Why expected discount factors yield incorrect expected present values

Szabolcs Szekeres

IID Gazdasági Tanácsadó Kft, Budapest, Hungary

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SZABOLCS SZEKERES*

ABSTRACT

Compound and discount factors determine the relationship between present and future values. When interest rates are stochastic, expected compound factors are computed by probability weighting all possible compound factors. It is customary to proceed likewise to compute expected discount factors. It has been noted that risk neutral certainty equivalent interest rates differ when computed from expected compound or expected discount factors, yielding alternative project rankings. This paper shows that expected discount factors yield incorrect expected present values because, unlike in the deterministic case, they are not the reciprocals of the corresponding expected compound factors.

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* IID Gazdasági Tanácsadó Kft., Budapest, Hungary

In “On Expected Value vs. Expected Future Value” Elisha A. Pazner and Assaf Razin (1975) assert that when the discount rate (cost of capital) is uncertain, the equivalency of the present value and future value criteria for investment project ranking no longer holds. That is, they lead to inconsistent investment project rankings. Their analysis is conducted from the point of view of risk neutral investors, only concerned about the expected value of their wealth, and the probabilities describing the interest rate uncertainty are real.

Pazner and Razin (1975) conclude: “As the two criteria discussed here are equally likely, on a priori grounds, to be used as guides to investment decision making, and as their use may provide different rankings of investment prospects, the question arises as to what is the correct way to approach the problem in general.”

The discrepancy between the two criteria has also been observed in the literature devoted to the choice of the social discount rate to be used in cost-benefit analysis. Martin L. Weitzman (1998) proposed adopting the expected present value criterion and noted that it would lead to a declining term structure[1] of certainty equivalent interest rates in the far distant future if interest rates are stochastic and subject to persistent, non mean-reverting shocks. Christian Gollier (2004) noted that using the future value criterion under the same assumptions would lead to an increasing term structure. This discrepancy became the “Weitzman-Gollier Puzzle,” which has never been truly solved in its own terms, that is, under the assumption that investors are risk neutral.

Commenting on this puzzle, Groom, Hepburn, Koundouri and Pierce (2005) state: “So, confusingly, whereas in the absence of uncertainty the two decision criteria are equivalent, once uncertainty regarding the discount rate is introduced the appropriate discount rate for us in CBA depends upon whether we choose ENPV or ENFV as our decision criterion. In the former case, discount rates are declining and in the latter they are rising through time. It is not immediately clear which of these criteria is correct.”

This paper explains the observed discrepancies by pointing out that expected discount factors computed by probability weighting all possible discount factors are not the reciprocals of the expected compound factors corresponding to the same probability distributions of

[1] Throughout this paper “term structure” refers to the time pattern of risk neutral certainty-equivalent rates that corresponds the probability distribution of interest rates assumed. No modeling of the term structure of real market interest rates is implied. In fact, in all the models considered in this paper, the expected values of the stochastic interest rates are constant through time.

interest rates, and therefore lead to computing wrong expected present values. In other words, what is generally called the expected present value criterion is not really what its name implies, for the customary method of computing expected present values is wrong. When the correct calculation is made, the discrepancy between the two decision criteria vanishes.

Section I summarizes the argument presented by Pazner and Razin (1975); Section II shows that the expected value of discount factors leads to the wrong expected present value; Section III provides another explanation of this fact, and an interpretation of what the incorrect method measures; Section IV explores the effect of autocorrelation of interest rates on the size of the discrepancy between correctly and incorrectly calculated certainty equivalent discount factors; and Section V presents conclusions, including a summary comparison of alternative calculation methods.

I. Summary of Pazner and Razin's (1975) analysis

For their analysis Pazner and Razin (1975) consider the evaluation of projects having an initial investment of B_0 and a stream of benefits $\{Bt\}$, where t ranges from 1 to T. If the cost of capital r used to discount benefits $\{B_t\}$ is stochastic, then the expected present value (EPV) and the expected future value (EFV) of the projects are given by the following expressions, in which the expectation operator applies to the random interest rate r, the probabilities of which are not explicitly shown:

$$EPV = \mathrm{E}\left[\sum_{t=0}^{T} \prod_{i=0}^{t} \frac{1}{1+r_i} B_t\right] \tag{1}$$

$$EFV = \mathrm{E}[\sum_{t=0}^{T} \prod_{i=t}^{T} (1 + r_i)\, B_t] \tag{2}$$

A project should be undertaken, under either criterion, if the expected value is greater than or equal to zero. It is well known that when r is deterministic, the two criteria are equivalent. Pazner and Razin (1975) show, however, that this equivalency breaks down when r is stochastic. To this end, they simplify the above expression by setting $T = 1$.

The certainty equivalent discount rate r^* can be calculated from the expected present value criterion as follows:

$$B_0 + \mathrm{l}/(1+r^*)\, B_1 = B_0 + \mathrm{E}\ [\ \mathrm{l}/(1+r)\ B_1] \equiv {}^{PR}EPV \tag{3}$$

The above is identical to expression (3) from Pazner and Razin (1975), except that their expected present value abbreviation *EPV* is replaced here by ^{PR}EPV for attribution, and to distinguish it from an alternative formulation to be presented below. Notice that the term $\mathrm{E}[1/(1+r)\, B_1] = \mathrm{E}[1/(1+r)]\, B_1$ as the B_t are not stochastic, and that $\mathrm{E}[1/(1+r)]$ is the probability weighted expected value of the discount factors corresponding to all possible values of the stochastic interest rate r.

Similarly, they define r^{**}, the certainty equivalent discount rate derived from the future value criterion, as follows:

$$B_0\,(1 + r^{**}) + B_1 = \mathrm{E}\,[(1 + r)\,B_0] + B_1 \equiv EFV \tag{4}$$

where *EFV* stands for expected future value.

Given that 1/(1+r) is strictly convex, Pazner and Razin observe[2] that by Jensen's Inequality

$$1 + r^{*} = \frac{1}{\mathrm{E}\left[\frac{1}{1+r}\right]} < \mathrm{E}[1 + r] = 1 + r^{**} \tag{5}$$

Therefore, it follows that $r^* < r^{**}$, which leads to the ranking discrepancy noted.

This derivation is perfectly correct, but the conclusion is predicated on the assumption that ^{PR}EPV, as defined in (1) above, is the correct expression for expected present value.

II. The expected value of discount factors leads to the wrong EPV

To analyze this question, the above example will be simplified even further by assuming that $B_0 = 0$ and $B_1 = 1$. In that case $EFV = 1$.

The EPV of 1 can be derived from the textbook definition of present value, namely, it is the amount that will compound to the *EFV* at the going (stochastic) market rate r:

$$\mathrm{E}[1+r]\; EPV \equiv 1 \tag{6}$$

Consequently

[2] Expression (5) is the same as Pazner and Razin's (1975) own expression (5), and is derived from the following instance of Jensen's inequality $\mathrm{E}\left[\frac{1}{1+r}\right] > \frac{1}{\mathrm{E}[1+r]}$.

$$EPV \equiv \frac{1}{\mathrm{E}[1+r]} \tag{7}$$

We can see that the correct *EPV* is different from ${}^{\mathrm{PR}}EPV$:

$$EPV \equiv \frac{1}{\mathrm{E}[1+r]} \neq \mathrm{E}\left[\frac{1}{1+r}\right] \equiv {}^{\mathrm{PR}}EPV \tag{8}$$

It is easier to see the nature of this inequality if we make the expectation operator explicit, with a simple two states of the world example. Let the stochastic r be one of $\{r_1, r_2\}$ with probabilities $\{p_1, p_2\}$. Expression (8) then becomes:

$$EPV \equiv \frac{1}{p_1(1+r_1)+p_2(1+r_2)} \neq \frac{p_1}{(1+r_1)} + \frac{p_2}{(1+r_2)} \equiv {}^{\mathrm{PR}}EPV \tag{9}$$

in which the basic reason for the inequality is the following:

$$\big(p_1(1+r_1)+p_2(1+r_2)\big)^{-1} \neq p_1(1+r_1)^{-1} + p_2(1+r_2)^{-1} \tag{10}$$

We have seen from (6) and (7) that *EPV* is the correct certainty equivalent discount factor (EPV of 1). That ${}^{\mathrm{PR}}$EPV is not the correct certainty equivalent discount factor is proven by the fact that ${}^{\mathrm{PR}}$EPV does not compound to the future value of 1. For our simple example this is shown by the following:

$$(p_1(1+r_1)^{-1} + p_2(1+r_2)^{-1})\big(p_1(1+r_1)+p_2(1+r_2)\big) \neq 1 \tag{11}$$

which can be generalized as follows:

$$\mathrm{E}\left[\frac{1}{1+r}\right] \mathrm{E}[1+r] \neq 1 \tag{12}$$

The startling conclusion that can be derived from this is that the probability weighted expectation of the scenario specific discount factors is not the correct risk neutral certainty equivalent discount factor! ${}^{\mathrm{PR}}EPV$ is an incorrect EPV measure, and the certainty equivalent rate r^* derived from it is also incorrect.

The correct *EPV* is given by expression (7), as it compounds to *EFV*, in compliance with the definition of present value. The correct certainty equivalent discount rate r^{**} can be derived from (7) as follows:

$$EPV \equiv \frac{1}{\mathrm{E}[1+r]} = \frac{1}{1+r^{**}} \tag{13}$$

from which we get that (1 + r**) = E [(1 + r)], the same as in (4) above. When EPV is correctly calculated, the certainty equivalent rates derived from both the EPV and EFV criteria are identical. There is no discrepancy between them.

Returning to the full example of Pazner and Razin (1975), we can now state that their expression (1) will not compute the correct EPV of any project. The correct result will be given only by the following expression:

$$EPV = \sum_{t=0}^{T} \frac{B_t}{\mathrm{E}[\prod_{i=0}^{t}(1+r_i)]} \tag{14}$$

Because the B_t are not stochastic, expression (2) can be rewritten as follows:

$$EFV = \sum_{t=0}^{T} \mathrm{E}[\prod_{i=t}^{T}(1+r_i)]\, B_t \tag{15}$$

Notice that both discounting and compounding is done with certainty equivalent discount and compound factors, respectively, which are not stochastic. Because the same certainty equivalent rate r** can equally be derived from both, the EPV and EFV rules will yield the same conclusion for any project. This is formally demonstrated for expressions (14) and (15) in the Appendix.

III. A more intuitive explanation of the preceding finding and an interpretation of the incorrect method

Since certainty equivalent compound factors are calculated as the probability weighted expectations of the scenario specific compound factors, it is understandable that the applicability of the same method to discount factors should be a widely held belief. Even though the previous section has proven this notion to be wrong, it is useful to provide another, perhaps more intuitive, explanation of why it is wrong, and also to provide an interpretation of what the incorrect calculation method actually is.

To do this we modify the simple two period model from the beginning of Section II by allowing t to vary, but keeping $B_0 = 0$, and $B_t = 1$. It will therefore serve to calculate the amount that a risk neutral investor would be willing to pay for a zero-coupon bond, with a face value of 1, due at time *t*. Using the continuous time formulation proposed by Weitzman (1998) will be most convenient.[3]

Weitzman proposed the following certainty equivalent discount factor, which, for this simplified model, is the equivalent of (1) above, and which is also the result of probability weighting scenario specific discount factors:

$$A = \mathrm{E}[e^{-rt}] \tag{16}$$

A is incorrect, because it is not the inverse of the certainty equivalent compound factor *C* that describes the assumed market conditions:

$$C = \mathrm{E}[e^{rt}] \tag{17}$$

The correct certainty equivalent discount factor *D* is, of course, the inverse of *C*:

$$D = \frac{1}{\mathrm{E}[e^{rt}]} \tag{18}$$

Notice, however, that *A* corresponds exactly to *C* when the product *rt* is negative. Having negative *r* would correspond to a capital market in which resources are stored for a fee, rather than being lent to someone willing to pay a positive interest rate. Having *t* negative would imply reversing the flow of time. Discounting with Weitzman's *A* is like compounding with the negatives of the assumed market interest rates, but from the future to the present. We could call it time reversed negative compounding.

To compare the behavior of *A* and *D* as a function of time, we assign numeric values to the simple two-scenario model. Let's assume that the states of the world are equiprobable, and that r_1=1% and r_2=5%. The difference between discounting and time reversed negative compounding will be explained with the help of two Figures.

[3] This Section is largely taken from Szekeres (2017), an unpublished working paper by the author.

Figure 1. Compound and discount factors
(5% interest p.a., logarithmic scale)

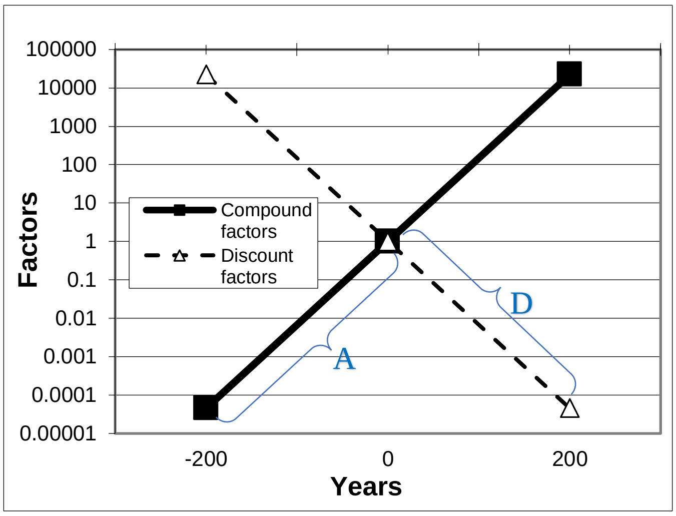

Figure 1 shows the compound and discount factors curves applicable to an investment of $1 made at time 0, in continuous time, with a deterministic annual interest rate of 5%, between years -200 and 200. (We have negative compounding and discounting to the left of year 0.) The equations being plotted are $e^{0.05t}$ for the compound factors curve, and $1/e^{0.05t}$ for the discount factors curve. The vertical scale in Figure 1 is logarithmic, which is why both the compound factors and discount factors curves are seen to be linear. The fact that one is the inverse of the other is evidenced by their symmetry with respect to the horizontal line passing through the value of 1. Note that the negative range of the compound factor curve (which is what A is) is symmetrical to the positive range of the discount factor curve D around the vertical axis (year 0), which means that in the deterministic case $A = D$ for any absolute value of t. In other words, discounting and time reversed negative compounding are equivalent if interest rates are not stochastic.

Figure 2. Compound factors at 1% and 5%, their expected value and the corresponding discount factors, logarithmic scale

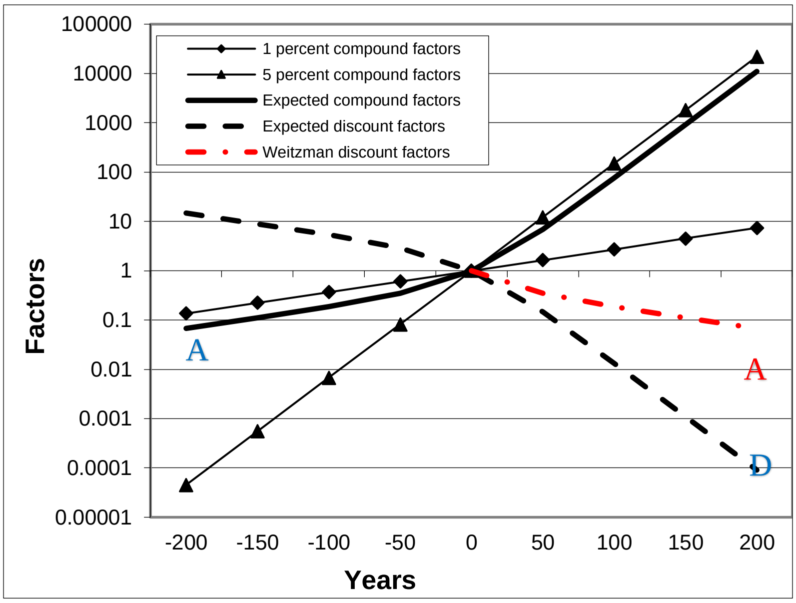

Figure 2 illustrates the stochastic case. It shows the compound factor curves corresponding to interest rates of 1% and 5%, both of which are linear in logarithmic terms. Their expectation is no longer linear, however. Moving forward in time (positive range of years), compound factors corresponding to the high interest rate grow proportionally larger relative to those of the low interest rate, thereby pulling their expected value ever closer to the compound factors curve of the high rate. The same happens moving backwards into the past (negative range of years), in which case it is the compound factors corresponding to the low interest rate that grow relatively larger, and it is therefore towards the compound factors curve of the low interest rate that their expected values tend asymptotically. In other words, the higher compound factors pull the expected compound factors upwards over the entire time range, this effect being stronger as the absolute value of time increases.

The immediate consequence of this is that the expected compound factors curve is no longer linear logarithmically. This is also true of the expected discount factors curve, which is the inverse of the expected compound factors curve. Because of this lack of linearity, the negative range of the expected compound factors curve is not symmetrical to the positive range of the expected discount factors curve with respect to the vertical axis, and cannot be used, therefore, to calculate EPVs correctly. As Figure 2 shows, the negative range of the compound factors curve is significantly higher than the positive range of the discount factors curve for all absolute values of time.

This is the reason why the probability weighted average of the conditional discount factors of alternative interest rate scenarios (which is what the negative range of the expected compound factors curve is) does not yield the correct EPV of amounts compounded to the future. To facilitate comparison with the correct discount factors, the negative range of the expected compound factors curve (A) is mapped to the positive range of years and labeled Weitzman discount factors in Figure 2. It significantly overstates the PV of future sums.

As Figure 2 shows, when interest rates are perfectly autocorrelated (because r is constant), the expected compound factors will be above the deterministic average of the low and high interest rates. This average is not shown in Figure 2 but would be represented by a straight line corresponding to 3%, half-way between the 1% and 5% lines. As this (peculiar) market's growing yields constitute the opportunity cost of any alternative long-term investment project, it is the role of discounting to see if such a project can do better than this market.

The incorrect discounting method singularly fails at this task. It misrepresents the assumed market conditions, because it implies, as shown in Figure 2, that the opportunity cost of long term investments is declining, when in fact it is growing through time. Investors using this method of computing EPVs do so at their peril. They will systematically overstate the EPVs of their projects.

IV. The effect of autocorrelation of interest rates

The nature of the market described in the previous Section results from the assumption of perfect autocorrelation of interest rates. In this Section we explore the effect of the degree of autocorrelation of interest rates on the size of the discrepancy between the correct and incorrect methods of discounting. To do so, we compute the value of zero-coupon bonds for different frequencies of compounding and different probability distributions of interest rates r.

In the first example time will be discrete (years) and the stochastic constant interest rate r will be compounded annually. For any time horizon t, the correct certainty equivalent discount factor will be $1/E[(1+r)^t]$, while the incorrect one, based on probability weighting the scenario specific discount factors, will be $E[1/(1+r)^t]$. To illustrate what happens in this case we use the numeric values of the two-scenario model already proposed: the states of the world are equiprobable, r_1=1% and r_2=5%. We obtain then the following illustrative results[4].

Table I. Alternative certainty equivalent discount factors and rates

t	$1/E[(1+r)^t]$	r^{**}	$E[1/(1+r)^t]$	r^*	Error
1	0.9709	3.00%	0.9712	2.96%	0.038%
10	0.7317	3.17%	0.7596	2.79%	3.819%
20	0.5163	3.36%	0.5982	2.60%	15.86%
30	0.3527	3.53%	0.4867	2.43%	37.96%

For year 1 – a single compounding period (row 1) – the certainty equivalent is $r^{**} = E[r]$. For later years, however, r^{**} is no longer $E[r]$, because the fast-growing compound factor associated with r_2 raises the expected compound factor above $E[r]$. In the case of r^* the converse is true, the expected discount factor is skewed towards that of the lower rate. The year 1 value is already wrong.

The large error that results from calculating with the wrong expression (a 37.96% overestimate of the value of a zero-coupon bond maturing in 30 years) is largely due to the assumption of perfect autocorrelation of interest rates. The term structures of certainty equivalent discount rates that results from the alternative calculations diverge markedly. The correct one is growing, as it should be, reflecting the accelerating effects of compound factors

[4] All numerical results presented in this paper can be found (and reproduced) in an Excel workbook available from the author. Send an email to szsz@iid.hu

associated with the higher rate, while the incorrect one is declining instead, divorced from the reality of the assumed market.[5]

Notice, incidentally, that the calculations in this Section also provide empirical corroboration of the main assertion of this paper. The correct certainty equivalent discount factors shown in the tables are the reciprocals of the certainty equivalent compound factors. Someone, who on the basis of the incorrect calculation, would be willing to invest \$0.4867 for the zero-coupon bond to get \$1 in year 30 would suffer a year 30 opportunity loss of 0.4867/0.3527–1= \$0.3799 by not investing in the market instead. This is the meaning of the error measures shown in the tables. One calculates EPVs to determine where to invest. It is therefore imperative to calculate correctly.

To explore what happens when the degree of autocorrelation is less than perfect, Monte Carlo simulations were conducted. This required an alteration of the model. Rather than there being a constant stochastic interest rate *r* for all years, we assume that there are as many interest rate variables as there are years. We assume that the yearly rates are identically distributed, with the same distribution as before, but with a degree of autocorrelation that will be specified in each examined case.

The correct discount factor for any time period *t* is[6]:

$$D = \frac{1}{\mathrm{E}[\prod(1+r)]} \tag{19}$$

where the *r* distribution is sampled independently for each year. With these assumptions, the following results were obtained, based on 10,000 simulations:

Table II. Alternative certainty equivalent discount factors and rates, coefficient of autocorrelation = 0

t	$\frac{1}{\mathrm{E}[\prod(1+r)]}$	r^{**}	$\mathrm{E}\left[\frac{1}{\prod(1+r)}\right]$	r^{*}	Error
1	0.9709	3.00%	0.9712	2.96%	0.038%
10	0.7449	2.99%	0.7465	2.97%	0.225%
20	0.5543	2.99%	0.5566	2.97%	0.418%
30	0.4127	2.99%	0.4152	2.97%	0.625%

[5] The frequency of compounding has no bearing on these results when *r* is constant, and therefore perfectly autocorrelated, provided that the periodic rate is such that the effective annual rate remains constant.

[6] We assume that values accrue at the end of the period to which the interest rates pertain. Indexing of *r* is omitted for simplicity, but it is implicitly from 1 to *t*.

As we can see the discrepancy in this case is not as serious as before, but it is still present. The term structure of certainty equivalent discount rates is flat with both calculation methods. Notice, however, that the first rows of Table I and Table II are identical. This shows the intrinsic error of the incorrect formula, over a single compounding period. In Table I the error is compounded due to the perfect autocorrelation of interest rates, while in Table II the accumulation of errors through time amounts to much less, because when a high *r* is followed by a low one, the acceleration effect in compounding is partially undone.

To map the effects of the autocorrelation assumption we calculated the results for two additional cases. The results for a coefficient of autocorrelation of 0.9 are the following:

Table III. Alternative certainty equivalent discount factors and rates, coefficient of autocorrelation = 0.9

t	$\frac{1}{\mathrm{E}[\prod(1+r)]}$	r^{**}	$\mathrm{E}\left[\frac{1}{\prod(1+r)}\right]$	r^{*}	Error
1	0.9709	3.00%	0.9712	2.96%	0.038%
10	0.7375	3.09%	0.7565	2.83%	2.577%
20	0.5444	3.09%	0.5735	2.82%	5.352%
30	0.4091	3.02%	0.4227	2.91%	3.341%

Even with this high degree of autocorrelation the term structures are basically flat in both cases. This reflects the fact that the effects of even such a high degree of autocorrelation are soon eroded, as autocorrelation across years diminishes with the power of the correlation coefficient. Notice that the year 1 error is the same as before, but correlation compounds errors, so that the valuation error of a 30-year zero-coupon bond becomes 3.341%.

It takes a very high degree of autocorrelation to approach the results of Table I. We obtain the following when the correlation coefficient is 0.99:

Table IV. Alternative certainty equivalent discount factors and rates, coefficient of autocorrelation = 0.99

t	$\frac{1}{\mathrm{E}[\prod(1+r)]}$	r^{**}	$\mathrm{E}\left[\frac{1}{\prod(1+r)}\right]$	r^{*}	Error
1	0.9709	3.00%	0.9712	2.96%	0.038%
10	0.7318	3.17%	0.7594	2.79%	3.775%
20	0.5177	3.35%	0.5978	2.61%	15.47%
30	0.3553	3.51%	0.4851	2.44%	36.52%

The term structures display the same characteristics as were seen for the perfect correlation case, and the 30-year bond valuation error is now 36.52%.

It is clear from the preceding examples that the degree of autocorrelation is a key determinant of the magnitude of the error that results from using the wrong certainty equivalent discount factor calculation. The error rates observed above pertain to the posited very simple didactic example, however, the purpose of which was primarily expository, so they might not be indicative of the errors that would be encountered in real life. To gauge the error incurred in a more realistic case, Monte Carlo simulations were conducted of a short-term interest rate model of the Cox, Ingersoll & Ross (CIR) type, with parameters that were calibrated by Yajie Zhao and Boru Wang (2017) with reference to a monthly data series of US three-month Treasury Bill rates spanning the period 1992 to 2017.

In this case we simulated monthly interest rates, so we had as many variables as there are months in the ten-year period simulated. These were generated by the CIR process, but only every third one was used in the calculation, as we assumed quarterly compounding. The rates are expressed on a per-annum basis, but the calculation uses the effective annual equivalent quarterly rates. The results obtained from 10,000 Monte Carlo simulations are as follows. In this table the expected values of the interest rates simulated for each year are shown as well.

Table V. Alternative certainty equivalent discount factors and rates with interest rate probabilities generated by a CIR type model. Annual data.

t	$\mathrm{E}[r]$	$\frac{1}{\mathrm{E}[\prod(1+r)]}$	r^{**}	$\mathrm{E}\left[\frac{1}{\prod(1+r)}\right]$	r^{*}	Error
1	4.31%	0.9577	4.42%	0.9586	4.31%	0.103%
2	4.31%	0.9163	4.47%	0.9199	4.26%	0.399%
3	4.32%	0.8758	4.52%	0.8835	4.22%	0.878%
4	4.32%	0.8363	4.57%	0.8490	4.18%	1.523%
5	4.32%	0.7980	4.62%	0.8165	4.14%	2.324%
6	4.32%	0.7609	4.66%	0.7858	4.10%	3.267%
7	4.32%	0.7250	4.70%	0.7566	4.07%	4.347%
8	4.31%	0.6905	4.74%	0.7288	4.03%	5.552%
9	4.31%	0.6573	4.77%	0.7025	4.00%	6.876%
10	4.30%	0.6253	4.81%	0.6773	3.97%	8.313%

The errors measured are not insignificant, and they grow appreciably with time. This is because the average autocorrelation coefficient of the simulated interest rates was 0.957, which is not too far from the coefficient observed in the dataset used by Zhao and Wang (2017), which is 0.996.

The errors corresponding to the perfect autocorrelation and the no autocorrelation cases were calculated for this example as well. Perfect autocorrelation was mimicked by using the interest rate simulated for the first month in all subsequent months of the analysis. In that case the error observed for the 10-year time horizon was 10.864%, slightly higher than the Table V value due to the increased autocorrelation. The no autocorrelation case was mimicked by repeating in each period the generation of the first period interest rate, but always using a new random seed. In that case the error observed for the 10-year time horizon was only 0.26%. As expected, this value is much lower due to the lack of autocorrelation, but it is still not zero.

In summary, the above examples show that the magnitude of the error produced by probability weighting scenario specific discount factors depends greatly on the degree of autocorrelation of interest rates. It should be noted that all errors reported are due to fact that probability weighting discount factors is wrong, and not to any Monte Carlo sampling error, because in all simulations the same interest rates were used for both the correct and the incorrect calculation methods.

The reason why errors grow with autocorrelation is that when autocorrelation is high, the likelihood of a high rate being followed by another high rate is high, which induces an acceleration effect that associates higher future values to higher interest rates, thus skewing their certainty equivalent upwards. When a high rate is followed by a low rate, however, the acceleration is partially revered, hence the lower the correlation the more subdued the acceleration effect. When autocorrelation is low, or zero, the frequency of compounding also matters, because the size of the error, which is always present, is lower over a shorter period. When autocorrelation is zero, it is plausible to expect the error to vanish when compounding frequency becomes instantaneous. In that case $r^{**} = \mathrm{E}[r]$, and EPVs can be computed as if r were deterministic, with a value of $\mathrm{E}[r]$. Then discounting and time reversed negative compounding will be equivalent.

It is interesting to notice that in expressions (1), (2), (14), (15) discounting and compounding is done by certainty equivalent discount or compound factors that are not stochastic. The discrepancy observed by Pazner and Razin (1975) between (1) and (2) was therefore not due to combining discounting with interest rate uncertainty, but rather to calculating the wrong certainty equivalent discount factors. The error is theoretically always there, but as we have seen its magnitude is fundamentally dependent on the autocorrelation of interest rates and (other than in the case of perfect autocorrelation) on the frequency of compounding.

It would be recommendable to review many commonly used expressions for EPV in the light of the foregoing. In some applications the errors committed might be small. But why not exclude the possibility of errors in the first place?

V. Conclusions

In formulating the conclusions of this paper, it is worth recalling that the entire analysis has been carried out from the point of view of risk neutral investors facing investment decisions under uncertainty described by real probabilities. The conclusions of this paper are as follows:

1. The intuitively appealing and widely held belief that the certainty equivalent discount factor is the probability weighted expectation of scenario specific discount factors is wrong because the EPVs thereby calculated will not compound to the corresponding EFVs, in defiance of the definition of present value.
2. The inconsistency between the EPV and EFV criteria observed by Pazner and Razin (1975) is due to their incorrect specification of *EPV*, expression (1) above. Likewise, the "Weitzman-Gollier Puzzle" was caused by Weitzman's (1998) incorrect specification of certainty equivalent discount factor *A*, expression (16) above. Using the correct EPV calculation the discrepancy vanishes, and the puzzle is solved.
3. The argument has been made in the literature of social discounting that long term discount rates should be declining functions of time because probability weighted expected discount factors are declining. They are, assuming a high enough degree of autocorrelation of interest rates, but they are not the correct risk neutral certainty equivalent discount factors. Consequently, as seen in Table I, the opposite conclusion follows from the underlying assumptions.
4. The order of magnitude of the error that results from using the wrong EPV calculation depends crucially on the degree of autocorrelation of interest rates and, when autocorrelation is less than perfect, on the frequency of compounding. For some common specifications, the right and wrong computational formulas are shown in the following table. Subscripts of r are omitted for simplicity, as in Pazner and Razin (1975). Some of the incorrect formulations in the right-hand column are quite often used. It would be worthwhile to revise them.
5. Notice that neither the usual formulations nor the correct ones proposed in Table VI allow for the specification of the degree of autocorrelation of interest rates, which, as was shown here, does affect the results. Consequently, strictly speaking, none will give correct results

if market interest rates show a high enough degree of autocorrelation. Under such circumstances only numerical methods will provide accurate results.

Table VI. Correct and incorrect formulations of the value of a zero-coupon bond to a risk neutral investor.

Case	**Expressions defining *EPV*, the expected present value of 1 due at time *t***	**Correct certainty equivalent discount factor *D*, derived from the definition EPV**	**Incorrect certainty equivalent discount factor *A*, based on probability weighting of scenario specific discount factors**
Discrete time, constant stochastic *r*	$EPV\ \mathrm{E}[(1+r)^t] \equiv 1$	$D = \frac{1}{\mathrm{E}[(1+r)^t]}$	$A = \mathrm{E}\left[\frac{1}{(1+r)^t}\right]$
Discrete time, variable stochastic *r(s)*, *s=1* to *t*	$EPV\ \mathrm{E}\left[\prod(1+r(s))\right] \equiv 1$	$D = \frac{1}{\mathrm{E}[\prod(1+r(s))]}$	$A = \mathrm{E}\left[\frac{1}{\prod(1+r(s))}\right]$
Continuous time, constant stochastic *r*	$EPV\ \mathrm{E}[e^{rt}] \equiv 1$	$D = \frac{1}{\mathrm{E}[e^{rt}]}$	$A = \mathrm{E}[e^{-rt}]$
Continuous time, variable stochastic *r(s)*	$EPV\ \mathrm{E}\left[\exp\left(\int_0^t r(s)ds\right)\right] \equiv 1$	$D = \frac{1}{\mathrm{E}\left[exp\left(\int_0^t r(s)ds\right)\right]}$	$A = \mathrm{E}\left[exp\left(-\int_0^t r(s)ds\right)\right]$

APPENDIX

To show that expressions (14) and (15) are congruent, we will first consider the EPV and EFV of any B_t and show how they are related. Subsequently, we will extend the result to all B_t, and then show that the EPV and EFV values of the entire benefit flow only differ from one another by a factor. Consequently, they will always be either both positive, both negative, or both zero.

Before proceeding, however, we will have to re-index expressions (14) and (15) because they are not coherently formulated. This is because Pazner and Razin (1975) assign an interest rate r_i to all time periods $i = 0, T$ in both (1) and (2)[7]. Calculating EPVs, it is natural to assume that benefits B_t are received at the end of the periods to which interest rates r_t pertain, because they can then be discounted with the interest rate corresponding to that period. Calculating EFVs, on the other hand, it is natural to assume that the B_t are received at the beginning of the periods for which the interest rates are given, as that rate can be used to compound the benefit of the period. This is what Pazner and Razin (1975) do separately in (1) and (2), but that is why these expressions are not coherent when they are looked at simultaneously. Notice also that, as in their formulation B_0 is discounted, *EPV* must lie in period $t = –1$. Similarly, as B_T is compounded, *EFV* must be in period $T + 1$.

To simultaneously calculate both *EPV* and *EFV*, with all the interest rates given, we will use interest rates r_t to compound values B_t, and $r_{t\text{ -}1}$ it to discount them. Therefore, we need an additional interest rate for $t = –1$, which was not contemplated in Pazner and Razin (1975). We also need to change the indexes in (14) to ensure congruence of the measures. To avoid negative indexes, we re-index benefits $\{B_t\}$, so that $t = 1, T$.

To clarify which interest rate applies to which benefit in which calculation, the following table shows how these concepts are related.

[7] Indexes i and t both refer to time from 0 to T.

Table VII. Interest rates used to compound and discount benefit stream $\{B_t\}$

Periods t	**0**	**1**	**2**		**t**		**T**	**T+1**
Interest rates r_t	r_0	r_1	r_2		r_t		r_T	
Amounts	*EPV*	B_1	B_2		B_t		B_T	*EFV*
Compounded by	r_0	r_1	r_2		r_t		r_T	
Discounted by		r_0	r_1		r_{t-1}		r_{T-1}	r_T

With this re-indexing, the EPV of B_t is valued at time $t = 0$, and its EFV at time $t = T + 1$. The EPV of any B_t is then:

$$EPV(B_t) = \frac{B_t}{\mathrm{E}[\prod_{i=1}^{t}(1+r_{i-1})]} \tag{20}$$

Expression (20) is summation term t of the re-indexed expression (14). Let's define the certainty equivalent discount factor used in (20) as follows:

$$D_0^t = \frac{1}{\mathrm{E}[\prod_{i=1}^{t}(1+r_{i-1})]} \tag{21}$$

Subscript 0 means that discounting is done to period 0, superscript t means that it discounts the value in period t.

Similarly, we can compute the EFV of B_t:

$$EFV(B_t) = \mathrm{E}[\prod_{i=t}^{T}(1+r_i)]B_t \tag{22}$$

Expression (22) is summation term t of the re-indexed expression (15). Let's define the certainty equivalent compound factor used in (22) as follows:

$$C_t^{T+1} = \mathrm{E}[\prod_{i=t}^{T}(1+r_i)] \tag{23}$$

which compounds B_t to period $T + 1$. Subscript t means that compounding is done for the value in period t, superscript $T + 1$ means that compounding is done to period $T + 1$. Notice that the last interest rate to be used in computing C_t^{T+1} is r_T.

We can use C_t^{T+1} (23) to compound *EPV(B_t)* (20) back to time t, thus obtaining again B_t.

$$C_0^t\, EPV(B_t) = \mathrm{E}[\prod_{i=0}^{t-1}(1+r_i)]\,\frac{B_t}{\mathrm{E}[\prod_{i=1}^{t}(1+r_{i-1})]} = B_t \qquad (24)$$

This shows the congruence between correct discounting and compounding. The certainty equivalent compound factor (C_0^t) and discount factor (D_0^t), being each other's reciprocals, cancel out. This is so because both span the same time period 0 to t. Notice that in (24) the r_i range from 0 to t -1in both products of factors (1 + r_i).

We can also make an alternative computation of *EFV*(B_t) by first compounding *EPV*(B_t) to period t, thereby reaching the value B_t, as already done in (24), and then compounding that further by C_t^{T+1}:

$$EFV[EPV(B_t)] = C_0^t\, EPV(B_t)\, C_t^{T+1} = \; EFV(B_t) \qquad (25)$$

Noticing that

$$C_0^t C_t^{T+1} = \; \mathrm{E}[\prod_{i=0}^{t-1}(1+r_i)]\; \mathrm{E}[\prod_{i=t}^{T}(1+r_i)] = \mathrm{E}[\prod_{i=0}^{T}(1+r_i)] = C_0^{T+1} \qquad (26)$$

we can state that

$$EPV(B_t)\, C_0^{T+1} = \; EFV(B_t) \qquad (27)$$

which means that compounding *EPV*(B_t) by the certainty equivalent compound factor C_t^{T+1} yields *EFV*(B_t), in compliance with the definition of present value. As this is true for all B_t, it follows that the total EFV of benefit flow $\{B_t\}$ will be its EPV times the constant C_t^{T+1}. Consequently, *EPV* and *EFV* will always be either both positive, both negative, or both zero. There is no discrepancy in the rankings given by the EPV and EFV criteria, provided that *EPV* is correctly calculated, as in (14).

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