_{t-j}, \qquad t \in \mathbb{Z}, \tag{2.6}$$

where the series in the right hand side of (2.6) *converges absolutely almost surely.*

ii) If (2.3) *admits a strictly periodically stationary solution then*

$$\rho \left(\prod_{v=0}^{S-1} \boldsymbol{\beta}_{S-v} \right) < 1, \tag{2.7}$$

where $\boldsymbol{\beta}_t$ *is the submatrix of* A_t *defined by*

$$\boldsymbol{\beta}_t = \begin{pmatrix} \beta_{t,1}\beta_{t,2}\dots\beta_{t,p-1} & \beta_{t,p} \\ I_{(p-1)\times(p-1)} & 0_{(p-1)\times 1} \end{pmatrix},$$

and $\rho(A)$ *denotes the spectral radius of the squared matrix* A, *i.e. the maximum modulus of the eigenvalues of* A.

In the special case $p = q = 1$, condition (2.5) reduces to

$$\frac{1}{S}\sum_{v=1}^{S} E\left(\log\left(\alpha_{0v+}(\eta_{v-1}^{+})^{\delta_{v-1}} + \alpha_{0v-}(\eta_{v-1}^{-})^{\delta_{v-1}} + \beta_{0v}\right)\right) < 0,$$

(cf. Aknouche et *al.*, 2018) while (2.7) is just $\prod_{v=0}^{S-1}\beta_{0v} < 1$. Now conditions for the existence of moments of the $PAP\text{-}GARCH_S(p,q)$ process are given as follows.

Theorem 2.2 i) *If* $\gamma^S(A) < 0$ *then there is* $\kappa > 0$ *such that for all* t

$$E\left(\sigma_t^{\kappa}\right) < \infty \quad \text{and} \quad E\left(|\epsilon_t|^{\kappa}\right) < \infty. \tag{2.8}$$

ii) *Let* $\{\epsilon_t, t\in\mathbb{Z}\}$ *be a strictly periodically stationary solution of* (2.1). *A necessary and sufficient condition for* $E\left(\epsilon_t^{m\delta_t}\right)$ *(*$m\in\mathbb{N}^*$, $1\leq t\leq S$*) to be finite is that*

$$\rho\left(\prod_{v=0}^{S-1} E\left(A_{S-v}^{\otimes m}\right)\right) < 1, \tag{2.9}$$

where $A^{\otimes m}$ *is the Kronecker product:* $A\otimes A\otimes\cdots\otimes A$ *with* m *factors.*

The following result shows that the S marginal distributions of the $PAP\text{-}GARCH_S(p,q)$ model are regularly varying provided that some limiting moment conditions are satisfied. Thus heavy-tailed marginals can be obtained for the $PAP\text{-}GARCH_S(p,q)$ model even when its input innovation sequence $\{\eta_t, t\in\mathbb{Z}\}$ has light-tailed distributions.

Theorem 2.3 *Assume model* (2.1) *satisfies the following three conditions: i)* $\gamma^{(S)} < 0$*, ii)* η_t *has* S *positive densities on* $\mathbb{R}$ *such that* $E|\eta_v|^{\tau_v} < \infty$ *for some* $\tau_v > 0$ $(1\leq v\leq S)$*, and iii)* $\min_{1\leq v\leq S}(\omega_v) > 0$. *Then for all* $1\leq v\leq S$,

$$P(\epsilon_v > x) \sim c_v x^{-\alpha\delta_v},$$

where $c_v > 0$ *and* α *is the unique solution of*

$$\lim_{n\to\infty}\frac{1}{n}\log E\left\|A_{nS+v}A_{nS+v-1}\dots A_{v+2}A_{v+1}\right\|^{\frac{\alpha}{2}} = 0.$$

3. Bayesian Griddy-Gibbs estimation

3.1. Gibbs sampler: prior and posterior analysis

Let $\boldsymbol{\epsilon}^T = (\epsilon_1, \ldots, \epsilon_T)'$ be a series generated from the $PAP\text{-}GARCH_S\,(p,q)$ model (2.1) with sample-size $T = NS$ $(N \geq 1)$. We first need to specify the distribution of the ipd_S innovation $\{\eta_t, t \in \mathbb{Z}\}$. Consider the following three cases:

Case i) *The pure Gaussian case*: $\eta_1, ..., \eta_S$ are normally distributed with mean zero and unit variance $(\eta_v \sim N(0,1))$, i.e.

$$f(\eta_v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\eta_v^2} \qquad (1 \leq v \leq S).$$

Case ii) *The pure Student case*: $\eta_1, ..., \eta_S$ are (standardized) Student distributed with degrees of freedom $\tau_1, ..., \tau_S > 2$, respectively $(\frac{\tau_v}{\tau_v - 2}\eta_v \sim t_{(\tau_v)})$, i.e.

$$f(\eta_v) = \frac{1}{\sqrt{\pi(\tau_v - 2)}} \frac{\Gamma\left(\frac{\tau_v + 1}{2}\right)}{\Gamma\left(\frac{\tau_v}{2}\right)} \left(1 + \frac{\eta_v^2}{(\tau_v - 2)}\right)^{-\frac{\tau_v + 1}{2}} \qquad (1 \leq v \leq S).$$

Case iii) *The mixed Gaussian-Student case*: it is assumed that for certain seasons $\{v_1, ..., v_k\} \subset \{1, ..., S\}$ $(1 \leq k \leq S-1)$ the innovations $\eta_{v_1}, ..., \eta_{v_k}$ are normally distributed with mean zero and unit variance. For the remaining seasons $\{v_{k+1}, ..., v_S\}$, η_{v_j} has a standardized Student distribution with degree of freedom $\tau_{v_j} > 2$ $(j = k+1, ..., S)$. Of course, Case i) and Case ii) are particular instances of Case iii) and they correspond to $k = S$ and $k = 0$ respectively.

Observe that the innovation $\{\eta_t, t \in \mathbb{Z}\}$ is *iid* in the first case and is ipd_S in the second and third cases. The choice of k and $\{v_1, ..., v_k\}$ is motivated by practical considerations such as the empirical Kurtosis of each season, the shapes of the seasonal empirical distributions, etc. Effective model selection measures such as the DIC may allow to select the best model. From now on the formal description of the method is shown for the general Case iii).

Assuming the power δ_v $(1 \leq v \leq S)$ and τ_{v_j} $(k+1 \leq j \leq S)$ unknown, the parameter vector to be estimated is denoted by $\theta = \left(\tau', \theta_1', \cdots, \theta_S'\right)' \in (0,\infty)^{r^*}$ where $r^* = (p + 2q + 2)\,S + (S-k)$, $\tau' = \left(\tau_{v_{k+1}}, ..., \tau_{v_S}\right)'$, $\theta_v' = \left(\omega_v, \alpha_{v+}', \alpha_{v-}', \beta_v', \delta_v\right)' \in (0,\infty)^{(p+2q+2)}$, $\alpha_{v+} = (\alpha_{v1+}, ..., \alpha_{vq+})'$, $\alpha_{v-} = (\alpha_{v1-}, ..., \alpha_{vq-})'$ and $\beta_v = \left(\beta_{v1}, ..., \beta_{vp}\right)'$ $(1 \leq v \leq S)$.

Adopting the Bayesian approach, the parameter vector θ is viewed as random with a prior distribution $f(\theta)$. The goal is thus to make inference about the posterior distribution $f\left(\theta/\boldsymbol{\epsilon}^T\right)$ which satisfies the following proportionality

$$f(\theta/\boldsymbol{\epsilon}^T) \propto f(\theta) f(\boldsymbol{\epsilon}^T/\theta), \tag{3.1}$$

where $f(\boldsymbol{\epsilon}^T/\theta)$ is the likelihood function given by

$$f(\boldsymbol{\epsilon}^T/\theta) = \prod_{t=1}^{T} f(\epsilon_t/\boldsymbol{\epsilon}^{t-1},\theta) = \prod_{n=0}^{N-1}\prod_{v=1}^{S} l_{v+nS}(\theta) \tag{3.2a}$$

$$l_{v+nS}(\theta) = \begin{cases} \frac{1}{\sigma_{v+nS}(\theta)} e^{-\frac{\epsilon^2_{v+nS}}{\sigma^2_{v+nS}(\theta)}} & \text{if } v \in \{v_1,...,v_k\},\ 0 \le k \le S \\ \frac{1}{\sigma_{v+nS}(\theta)\sqrt{(\tau_v-2)}} \frac{\Gamma\left(\frac{\tau_v+1}{2}\right)}{\Gamma\left(\frac{\tau_v}{2}\right)} \left(1 + \frac{\epsilon^2_{v+nS}}{(\tau_v-2)\sigma^2_{v+nS}(\theta)}\right)^{-\frac{\tau_v+1}{2}} & \text{if } v \in \{v_{k+1},...,v_S\} \end{cases} \tag{3.2b}$$

$$\sigma_{v+nS}(\theta) = \left(\omega_v + \sum_{i=1}^{q} \alpha_{vi+}(\epsilon^+_{Sn+v-i})^{\delta_{v-i}} + \alpha_{vi-}(\epsilon^-_{Sn+v-i})^{\delta_{v-i}} + \sum_{j=1}^{p} \beta_{vj}\sigma^{\delta_{v-j}}_{Sn+v-j}(\theta)\right)^{\frac{1}{\delta_v}}. \tag{3.2c}$$

Because of the periodic structure of the $PAP\text{-}GARCH_S\,(p,q)$ model, it is natural to assume that the parameters $\theta_1, \cdots, \theta_S$ are independent of each other. The joint posterior distribution $f\left(\theta/\boldsymbol{\epsilon}^T\right)$ can then be estimated using the Gibbs sampler provided we can draw samples from any of the r^* conditional posterior distributions: $f(\tau_{v_j}/\theta_{-\{\tau_{v_j}\}}, \boldsymbol{\epsilon}^T)$ $(k+1 \le j \le S)$, $f(\delta_v/\theta_{-\{\delta_v\}}, \boldsymbol{\epsilon}^T)$, $f(\omega_v/\theta_{-\{\omega_v\}}, \boldsymbol{\epsilon}^T)$, $f(\alpha_{vi+}/\theta_{-\{\alpha_{vi+}\}}, \boldsymbol{\epsilon}^T)$, $f(\alpha_{vi-}/\theta_{-\{\alpha_{vi-}\}}, \boldsymbol{\epsilon}^T)$ $(i = 1,...,q)$ and $f(\beta_{vj}/\theta_{-\{\beta_{vj}\}}, \boldsymbol{\epsilon}^T)$ $(j = 1,...,p)$ $(1 \le v \le S)$, where $x_{-\{\theta_j\}}$ denotes the vector obtained from x after removing the parameter component θ_j.

To get the latter conditional posterior distributions, prior distributions of the parameter components are to be determined. In general, the choice of conjugate priors is appealing as it simplifies the analysis. However, since the volatility $\sigma_t(\theta)$ is a nonlinear function of θ with a feedback mechanism, it is difficult to find conjugate priors for this model and the conditional posterior distributions would be of complicated expressions that are difficult to sample directly (see e.g. Bauwens and Lubrano, 1998; Xia et *al.*, 2017). This is why noninformative priors are used. They are given for all $1 \le v \le S$ and $k+1 \le j \le S$ as follows

$$\begin{aligned} \omega_v &\sim U_{(0,A)},\ \alpha_{v+} \sim U_{(0,A)^q},\ \alpha_{v-} \sim U_{(0,A)^q}, \\ \beta_v &\sim U_{(0,A)^p},\ \delta_v \sim U_{(a,A)},\ \tau_{v_j} \sim U_{(2,A)}, \end{aligned}$$

where U_D denotes the uniform distribution on the set D, A is a fairly large positive number and $a < A$ is a quite small positive number. To simplify the computation we reparametrize the standardized Student distribution with $\psi_{v_j} = \frac{1}{\tau_{v_j}}$. The prior distribution for ψ_v is then

$$\psi_{v_j} \sim U_{\left(\frac{1}{A},0.2\right)},\ (1 \le v \le S).$$

Based on the above priors, the conditional posterior distributions of ψ_{v_j}, δ_v, ω_v, α_{v+}, α_{v-} and β_v can easily be derived from (3.1), except for a scale factor. For example, the kernel of the conditional posterior of δ_v is written as follows

$$f(\delta_v/\theta_{-\{\delta_v\}}, \boldsymbol{\epsilon}^T) \propto \prod_{n=0}^{N-1}\prod_{v=1}^{S} l_{v+nS}(\theta), \qquad 1 \le v \le S, \tag{3.3}$$

where $l_{v+nS}(\theta)$ is given by (3.2b)-(3.2c). The kernels of the remaining distributions $f(\psi_{v_j}/\theta_{-\{\tau_{v_j}\}},\boldsymbol{\epsilon}^T)$, $(k+1\leq j\leq S)$, $f(\omega_v/\boldsymbol{\epsilon}^T,\theta_{-\{\omega_v\}})$, $f(\alpha_{vi+}/\boldsymbol{\epsilon}^T,\theta_{-\{\alpha_{vi+}\}})$, $f(\alpha_{vi-}/\boldsymbol{\epsilon}^T,\theta_{-\{\alpha_{vi-}\}})$ $(i=1,...,q)$ and $f(\beta_{vj}/\boldsymbol{\epsilon}^T,\theta_{-\{\beta_{vj}\}})$ $(1\leq v\leq S)$ are obtained as in (3.3). However, the parameters ω_v, α_{v+}, α_{v-} and β_v are restricted to lie in the periodic stationarity domain described by (2.5).

3.2. Griddy-Gibbs estimate

Once determining the kernel of $f\left(\theta_j/\boldsymbol{\epsilon}^T,\theta_{-\{\theta_j\}}\right)$ $(j=1,...,r^*)$, we may use some indirect sampling algorithms to draw each component of θ. We chose the Griddy-Gibbs sampler (Ritter and Tanner, 1992) whose implementation seems simple in our context. We illustrate its principle on the power parameter δ_v $(1\leq v\leq S)$. The same scheme may be done for the remaining parameters $\psi_{v_j}=\frac{1}{\tau_{v_j}}$ $(k+1\leq j\leq S)$ and $\left(\omega_v,\alpha'_{v+},\alpha'_{v-},\beta'_v\right)'$ $(1\leq v\leq S)$.

Griddy scheme

1) Select a grid of g points $\delta_{v1}\leq\delta_{v2}\leq...\leq\delta_{vg}$ from a given interval $[\delta_{v1},\delta_{vg}]$; then evaluate the conditional posterior $f(\delta_v/\theta_{-\{\delta_v\}},\boldsymbol{\epsilon}^T)$ at each one of these points, giving $f_{vs}=f(\delta_{vs}/\boldsymbol{\epsilon}^T,\theta_{-\{\delta_v\}})$, $(1\leq s\leq g)$.

2) From the values $f_{v1},f_{v2},...,f_{vg}$, build the discrete distribution $p_v(.)$ defined at δ_{vs} $(1\leq s\leq g)$ by $p(\delta_{vs})=\frac{f_{vs}}{\sum\limits_{j=1}^{g}f_{vj}}$. This may be seen as an approximation to the inverse cumulative distribution of $f(\delta_v/\theta_{-\{\delta_v\}},\boldsymbol{\epsilon}^T)$.

3) Generate a number from the uniform distribution on $(0,1)$ and transform it using the discrete distribution $p(.)$ obtained in 2) to get a random draw for δ_v.

The following algorithm summarizes the Griddy-Gibbs sampler for drawing from the conditional posterior distribution $f\left(\theta/\boldsymbol{\epsilon}^T\right)$. For $l=0,1,...,L$, let $\theta^{(l)}$ be the Griddy-Gibbs draw of θ at the l-th Gibbs iteration.

Algorithm 3.1 (Griddy Gibbs sampler for the $PAP\text{-}GARCH_S(p,q)$ model)

Step 0 Specify starting values $\psi_{v_j}^{(0)}$, $\theta_v^{(0)}=\left(\omega_v^{(0)},\alpha_{v+}^{(0)\prime},\alpha_{v-}^{(0)\prime},\beta_v^{(0)\prime},\delta_v^{(0)}\right)'$ $(k+1\leq j\leq S,\ 1\leq v\leq S)$.

Step 1 Repeat for $l=0,1,...,L-1$.

a) For $1\leq v\leq S$, sample δ_v from $f(\delta_v/\theta_{-\{\delta_v\}},\boldsymbol{\epsilon}^T)$ using the following Griddy scheme:

a1) Select a grid $\delta_{v1}^{(l+1)}\leq\delta_{v2}^{(l+1)}\leq...\leq\delta_{vg}^{(l+1)}$.

a2) For $1\leq s\leq g$ calculate $f_{vs}^{(l+1)}=f(\delta_{vs}^{(l+1)}/\boldsymbol{\epsilon}^T,\theta^{(l)}_{-\{\delta_v^{(l)}\}})$ from (3.3) and define the inverse distribution $p\left(\delta_{vs}^{(l+1)}\right)=\dfrac{f_{vs}^{(l+1)}}{\sum_{j=1}^{g}f_{vj}^{(l+1)}}$, $1\leq s\leq g$.

a3) Generate a number u from the uniform $(0, 1)$ distribution and transform it using the inverse distribution $p\,(.)$ to get $\delta_v^{(l+1)}$.

b) Using a Griddy step similarly to a), sample from $f(\omega_v/\boldsymbol{\epsilon}^T, \theta^{(l)}_{-\{\omega_v^{(l)}\}})$ to get $\omega_v^{(l+1)}$.

c) Using a Griddy step similarly to a), sample from $f(\alpha_{vi+}/\boldsymbol{\epsilon}^T, \theta^{(l)}_{-\{\alpha_{vi+}^{(l)}\}})$ to get $\alpha_{vi+}^{(l+1)}$, $1 \leq i \leq q$.

d) Using a Griddy step similarly to a), sample from $f(\alpha_{vi-}/\boldsymbol{\epsilon}^T, \theta^{(l)}_{-\{\alpha_{vi-}^{(l)}\}})$ to get $\alpha_{vi-}^{(l+1)}$, $1 \leq i \leq q$.

e) Using a Griddy step similarly to a), sample from $f(\beta_{vj}/\boldsymbol{\epsilon}^T, \theta^{(l)}_{-\{\beta_{vj}^{(l)}\}})$ to get $\beta_{vj}^{(l+1)}$, $1 \leq j \leq p$.

f) Using a Griddy step similarly to a), sample from $f(\psi_{v_j}/\boldsymbol{\epsilon}^T, \theta^{(l)}_{-\{\psi_{v_j}^{(l)}\}})$ to get $\psi_{v_j}^{(l+1)}$, $k+1 \leq j \leq S$.

Step 3 Return $\theta^{(l)}$, $l = 1, ..., L$.

The Griddy-Gibbs estimate $\widehat{\theta}_G$ of θ is obtained by averaging the posterior draws of θ giving

$$\widehat{\theta}_G = \tfrac{1}{L}\sum_{l=1}^{L} \theta^{(l)}. \tag{3.4}$$

It is important to note that the efficiency of the Griddy scheme is very sensitive to the choice of the grid $\{\delta_{v1}, ..., \delta_{vg}\}$. We follow here a similar choice by Tsay (2010) which, at the l-th Gibbs iteration, consists in taking the range of δ_v to be $\left[\delta_{vl}^1, \delta_{vl}^2\right]$ where

$$\delta_{vl}^1 = 0.6\max(\delta_v^{(0)}, \delta_v^{(l-1)}), \quad \delta_{vl}^2 = 1.4\min(\delta_v^{(0)}, \delta_v^{(l-1)}), \tag{3.5}$$

$\delta_v^{(l-1)}$ and $\delta_v^{(0)}$ being, respectively, the estimate of δ_v at the $(l-1)$-th Gibbs iteration and the initial value. Even if this choice may greatly depend on the initial parameter draws, it gives, however, quite satisfactory results that are not case-sensitive as indicated by the used $MCMC$ diagnostic tools (cf. Section 3.2 below). On the other hand, it is well-known that the Griddy algorithm could be enhanced by considering a trapezoidal integration or the Simpson rule (Bauwens and Lubrano, 1998; Bauwens et *al.*, 2013). However, in our context, the use of the griddy Step a2) in Algorithm 3.1 and the variable grid (3.5) seems suitable, from which fairly good estimates can be obtained (cf. Section 4) without using too sophistical computational devices that could render the computation infeasible.

3.3. *MCMC* Diagnostic tools

In order to assess the convergence of the BGG algorithm, some $MCMC$ diagnostic tools such as the autocorrelation of posterior draws, the Relative Numerical Inefficiency (RNI, Geweke, 1989) and the Numerical Standard Error (NSE, Geweke, 1989) are considered. The RNI gives a broad indication on the inefficiency due to the serial correlation of the BGG draws. It is a complementary tool to the autocorrelations of parameter draws and indicates how the posterior draws mixe well. It is given by

$$RNI = 1 + 2\sum_{h=1}^{B} K\left(\tfrac{h}{B}\right)\widehat{\rho}_h,$$

where $B = 500$ is the bandwidth, $K(.)$ is the Parzen kernel (e.g. Kim et *al.*, 1998) and $\widehat{\rho}_h$ the sample autocorrelation for the lag h of the BGG parameter draws.

The NSE is the square-root of the estimated asymptotic variance of the $MCMC$ estimator. It is given by

$$NSE = \sqrt{\frac{1}{L}\left(\widehat{\gamma}_0 + 2\sum_{h=1}^{B} K\left(\tfrac{h}{B}\right)\widehat{\gamma}_h\right)},$$

where $\widehat{\gamma}_h$ is the sample autocovariance at lag h of the BGG parameter draws.

3.4. Model selection via the Deviance Information Criterion

Selecting the best $PAP\text{-}GARCH_S(p,q)$ model among several other candidates, which includes order and period selection, is carried out using the DIC (Spiegelhalter et *al.*, 2002). This criterion which may be seen as a Bayesian generalization of the AIC (Akaike Information Criterion) is easily obtained from $MCMC$ draws, needing no extra-calculations. In the context of the $PAP\text{-}GARCH_S(p,q)$, $DIC := DIC(p,q,S)$ is defined to be

$$DIC = -4E_{\theta/\boldsymbol{\epsilon}^T}\left(\log\left(f\left(\boldsymbol{\epsilon}^T/\theta\right)\right)\right) + 2\log\left(f\left(\boldsymbol{\epsilon}^T/\overline{\theta}\right)\right),$$

where $f\left(\boldsymbol{\epsilon}^T/\theta\right)$ is the likelihood given by (3.2) and $\overline{\theta} = E\left(\theta/\boldsymbol{\epsilon}^T\right)$ is the posterior mean of θ. From the Griddy-Gibbs draws, the expectation $E_{\theta/\boldsymbol{\epsilon}^T}\left(\log\left(f\left(\boldsymbol{\epsilon}^T/\theta\right)\right)\right)$ can be estimated by averaging the conditional log-likelihood, $\log f\left(\boldsymbol{\epsilon}^T/\theta\right)$, over the posterior draws of θ. Moreover, the joint posterior mean estimate of $\overline{\theta}$ can be approximated by the mean of the posterior draws of $(\theta^{(l)})_{1\leq l\leq L}$.

3.5. Bayesian forecasting

Now we show how to get in-sample and out-of-sample predictions of the volatility and the Value at Risk (VaR) in $PAP\text{-}GARCH_S(p,q)$ model using a simulation-based approach (see e.g. Chen and So, 2006; Hoogerheide and van Dijk, 2010; Xia et *al.*, 2017 among others). Once generating the posterior draws $\theta^{(l)}$ $(l = 1, ..., L)$ from Algorithm 3.1, we can use them to readily generate in-sample volatilities $(\sigma_t^{2(l)}, t = 1, ..., T)$ according to $(3.2c)$ while replacing θ by $\theta^{(l)}$, i.e.

$$\begin{aligned}
\sigma_{nS+v}^{2(l)} &= \sigma_{nS+v}^{2}\left(\theta^{(l)}\right), \; l = 1, ..., L, \\
&= \left(\omega_v^{(l)} + \sum_{i=1}^{q}\alpha_{vi+}^{(l)}(\epsilon_{Sn+v-i}^{+})^{\delta_{v-i}^{(l)}} + \alpha_{vi-}^{(l)}(\epsilon_{Sn+v-i}^{-})^{\delta_{v-i}^{(l)}} + \sum_{j=1}^{p}\beta_{vj}^{(l)}\sigma_{Sn+v-j}^{\delta_{v-j}^{(l)}}(\theta^{(l)})\right)^{\frac{2}{\delta_v}}.
\end{aligned} \tag{3.6}$$

Thus $\left(\sigma_t^{2(l)}\right)_{1\leq l\leq L}$ may be seen as a posterior $MCMC$ sample from $f\left(\sigma_t^2/\boldsymbol{\epsilon}^T\right)$, so a Bayesian in-sample estimate of σ_t^2 is given by

$$\widehat{\sigma}_t^2 = \frac{1}{L}\sum_{l=1}^{L}\sigma_t^{2(l)}, \ \ t = 1, ..., T. \tag{3.7}$$

To forecast future volatilities σ_{T+h}^2 $(h = 1, 2, ...)$ we use a sequential method on h as follows:

i) For $h = 1$, since σ_{T+1}^2 depends on $\epsilon_T, \epsilon_{T-1}, ...$ which are available in the sample, we can easily compute $\sigma_{T+1}^{2(l)}$ $(1 \leq l \leq L)$ from (3.6). This may be seen as a posterior sample from the predictive distribution $f\left(\sigma_{T+1}^2/\boldsymbol{\epsilon}^T\right)$. Therefore, the volatility forecast $\widehat{\sigma}_{T+1}^2$ is given as in (3.7) by $\widehat{\sigma}_{T+1}^2 = \frac{1}{L}\sum\limits_{l=1}^{L}\sigma_{T+1}^{2(l)}$. Then, $\epsilon_{T+1}^{(l)}$ $(1 \leq l \leq L)$ can be generated from $f\left(\epsilon_{T+1}/\boldsymbol{\epsilon}^T, \theta^{(l)}\right)$ using $(3.2a)$ and the one-step ahead predicted return is given by $\widehat{\epsilon}_{T+1} = \frac{1}{L}\sum\limits_{l=1}^{L}\epsilon_{T+1}^{(l)}$.

ii) For $h = 2$, with $\epsilon_{T+1}^{(l)}$ available, we can generate $\sigma_{T+2}^{2(l)}$ $(1 \leq l \leq L)$ using (3.6). Then $\epsilon_{T+2}^{(l)}$ $(1 \leq l \leq L)$ can be generated from $f\left(\epsilon_{T+2}/\boldsymbol{\epsilon}^{T+1}, \theta^{(l)}\right)$ using again $(3.2a)$. This may be utilized to sample $\sigma_{T+3}^{2(l)}$ $(1 \leq l \leq L)$ in the following step.

iii) For $h \geq 3$ we can sequentially repeat the above steps i) and ii).

As a by-product of the above volatility prediction scheme, forecasting the VaR is useful for evaluating risk market (e.g. Francq and Zakoïan , 2010). Under the $PAP\text{-}GARCH_S\,(p, q)$ model, the one-step VaR_{T+1} at the significance level ϕ is the quantile of ϵ_{T+1} (to within a sign "$-$") at level ϕ, i.e., $\phi = P\left(\epsilon_{T+1} \leq -VaR_{T+1}\right)$. More explicitly, it is given by

$$VaR_{T+1} = -F_{T+1}^{-1}(\phi)\sqrt{\sigma_{T+1}^2(\theta)},$$

where $F_{T+1}^{-1}(.)$ is the inverse of the probability cumulative function of η_{T+1} and is S-periodic over time due to the periodic stationarity of the innovation $\{\eta_t, t \in \mathbb{Z}\}$. The l-th draw of VaR is then given by $VaR_{T+1}^{(l)} = -F_{T+1}^{-1}(\phi)\sqrt{\sigma_{T+1}^2\left(\theta^{(l)}\right)}$ $(1 \leq l \leq L)$ and is readily obtained from the volatility forecast sample $\left(\sigma_{T+1}^2\left(\theta^{(l)}\right)\right)_{1\leq l\leq L}$. Hence, the estimated one-period ahead VaR for the time $T+1$ is given by

$$\widehat{VaR}_{T+1} = \frac{1}{L}\sum_{l=1}^{L}VaR_{T+1}^{(l)}.$$

More generally, the h-step VaR_{T+h} $(h \geq 1)$ is defined to be the ϕ-quantile of $\epsilon_{T+1} + ... + \epsilon_{T+h}$, i.e. a solution of the equation

$$\phi = P\left(\epsilon_{T+1} + ... + \epsilon_{T+h} \leq -VaR_{T+h}\right).$$

Evaluating the h-step ahead VaR_{T+h} then requires the estimation of the ϕ-quantile of $\epsilon_{T+1} + ... + \epsilon_{T+h}$ which may be easily obtained from the $MCMC$ sample $\left(\epsilon_{T+1}^{(l)} + ... + \epsilon_{T+h}^{(l)}\right)_{1\leq l\leq M}$ given as above.

4. Simulation study

This Section examines the finite-sample performance of the Griddy-Gibbs estimate, $\widehat{\theta}_G$, through simulated series generated from the 5-periodic $PAP\text{-}GARCH_5(1,1)$ model (2.1). The choice of $S = 5$ is motivated by computational as well as practical considerations. Two cases are considered for the distribution of the innovation. The first one (cf. Table 4.1) is the standard Gaussian Case i) in which the innovation sequence is iid with $\eta_v \sim N(0,1)$, $(1 \leq v \leq S)$. The second one (cf. Table 4.2) corresponds to the standardized Student Case ii) for which the innovation is ipd_S with $\eta_v \sim t\left(\frac{1}{\psi_v}\right)$ $(1 \leq v \leq S)$. The parameter θ is fixed for each instance so that to be in accordance with empirical evidence while satisfying the stability conditions for the model (cf. Table 4.1, Table 4.2). In particular, the models considered are characterized by high persistence while satisfying the strict periodic stationarity condition (2.5). Moreover, at each season, the α_{v+} is significantly different from α_{v-}, so asymmetry of the models is ensured. In addition, different power values across seasons are allowed.

For each instance, we consider $Rep = 1000$ replications of $PAP\text{-}GARCH_5\,(1,1)$ series with a sample size $T = 1000$, for which we calculate the Griddy Gibbs estimate $\widehat{\theta}_G$. In evaluating $\widehat{\theta}_G$, we use $L = 1000$ Gibbs iterations from which we discard the first 400 iterations. The initial parameter draw $\theta^{(0)}$ is taken to be the true value of θ. In the Griddy Gibbs iteration, the range of the grid is taken as in (3.5) and each component of θ is generated using $g = 300$ grid points.

Mean of estimates $\widehat{\theta}_G$ and their standard deviations (Std) over the 1000 replications for both the Gaussian and Student cases are reported in Table 4.1 and Table 4.2 respectively.

Season v		ω_v	α_{v+}	α_{v-}	β_v	δ_v
1	$True$	0.2	0.25	0.35	0.4	1.3
	$Mean$	0.2120	0.2786	0.3827	0.4108	1.3071
	Std	0.0501	0.0815	0.1062	0.1155	0.2894
2	$True$	0.1	0.15	0.3	0.2	1.2
	$Mean$	0.1050	0.1654	0.3166	0.1952	1.1970
	Std	0.0253	0.0479	0.0897	0.0508	0.1712
3	$True$	0.15	0.2	0.1	0.25	0.8
	$Mean$	0.1525	0.2167	0.1068	0.2407	0.8046
	Std	0.0265	0.0625	0.0298	0.0523	0.0868
4	$True$	0.4	0.3	0.2	0.15	1
	$Mean$	0.4115	0.3369	0.2241	0.1627	1.0294
	Std	0.0661	0.0982	0.0618	0.0467	0.2238
5	$True$	0.15	0.1	0.18	0.1	1.6
	$Mean$	0.1586	0.1125	0.1987	0.1088	1.5983
	Std	0.0381	0.0324	0.0563	0.0316	0.2562

Table 4.1. Mean and standard deviation (Std) of $\widehat{\theta}_G$ for the 5-periodic Gaussian $PAP\text{-}GARCH_5(1,1)$ series with $T = 1000$, $L = 1000$, $g = 300$ and $Rep = 1000$.

Season v		ω_v	α_{v+}	α_{v-}	β_v	δ_v	ψ_v
1	$True$	0.2	0.25	0.35	0.4	1.3	0.2
	$Mean$	0.2295	0.2875	0.3722	0.4210	1.3420	0.1884
	Std	0.0423	0.0732	0.0608	0.0802	0.2952	0.0420
2	$True$	0.1	0.15	0.3	0.2	1.2	0.25
	$Mean$	0.1144	0.1701	0.3156	0.1951	1.2202	0.2355
	Std	0.0241	0.0354	0.0713	0.0407	0.2208	0.0410
3	$True$	0.15	0.2	0.1	0.25	0.8	0.3333
	$Mean$	0.1602	0.2135	0.1202	0.2281	0.8135	0.3140
	Std	0.0280	0.0534	0.0206	0.0312	0.0954	0.0720
4	$True$	0.4	0.3	0.2	0.15	1	0.25
	$Mean$	0.3859	0.3409	0.2388	0.1649	1.0584	0.2323
	Std	0.0570	0.0841	0.0530	0.0291	0.2014	0.0489
5	$True$	0.15	0.1	0.18	0.1	1.6	0.2
	$Mean$	0.1640	0.1233	0.2021	0.1203	1.6424	0.1896
	Std	0.0288	0.0287	0.0437	0.0258	0.3620	0.0411

Table 4.2. Mean and standard deviation (Std) of $\widehat{\theta}_G$ for the 5-periodic Student $PAP\text{-}GARCH_5(1,1)$ series with $\psi_v = \frac{1}{\tau_v}$, $T = 1000$, $L = 1000$, $g = 300$ and $Rep = 1000$.

From Table 4.1 and Table 4.2 it can be seen that the parameters are well estimated with quite small bias and small standard deviations.

We are also interested in Bayesian volatility and VaR forecasting. We first generate a 5-periodic $PAP\text{-}GARCH_5\,(1,1)$ series with parameters given by Table 4.1 for the Gaussian case and Table 4.2 for the Student innovation assumption. Then we get the true volatility σ_t^2 for $t = 1, ..., 1000 + h$, where the horizon of prediction h is taken in the set $\{1, ..., 8\}$. Finally, we compute the Griddy-Gibbs estimate $\widehat{\theta}_G$ from which, using (3.7), we obtain the prediction $\widehat{\sigma}_t^2$ and its standard deviation over the L Gibbs draws, for $t = 1, ..., 1000 + h$. For the Gaussian and student cases, Table 4.3 and Table 4.4 show, respectively, the true volatility σ^2_{1000+h} for all $h \in \{1, ..., 8\}$, the mean of Griddy-Gibbs sampled volatilities $\left(\sigma^{2(l)}_{1000+h}\right)$ and their standard deviation (Std).

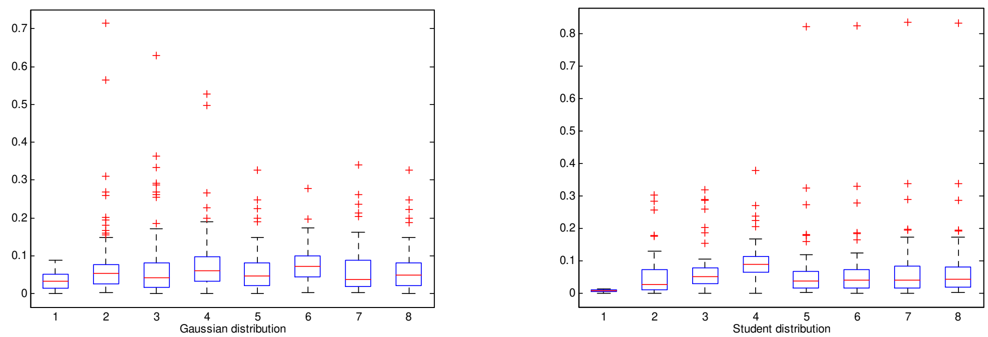
	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$
True value σ^2_{1000+h}	0.2487	0.2576	0.2556	0.2481	0.2777	0.2273	0.2370	0.2493
Mean : $\widehat{\sigma}^2_{1000+h}$	0.2321	0.2541	0.2550	0.2571	0.2480	0.2526	0.2521	0.2468
Std	0.0282	0.0695	0.0672	0.0801	0.0618	0.0887	0.0696	0.0642

Table 4.3. Volatility forecasts from the 5-periodic Gaussian $\text{PAP-GARCH}_5\,(1,1)$ in Table 4.1 for the horizon h with $h = 1, ..., 8$.

	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$
True value σ^2_{1000+h}	0.3372	0.3269	0.4194	0.3442	0.2907	0.3570	0.3171	0.2771
Mean : $\widehat{\sigma}^2_{1000+h}$	0.3308	0.3741	0.3911	0.3726	0.3729	0.3958	0.3701	0.3583
Std	0.035	0.1255	0.1245	0.1178	0.1145	0.1499	0.1021	0.0849

Table 4.4. Volatility forecasts from the 5-periodic Student $\text{PAP-GARCH}_5\,(1,1)$ in Table 4.2 for the horizon h with $h = 1, ..., 8$.

On the basis of 100 replications of this generated series, we calculate the mean absolute error (MAE) $\left|\widehat{\sigma}^2_{1000+h} - \sigma^2_{1000+h}\right|$ for $h = 1, ..., 8$ in both Gaussian and Student cases and obtain the corresponding boxplots (cf. Figure 4.1). It may be observed that the MAE's are small enough for all time horizons. Of course, the MAE of volatilities prediction becomes large as long as the time horizon h increases.



(a) (b)

Figure 4.1. Boxplot of the MAE of $\left|\widehat{\sigma}^2_{1000+h} - \sigma^2_{1000+h}\right|$ $(h = 1, ..., 8)$ for the Gaussian (panel (a)) and Student (panel (b)) cases.

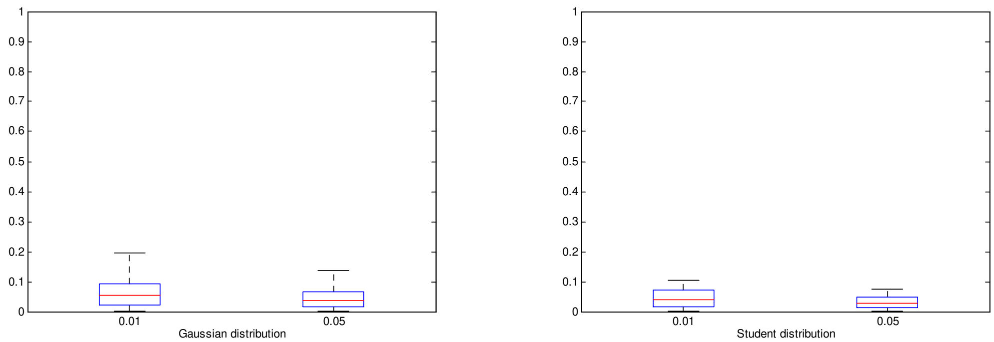
Concerning VaR forecasting, Table 4.5 and Table 4.6 show respectively for the Gaussian and Student cases the mean and the standard deviation of the h-step VaR_{1000+h} estimate $(h = 1, ..., 8)$ at the probability levels $\phi = 0.01$ (Table 4.5) and $\phi = 0.05$ (Table 4.6). For the one-step ahead prediction corresponding to $h = 1$, the boxplot of MAE of $\left|\widehat{VaR}_{1001} - VaR_{1001}\right|$ for both Gaussian and Student cases are displayed in Figure 4.2.

$\left(VaR^{(l)}_{1000+h}\right)_l$	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$
Mean $(\phi = 0.01)$	1.1654	1.6949	2.0750	2.3448	2.6313	2.8606	3.0818	3.2881
Std $(\phi = 0.01)$	0.0104	0.0202	0.0607	0.0494	0.0426	0.0477	0.0841	0.0755
Mean $(\phi = 0.05)$	0.8083	1.1663	1.4353	1.6113	1.8355	1.9619	2.1566	2.2774
Std $(\phi = 0.05)$	0.0118	0.0286	0.0340	0.0183	0.0304	0.0300	0.0298	0.0507

Table 4.5. Mean and standard deviations of VaR_{1000+h} forecast at the levels $\phi = 0.01$ and $\phi = 0.05$ for $h = 1, ..., 8$ using the 5-periodic Gaussian PAP-GARCH$_5(1,1)$ in Table 4.1.

$\left(VaR^{(l)}_{1000+h}\right)_l$	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$
Mean $(\phi = 0.01)$	1.6320	2.5928	3.0001	3.5134	3.7597	4.2026	4.3246	4.7986
Std $(\phi = 0.01)$	0.4251	0.2396	0.2364	0.1014	0.0861	0.1581	0.0655	0.0978
Mean $(\phi = 0.05)$	0.9040	1.4366	1.8599	2.1883	2.4368	2.6574	2.8440	3.0553
Std $(\phi = 0.05)$	0.0285	0.0324	0.0515	0.0608	0.0864	0.1102	0.0602	0.1439

Table 4.6. Mean and standard deviations of VaR_{1000+h} forecast at the levels $\phi = 0.01$ and $\phi = 0.05$ for $h = 1, ..., 8$ using the 5-periodic Student PAP-GARCH$_5(1,1)$ in Table 4.2.



(a) (b)

Figure 4.2. Boxplot of the MAE of $\left|\widehat{VaR}_{1001} - VaR_{1001}\right|$ for the Gaussian (panel (a)) and Student (panel (b)) cases at the levels $\phi = 0.01$ and $\phi = 0.05$.

From Table 4.5 and Table 4.6, it may be observed that in both Gaussian and Student cases the standard deviations of the VaR forecasts are quite small for all time horizon h and naturally increase with respect to h. In the Student case, the VaR estimates seem slightly more precise than those obtained for the Gaussian distribution. The boxplots in Figure 4.2 also show that the MAE's are quite small, so the estimates have a good predictive performance.

From the above simulation analysis, it may be concluded that the Griddy-Gibbs sampler can be considered as a useful tool in modeling and predicting the PAP-$GARCH$ volatility despite the larger number of parameters involved compared to the AP-$GARCH$ equation.

5. Real applications to daily S&P500 returns

In this Section we fit a PAP-$GARCH_S\,(1,1)$ model with period $S = 5$ to daily returns on the $S\&P$ 500 (closing value). It is often argued that daily financial asset returns are characterized by the day-of-the-week effect which suggests the existence of periodicity in volatility with period $S = 5$ (e.g. Bollerslev and Ghysels, 1996; Franses and Paap, 2000; Tsiakas, 2006; Osborn et *al.*, 2008; Regnard and Zakoïan, 2011). Because of the presence of holidays, model (2.1) in which $\theta_v = \theta_{nS+v}$ $(1 \leq v \leq S,\ n \in \mathbb{Z})$ seems not suitable. This is because with model (2.1), each day of a week may have a different specification than the same day of the week before. So when $S = 5$ we rather consider the following variant of model (2.1) (denoted by PAP-$GARCH_5^*\,(1,1)$):

$$\begin{cases} \epsilon_t = \sigma_t^{\delta_{d(t)}} \eta_t \\ \sigma_t^{\delta_{d(t)}} = \omega_{d(t)} + \alpha_{d(t)+} (\epsilon_{t-1}^{+})^{\delta_{d(t-1)}} + \alpha_{d(t)-} (\epsilon_{t-1}^{-})^{\delta_{d(t-1)}} + \beta_{d(t)} \sigma_{t-1}^{\delta_{d(t-1)}} \end{cases}, \tag{5.1}$$

in which $d(t)$ is defined to be

$$d\left(t\right)=\begin{cases}1 \text{ if the day corresponding to } t \text{ is a Monday}\\ 2 \text{ if the day corresponding to } t \text{ is a Tuesday}\\ \vdots \\ 5 \text{ if the day corresponding to } t \text{ is a Friday.}\end{cases}$$

The $PAP\text{-}GARCH_5^*(1,1)$ specification (5.1) with missing values (see e.g. Franses and Paap, 2000; Regnard and Zakoïan, 2011; Aknouche, 2017) seems able to accomodate the day-of-the-week effect.

Two instances of the distribution of innovation $\{\eta_t, t \in \mathbb{Z}\}$ in (5.1) are assumed. In the first one, $\eta_1, ..., \eta_S$ are normally distributed with mean zero and unit variance ($\eta_v \sim N(0,1)$), whereas in the second $\eta_1, ..., \eta_S$ are (standardized) Student distributed with degrees of freedom $\tau_1, ..., \tau_S > 2$ respectively.

For the two instances we use the Bayesian Griddy Gibbs estimate with number of iterations $L = 1000$ and burn-in 400. The initial parameter estimate $\theta^{(0)}$ is taken as follows. The initial power parameter δ_v at a day v is taken to be inversely proportional to the Kurtosis relative to that day. For the remaining parameters, we take the values obtained while estimating a $GARCH(1,1)$ model for the series of each day. In the Griddy step, 500 grid points are used and the range of parameters at the l-th Gibbs iteration is taken as in (3.5).

5.1. The data and the day-of-the-week effect

The dataset covers the period starting from January, 01, 2007 to December, 31, 2012, with a total of $T = 1509$ observations. The time series plots of the index (panel (a)) and its return (panel (b)) are presented in Figure 5.1. The same dataset was considered by Chan and Grant (2016) and Aknouche (2017).

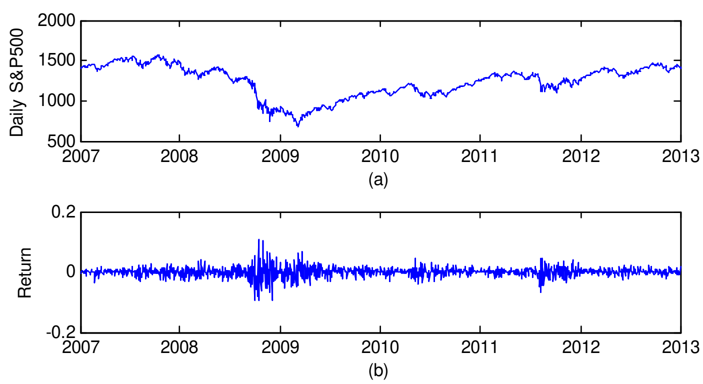


Figure 5.1. Daily $S\&P$ 500 from January 2007 to December 2012 : (a) level, (b) return.

Some descriptive statistics concerning the $S\&P500$ returns, the absolute returns, the squared returns and the log-absolute returns can be found in Aknouche (2017). To highlight the day-of the week effect, Table

5.1 shows for each (trading) day the average return, the volatility (approximated by the absolute value), the kurtosis, and skewness where it may be seen that these measures are quite different from a day to another.

		Sample size	Mean of (ϵ_t)	Mean of ($\|\epsilon_t\|$)	Mean of (ϵ_t^2)	Kurtosis	Skewness
	Full series	1509	$4.4711e-06$	0.0102	$2.4646e-04$	10.4975	-0.2643
1	Monday	284	0.0012	0.0108	$2.6627e-04$	9.4713	1.0406
2	Tuesday	308	-0.0003	0.0099	$2.2774e-04$	10.2425	-1.5226
3	Wednesday	311	0.0001	0.0109	$2.5983e-04$	7.0415	-0.7051
4	Thursday	305	-0.0002	0.0088	$1.5129e-04$	5.5615	0.0655
5	Friday	301	-0.0007	0.0108	$3.2866e-04$	12.7415	-0.1795

Table 5.1. Day of the week effect in daily S&P 500 returns.

This is also confirmed by Figure 5.2a which shows the kernel estimate of the distribution of return for each trading day together with the full series. These distributions seems to have different shapes. The same finding may be observed in the boxplot of each day (cf. Figure 5.2b). This reinforces the intuition that a periodic model with periodic innovation might be better in explaining the day specificities than a non-periodic model.

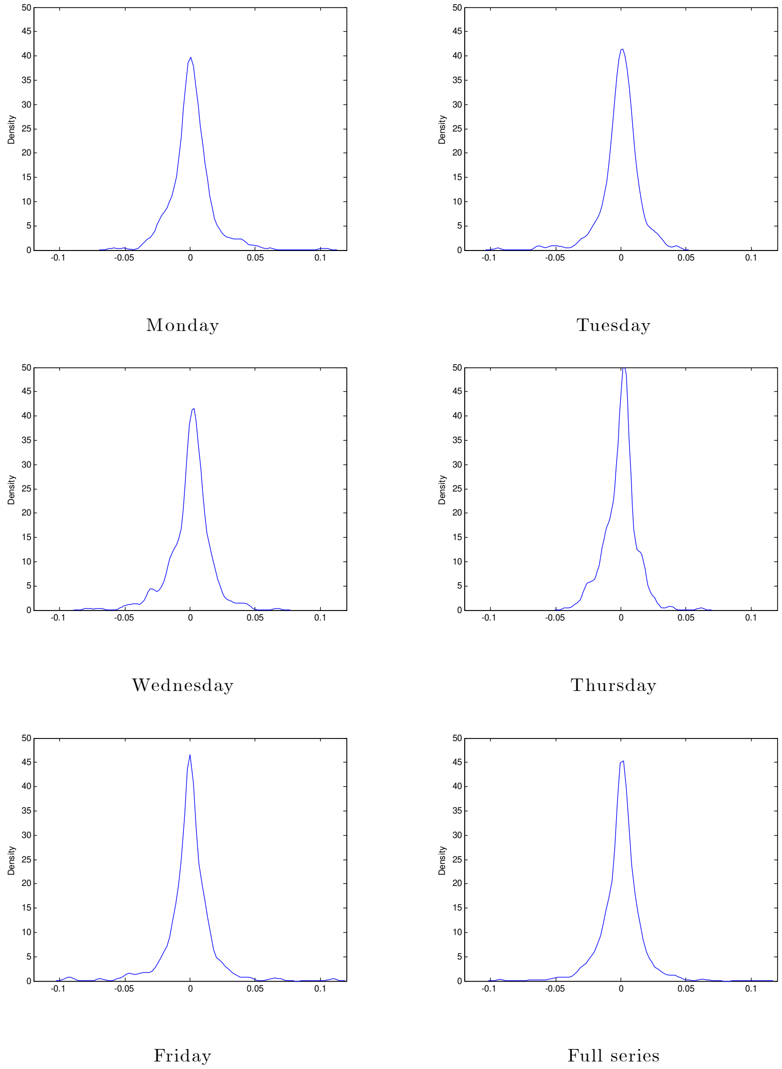


Figure 5.2a. Kernel density estimate of the distribution of returns for the full series and each trading day.

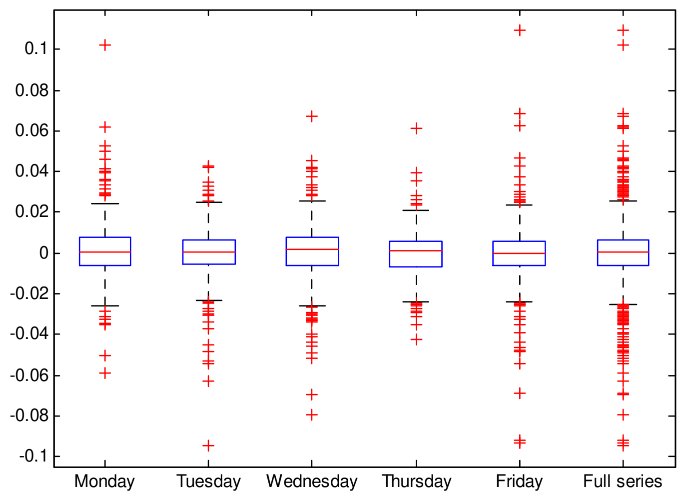


Figure 5.2b. Boxplots of of the distribution of returns for the full series and each trading day.

5.2. Gaussian $PAP\text{-}GARCH_5^*(1,1)$ estimated model

In order to identify the best fitting model, we estimate the DIC of eight subclasses of the $PAP\text{-}GARCH_S(1,1)$ and $PAP\text{-}GARCH_S(1,1)$ models, namely the $PAP\text{-}GARCH_5^*(1,1)$ given by (5.1), the non-periodic asymmetric power $GARCH(1,1)$ ($AP\text{-}GARCH(1,1)$) model corresponding to $S=1$, the standard $GARCH(1,1)$ model, the S-periodic asymmetric power $GARCH$ model ($PAP\text{-}GARCH_S(1,1)$) given by (2.1) for $S \in \{2,...,4\}$, the 5-periodic $GARCH_5^*(1,1)$ ($P\text{-}GARCH_5^*(1,1)$), and the 5-periodic threshold $GARCH_5^*(1,1)$ ($PT\text{-}GARCH_5^*(1,1)$) model. For all models the innovation is iid having a standard Gaussian distribution.

Model	$PAP\text{-}GARCH_1(1,1)$	$PAP\text{-}GARCH_2(1,1)$	$PAP\text{-}GARCH_3(1,1)$	$PAP\text{-}GARCH_4(1,1)$
DIC	-7030.6480	-6942.8207	-7019.6105	-7000.7523
Rank	2	7	4	5
Model	$PAP\text{-}GARCH_5^*(1,1)$	$PT\text{-}GARCH_5^*(1,1)$	$P\text{-}GARCH_5^*(1,1)$	$GARCH(1,1)$
DIC	-7034.3004	-6930.7031	-6971.5962	-7025.0591
Rank	1	8	6	3

Table 5.2. Estimated DIC for various conditionally Gaussian PAP-GARCH$_S$(1,1) and PAP-GARCH$_5^*$(1,1) models.

Table 5.2 shows the DIC and the rank of each one of the compared models. It may be observed that the DICs are significantly far from each other. In fact, the best model in the DIC sense is the $PAP\text{-}GARCH_5^*(1,1)$ with DIC equaling -7034.3004. The second ranked model is the non-periodic $AP\text{-}GARCH(1,1)$ and surprisingly is followed by the standard $GARCH(1,1)$ model which surpasses the threshold and periodic $GARCH$ models. Thus we retains the $PAP\text{-}GARCH_5^*(1,1)$ model.

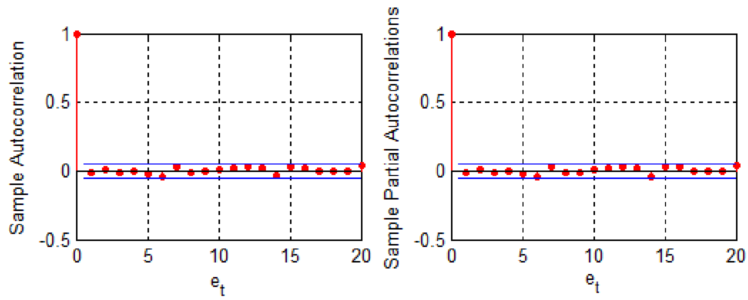
Table 5.3 displays the BGG estimates of the $PAP\text{-}GARCH_5^*(1,1)$, their $MCMC$ standard deviations

(Std), their NSE and their RNI.

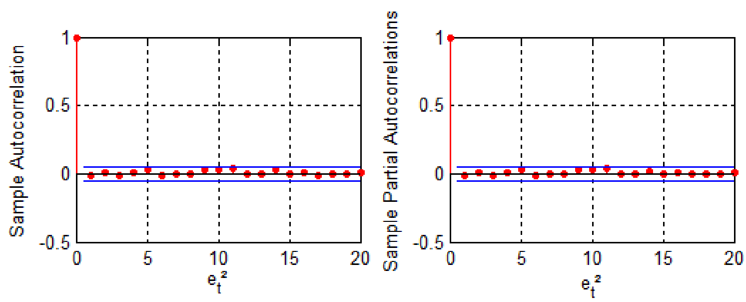
Day	Parameters	*Mean*	*Std*	*RNI*	*NSE*
Monday	δ_1	0.6183	0.0258	3.8105	0.0016
	ω_1	0.2830 $e-04$	0.0207.$e-04$	3.0642	0.1146.$e-06$
	α_{1+}	0.3056	0.0630	3.2201	0.0037
	α_{1-}	0.5115	0.0391	3.7839	0.0020
	β_1	0.7466	0.0181	2.9539	0.0012
Tuesday	δ_2	0.6239	0.0245	4.2003	0.0016
	ω_2	0.2023 $e-04$	0.0125.$e-04$	3.0120	0.0687.$e-06$
	α_{2+}	0.2028	0.0379	3.1586	0.0022
	α_{2-}	0.3265	0.0468	3.2612	0.0024
	β_2	0.8536	0.0206	2.4796	0.0015
Wednesday	δ_3	0.9222	0.0371	3.5016	0.0022
	ω_3	0.4265 $e-04$	0.0709 $e-04$	2.9078	0.3821 $e-06$
	α_{3+}	0.2688	0.0646	2.9837	0.0034
	α_{3-}	0.4649	0.0423	3.4362	0.0021
	β_3	0.6673	0.0191	2.4585	0.0013
Thursday	δ_4	0.8139	0.0595	3.7871	0.0037 $e-06$
	ω_4	0.1062 $e-04$	0.0343	3.1505	0.1923
	α_{4+}	0.1244	0.0241	2.7845	0.0013
	α_{4-}	0.2627	0.0415	3.1916	0.0023
	β_4	0.7912	0.0276	2.6221	0.0021
Friday	δ_5	0.4589	0.0224	4.0096	0.0014 $e-06$
	ω_5	0.2352	0.0462	2.8572	0.2470
	α_{5+}	0.3220	0.0686	2.1526	0.0041
	α_{5-}	0.3935	0.0647	2.9243	0.0035
	β_5	0.6785	0.0203	2.4539	0.0015

Table 5.3. Bayesian Griddy-Gibbs estimates of the Gaussian $PAP\text{-}GARCH_5^*(1,1)$ model for the S&P500 returns.

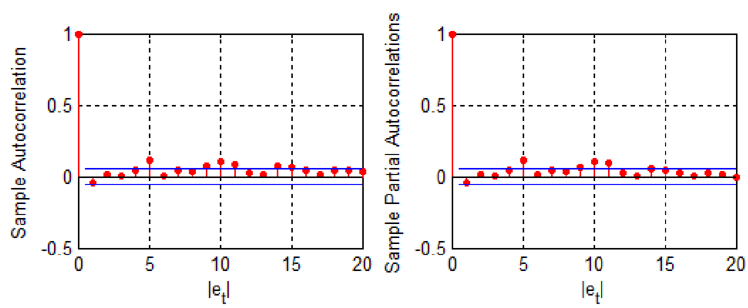
From Table 5.3, it can be seen that the parameters are quite well estimated as shown by their low $MCMC$ standard deviations, low RNI and small NSE. Moreover, the parameters are quite different from a day to another. In particular, α_{v-} is fairly different from α_{v+} for all $v \in \{1, ..., S\}$, which implies an asymmetry of the model volatility. On the other hand, the estimated model is characterized by high persistence and in overall the estimates are comparable with similar models in the literature when $S = 1$ (e.g. Pan et *al.*, 2008; Xia et *al.*, 2017). In addition, simple and partial (sample) autocorrelations of the residuals given by $e_t = \frac{\epsilon_t}{\sigma_t(\hat{\theta})}$ $(t = 1, ..., T)$, the squared residuals (e_t^2) and the absolute residuals $(|e_t|)$ are shown in Figure 5.3. It may be concluded that the residuals look like an independent white noise, which validates the estimated model. Note that these results are stable enough to using different initial values and different numbers of iterations for the Griddy-Gibbs sampler.



(a)



(b)



(c)

Figure 5.3. Sample autocorrelations of e_t (panel (a)), e_t^2 (panel (b)) and $|e_t|$ (panel (c)) under the Gaussian innovation.

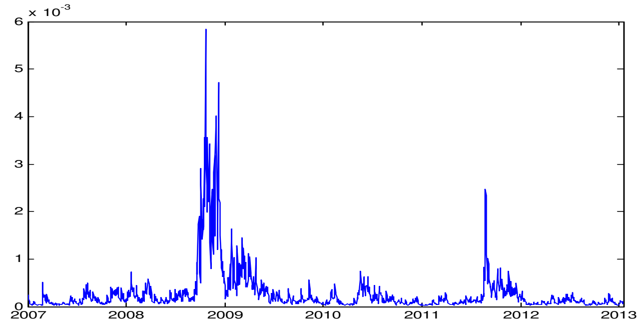
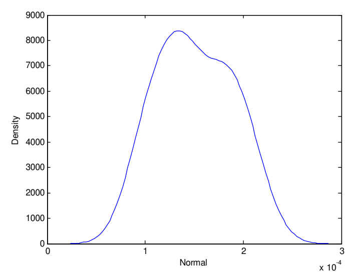


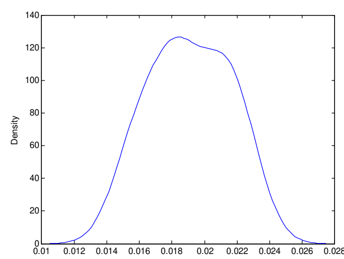
Figure 5.4. Volatility induced by the Gaussian $PAP\text{-}GARCH_5^*(1,1)$ model.

The volatility induced by the Gaussian $PAP\text{-}GARCH_5^*\,(1,1)$ is displayed in Figure 5.4 showing a similar pattern to the actual variability of the $S\&P500$ returns.

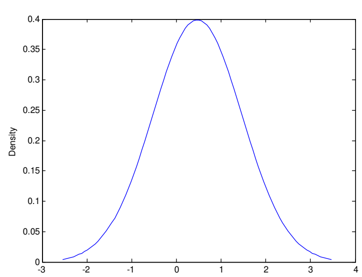
We finally exploit the estimated $PAP\text{-}GARCH_5^*\,(1,1)$ model to get one-step ahead predictive distribution (cf. Figure 5.5) of the volatility (panel (a)), the return (panel (b)) and the value at risk at levels 5% (panel (c)) and 1% (panel (d)). The posterior means of these predictive distributions together with their $MCMC$ standard deviations are reported in Table 5.4. In summury, the estimated $PAP\text{-}GARCH_5^*(1,1)$ model seems to give good results.



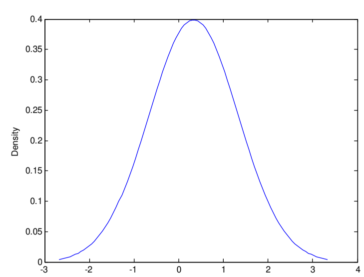
(a) Volatility $\widehat{\sigma}_{T+1}$.



(b) Return $\widehat{\epsilon}_{T+1}$.



(c) 5% $\widehat{VaR}_{T+1}$.



(d) 1% $\widehat{VaR}_{T+1}$.

Figure 5.5. The one-step-ahead predictive distribution of the volatility (panel (a)), the return (panel (b)) and the value at risk at 5% (panel (c)) and 1% (panel (d)) for the Gaussian case.

	$\widehat{\epsilon}_{T+1}$	$\widehat{\sigma}^2_{T+1}$	$\alpha = 0.01$ $\widehat{VaR}_{T+1}$	$\alpha = 0.05$ $\widehat{VaR}_{T+1}$
$Mean$	0.0191	$1.5111e-04$	0.4343	0.3071
Std	0.0024	$3.6861e-05$	$0.8348e-4$	$0.5009e-4$

Table 5.4. One day-ahead prediction of the volatility, the return and the value at risk at levels 5% and 1% for the S&P500 under the Gaussian assumption.

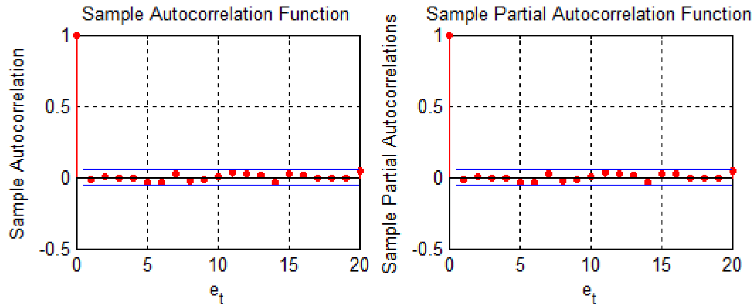
5.2. Student PAP-$GARCH_5^*(1,1)$ estimated model

We now estimate a PAP-$GARCH_5^*(1,1)$ given by (5.1) based on the same $S\&P500$ series but under the assumption that the innovation $\{\eta_t, t \in \mathbb{Z}\}$ is ipd_5 with a standardized Student distribution, i.e. $\frac{\tau_v}{\tau_v - 2}\eta_v \sim t_{\left(\psi_v^{-1}\right)}$ $(1 \leq v \leq 5)$. The means of the BGG estimates, their $MCMC$ standard deviations, their RNI and their NSE are reported in Table 5.5. The same conclusions may be drawn. The estimates are quite good considering their small *std*, RNI and NSE. The periodicity of the model is significant in view of the estimates along seasons which are quite different. Moreover, the model seems able to absorb asymmetry since α_{v-} and α_{v+} are quite different for all $v \in \{1, ..., 5\}$. Finally, in view of the sample autocorrelations of (e_t), $\left(e_t^2\right)$ and $(|e_t|)$ (cf. Figure 5.6), the residuals seem compatible with the independence assumption. Therefore, the Student PAP-$GARCH_5^*(1,1)$ model may be validated despite the high value of the corresponding DIC

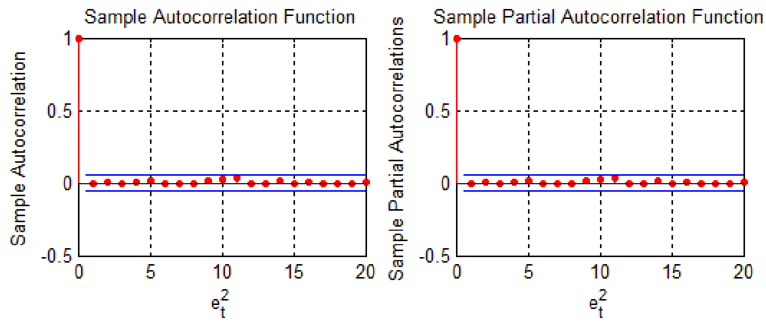
which is equal to 13421.0183.

Day	Parameters	*Mean*	*Std*	*RNI*	*NSE*
Monday	δ_1	0.5680	0.0146	4.2532	0.0078
	ω_1	$0.1441e-04$	$0.0038e-04$	2.1617	$0.7803e-07$
	α_{1+}	0.2785	0.0360	4.2542	0.0019
	α_{1-}	0.3052	0.0117	4.1500	0.0063
	β_1	0.7001	0.0900	4.1542	0.0048
	ψ_1	0.0910	0.0082	4.2376	0.0044
Tuesday	δ_2	0.6438	0.0166	4.3200	0.0088
	ω_2	$0.1027e-04$	$0.0037e-04$	2.2259	$0.7852e-07$
	α_{2+}	0.1798	0.0230	4.4200	0.0012
	α_{2-}	0.2227	0.0860	4.1011	0.0046
	β_2	0.8098	0.0104	4.3542	0.0056
	ψ_2	0.0993	0.0090	4.3654	0.0048
Wednesday	δ_3	0.9467	0.0243	4.5312	0.0130
	ω_3	$0.2395e-04$	$0.0018e-04$	2.7221	$0.4291e-07$
	α_{3+}	0.2918	0.0037	4.2200	0.0020
	α_{3-}	0.2868	0.0110	4.2104	0.0059
	β_3	0.6326	0.0810	4.5425	0.0043
	ψ_3	0.1241	0.0112	4.1298	0.0060
Thursday	δ_4	0.8521	0.0219	4.4465	0.0117
	ω_4	$0.0917e-04$	$0.0010e-04$	2.3329	$0.2066e-07$
	α_{4+}	0.1689	0.0220	4.5420	0.0012
	α_{4-}	0.2003	0.0770	4.2507	0.0041
	β_4	0.7988	0.0103	4.2542	0.0055
	ψ_4	0.1655	0.0149	4.2145	0.0080
Friday	δ_5	0.4734	0.0122	4.0213	0.0065
	ω_5	$0.1376e-04$	$0.0024e-04$	2.0988	$0.5007e-07$
	α_{5+}	0.3592	0.0460	4.2425	0.0025
	α_{5-}	0.3471	0.0134	4.2762	0.0071
	β_5	0.6664	0.0860	4.2114	0.0046
	ψ_5	0.0828	0.0075	4.2376	0.0040

Table 5.5. Bayesian Griddy-Gibbs estimates of the Student $PAP\text{-}GARCH_5^*(1,1)$ model for the S&P500 returns.



(a)



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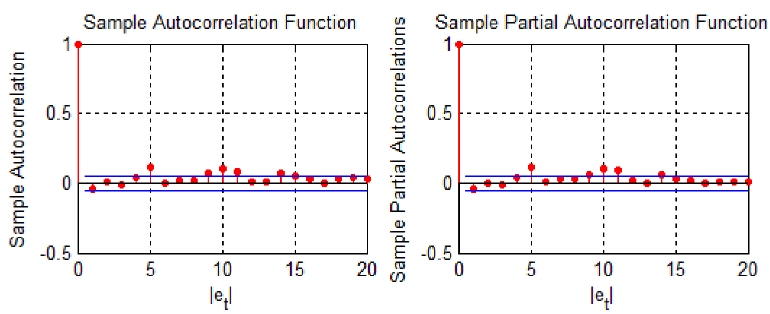


(c)

Figure 5.6. Sample autocorrelations of e_t (panel (a)), e_t^2 (panel (b)) and $|e_t|$ (panel (c)) under Student innovations.

The volatility estimated by the Student $PAP\text{-}GARCH_5^*(1,1)$ model is showed in Figure 5.7. Under the Student assumption, we calculate the one-step ahead predictive distribution (cf. Figure 5.8) of the volatility (panel (a)), the return (panel (b)) and the value at risk at levels 5% (panel (c)) and 1% (panel (d)). The posterior means of these predictive distributions together with their $MCMC$ standard deviations are reported in Table 5.6. As in the Gaussian case, it may be concluded that the $PAP\text{-}GARCH_5^*(1,1)$ model with Student innovations seems useful in modeling $S\&P500$ return series.

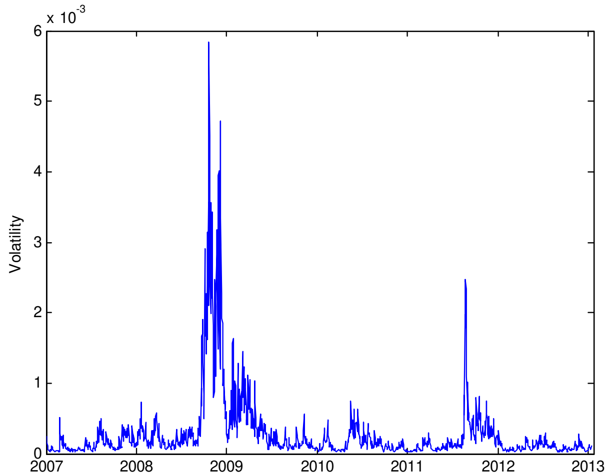
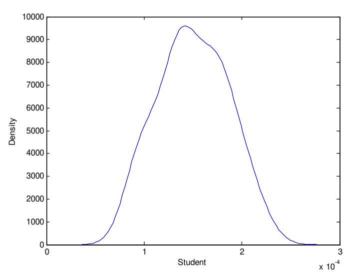
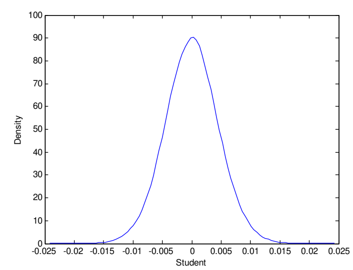


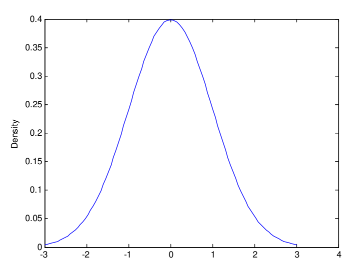
Figure 5.8. Volatility induced by the Student $PAP\text{-}GARCH_5^*(1,1)$ model.



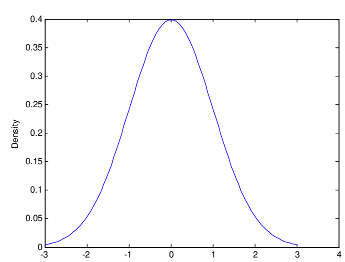
(a) Volatility $\widehat{\sigma}_{T+1}$.



(b) Return $\widehat{\epsilon}_{T+1}$.



(c) 5% $\widehat{VaR}_{T+1}$.



(d) 1% $\widehat{VaR}_{T+1}$.

Figure 5.8. The one-step-ahead predictive distribution of the volatility (panel (a)), the return (panel (b)) and the value at risk at 5% (panel (c)) and 1% (panel (d)) for the Student case.

	$\widehat{\epsilon}_{T+1}$	$\widehat{\sigma}^2_{T+1}$	$\alpha=0.01$ $\widehat{VaR}_{T+1}$	$\alpha=0.05$ $\widehat{VaR}_{T+1}$
$Mean$	-0.0020	$1.1425e-04$	0.4410	0.3118
Std	$1.5515e-04$	$1.7553e-05$	$0.3278e-04$	$0.1945e-04$

Table 5.6. One day-ahead prediction of the volatility, the return and the value at risk at levels 5% and 1% for the S&P500 under the Student assumption.

6. Conclusion

The main trust of this paper has been to propose a simple Bayesian Griddy Gibbs estimate ($BGGE$) for the $PAP\text{-}GARCH_S\,(p,q)$ model under various distributional assumptions on the periodic innovation sequence. This model is designed to account for periodicity in volatility as well as other important features of volatility including asymmetry and periodic power dependence. We highlighted the structure of the $PAP\text{-}GARCH_S\,(p,q)$ model which is needed in the estimation stage. We also showed that the proposed estimate is simple to implement, not case-sensitive and applies to a general framework. In particular, the $BGGE$ accommodates with the prediction of the volatility and the Value at Risk. Real applications showed that in overall (low-frequency) daily $S\&P500$ returns are compatible with the proposed model.

As any periodic model, the principal drawback of the $PAP\text{-}GARCH_S\,(p,q)$ model is that it retains a large number of parameters when the period S tends to be large, thereby making their estimation and their interpretation very difficult. An important issue is then the adaptation of the $BGGE$ in the context of high frequency periodicity and non-integer periodicity such as intraday high frequency returns (see Aknouche et *al.*, 2018 for the $GQMLE$).

7. Appendix: Proofs

Proof of Theorem 2.1 i) *Sufficiency*: Equation (2.3) may be cast in the following system of S recurrence equations

$$Y_{nS+v} = \mathbb{A}_{nS+v} Y_{(n-1)S+v} + \mathbb{B}_{nS+v},\ n \in \mathbb{Z},\quad v \in \{0,...,S-1\}, \tag{7.1}$$

where $\mathbb{A}_{nS+v} = \prod_{i=0}^{S-1} A_{nS+v-i}$ and $\mathbb{B}_{nS+v} = \sum_{j=0}^{S-1} \prod_{i=0}^{j-1} A_{nS+v-i} B_{nS+v-j}$, so $\{(\mathbb{A}_{nS+v}, \mathbb{B}_{nS+v}), n \in \mathbb{Z}\}$ is iid for all $v \in \{0,...,S-1\}$. The top Lyapunov exponent $\gamma_v^{(S)}$ associated with (7.1) is given for all $v \in \{0,...,S-1\}$ by (cf. Bougerol and Picard, 1992)

$$\begin{aligned} \gamma_v^{(S)} &= \inf\left\{\frac{1}{n} E\log\left\|\mathbb{A}_{nS+v}\mathbb{A}_{(n-1)S+v}...\mathbb{A}_{S+v}\right\|,\ n\geq 1\right\} \\ &= \inf\left\{\frac{1}{n} E\log\left\|A_{nS+v}A_{nS+v-1}...A_{v+1}\right\|,\ n\geq 1\right\}, \\ &= \lim_{n\to\infty}\frac{1}{n} E\log\left\|A_{nS+v}A_{nS+v-1}...A_{v+1}\right\|\ \ a.s. \end{aligned} \tag{7.2}$$

Since $E\log|\eta_v|^{\delta_v} < \infty$ for all $0 \leq v \leq S-1$, it follows that $E\log^+\|\mathbb{A}_v\| < \infty$ and $E\log^+\|\mathbb{B}_v\| < \infty$. Therefore, by Theorem 2.5 of Bougerol and Picard (1992), equation (7.1) admits a unique nonanticipative strictly stationary and ergodic solution $\{Y_{nS+v}, n \in \mathbb{Z}\}$ provided that $\gamma_v^{(S)} < 0$. The solution is given for all $v \in \{0, ..., S-1\}$ by

$$Y_{nS+v} = \sum_{j=0}^{\infty}\prod_{i=0}^{j-1}\mathbb{A}_{(n-i)S+v}\mathbb{B}_{(n-j)S+v}, \quad n \in \mathbb{Z}, v \in \{0, ..., S-1\}, \tag{7.3}$$

where the series in equality (7.3), which is exactly (2.6), converges absolutely $a.s.$ This shows that $\{Y_t, t \in \mathbb{Z}\}$ is the unique causal strictly periodically stationary and periodically ergodic solution of (2.3). Note finally that by a sandwitching argument, it is easy to see that for all $v \in \{0, ..., S-1\}$

$$\gamma_v^{(S)} = \lim_{n\to\infty}\frac{1}{n}E\log\|A_{nS+v}A_{nS+v-1}...A_{v+1}\| = \lim_{n\to\infty}\frac{1}{n}E\log\|A_{nS}A_{nS-1}...A_1\| := \gamma^{(S)}.$$

Necessity: Assume that model (2.3) admits a nonanticipative strictly periodically stationary solution $\{Y_t, t \in \mathbb{Z}\}$. From the non-negativity of the coefficients of A_t in (2.3) it follows that for all $k > 1$,

$$Y_v \geq \sum_{j=0}^{k}\prod_{i=0}^{j-1}A_{v-i}B_{v-j}, \quad a.s.,$$

implying that the series $\sum_{j=0}^{\infty}\prod_{i=0}^{j-1}A_{v-i}B_{v-j}$ converges $a.s.$ Therefore,

$$\prod_{i=0}^{j-1}A_{v-i}B_{v-j} \to 0, \quad a.s. \text{ as } j \to \infty,$$

from which we have to show

$$\prod_{i=0}^{j-1}A_{v-i} \to 0, \quad a.s. \text{ as } j \to \infty. \tag{7.4}$$

This holds whenever

$$\lim_{j\to\infty}\prod_{i=0}^{j-1}A_{v-i}e_m = 0, a.s. \text{ for all } 1 \leq m \leq r, \tag{7.5}$$

where $r = p + 2q - 2$ and $(e_m)_{1\leq m\leq r}$ is the canonical basis of $\mathbb{R}^r$. Since A_t has the same "sparsity" as the matrix A_t in Pan et *al.* (2008, p. 373), then (7.5) follows from their result using similar arguments (see also Aknouche and Bibi, 2009 for the particular *P-GARCH* case).

ii) Since $\{A_t, t \in \mathbb{Z}\}$ is nonnegative then

$$\gamma^S(A) \geq \gamma^S(\boldsymbol{\beta}) := \log\rho\left(\prod_{v=0}^{S-1}\boldsymbol{\beta}_{S-v}\right). \tag{7.6}$$

If (2.3) has a strictly periodically stationary solution, then $\gamma^S(A) < 0$. In view of (7.6), it follows that $\gamma^S(\boldsymbol{\beta}) < 0$ establishing (2.7).

Proof of Theorem 2.2 i) The proof is similar to that of Lemma 2.3 of Berkes et *al.* (2003). First, we have to show that if $\gamma^S(A) < 0$ then there is $\delta > 0$ and n_0 such that

$$E\left(\|A_{n_0S}A_{n_0S-1}...A_1\|^{\delta}\right) < 1. \tag{7.7}$$

Since $\gamma^S(A) = \inf_{n\in\mathbb{N}^*}\left\{\frac{1}{n}E\left(\log\|A_{nS}A_{nS-1}...A_1\|\right)\right\}$ is strictly negative, there is a positive integer n_0 such that

$$E\left(\log\|A_{n_0S}A_{n_0S-1}...A_1\|\right) < 0.$$

On the other hand, working with a multiplicative norm and by the ipd_S property of the sequence $\{A_t, t\in\mathbb{Z}\}$ we have

$$\begin{aligned} E\left(\|A_{n_0S}A_{n_0S-1}...A_1\|\right) &= \|E\left(A_{n_0S}A_{n_0S-1}...A_1\right)\| \\ &= \|E\left(A_SA_{S-1}...A_1\right)^{n_0}\| \\ &\leq \|E\left(A_SA_{S-1}...A_1\right)\|^{n_0} < \infty. \end{aligned}$$

Let $f(x) = E\left(\|A_{n_0S}A_{n_0S-1}...A_1\|^x\right)$. Since $f'(0) = E\left(\log\|A_{n_0S}A_{n_0S-1}...A_1\|\right) < 0$, $f(x)$ decrease in a neighborhood of 0 and since $f(0) = 1$, it follows that there exists $0 < \delta < 1$ such that (7.7) holds. Now from (2.6) we have for some $v \in \{1, ..., S\}$

$$\|Y_v\| \leq \sum_{k=1}^{\infty}\left\|\prod_{j=0}^{k-1}A_{v-j}\right\|\|B_{v-k}\| + \|B_v\|.$$

Since $0 < \kappa < 1$, then

$$\|Y_v\|^{\kappa} \leq \sum_{k=1}^{\infty}\left\|\prod_{j=0}^{k-1}A_{v-j}\right\|^{\kappa}\|B_{v-k}\|^{\kappa} + \|B_v\|^{\kappa},$$

which, by the independence of A_{v-j} and B_{v-k} for $j < k$, implies that

$$\begin{aligned} E\|Y_v\|^{\kappa} &\leq \sum_{k=1}^{\infty}E\left(\left\|\prod_{j=0}^{k-1}A_{v-j}\right\|^{\kappa}\right)E\left(\|B_{v-k}\|^{\kappa}\right) + E\left(\|B_v\|^{\kappa}\right) \\ &\leq B(\kappa)\sum_{k=1}^{\infty}E\left(\left\|\prod_{j=0}^{k-1}A_{v-j}\right\|^{\kappa}\right) + E\left(\|B_v\|^{\kappa}\right), \end{aligned}$$

where $B(\kappa) = \max\limits_{0\leq v\leq S-1}E\left(\|B_{v-k}\|^{\kappa}\right)$. In view of (7.7) there exist $a_v > 0$ and $0 < b_v < 1$ such that

$$E\left(\left\|\prod_{j=0}^{k-1}A_{v-j}\right\|^{\kappa}\right) \leq a_vb_v^k \leq ab^k,$$

where $ab^k = \max\limits_{0\leq v\leq S-1}\left\{a_vb_v^k\right\}$. This proves that $E\|Y_v\|^{\kappa} < \infty$, showing (2.8).

ii) Define $\left\{\widetilde{Y}_t, t \in \mathbb{Z}\right\}$ by

$$\begin{cases} \widetilde{Y}_t = A_t \widetilde{Y}_{t-1} + B_t & t \geq 1 \\ \widetilde{Y}_t = 0 & t \leq 0, \end{cases} \tag{7.8}$$

and let $Y^{(v)}$ $(0 \leq v \leq S-1)$ be a random variable having the same distribution as the term Y_{nS+v} of the unique periodically stationary solution given by (7.3). It is clear that $\widetilde{Y}_{nS+v} \xrightarrow{\mathcal{L}} Y^{(v)}$ as $n \to \infty$. Let $m = 2$. From the weak convergence theory (Billingsley, 1968), to show that $E\left(vec\left(Y^{(v)}Y^{(v)\prime}\right)\right)$ is finite for all v, it is sufficient to show that $\lim \inf_{n\to\infty} E\left(vec\left(\widetilde{Y}'_{nS+v}\widetilde{Y}_{nS+v}\right)\right) < \infty$ for all v. Set $V_{nS+v} = E\left(vec\left(\widetilde{Y}'_{nS+v}\widetilde{Y}_{nS+v}\right)\right)$. From (7.8) we get the following first-order S-periodic difference equation

$$V_{nS+v} = E\left(A_v^{\otimes 2}\right) V_{nS+v-1} + \left[E(A_v \otimes B_v) + E(B_v \otimes A_v)\right] E\left(\widetilde{Y}_{nS+v}\right) + vec(E\left(B_v B_v'\right)), \tag{7.8}$$

where $E\left(A_t^{\otimes 2}\right)$, $E(A_t \otimes B_t)$ and $vec(E\left(B_t B_t'\right))$ are finite S-periodic matrices in t. Since, the last two terms of the right-hand side of (7.8) are bounded, it follows that $\lim_{n\to\infty} V_{nS+v}$ exists for every $1 \leq v \leq S$ whenever (2.9) holds, which completes the proof for $m = 2$. For general m the proof is similar.

Proof of Theorem 2.3 The proof is very similar to that of Corollary 3.5 of Basrak et *al.* (2002).

References

[1] Aknouche, A. (2017). Periodic autoregressive stochastic volatility. *Statistical Inference for Stochastic Processes,* **20**, 139-177.

[2] Aknouche, A. and Al-Eid, E. (2012). Asymptotic inference of unstable periodic *ARCH* processes. *Statistical Inference for Stochastic Processes*, **15**, 61-79.

[3] Aknouche, A. and Bibi, A. (2009). Quasi-maximum likelihood estimation of periodic *GARCH* and periodic *ARMA-GARCH* processes. *Journal of Time Series Analysis*, **30**, 19-46.

[4] Aknouche, A. and Touche, N. (2015). Weighted least squares-based inference for stable and unstable threshold power *ARCH* processes. *Statistics & Probability Letters*, **97**, 108-115.

[5] Aknouche, A., Al-Eid, E. and Demouche, N. (2018). Generalized quasi-maximum likelihood inference for periodic conditionally heteroskedastic models. *Statistical Inference for Stochastic Processes*, http://dx.doi.org/10.1007/s11203-017-9160-x (forthcoming).

[6] Ambach, D. and Croonenbroeck, C. (2015). Obtaining superior wind power predictions from a periodic and heteroscedastic wind power prediction tool. *In Stochastic Models, Statistics and Their Applications, edt*, 225-232.

[7] Ambach, D. and Schmid, W. (2015). Periodic and long range dependent models for high frequency wind speed data. *Energy*, **82**, 277-293.

[8] Ardia, D. (2008). Bayesian estimation of a Markov-switching threshold asymmetric $GARCH$ model with Student-t innovations. *The Econometrics Journal,* **12**, 105-126.

[9] Basrak, B., Davis, R.A. and Mikosch, T. (2002). Regular variation of $GARCH$ processes. *Stochastic Processes and Their Applications,* **99**, 95-115.

[10] Bauwens, L. and Lubrano, M. (1998), Bayesian inference on $GARCH$ models using Gibbs sampler. *Journal of Econometrics*, **1**, 23-46.

[11] Bauwens, L., Dufays, A. and Rombouts, J.V.K. (2014). Marginal likelihood for Markov-switching and change-point $GARCH$ models. *Journal of Econometrics*, **178**, 508-522.

[12] Berkes, I., Horvàth, L. and Kokoskza, P. (2003). $GARCH$ processes: structure and estimation. *Bernoulli*, **9**, 201-227.

[13] Billingsley, P. (1968). *Probability and measure*. Wiley, New York.

[14] Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, **31**, 307-327.

[15] Bollerslev, T. and Ghysels, E. (1996). Periodic autoregressive conditional heteroskedasticity. *Journal of Business & Economic Statistics,* **14**, 139-152.

[16] Bollerslev, T., Cai, J. and Song, F.M. (2000). Intraday periodicity, long memory volatility, and macroeconomic announcement effects in the US Treasury bond market. *Journal of Empirical Finance*, **7**, 37-55.

[17] Bougerol, P. and Picard, N. (1992). Stationarity of $GARCH$ processes and some nonnegative time series. *Journal of Econometrics*, **52**, 115-127.

[18] Chen, C.W.S. and So, M.K.P. (2006). On a threshold heteroscedastic model. *International Journal of Forecasting*, **22**, 73-89.

[19] Ding, Z., Granger, C.W.J. and Engle, R.F. (1993). A long memory property of stock market returns and a new model. *Journal of Empirical Finance*, **1**, 83-106.

[20] Engle, R.F. (1982). Autoregressive Conditional Heteroskedasticity with estimates of variance of U.K. Inflation. *Econometrica*, **50**, 987-1008.

[21] Franses, P.H. and Paap, R. (2000). Modeling day-of-the-week seasonality in the S&P 500 Index. *Applied Financial Economics*, **10**, 483-488.

[22] Francq, C. and Zakoïan, J.M. (2013). Optimal predictions of powers of conditionally heteroskedastic processes. *Journal of Royal Statistical Society*, **B75**, 345-367.

[23] Francq, C. and Zakoïan, J.M. (2010). *GARCH models: Structure, statistical inference and financial applications.* John Wiley.

[24] Francq, C. and Zakoïan, J.M. (2008). Deriving the autocovariances of powers of Markov-switching $GARCH$ models, with applications to statistical inference. *Computational Statistics & Data Analysis*, **52**, 3027-3046.

[25] Geweke, J. (1989). Bayesian inference in econometric models using Monte Carlo integration. *Econometrica,* **57**, 1317-1339.

[26] Granger, C.W.J. (2005). The past and future of empirical finance: Some personal comments. *Journal of Econometrics*, **129**, 35-40.

[27] Haas, M. (2009). Persistence in volatility, conditional kurtosis, and the Taylor property in absolute value $GARCH$ processes. *Statistics & Probability Letters*, **79**, 1674-1683.

[28] Haas, M., Mittnik, S. and Paolella, M.S. (2004). A New Approach to Markov Switching $GARCH$ Models. *Journal of Financial Econometrics,* **4**, 493-530.

[29] Hamadeh, T. and Zakoïan, J.M. (2011). Asymptotic properties of LS and QML estimators for a class of nonlinear $GARCH$ processes. *Journal of Statistical Planning and Inference*, **141**, 488-507.

[30] Hoogerheide, L. and van Dijk, H.K. (2010). Bayesian forecasting of value at risk and expected shortfall using adaptive importance sampling. *International Journal of Forecasting*, **26**, 231-247.

[31] Hwang, S.Y. and Basawa, I.V. (2004). Stationarity andmoment structure for Box-Cox transformed threshold $GARCH(1,1)$ processes. *Statistics and Probability Letters*, **68**, 209-220.

[32] Kim, S., Shephard, N. and Chib, S. (1998). Stochastic volatility: likelihood inference and comparison with $ARCH$ models. *The Review of Economic Studies.* **65**, 361-393.

[33] Osborn, D.R., Savva, C.S. and Gill, L. (2008). Periodic dynamic conditional correlations between stock markets in Europe and the US. *Journal of Financial Econometrics*, **6**, 307-325.

[34] Pan, J., Wang, H. and Tong, H. (2008). Estimation and tests for power-transformed and threshold $GARCH$ models. *Journal of Econometrics*, **142**, 352-378.

[35] Regnard, N. and Zakoïan, J.M. (2011). A conditionally heteroskedastic model with time-varying coefficients for daily gas spot prices. *Energy Economics*, **33**, 1240-1251.

[36] Ritter, C. and Tanner, M.A. (1992). Facilitating the Gibbs sampler: the Gibbs stopper and the Griddy-Gibbs sampler. *Journal of the American Statistical Association,* **87**, 861-870.

[37] Rossi, E. and Fantazani, D. (2015). Long memory and periodicity in intraday volatility. *Journal of Financial Econometrics*, **13**, 922-961.

[38] Smith, M.S. (2010). Bayesian inference for a periodic stochastic volatility model of intraday electricity prices. *In Statistical Modelling and Regression Structures*, edt, 353-376.

[39] Spiegelhalter D.J., Best, N.G., Carlin, B.P. and van der Linde, A. (2002). Bayesian measures of model complexity and fit. *Journal of Royal Statistical Society*, **B64**, 583-639.

[40] Tsay, R.S. (2010). *Analysis of financial time series: financial econometrics*, 3rd edn. Wiley, New York

[41] Tsiakas, I. (2006). Periodic stochastic volatility and fat tails. *Journal of Financial Econometrics*, **4**, 90-135.

[42] Xia, Q., Wong, H., Liu, J. and Liang, R. (2017). Bayesian Analysis of Power-Transformed and Threshold *GARCH* Models : A Griddy-Gibbs Sampler Approach. *Computational Economics*, **50**, 353-372.

[43] Ziel, F., Steinert, R. and Husmann, S. (2015). Efficient modeling and forecasting of electricity spot prices. *Energy Economics*, **47**, 98-111.

[44] Ziel, F., Croonenbroeck, C. and Ambach, D. (2016). Forecasting wind power modeling periodic and non-linear effects under conditional heteroscedasticity. *Applied Energy*, **177**, 285-297.