A new Model for Stock Price Movements

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Remark: I am looking for a.) support to write a Ph.D. thesis about the topic of this paper and b.) possibilities of publishing this paper or parts of it and c.) coauthorships. Anybody interested is encouraged to contact me.

* Risk Analyst @ Dresdner Kleinwort Investment Bank / Disclaimer: The presented viewpoints are solely the personal viewpoints of the author and not of the aforementioned firm.
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Abstract
This paper aims to present a new alternative diffusion model for asset price movements. In contrast to the popular approach of Brownian motion it proposes deterministic diffusion for the modelling of stock price movements. These diffusion processes are a new area of physical research and can be created by the chaotic behaviour of rather simple piecewise linear maps, but can also occur in chaotic deterministic systems like the famous Lorenz system. The reason for the investigation on deterministic diffusion processes as suitable model for the behaviour of stock prices is, that their time series can obey certain stylized facts of real world stock market time series. For example they can show fat tails of empirical log returns in union with varying volatility i.e. heteroscedasticity as well as slowly decaying autocorrelation of squared log returns. These phenomena could not be explained by a simple Brownian motion and have been the most criticism to the lognormal random walk. The scope of this paper is to show that deterministic diffusion models can explain the occurrence of those empirical observed stylized facts and to discuss the implications for economic theory with respect to market efficiency and option pricing.

1. Introduction
Despite its popularity the model of normal distributed log returns for the movement of stock prices there has been a large discussion in literature whether it is appropriate or not. A lot of empirical testing has been done which mainly concluded that stock market returns are not normally and independently distributed and hence do not follow random walks. Lo and MacKinlay present a simple specification test in their 1988 paper REF001 that enabled them to reject the Random Walk hypothesis for stock market prices. Le Baron, Scheinkman and William A. Brock present their measure of the BDS-Statistic in REF021 based on the correlation dimension, a popular characterization of chaotic systems. The BDS-Statistic makes it possible to discern iid. from other correlated (mostly chaotic, nonlinear) time series. The BDS Statistic has been applied by REF0019 et. Al. to test time series weather have a chaotic or stochastic origin. Not necessary to mention, that obtained results are ambiguous and yet there has not been a final judgement in favour to one or the other model. Hans Malmsten, Timo Teräsvirta review the mostly observed stylized facts of financial time series against three popular Volatility Models: GARCH, EGARCH, and ARSV. See REF010. They find, that those models never show the stylized facts at a time and therefore have unsatisfactory explanation power.

The subject of this paper is the investigation on the reasons for these empirical results and the proposal of the deterministic diffusion model for stock market prices. Furthermore the implications of this new model on economic theory e.g. market efficiency and option pricing

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are discussed. In section 2. an introduction to the phenomenon of deterministic diffusion will be given. In section 3. a simple dynamic asset pricing model is proposed to justify the usage of deterministic diffusion as the appropriate approach. Section 4. compares the stylized facts of the time series generated by the model and real world time series of the german equity index DAX™ and relates to them to the classical stylized facts of economic time series mentioned above. Section 5 discusses the implications on economic theory regarding asset pricing, market efficiency and option pricing. Finally section 6. reviews the approach and gives summary of the main results as well as an outlook on what should still be the object of future research.

2. Deterministic Diffusion
Firstly we shall come about with a formal definition of deterministic diffusion and a simple piecewise linear map that obeys such behaviour.

Definition 2.1(Deterministic Diffusion):
Deterministic diffusion is the displacement of a particle X on the real line in time according to a deterministic law:

\[ X(t+1) = M(X(t)) \]

X(t) denotes the position of the particle at time t and M is an expanding Mapping M: \( \mathbb{R} \rightarrow \mathbb{R} \) (expanding means \( M' > 1 \))

In the context of our further studies we will be concerned with normal and anomalous diffusion. Thus it is convenient to give two more formal definitions.

Definition 2.2a:
A diffusion process is called normal if its mean square displacement scales linearly with factor one in time, i.e. for any times \( t_1 < t_2 \) holds:

\[ \langle \Delta X_{t_2}^2 \rangle = \frac{(t_2-t_1)}{t_1} \langle \Delta X_{t_1}^2 \rangle \]

and the diffusion coefficient D:

\[ D = \lim_{t \to \infty} \frac{\langle \Delta X_{2t}^2 \rangle}{2t} \]

exists. \( <> \) denotes the average of the squared displacement \( \Delta X_{t}^2 = (X(t)-X(0))^2 \) at time t with respect to all starting values \( X(0) \).

Definition 2.2b:
A diffusion process is called anomalous if its mean square displacement scales according a power law with respect to a coefficient \( \alpha <> 1 \) in time, i.e. for any times \( t_1 < t_2 \) holds:

\[ \langle \Delta X_{t_2}^2 \rangle = \left[ \frac{(t_2-t_1)}{t_1} \right]^\alpha \langle \Delta X_{t_1}^2 \rangle \]

If \( \alpha > 1 \) the diffusion is called super diffusion if \( \alpha < 1 \) it is called sub diffusion. Since the reader is now familiar with the basic concept of deterministic diffusion it is now time to
introduce a family of simple 1-d maps which will be mainly under study in the remainder of this paper obeying deterministic diffusion for a wide range of parameters.

Call \( \mathcal{I} \) the family of piecewise linear maps \( M : \mathbb{R} \to \mathbb{R} \) with uniform slope \( a \) having the properties:

1.) \( M \) is expanding: \( a > 1 \)
2.) \( M \) is lifting: \( M(X-n) + n = M(X) \) for any real number \( X \) an integer \( n \) with \( n = \text{int}(X) \)
3.) \( M \) is chaotic i.e. for its Lyapunov exponent\(^2 \) \( \lambda = \ln(a) \) holds \( \lambda > 0 \)

Since in the context of modelling economic time series we are interested only in a diffusion processes with non negative outcome we shall specify another subclass \( \mathcal{I}_{>0} \subset \mathcal{I} \) with the same properties like Maps of \( \mathcal{I} \) with only positive values permitted.

Call \( \mathcal{I}_{>0} \) the family of piecewise linear maps \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) with uniform slope \( a \) having the properties:

4.) \( M \) is expanding: \( a > 1 \)
5.) \( M \) is lifting: \( M(X-n) + n = M(X) \) for any positive real number \( X \) and positive integer \( n \) with \( n = \text{int}(X) \), \( n > 0 \)
6.) \( M \) is chaotic i.e. for its Lyapunov exponent \( \lambda = \ln(a) \) holds \( \lambda > 0 \)

The example Map \( S \subset \mathcal{I} \) we want to consider in the following is the saw tooth map \( S(X) \) defined by:

\[
S(X) = \begin{cases} 
 a \left[ x - \text{int}(x) \right] + \text{int}(x) & 0 < x - \text{int}(x) < \frac{1}{2} \\
 a \left[ x - \text{int}(x) \right] - a + 1 + \text{int}(x) & \frac{1}{2} < x - \text{int}(x) < 1 
\end{cases}
\]

with parameter values \( a > 2 \). Note for \( 1 > a > 2 \) \( S \) is chaotic but its iterates do not leave the unit interval. To find its analogon \( S_{>0}(X) \) in \( \mathcal{I}_{>0} \) one has to modify (2.5) to require \( S(X) \gg 0 \).

\[
S_{>0}(X) = \begin{cases} 
 a \left[ x - \text{int}(x) \right] + \text{int}(x) & 0 < x - \text{int}(x) < \frac{1}{2} \\
 a \left[ x - \text{int}(x) \right] - a + 1 + \text{int}(x) & \frac{1}{2} < x - \text{int}(x) < 1; a \leq 2 + 2 \text{int}(x) \\
 a \left[ x - \text{int}(x) \right] - \frac{1}{2}a + \text{int}(x) & \frac{1}{2} < x - \text{int}(x) < 1; a > 2 + 2 \text{int}(x) 
\end{cases}
\]

To get a better understanding on how the dynamics of \( S \) and \( S_{>0} \) work in Fig. 1 the maps and a trajectory of the maps are shown schematically and as time series. As one can see the behaviour of the maps equals in particular characteristics the behaviour of stock market prices. Small changes are followed by small changes and large changes are followed mostly by large changes and occur after a regime of small changes in sequence.

---

\(^1\) \( n \) is the smallest integer smaller than \( X \)
\(^2\) For a definition see section 4.1) / Definition 4.1.1)
Before we turn to relating the model of deterministic diffusion to stock market prices we will have to give again some definitions, that will ease later work.

Definition 2.3 (ergodic / invariant measure)
A Map $M: \mathbb{R} \rightarrow \mathbb{R}$ is called ergodic if there exists a measure (density) $0 \leq \rho(x) \leq 1$ such that:

$$\rho(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{0}^{n} \delta(M^n(x_0) - x) \rho(x_0) dx_0 \quad (2.7)$$

$\delta$ is the Dirac delta function, $M^n$ denotes the nth iterate of the map $M$, the integral is over all initial conditions $x_0$ in $\mathbb{R}$. The measure $\rho(x)$ is called the invariant measure of $M$.

Note, since we are dealing with a deterministic process $\rho$ is not a probability measure but rather measures the frequency of appearances of a value $x$ in $\mathbb{R}$ in average. The kernel of (2.7) $\delta(M^n(x_0)-x)$ is called the Frobenius Perron Operator.

Right at this time we are at the heart of statistical mechanics, but didn’t we want to talk about stock market prices?
3. Chaotic Stock Pricing

One motivation for the choice of deterministic diffusion as model for stock price movements is that it can be reproduced in an extended model framework of Day and Huang REF016 which is presented in the following. The original model did not allow the stock price to leave a certain interval. Therefore it will be enriched so that the stock price is not restricted to a certain volume in phase space anymore. Finally the discussion will be simplified in terms of considering qualitative arguments which allow for a much simpler reasons to model the asset price movements as deterministic diffusion.

Consider three stylized phenotypes of investors:

1.) $\alpha$-investors
The so called sophisticated investors have the following demand function:

$$D_\alpha(p, p^\alpha_F, d, u) = \left(a\left(p^\alpha_F - p\right)\right)\left(p - d + 0.01\right)^{-\delta_1} \left(u - p + 0.01\right)^{\delta_2}, a > 0$$

Once the stock price is above the level they assume to be the fundamental price $p^\alpha_F$ according to some public information they want to sell the stock because they expect it to decline towards the fundamental value. On the other hand they buy the stock when it is cheaper then the fundamental price they assume. The strength of their reaction is determined by the parameter $a > 0$ and the chance that the stock price will fall or rise respectively it is above or beneath $p^\alpha_F$ expressed by the chance function:

$$\Theta(p, d, u) = \left(p - d + 0.01\right)^{-\delta_1} \left(u - p + 0.01\right)^{\delta_2}$$

The chance of a price fall or rise will be judged by $\alpha$-investors more likely the more the price is distinct of from $p^\alpha_F$. Further more the demand of an investor is bounded by the levels $u$ and $d$ which represent the highest price he would sell and the lowest price he would buy. When prices are above $u$ the chance of loosing money on a crash is perceived as to high therefore the asset is not bought. If the price is lower than $d$ than the chance that it will ever rise again will be perceived as too little due to the fact that other investors in the market do not seem to be rational enough to allow for a good stock pricing. The parameters $\delta_1$ and $\delta_2$ represent the relative strength of the bottoming or topping price $d$ and $u$ respectively in the chance function.

2.) $\beta$-investors
Are the less sophisticated investors they expect the prices to rise when they are above the fundamental value and they estimate them to decline, when they are beneath the fundamental value. One could call them also trend-investors. Their demand function is represented as:

$$D_\beta(p, p^\beta_F) = b\left(p - p^\beta_F\right), b > 0$$

The parameter $b > 0$ represents the strength of their demand reacting to the difference between $p$ and $p^\beta_F$.

3.) $\gamma$-Investors
Simply make the market in the sense that they to buy excess supply and sell from their own stock in case of excess demand $E(p)$. In case of excess supply they lower the price and in case
of excess demand the rise the price in order to sell or buy not too much from or to their inventory in line to keep their stock on a reasonable average over time. Also they might have some speculative motives as well, but this may not be of interest here. The demand and price adjustment function $\theta$ will be given as:

$$D_p(p, E(p)) = -E(p)$$

$$\theta(p) = p + cE(p)$$

Where $c > 0$ is the adjustment parameter for the price.

In our simple dynamical model one yields the following iterative price formula $\Theta$:

$$p_{t+1} = \theta(p_t) = p_t + c\left[a\left(p_{F_{\alpha}}^n - p_t\right)\right] \left[p_t - d + 0.01\right]^{-\delta} \left[u - p_t + 0.01\right]^{-\delta} + b\left(p - p_{F_{\alpha}}^n\right)$$

(3.1)

In REF016 the parameters for a numerical experiment are chosen to be:

$$d = 0, \delta_1 = \delta_2 = 0.5, u = 1,$$

$$a = 0.3, b = 0.88, c = 2, p_{F_{\alpha}}^n = p_{F_{\beta}}^n = p_F = 0.5$$

(3.2)

Within this parameter setting the system has no stable cycles, is restricted to the interval $[d, u]$ and the global fixed point $p_F$ is unstable, thus fixed points and stable cycles are dense in $[d, u]$ and deterministic chaotic motion is generated intrinsically by the model see REF016. Fig 2.1 shows a trajectory and the phase diagram of the price adjustment equation (3.1). The trajectories do not look realistic and the price does not diffuse.

To improve the model, consider the following modifications:

Let there be not just three groups of investors that are alike but rather $N$ different groups of investors of each category $\alpha$-investors, $\beta$-investors and $\gamma$-investors. $\alpha_n, \beta_n, \gamma_n: n=1,2,3,\ldots,N$ with the same parameters $a, b, c$ $\forall n$ and with topping and bottoming prices $u_n, d_n$ satisfying $1 < n < N: u_n = n * u_1, d_n = n * d_1, p_{F_n}^n = n -1 + p_{F_1}$. Furthermore claim that $\max(\Theta(p)) > u_n$ $p \in [d_n, u_n]$ and $\min(\Theta(p)) < d_n$ $p \in [d_n, u_n]$ whenever $\Theta(d_n) >> 0$ and $\Theta(u_n) < u_n$ and claim $0 << \Theta(p) << u_N$. Assume $|\Theta'(p)| > 1$ and that at price levels $p \in [u_{n-1}, d_{n+1}]$ only the group $n$ of
investors commits trading. Finally let the parameters $d_1$ and $d_2$ of the chance function of the $\alpha$-investors be zero.

On the basis of these assumptions one can construct a chain of chaotic maps that obey deterministic diffusion and the iterative pricing formula $\Theta(p)$ is piecewise linear in $p$ (see Fig 3.2). The model can now be expressed as:

\[
\begin{align*}
  p_{n+1} &= \Theta(p) = p_n + c\left[a(p_n - p_{n-1}) + b(p_{n-1} - p_{n-2})\right] \quad \text{for } a + c > 0 ; \\
  p_{n+1} &= \Theta(p) = p_{n-1} - c\left[a(p_{n-1} - d_n) + b(p_{n-1} - d_{n-1})\right] \quad \text{for } a - c < 0 ; \\
  p_{n+1} &= \Theta(p) = p_{n-1} - c\left[a(p_{n-1} - u_n) + b(p_{n-1} - u_{n-1})\right] \quad \text{for } a + c > 0.
\end{align*}
\]

As one can see it is possible to derive an expectation driven asset pricing model with only a few assumptions regarding the pricing process and can yield stochastic looking time series despite they are generated by a deterministic process. Note that the model permits rational and irrational behaviour as well as disagreement between investor groups regarding the fundamental value of an asset.

The time series of the models also exhibit behaviour of real world stock price time series where large changes are followed subsequently by large changes and small changes by small changes. The patterns arise when investors of one group trade among each other (small change after small) and suddenly $\Theta(p) > u_n$ or $\Theta(p) < d_n$ and the price gets adjusted so that the next group involves in trading or this happens for a few subsequent groups (large change after large change). This behaviour is typical for stock market prices and hence one more argument to use deterministic scattering maps to model stock market price movements.

In context of the presented we can conclude that expectations and behavioural patterns might drive the price in the context of deterministic diffusion and the behaviour of those artificial time series seems to mime the real world very well. In general the stock market could be presumably better understood as a deterministic scattering mechanism where one event depends on the previous. The independency assumption of Gaussian white noise is hence to restrictive and too naive.

In the next section we will examine weather the model is suitable to explain mostly observed so called “stylized” facts of financial time series that could not be explained by the standard Gaussian noise models.

![Iterative Pricing Function](image1)

![200 Iterates of $\Theta(p)$ at parameter values $a = 0.22, b = 1, c = 4, d_1=0, u_1=1, p_1=0.5$, starting value $p = 9.7$](image2)

Fig. 3.2 (a.) Chain of piecewise linear maps (b.) 200 Iterates of $\Theta(p)$ at parameter values $a = 0.22, b = 1, c = 4, d_1=0, u_1=1, p_1=0.5$, starting value $p = 9.7$
4. Stylized facts of stock price time series

With in this section we want to present the most popular empirically observed stylized facts of stock return distributions that are contradictory to the assumption of Brownian motion. Each fact gets exemplified with real world data of the German equity index DAX™ and the model of Section 3 time series. The stylized facts commonly observed on stock return distribution are:

i. Fat tails
ii. Heteroscedasticity
iii. Long range dependence
iv. Sensitivity to initial conditions

It will be shown that deterministic diffusive processes like the model of Section 3 have similar features and can therefore in contrast to simple random walks give better explanation to real world behaviour of stock prices. Subject to the forthcoming analysis in this section where 2000 model iterates and 2631 consecutive daily closing prices of the German equity index DAX™ from 11.06.1996 to 24.10.2006.

4.1 Fat tails

Definition 4.1.1 (Fat tailed probability distribution)

A (probability) distribution P is called fat tailed if the probability of extreme events vanishes exponentially with the magnitude of the event thus the following scaling law holds:

\[ P[X > x] \approx x^{-\alpha} \alpha > 0 \quad (4.1.1) \]

Note that if \( \alpha < 2 \) the variance and all higher moments of the distribution does not exists. We will examine our model log returns by plotting \( \log[P(X>x)] \) against \(-\log(x)\). The slope of the regression curve is used as estimate for \( \alpha \). An extreme event was assumed to be at least two standard deviations away from the centre of the distribution. The parameter values of the model used for this numerical investigation were:

\[
\begin{align*}
d_t &= 0, \ u_t = 1, a = 0.2 + 0.01*\sqrt{2}, b = 1 + 0.01*\sqrt{10} \\
c &= 4.25 + 0.0001*\sqrt{5}, \ p_a = p_b = p_c = 0.5 \\
\end{align*}
\] (4.1.2)

The choice of the irrational parameter settings (4.1.2) was motivated by the fact, that even more strange patterns should emerge in the time series if the model parameters get set with irrational numbers and the fact that most measurements in the real world would be irrational as well. In Fig. 4.1.1 (a.) the long term behaviour of the models at parameters (4.1.2) is shown for 2000 iterates. Additionally the volatility of 50 consecutive values is implemented in the same graph. In Fig. 4.1.1 (b.) the log return distribution of the model is compared to a standard normal distribution and finally Fig. 4.1.1. (c.) shows \( \log[P(X>x)] \) against \(-\log(x)\) plot of the model. The slope estimated from plotting \( \log[P(X>x)] \) against \(-\log(x)\) was \( \alpha \approx 2.18 \) indicating, that the variance of the distribution barely exists and that it has indeed fat tails.

One can see that there seem to exists two scaling regimes for \( \log[P(X>x)] \) against \(-\log(x)\) for values of \(-\log(x)\) being smaller than 0.8 and one for being larger than 0.8 when assuming a piece wise linear tail scaling function.
The author conducted regressions for values before 0.8 one afterwards yielding two distinct values $\alpha_{<0.8} \approx 1.61$ and $\alpha_{>0.8} \approx 22.93$. These results give evidence, that there exists two regimes in the model, one for which even the variance of the iterates is infinite and one for which the most of the moments exists. But both regimes have in common, that their distributions are fat tailed.

Since $10^{0.8} \approx 6.4$ the regime for $\alpha_{>0.8} \approx 22.93$ corresponds to large relative price changes of more than six times of the standard deviation. Those events happen, whenever absolute values of the stock price are low. In particular in the numerical experiment presented here there have been only 1.2% of the observations below this level of stock prices found to be $p=6.37$, indicated by the dashed line in Fig. 4.1.1. (a.). The tails of the distribution obey a strong scaling behaviour expressed by the high value of $\alpha_{>0.8} \approx 22.93$ due to the bound of the motion of prices by the level of zero. Scaling before $10^{0.8}$ with $\alpha_{<0.8} \approx 1.61$ can be interpreted as the free flow of prices that is not bounded. Note that for a value of $\alpha_{<0.8} \approx 1.61$ the volatility and all higher moments than the order two of the distribution would not exist. Thus the bound zero introduced to prices saves the distribution to be too awkward.
Finally Figure 4.1.2 shows the same graphs for the German equity Index DAX\textsuperscript{TM}. There are obvious parallels between the model time series and the DAX\textsuperscript{TM} time series. Both do have fat tails and show strong heteroskedastic behaviour. The value of $\alpha$ was estimated to be 3.4. Thus for the DAX\textsuperscript{TM} time series there seem to exist one more moments compared to the model time series. Also there could be two regimes of $\alpha$ be discovered, one $\alpha_{>0.6} \approx 2.82$ and one $\alpha_{>0.6} \approx 13.18$ when assuming a piecewise linear scaling function. In general all the scaling behaviour for both time series, the DAX and the model time series does not behave like a straight line in the far out tail.

From this section the reader should have been enabled to get an idea of how fat tails and volatility clustering of stock returns can be related by a deterministic law generating them. In the next section we will return more in detail to the phenomenon of volatility clustering also named heteroskedastic.
4.2 Heteroscedasticity

Heteroscedasticity is a common feature observed in stock market time series. It happens to occur when a lot of large changes follow abruptly a series of moderate changes. As like in the last section let us start with a brief definition what we mean by the term heteroscedasticity.

Definition 4.2.1 (Heteroscedasticity)
Define the m-sample variance estimator at sample point k of a sample of N realizations of a variable \(x_1, x_2, \ldots, x_N\) as:

\[
\hat{\sigma}_{m,k}^2 = \frac{1}{m-1} \sum_{i=k}^{m+k} (x_i - \hat{\mu}_{m,k}) \quad \text{where} \quad \hat{\mu}_{m,k} = \frac{1}{m} \sum_{j=k}^{m} x_j \quad \text{with} \quad 1 < k < N \quad \text{and} \quad n + k < N \quad \forall \quad k
\]

is defined as the m-sample mean estimator at sample point k. And define the sample variance by:

\[
\hat{\sigma}_N^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \hat{\mu}_N) \quad \text{where} \quad \hat{\mu}_N = \frac{1}{N} \sum_{j=1}^{N} x_j
\]

is defined as the sample mean estimator. In this context heteroscedasticity means that for N samples there exist significant many values k so that the m-sample estimators of non overlapping sample buckets differ significantly from the sample variance.

From Fig 4.1.1(a) we can see clearly that the model generated price time series should obey heteroscedasticity. But of course one needs to test for heteroscedasticity by generating a test statistic, so that we know what it means that the there exist significantly many sample estimators that are distinct from the sample variance. In the following the hypothesis of heteroscedasticity will be tested against the Null Hypothesis of a homoscedastic Gaussian random process. To test for heteroscedasticity in the time bucket \(\tau\) the following test statistic was applied:

\[
T(m, \alpha) = \frac{m \sum_{k=1}^{\text{int}(N/m)} I_{k,m}(\alpha)}{N}
\]

Where \(I_k\) is an indicator function for the non overlapping buckets \(k = 1, 2, 3, \ldots N/m\) indicating:

\[
I_{k,m}(\alpha) = \begin{cases} 
1 & \text{if} \quad \frac{\hat{\sigma}_{k,m}^2}{\hat{\sigma}_N^2} < \frac{1}{m-1} X_{\alpha/2}(m) \quad \text{or} \quad \frac{\hat{\sigma}_{k,m}^2}{\hat{\sigma}_N^2} > \frac{1}{m-1} X_{1-\alpha/2}(m) \\
0 & \text{if} \quad \frac{1}{m-1} X_{\alpha/2}(m) < \frac{\hat{\sigma}_{k,m}^2}{\hat{\sigma}_N^2} < \frac{1}{m-1} X_{1-\alpha/2}(m)
\end{cases}
\]

Where \(X_{\alpha}(m)\) is the \(\alpha\)-Quantile of the \(\chi^2\) distribution with m degrees of freedom.
The test is motivated by the fact that one can show under the assumption of a homoscedastic random process that:

$$\frac{\hat{\sigma}_m^2}{\sigma^2_N} \sim \chi^2(m)$$

where \( \hat{\sigma}_m^2 \) means is distributed as. I.e. under the assumption of a homoscedasticity gaussian random process the ratio of the sample variance and a m-sample variance should behave like a \( \chi^2 \) distributed random variable with m degrees of freedom multiplied by \( 1/(m-1) \).

Thus the Indicator function always indicates on the test level \( \alpha \) if the m-sample variance differs significantly from its expected value at either the upside or the downside that should be measured best by the sample variance. Finally we conclude that:

$$T \equiv \frac{1}{n} B(n, p) : n = \text{int}(N/m), p = \alpha$$

I.e. T follows a binominal distribution multiplied by 1/n with parameters \( n=\text{int}(N/m) \), which is the number of time buckets yielded by the choice of time bucket length m, and \( \alpha \) is the chosen test level. To commit a statistical test regarding \( T \), one has to check whether the observed quantity of as significant indicated variance changes exceeds a certain 1-\( \beta \)-quantile of the binominal distribution multiplied by 1/N or not:

$$T > \frac{1}{n} B_{\beta-1}(\text{int}(N/m), \alpha) \text{ reject null}$$

$$T < \frac{1}{n} B_{\beta-1}(\text{int}(N/m), \alpha) \text{ not reject null}$$

The value \( \beta \) will then be the level of confidence of the test.

As like in the last section the features of the model time series as well as to real world time series of the German equity index DAX were compared. Table 4.2.1 and Table 4.2.2 in Appendix B1 show the test results for the model and the DAX time series data respectively for various time buckets observed. Test parameters chosen where \( \alpha=\beta=0.01 \). The null hypothesis of a homoscedastic i.i.d Gaussian random process is rejected for every time bucket significantly at a confidence level of one percent.

Finally in Fig. 4.2.1 and Fig 4.2.2 the log returns of the model and the DAX time series are plotted against time to give a more illustrative impression of the phenomenon of volatility clustering. As can be seen both time series share the behaviour of heteroscedasticity significantly. The Model time series is more erratic and the heteroscedasticity seems to be driven mainly by the fact that modelling the time series in means of logarithmic returns does not suite the absolute changes caused by the deterministic scattering mechanism. It is obvious that whenever the price of the asset is low the absolute scattering relative to the lower price causes lager impact on log-volatilities. The same Effect can be found in the DAX time series, but nevertheless there seem to be still other mechanisms driving the volatility changes.
Fig. 4.2.1 Time Series of the model together with its log returns.

Fig. 4.2.2 Time Series of the german equity index DAX\textsuperscript{TM} together with its log returns.
4.3 Long range dependence

One striking feature of Brownian motion is that it has got no memory. Thus all realizations are independent from one another in time. Real world stock market returns show exactly the opposite as our model time series does. The reader may ask himself in a justified manner what long range dependence is and how it should be measured.

Definition 4.3.1 Long range dependence
A time dependent process \( x(t) \) is said to be long range dependent, if the autocorrelation of its absolute time lagged values raised by any power \( k \geq 1 \) is greater than zero and decays in time by a power law with the rate \( \delta^k \).

\[
\forall t > 0 \ s > 0, \ k \geq 1: \rho(s, k) = corr\left( |x(t)|^k, |x(t+s)|^k \right) < 0 \text{ and } \\
\rho(s, k) \approx corr\left( |x(t)|^k, |x(t+l)|^k \right)s^{\delta_k}
\]

the lower the absolute value of \( \delta_k \) the less the decay of dependence.

To examine the DAX\(^\text{TM}\) and the model time series of log returns \( \rho(n,2) \ n \in \mathbb{N}^+ \) got calculated as well as the ordinary autocorrelation function. The results are shown in Fig 4.3.1. Both time series show the same qualitative behaviour, but with different quantitative peculiarity. The model autocorrelations of squared returns start at a very high level and decay very fast where as the real world time series correlations start at a lower level and decay more slowly. For the model series \( \delta_2 = -0.62 \) and for the DAX\(^\text{TM} \) \( \delta_2 = -0.21 \) got computed showing a faster decay and loss of dependence in the model time series than in the DAX\(^\text{TM} \) returns.

Another measure of long range dependence or persistence is the Hurst Exponent, denoted in the following with \( H \). It is named after its inventor the hydrologist Harold Edwin Hurst. He invented it when analysing yearly water run offs of the Nile river. Consider \( n \) observations of a variable \( x: x_1, x_2, x_3, \ldots, x_n \) and the cumulated values \( X_k = x_1 + x_2 + x_3 + \ldots + x_k \). The value \( X_k - (k/n)X_n \) measures the divergence of the cumulated values of a time series of length \( k \) from the cumulated value of the whole time series. Define the Range \( R_n \) as follows:

\[
R_n = \max_{1 \leq k \leq n} \left( X_k - \frac{k}{n}X_n \right) - \min_{1 \leq k \leq n} \left( X_k - \frac{k}{n}X_n \right)
\]

The empirical Standard deviation is given by:

\[
S_n = \sqrt{\frac{1}{n} \sum_{k=1}^{n} \left( x_k - \frac{X_n}{n} \right)^2}
\]

Hurst found that:

\[
\frac{R_n}{S_n} \approx cn^H
\]
with $\delta^2 = -0.62$ (b.) Autocorrelation Correlation Decay for normal and squared DAX™ time series of log returns with $\delta^2 = -0.21$. (c.) log/log plot for the volatility scaling of the model time series, slope estimated $H=0.40$ from the first 30 samples. (d.) log/log plot for the DAX™ log return volatility scaling from the first 30 samples Yielding $H=0.48$.

Where the values $X_t$ have the same distribution like $n^{H}X_1$ with $H=0.7$, indicating that the Nile River run-offs are not i.i.d. random events, but rather depend on one another in a persistent manner. A process that exhibits such a scaling behaviour is called statistically self-similar. Formal:

**Definition 4.3.1 (statistical self similarity)**

A statistical self similar process $x$ is defined by:

$$x(\alpha t) \cong x(t)\alpha^{2H}$$

where $x(t)$ is the value of the process after $t$ time steps, $\alpha \in \mathbb{R}^+$, $H \in [0,1]$ is the Hurst coefficient and the operator $\cong$ means congruency in distribution. If $H > 0.5$ a process is called persistent, if $H<0.5$ anti-persistent because for $H>0.5$ ($H<0.5$) the increments of a statistical self similar process are positively (negatively) correlated. (See upcoming Theorem 4.3.1)

The estimation of $H$ from empirical data is straight forward. One plots $\ln(S_k)$ against $\ln(k)$. The slope of the regression line holds as estimate for $H$. Fig 4.3.1 (c.) and (d.) show the resulting log/log plots for the model time series and the DAX™ time series respectively. The most striking observation from the plots is, that after some autonomous value $\ln(k)$ the logarithm of the lag $k$ volatilities begins to flutter and doesn’t allow for a clear estimation for $H$ since a straight line wouldn’t be consistent with the observed behaviour.
Therefore only the first 30 samples of \( \ln(S_k) / \ln(k) \) were used to estimate \( H \) respectively because for those values a straight line could be nicely fitted. For the model time series \( H = 0.40 \) was estimated and for the DAX\(^\text{TM} \) data \( H = 0.48 \) was estimated.

Surprisingly neither the DAX\(^\text{TM} \) nor volatility scaling plots of the model log returns allow for a good estimation of \( H \) and the estimated \( H \) values don’t show any incident for persistence in the above defined sense. From the common understanding a value of \( H \) clearly above 0.5 would be the expected outcome. But it is also not too difficult too get an intuitive understanding of the results observed.

Recall that the autocorrelations of the log returns of both time series were closed to zero but only for the model time series the autocorrelation of time lag one was significantly negative. From this we can deduce that only a week statistical dependence of linear order exists in all time lags except of time lag one for the model series. If this is the case it is likely that \( H \approx 1/2 \) for the DAX\(^\text{TM} \) series and \( H \not< 0.5 \) for the model because one can show if and only if \( H = 1/2 \) then the increments of a self similar stochastic process are independent. Furthermore it holds, that if \( H < 0 \ (>0 \) ) the increments of a statistical self similar are negatively (positively) correlated. This is pretty much in line with Fig 4.3.1 (a.). A Hurst Exponent of \( H = 0.4 \) is measured for the model and the autocorrelation for the time lag \( t = 1 \) is negative!

**Theorem 4.3.1**
For any statistical self similar stochastic process \( x(t) \) with \( 0 < H < 1 \) holds that:

i.) the autocorrelation function of its increments \( dx(t) \) is zero for all time lags and all increments are independent if and only if \( H = \frac{1}{2} \)

\[
\rho_{dx}(t,k) = \frac{\text{cov}(dx(t),dx(t+k))}{\text{var}(dx(t))} = 0 \Leftrightarrow H = \frac{1}{2}, k = 1,2,....N
\]

ii.) If \( H > 0.5 \) the increments are positively correlated, if \( H < 0.5 \) they are negatively correlated.

\[
\rho_{dx}(t,k) = \frac{\text{cov}(dx(t),dx(t+k))}{\text{var}(dx(t))} > 0 \Leftrightarrow H > \frac{1}{2}, k = 1,2,....N
\]

\[
\rho_{dx}(t,k) = \frac{\text{cov}(dx(t),dx(t+k))}{\text{var}(dx(t))} < 0 \Leftrightarrow H < \frac{1}{2}, k = 1,2,....N
\]

iii.) If \( H \not< 0.5 \) the a self similar stochastic process obeys long range dependence.

**Proof:**
See Appendix C1

Finally the author would like to introduce the information entropy of a distribution and analyze its development over time.
Definition 4.3.2 (The Information Entropy)

The Information Entropy for a random variable $x$ with density $h(x)$ according to Shannon is defined by:

$$
\Pi = -\int h(x) \log_2(h(x)) dx
$$

It gives the maximum Information in bits one learns from one outcome of the random variable.

Thus the higher the information entropy is the more certain is an experiment because any additional outcome gives more information about the underlying process. For example consider a coin toss. The information entropy of it is $-2*(1/2)*\log_2(1/2) = -\log_2(1/2)=1$. Now consider a skew of the coin toss so that the probability of the one side of the coin turns $1/4$ and that of the other $3/4$. The information entropy then is $-(1/4)*\log_2(1/4)- (3/4)*\log_2(3/4) = 0.81$. Thus the Experiment needs to be repeated more often to get the same Information than one outcome of the not skewed coin toss produces.

For the following investigation about 6400 Daily DAX Returns from 1996 to 2006 were used and approx 10000 Model returns. Fig 4.3.2 shows the development of the information entropy over different time horizons for the model time series and the DAX time series computed for the normalized distributions of log returns.

From observing Fig. 4.3.2 it is clear, that we cannot be dealing with a self similar stochastic process in both cases because for such a process the normalized information entropy would be the same for each time horizon. It can also be seen that the information entropy increases with time, meaning, that the riskiness or uncertainty involved in the process decreases since one learns more about the world by one experiment on a longer time horizon.

From the previous observations naturally the question arises how the distribution of the processes may evolve over time. To investigate on this issue the normalized distributions for the buckets 1,3,5 and 10,30,60 samples were drawn into one graph shown in Appendix A.1. Again both time series obey the same peculiarities in a different manner. The greater the time horizon the less the distribution resembles a probability distribution in the classical sense. It shows many humps and gets fuzzy. This can be understood as cause for the fluttering of the volatility in the Hurst Coefficient estimation process in 4.3.2. It is clearly evident that we are not dealing with a self similar process since none of the distributions matches each other.

The results observed can be interpreted as follows. In the short term up to 30 realizations the deterministic distribution can be interpreted as a log stochastic process, that has a very complex dependency between succeeding realizations, that cannot be detected by the simple autocorrelation function. Only the autocorrelation of squared returns reveals the long range dependency between succeeding iterations. It decays very slowly for longer time horizons above 10 samples, indicating the strong long memory in the time series. As time progresses risk declines measured in the form of the information entropy. Thus deterministic forces take toll as time evolves making the density become “less” random looking very much in contrary to the implications of any probabilistic model.
Fig 4.3.2 development of the information entropy of model and DAX time series log returns.
4.4 Sensitivity to initial conditions

Apart from the classical stylized facts the author would like to add this section to make the understanding of the following more clear. Sensitivity to initial conditions is the property of a dynamic system that prescribes how it reacts on a small difference in the starting value on the long run. A popular measure of this kind of behaviour is the Lyapunov exponent. It prescribes the average exponential expansion rate of a small error in the initial conditions.

**Definition 4.1.1 (Sensitivity to initial conditions)**

A dynamical system is said to be sensitive to its initial conditions if a small error $\delta x$ expands on the average exponential with rate $\lambda$, called Lyapunov exponent. The formal definition of $\lambda$ is given by:

$$
\lambda = \lim_{t \to \infty} \lim_{\delta x_0 \to 0} \frac{1}{t} \sum_{n=1}^{t} \ln \left( \frac{\delta x_n}{\delta x_0} \right)
$$

where $\delta x_0$ is the error in the initial condition of the iterates $x(t)$ of the system and $\delta x_t$ is the error after $t$ time steps.

To illustrate the meaning of this sensitivity, in Fig. 4.4.1 three trajectories of the model all only one hundredth i.e. 0.01 apart from each other are shown, yielding significantly different trajectories.

Furthermore the Lyapunov exponents for the model and the DAX$^\text{TM}$ have been computed. The following method was applied: Denote $x_{\text{NN}}$ the nearest neighbour of the starting value of the time series $x(0)$ in the sense of:

$$
|x_{\text{NN}} - x(n)| = \inf |x(0) - x(n)| n = 1, 2, 3, \ldots, T
$$

where $T$ is the length of the Time series. Then the following quantity holds as estimate for $\lambda$:

$$
\tilde{\lambda} = \frac{1}{T} \sum_{n=1}^{T-I_{\text{NN}}} \ln \left( \frac{|x(n) - x(I_{\text{NN}} + n)|}{|x_{\text{NN}} - x(0)|} \right)
$$

Where $I_{\text{NN}}$ is the time index of the nearest neighbour value $x_{\text{NN}}$. For the DAX$^\text{TM}$ time series Lyapunov exponent of $\hat{\lambda}_{\text{DAX}}=3.27$ was calculated. The $\lambda$ of the model was estimated to be $\hat{\lambda}_{\text{model}}=4.2$. Thus the model has less forecast ability than the DAX$^\text{TM}$. To illustrate this statement consider the Lyapunov time, defined as the maximal time of forecast ability of a dynamical system. It is given by:

$$
T_{\lambda} = -\frac{1}{\lambda} \ln \left( \frac{\delta x_0}{\epsilon} \right)
$$

Where $\epsilon$ denotes the maximal extend of the system. To get an estimate for the forecast ability of the DAX$^\text{TM}$ and the model respective an initial error of 0.5 and 0.01 as well as an extend of 10000 and 50 respectively were used. The Dax obeys a Lyapunov time of 3.46 days where as the model has a $T_{\lambda} = 3.08$ itera tes. The results are summarized in Table 4.4.1.
turns out, that if we new the true model only a limited forecast of up to 3 Days would be possible.

![Model Time Series](image)

**Fig. 4.4.1** Model time Series for initial conditions 9.7; 9.71; 9.72 respectively yielding extremely different trajectories.

<table>
<thead>
<tr>
<th>Model</th>
<th>DAX</th>
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<tr>
<td>Lyapunov Exponent:</td>
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<tr>
<td>Lyapunov Time</td>
<td>3.08</td>
</tr>
<tr>
<td>Initial Error</td>
<td>0.01</td>
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<tr>
<td>System Extend</td>
<td>50</td>
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</tbody>
</table>

**Table 4.41** Results of the sensitivity analysis of the DAX™ and model time series.
5. Implications for economic theory

In the following paragraphs the implications of the so far introduced model of deterministic diffusion shall be discussed and reviewed against classical results of capital markets theory like the Market Efficiency Hypothesis, the CAPM and the Black Scholes option pricing model. In the subsection Option Pricing we also discuss the applicability of alternative Option pricing models like that of Fractional Brownian Motion and the Option Models of McCulloch for $\alpha$-stable distributions. Further more statistical mechanic option pricing proposals are made and exemplified.

5.1 Market Efficiency / CAPM

In the beginning we would like to recall the classical Market Efficiency Hypothesis and afterwards we would like to relate the deterministic diffusion model to it.

**Market Efficiency Hypothesis (MEH)**

The Markets are:

1.) **Semi efficient**
If all information about price histories is contained in the prices.

2.) **Efficient**
If they are semi efficient and all public information is contained in the prices.

3.) **Strong efficient**
If they are efficient and all non public information is contained in the prices.

In strong efficient markets all prices are assumed to follow random walks of geometric Brownian motion in classical capital market theory, since only the occurrence of new information changes the price and the price and its history do not contain any information about its future development. The implication for capital asset pricing is thus that in equilibrium, assuming risk neutral investors, an asset $i$ is priced according that the expected excess return against the risk free interest rate $r_i - r_f$ can be expressed in terms of the excess return of a market portfolio $M$ $r_M - r_m$ times a beta factor $\beta_{i,M}$ of a stock. The asset’s price therefore reflects the risk premium to be paid to a risk neutral investor in equilibrium relative to the Market $M$:

$$r_i - r_f = \beta_{i,M} \left[ r_M - r_f \right] \text{ where } \beta_{i,M} = \frac{\text{Cov}(r_i, r_M)}{\sigma_M^2}$$

The just stated equation forms the heart of the CAPM (Capital Asset Pricing Model). There have been numerous articles and empirical investigations on whether the CAPM holds or not. The focus of the following will be rather a theoretical reasoning about the validity of the CAPM in the framework of deterministic diffusion. But first we turn to the market efficiency…

In general the presence of deterministic diffusion doesn’t require the markets to be efficient. But also doesn’t exclude in general efficiency. Furthermore efficiency may not require
Gaussian random walks. The reason here for is that the change in price can be understood as a hit of a particle with another one and the there from induced change in direction and speed. In a liquid or gas performing Gaussian movements all particles are equal and the temperature is constant, but what if there are particles with different size and obstacles are also present and temperature changes endogenously or other forces influence the movement of the particle? Still if nobody does know about the obstacles and the possible future paths of the particle, the movement is not Gaussian, but the market would be still strong efficient.

On mayor drawback of the MEH is that it doesn’t explain how the information gets into prices. For example if the world has got a long memory and all events depend on each other on a long range, than information flows according to a long range dependent process that may be chaotic and is therefore reflected in prices. This means prices show the evolving long memory process of the development of the economy and can be deterministic diffusion regardless if the markets are efficient or not.

Also the MEH doesn’t allow for irrational and speculative investors. These types are modelled as Beta-Investors in our model. Recall Section 3. Furthermore all investors should have the same opinion and information in the MEH and CAPM framework. This is definitely not true in the real world, especially when it comes to the opinion about the future development of prices. It also should be obvious that limited rationality also plays a significant role in the real world since the future is mostly unknown and is not possible to be forecasted. Therefore no exact pricing of an asset is possible since one may need at least the probability distribution of the future cash flows to derive a price for the asset. Recall also our investigations of paragraph 4.3 that suggest that probability like laws of asset price time series start diminishing after 30 days, are not stable stochastic processes and determinism prevails on the long run, suggesting an inappropriateness of the random walk model.

To outline the different implications and assumptions in the preceding text the author would like to give a Chaotic Market Hypothesis CMH, that can be understood as modification or extension of the Fractal Market Hypothes stated earlier by Peters in REF 008.

Chaotic Market Hypothesis

1.) Efficiency
Markets may or may be not semi efficient, efficient or strongly efficient in the sense of the MEH. Information gets incorporated into prices or does not depending on the investor behaviour. If it does it reflects a long memory process of a large deterministic system. Every price is right as long as investors are willing to pay it. (The market is always right)

2.) Investor Behaviour
Investors can act rational as well as irrational according to their personality. They have different investment horizons and different opinions about the future development of a price and have limited knowledge about the future.

3.) Evolution of Prices
Prices diffuse according to deterministic laws, that can be to a certain extend interpreted as random. The diffusion is caused by a deterministic scattering process driven by news and behavioural patterns of investors. On the long run starting from only 30 days the long term history of the world dominates the evolution of Prices. The Evolution Law of prices shall be called “Deterministic Diffusion”. It has infinite long memory and is thus not Markov.
When considering the CMH, the CAPM is only valid in a deterministic diffusion environment for short time horizons when markets can be interpreted as random walks, when markets are calm and efficient and the volatility is constant. Therefrom we conclude that classical asset pricing models give a good understanding of how prices should be, but only capture certain aspects of deterministic diffusion that are only a limited part of it. Hence deterministic diffusion seems to be the more appropriate model for stock price histories than a geometric Brownian motion is.

At this point the discussion could be extended further, but of course there are other truly interesting aspects of the deterministic diffusion model like for example option pricing....
5.2 Option pricing

The proceeding in this will be as follows. First the to the author known alternative option pricing models that have been proposed in recent literature in presence of long range dependence and fat tails are outlined. Namely the Option pricing in fractional Brownian motion REF004 Necula 2001 and an Option pricing formula for $\alpha$-stable distributions REF012/13 McCullogh 2005/1996 are reviewed. Afterwards the applicability to deterministic diffusive processes is discussed. Following this a statistical dynamics approach to option pricing is sketched, since its full development has not yet been finished by the author and will be the scope of further papers than the one of this paper. Due to numerical simulations the difference between the classical Black Scholes Model and so far existing approaches is shown.

5.2.1 Alternative Option Pricing Models

5.2.1.1 Option Pricing under Fractional Brownian Motion

Firstly the model for fractional Brownian Motion shall be discussed.

**Definition 5.2.1.1 (Fractional Brownian Motion)**
A fractional Brownian motion FBM stochastic process $B_H(t)$ is a Gaussian Process processes $B_H(t) \approx N(\sigma^H(t), \mu^H(t))$ with conditional moments:

$$\sigma^2_H(t,T) = \left( T^{2H} - t^{2H} \right) \sigma^2 ; \mu^H(t) = B^H_H(t) \ (5.2.1.1.1)$$

and its values are correlated by the covariance function:

$$E[B^H_H(t)B^H_H(s)] = \frac{1}{2} \left( |s|^{2H} + |t|^{2H} - |t-s|^{2H} \right) \sigma^2$$

$\sigma$ can be interpreted as the instantaneous volatility per $H$ weighted unit of time. The time evolution of FBM given by:

**Theorem 5.2.1.1 (Time Evolution of Fractional Brownian Motion)**

The time evolution of Fractional Brownian Motion is given by the equation:

$$B^H_H(t) = \frac{1}{C(H)} \int \left[ \left( \left( t-s \right)^{H-\frac{1}{2}} - \left( -s \right)^{H-\frac{1}{2}} \right) dB_s \right]$$

where $B_s \approx N(0,1)$ is a standard normally distributed Gaussian Process and the integral is over the whole real line and:

$$C(H) = \left( \int_0^\infty \left( 1 + s \right)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds + \frac{1}{2H}$$

**Proof:**
See REF006 D. Nualart 2001
Note, that in the case \( H=(1/2) \) the Fractional Brownian Motion becomes a standard Brownian Motion. The main features of FBM are:

(i) FBM is a stochastic process that is self similar according Definition 4.3.1.

(ii) FBM the increments are positively correlated if \( H>0.5 \) and negatively correlated if \( H<0.5 \)

(iii) FBM obeys long range dependence.

(iv) FBM is a semi Martingale

The properties (ii) and (iii) follow from directly from Theorem 4.3.1

For the proof of Property (iv.) the reader is prompted to See REF006 D. Nualart.

If one defines geometric fractional Brownian motion with its stochastic differential equation and its solution:

**Definition 5.2.1.1 (Geometric Fractional Brownian motion)**

The geometric fractional Brownian motion is given by the stochastic differential equation:

\[
(5.1) \ d\tilde{B}_H(t) = \mu\tilde{B}_H(t)dt + \sigma\tilde{B}_H(t)dB_H(t)
\]

**Theorem 5.2.1.2 (Solution of the stochastic differential equation of geometric Fractional Brownian motion)**

The solution of (5.1) is:

\[
\tilde{B}_H(t) = \tilde{B}(0)\exp\left(\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right)
\]

**Proof:**

see REF022 Daye.

With the help of the geometric FBM one can yield a fractional Black Scholes formula.
**Theorem 5.2.1.2 (Fractional Black Scholes Formula)**

The fractional Black Scholes Formula under FBM without dividend payment is given by:

\[
C(t, T, r, S(t), X) = S(t)N(d1) - Xe^{-(T-t)r}N(d2)
\]

\[
P(t, T, r, S(t), X) = Xe^{-(T-t)r}N(-d1) - S(t)N(-d2)
\]

where:

\[
d1 = \ln\left(\frac{S(t)}{K}\right) + r(T-t) + \frac{\sigma^2}{2}(T^{2H} - t^{2H})
\]

\[
d2 = \ln\left(\frac{S(t)}{K}\right) + r(T-t) - \frac{\sigma^2}{2}(T^{2H} - t^{2H})
\]

C is the price of a European call option and \(P\) is the price of a European put option respectively. \(T > t\) is the date before the evaluation date \(t\) where the option matures, \(X\) is the strike price of the option, \(r\) is the risk free zero interest rate from the evaluation date until the date of maturity.

**Proof:**

See REF004 Necula.

Note that in the Case \(H=0.5\) the Black Scholes Pricing Formula results. Firstly we want to examine how the Hurst parameter \(H\) takes influence on the option price. Secondly we want to relate the model of FBM to the properties of real world stock market time series and discuss its relevancy in context of the findings in section 4.3.

For simplicity in the following example we set \(t=0\), because this resolves the time relevancy property of option prices that’s given under FBM. The time relevancy property can be stated as follows: Due to the time dependence of volatility in the FBM model an option with the same time to maturity at time \(t_1\) has another price than another option with the same time to maturity at another time \(t_2\). This follows from Theorem 5.2.1.2 and can be easily understood by the property of the influence of past price changes on present price changes. The knowledge of the past time series gives information about the future time series and is hence priced in the value of an option. Since the variance grows with less (more) than the factor of time according to whether \(H < 0.5\) (\(H > 0.5\)) the risk of holding the underlying of the option contract shrinks (grows) in time in comparison to a normal Brownian motion and is different for the same time horizon at different points in time, recall equation (5.2.1.1.1).

To analyze the impact of \(H\) on the option price an example of a call a put option on a stock with strike of 70 EUR and annualized log return volatility 20% and 0.75 years time to maturity was examined for different Hurst parameters \(H \in (0.3;0.4;0.6;0.7)\) relative to a PV= plain vanilla option. A Zero Rate of 0% was assumed for simplicity. The resulting option prices depending on the underlying price are shown in Fig. 5.2.1 a.) and b.) .

It can be clearly seen for \(H>0.5\) (\(H<0.5\)) the option price of a option is strictly greater (smaller) than that of a plain vanilla option. Furthermore for Values \(H>0.5\) the gamma of the option price is smaller (larger) for in (out of) the money options in comparison to the plain vanilla option and for values \(H < 0.5\) the converse is true.

---

3 throughout the paper for simplification the dividends are not taken into account in all option pricing formulas.
Fig. 5.2.1 a.) and b.) Price function of a call respectively put option depending on the underlying price for different setting of the Hurst parameter $H \in (0.3;0.4;0.6;0.7)$ and a PV = plain vanilla Black Scholes option. Time to maturity 0.75 Years, annualized log vola of underlying price = 20% , Strike = 70 EUR. A Zero Rate of 0% was assumed for simplicity.
These effects result from the recently discussed properties of long range dependent self-similar diffusion processes. Since if $H > 0.5$ the volatility grows faster than the square root of time and a contingent claim value rises (shrinks) with the magnitude of $H > 0.5$ ($H < 0.5$) above (beneath) the value of a plain vanilla option in the Black Scholes world. The probability that the option ends in the money is greater if $H > 0.5$ than in the case of normal Brownian motion and smaller if $H < 0.5$. The described Gamma effects arise since as $H > 0.5$ the time reversibility of any movement out of the money is more likely as for values $H$ smaller than 0.5. Then any movement out of the money may be less likely reversible in time. As $H < 0.5$ the option price converges to the price of a knock out option with strike equal to knock out level and endless maturity. For this kind of contracts volatility does not matter.

To analyse the relevance for practical option pricing the Hurst parameter was measured for 28 shares of the DAX™ index according to the method described in section 4.3. For the test 1023 daily closing prices of the period 02.01.2003 to 28.12.2006 were used. The results are presented in table 5.2.1.1. The procedure was carried out for the period of log volatilities from 1 to 30 days and from 20 to 40 days respectively.

For log volatilities of 30 days and less a stable linear relationship can be established for all shares showing slight non-persistency with values of $H < 0.5$ and mostly $H > 0.4$. The coefficient of determinism of the regression of $\ln(S_k)$ against $\ln(k)$ is closed to 1 underlining a good linear fit. For log volatilities of periods 20 to 50 days the linear relationship gets lost showing worse coefficients of determinism for the linear regression more apart from one. Also the estimated Hurst coefficients for 20 to 50 days volatilities show very strong non-persistency. But still the bad regression quality doesn’t suggest a good proxy quality for the Hurst coefficient or even that a linear relationship between $\ln(S_k)$ against $\ln(k)$ doesn’t exist.

These results are pretty much in line with the findings in 4.3 and even better fit to the results of the Hurst coefficients obtained for the deterministic diffusion model showing non-persistency and a breakdown of probabilistic structures after around 30 days in the future.

From the above findings we conclude that the fractional Brownian motion model can be used to evaluate short term maturing options, but only to the limited extend that its conditional distribution is still a standard normal distribution with scaled volatility and doesn’t posses therefore fat tails. Furthermore as going to be shown in the next section no statistical self-similar structures of log returns exist in the deterministic diffusion model and also not in real world time series, which accounts also for a limited applicability of the FBM model. But one can see from FBM that long range dependency should have a strong influence on option prices.
Guido Venier: A new Model for Stock Price Movements

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Table 5.2.1.1 Hurst Exponent estimation for 28 DAX shares
Quality denoting the constant of determinism of the regression

5.2.1.1 Option Pricing under $\alpha$-stable distributions

The idea of pricing options under the assumption of $\alpha$-stable distribution stems from the fact that $\alpha$-stable distributions are the limiting distribution in the framework of a generalized central limit theorem that reads:

**Theorem 5.2.1.1.1 Generalized Central Limit Theorem**

Let $X_1, X_2, X_3, \ldots, X_n$ be i.i.d random variables. There exists constants $a>0$, $b>0 \in \mathbb{R}$ and non degenerate $Z$ such that:

$$a(X_1 + X_2 + X_3 + \ldots + X_n) \xrightarrow{\text{dist}} b \rightarrow Z$$

If and only if $Z$ is $\alpha$-stable random variable for some $0<\alpha<2$. $\xrightarrow{\text{dist}}$ means converges in distribution.

Proof:
(see REF014 Nolan)

For $\alpha$-stable distributions the second moment doesn’t exists for all $\alpha \in (0;2]$ and for $\alpha \in (0;1]$ even the expected value does not exist. Furthermore there does not exist a closed form solution in terms of a formula for their density function. They can only be defined by their moment generating function.
Definition 5.2.1.1.1($\alpha$-stable distributions)
A random variable $X$ is stable $\alpha$ if and only if $X \equiv \gamma Z + \delta$, $\gamma > 0$, $\delta \in \mathbb{R}$ where $Z$ has a characteristic function:

$$E[\exp(isZ)] = \begin{cases} \exp\left(-|s|^{\alpha} \left[1 - i\beta \tan \frac{\pi \alpha}{2} (\text{sign}[s])\right]\right) & \alpha \neq 1 \\ \exp\left(-|s| \left[1 + i\beta \frac{2}{\pi} (\text{sign}[s]) \log(|s|)\right]\right) & \alpha = 1 \end{cases}$$

with $\alpha \in (0,2], \beta \in [-1,1]$.

Note that for $\alpha = 2$ $X$ is distributed with a Normal distribution that has volatility $2^{0.5}\gamma$ and expected value $\delta$. Thus the Normal Distribution is a special case of $Z$ with $\alpha = 2$. The parameter $\delta$ is called the location parameter and is identical with the expected value of the distribution for $1 \leq \alpha \leq 2$. The parameter $\gamma$ is called the scale parameter and is for $\alpha = 2$ identical with half the variance of the distribution. The parameter $\beta$ describes the skewness of the distribution. For $\beta = -1$ the distribution is skewed completely to the left and for $\beta = 1$ skewed completely to the right. The parameter $\alpha$ is called the characteristic exponent and describes the tail behaviour of the distribution according to equation (4.1.1).

Economic concerns arose e.g. by REF018 Merton that with the use of $\alpha$-stable distributions as model for the return of a price of an asset the return could be infinite if $\alpha < 1$. Thus the risk free discount rate had to be infinite to yield a finite option price. Another drawback of stable distributions is that no closed form solution for the density function can be obtained. But there do exist numerical algorithms to compute the density for various parameters. See REF014 Nolan.

So far only a number of limited approaches exist to yield an option pricing formula, and yet they are quite technical and need furrier inversion to compute prices in the end. The most common known are those of McCulloch see REF012 REF013 2001 / 1996 and Carr and Wu REF015 2003. Summarizing them all one would yield the following Theorem:

Theorem 5.2.1.1.2 Option Prices for $\alpha$-stable distributed log returns (Part1)
In the case of $\alpha$-stable distributed log returns finite and positive option prices exists, if the following criteria are fulfilled:

(i.) Both the expected log utility $v_1$ of the numeraire good for which the option is bought (mostly money) and the log utility $v_2$ of the underlying of the option are maximally negatively skewed with $\beta = -1$ and have both the same parameter $\alpha$

(ii.) Investors are risk neutral and the log distribution of the asset returns are maximally negatively skewed with $\beta = -1$

Proof:
See REF012 and REF013 McCullogh 2001 / 1996 and REF015 Carr and Wu 2003
Theorem 5.2.1.1.2 Option Prices for $\alpha$-stable distributed log returns (Part 2)

In the case of $\alpha$-stable distributed log returns finite and positive option prices exists, if the following criteria are fulfilled:

(i) Investors are risk neutral

(ii) For the parameter $\alpha$ holds $1 \leq \alpha < 2$

The price is then obtained by numerical integration of the density $f(x|\alpha;\beta;\gamma;\delta)$

$$P = \Phi \int_{-\infty}^{\infty} \Phi(x)f(x|\alpha;\beta;\gamma;\delta)dx$$

Where $\Phi(x)$ is the payoff profile function given by:

$$\Phi(x) = e^{-r(T-t)} \max\{S \exp(\gamma x + \delta) - X; 0\} \text{ if Call}$$

$$\Phi(x) = e^{-r(T-t)} \max\{X - S \exp(\gamma x + \delta); 0\} \text{ if Put}$$

$r$ is the risk free interest rate, $S$ the spot price and $X$ the Strike of the option.

Proof:
See Appendix

It is obvious that under few restrictive assumption like in theorem 5.2.1.1.2 the option price of an asset in an $\alpha$-stable can be easily obtained. At this point the reader has to be disappointed since no more detailed analysis of such option prices will follow since a.) In due curse the limited applicability of stable distributions to stock returns will be discussed and b.) In section 5.2.1.2 fractional Brownian motion option prices and statistical dynamic option prices will be compared.

At this point we will give a simple argument of the limited applicability of the $\alpha$-stable distribution approach. 1.) In praxis on can show that the required self similarity of the distribution does not exist. 2.) Parameter estimation will therefore not be possible since one has always to assume that a $\alpha$-stable distribution generated the data to do correct estimates. 3.) The parameter estimates of $\alpha$ done so far in this part are either closed to 2 or above 2 or even much greater than 2 (See section 4.1).

The most striking argument is the lacking self similarity in praxis. In Figure 5.2.1.1.1a the log density of daily DAX$^{TM}$ returns computed from closing prices are shown for periods 1980-1992 and 1993-2007 respectively. The estimated densities look very different and also the estimated values of $\alpha$ diverge significantly. The same observation has been done with the iterates of the model of section 3. shown in figure 5.2.1.1.1b.

The reason for the observed behaviour is most possible, that one draws from a population, that’s not independent, but treats the samples as if one was drawing from an independent distribution. Even though for the model the time series generating process is always the same, the log densities are not stable at all.
5.2.1.2 Statistic Dynamics Option Pricing.

So far we have discussed alternative stochastic option pricing models, but which do have, as argued before, limited relevance. Fractional Brownian motion captures long range dependence, but the conditional densities are normal and therefore don’t have fat tails. \( \alpha \)-stable Distributions have fat tails, but don’t have long range dependence. Last but not least the DAX\( ^{TM} \) and model log returns don’t seem to have a stable probability law generating them.

In the case of deterministic diffusion no such probability law prevails and the empirical distributions of log increments still have fat tails and are not independent at the same time, matching empirical facts very well. The reader may now ask the justified question: “How does uncertainty arises in a deterministic world?”. The answer is as simple as the question itself… Uncertainty arises in deterministic diffusion for two reasons:

1.) Uncertainty of the measurement of the initial conditions and
2.) Limited knowledge of the deterministic model generating the time series.

Aspect 2.) can unfortunately not be scope of this paper due to its complexity and will therefore be subject to further research.

To remember the impact of aspect 1.) the reader may review Fig. 4.1, where for a few slightly different initial conditions trajectories of the model of section 3 have been shown, each going a quiet different path. In the following we assume firstly, that the deterministic law of motion is known to the market participant, and that only the uncertainty of initial conditions plays a role. Under this assumption a general option pricing formula gets derived. This is quite unrealistic of course and therefore the obtained results have got only illustrative value. But the Author is quiet confident that there exist treatments that allow for the generalization of the results and the incorporation of aspect 2.) above.
Definition 5.2.1.2.1 (Conditional measure)
The conditional measure $\rho_n(x \parallel x_t; \varepsilon)$ given the state $x_t$ of a Map $M$ and measurement error $\varepsilon$ of initial conditions is defined by:

$$
\rho_n(x \parallel x_t; \varepsilon) = \frac{1}{\rho(M^n(I))} \int_I \delta\left(M^n(z) - x\right) \rho(z) dz \quad (5.2.1.2.1.1)
$$

Where $I \in \mathbb{R}$ is in interval such that: $I = [x_t-\varepsilon/2; x_t + \varepsilon/2]$ , $\rho(x)$ is the invariant measure of the map. and $\rho(M^n(I))$ is the “Mass” of the invariant measure on the n times iterated interval $I$ by the Map $M$:

$$
\rho(M^n(I)) = \int_{M^n(I)} \delta(z) \rho(z) dz
$$

Thus the conditional measure $\rho_n(x \parallel x_t; \varepsilon)$ assigns a probability mass to every point $x \in \mathbb{R}$ that it can be reached by iterating the Map $M$ n steps forward in time given an initial state $x_t$ and a measurement error $\varepsilon$.

Since the preceding definition did not give an applicable Formula, consider to partition $I$ into $k$ subintervals of length $I/k$. Furthermore assign the density $1/k$ to each point of $I$. Then a discrete approximation to (5.2.1.1.1.1) is:

$$
\overline{\rho}_k(x \parallel x_t; \varepsilon; k) = \frac{1}{k} \sum_{i=1}^{k} \chi\left(M^n\left(x_t + \frac{sI}{k} - \frac{\varepsilon}{2}\right) - x\right) \quad (5.2.1.2.1.2)
$$

where $\chi$ is a characteristic function defined as:

$$
\chi(0) = 1 \quad ; \quad \chi(x < 0) = 0
$$

It is obvious, that (5.2.1.1.1.2) can be computed when the equations of driving the diffusive process are known, at least by numerical simulations. The derivation of a Statistic Dynamics option pricing formula is basically straightforward:

Theorem 5.2.1.2.1 (Statistic Dynamics option pricing formula)
Given an ergodic Map $M$ with invariant measure $\rho(x)$, conditional measure $\rho_n(x \parallel x_t; \varepsilon)$ given the state $x_t$ of a Map $M$ at time $t$ and measurement error $\varepsilon$ of initial conditions. An option with time to maturity $T$, and risk free interest rate $r$ until maturity can be priced as:

$$
\text{Call}(x_t; x_t; T; r) = \int_0^\infty e^{-rT} \text{Max}\left\{S - X; 0\right\} \rho_T(S \parallel x_t; \varepsilon) dS \quad (5.2.1.2.1.3a)
$$

$$
\text{Put}(x_t; x_t; T; r) = \int_0^\infty e^{-rT} \text{Max}\left\{X - S; 0\right\} \rho_T(S \parallel x_t; \varepsilon) dS \quad (5.2.1.2.1.3b)
$$

Assuming risk neutral individuals, that can compute $\rho_n(x \parallel x_t; \varepsilon)$ approximately.
To give the reader an idea on how equations (5.2.1.2.1.3) influence the option price, we computed $\rho_n(x \mid x_t; \varepsilon)$ by numerical simulations using the approximation (5.2.1.2.1.2). Subject of the simulation were 2000 itera-\(\varepsilon\)es of the model of section 3). An approximation error of $\varepsilon=0.01$ was assumed, equivalent to that an accurate measurement only possible up to one hundredth. The interval $I$ was partitioned into ten equally long pieces of length $I/10$ each being assigned a density of $1/10$, the density was then evolved until time to maturity and Theorem 5.2.1.2.1 was used to price the option. Additional to the Statistic Dynamics option price the Hurst Option Price and the traditional Black Scholes Option price was computed. Two distinct experiments were conducted. In the first experiment on the first 250 itera-\(\varepsilon\)es the parameter estimation of the volatility for the Black Scholes Model and Hurst Model was performed. In the second experiment the parameter estimation of the volatility for the two option pricing models was done over the whole 2000 itera-\(\varepsilon\)es. The Hurst Coefficient was estimated always over all 2000 itera-\(\varepsilon\)es to give a better estimation quality with value $H = 0.32$. The two experiments were chosen to find out about the impact of the parameter estimation error that occurs when calibrating a homoscedastic model in a heteroskedastic environment. In both numerical investigations the risk free interest rate $r$ was assumed to be zero.

The results are shown in Appendix B2 in tables 5.2.1.2.1 a-c.) and 5.2.1.2.2 a-c.). The results can be summarized as follows:

1.) Effects of heteroscedasticity

Firstly the parameterisation with the overall volatility of 21% estimated for all 2000 itera-\(\varepsilon\)es is generally superior (inferior) for in (out of) the money options to that when using only the first 250 with a volatility estimator of 9%. The prices for in the money options come closer to the prices in the statistic dynamics world when the higher overall volatility is used. For out of the money options the Prices are generally to high given by the Black Scholes model compared to the statistic dynamics prices. This fact leads to the next point.

2.) Hurst Effect

Secondly the low Hurst Exponent of 0.32 indicates a strong anti-persistence effect that causes the log price not to diffuse unbounded into all directions equally, but rather to return to where it has come from more likely. Therefore out of the money options appear to be priced to high by the Black Scholes model in the presence of the deterministic diffusion model of section 3. The Hurst Option prices match those of the statistic dynamics option prices far better than the Black Scholes prices, since the Hurst model captures the anti persistency effect.

3.) Time to maturity

Thirdly the larger the time to maturity, the larger the divergence to the pricing of the Black Scholes model of the statistics dynamics pricing. This observation goes well in line with the observations made in sections 4.3 and 5.2.1.1 i.e. that the process of deterministic diffusion does not posses a statistic self similarity and the random structure in term of its empirically measured probability density vanishes in time just after only 30 time steps.

So far we can conclude that the Black Scholes Model can lead to large price deviations from the statistic dynamic price when the underlying price driving process is not that was assumed, a plain geometric Brownian motion. The effect becomes more severe in the presence of deterministic diffusion as the time evolves. The Hurst Option pricing model gives a good approximation to the statistic dynamics option prices, but needs a large sample of the
time series to be estimated efficiently regarding the Hurst coefficient as well as the input volatility. Up to 30 days the option prices of the Hurst Model diverge more and more from that of the statistical dynamic prices.

Of course in praxis the approach as presented here is not applicable because one does not know the true process driving the dynamics of the stock price. But there should be possible methods yielding approximately comparable results. This will be part of future research of the author and should come up in due course.
6 Summary and Conclusions

In the foregoing paper a new model named deterministic diffusion was introduced to model stock price processes. The model can be motivated by simple behavioural models of the stock market and does not need too many restrictive assumptions to be reasonable, like complete rationality of all market participants and market efficiency. Furthermore it helps in understanding on how randomness comes about and how typical stylized facts like i.) heteroscedasticity ii.) long range dependency iii.) fat tailed frequency distributions in real world stock market data can be explained.

Comparisons throughout the paper to real world DAX™ time series show obvious parallel features that are not neglect able and give evidence for the appropriateness of the approach. Both time series have fat tailed frequency distributions of their log returns, slowly decaying autocorrelations of they squared returns and show a large degree of heteroscedasticity. Neither the DAX™ nor the model time series showed stochastic self similarities and have a Hurst coefficient of \( H<0.5 \) for their log prices. I.e. they diffuse anti persistent and therefore not arbitrary equally to every direction. This means the space they cover over time is in average less than that of a geometric Brownian motion. Basically the story of deterministic diffusion is much about a random looking process that certainly are not random since the randomness vanishes as time evolves.

The implications on option prices have been sketched in a model framework and show how important differences arise to the classical Black Scholes model compared with a statistical dynamics approach to option pricing. Other alternative option models have been introduced and the applicability in the deterministic diffusion framework has been discussed. The Option pricing for fractional Brownian motion model was recommended for short time horizons. In case of good parameter estimation it can give better prices than the Black Scholes model in presence of deterministic diffusion because it can cope with anti persistent diffusive processes.

Future research should be concerned with a more precise description of the deterministic diffusion process in terms of variables like the Hurst coefficient or scattering maps, empirical fitting of parameters of those maps and processes to model existing time series, empirical detection methods of deterministic diffusion and easy applicable option pricing formulas. Furthermore other economic time series like exchange rates and interest rates could be analyzed to see if the deterministic diffusion model would be reasonable for them as well. The implications for Risk management surely should also not be out of scope of further investigations.
Appendix

A Additional Figures

A1 Figures for 4.3.2

Fig A.1.1 (a.) and (b.) the densities for the model log returns for different time horizons.
Fig A.1.2 (a.) and (b.) the densities for the DAX™ log returns for different time horizons.
### B. Additional Tables

#### B1. Tables for 4.1

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Table 4.2.2

Table 4.2.1, 4.2.2 Results for model time series of the test statistic T for various time buckets by observing N=2000 sample date points for the model time series and DAX time series respectively. Upper-T, Lower-T percentage of samples for which the m-sample bucket variance differs significantly from its expected value indicated by I at either the upside or the downside respectively. Sum-T total percentage of m-sample bucket variance being significantly from its expected value. P-Value if the 1-β Quantil of the Binomial Distribution B(n,p) Multiplied by 1/N with β = 0.01
### B2 Tables for 5.2.1.2

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(c.)

Tables 5.2.1.2.2 a-c.) showing the numerical results for the three option models in comparison for the 1st experiment. Estimated H=0.32, Volatility 21%. Times to maturity 30, 100, 250 timesteps respectively.
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(c.)

Tables 5.2.1.2.2 a-c.) showing the numerical results for the three option models in comparison for the 2nd experiment. Estimated H=0.32, Volatility 21%. Times to maturity 30, 100, 250 timesteps respectively.
C Proofs of Theorems

C 1 Proof of Theorem 4.3.1:

(i.) Part I.)

H=1/2 =>

Write:

\[
\text{cov}(dx(t), dx(t+k)) = \text{cov}(x(t+\Delta) - x(t), x(t+k+\Delta) - x(t+k))
\]

\[
\text{cov}(dx(t), dx(t+k)) = \text{cov}(x(t+\Delta), x(t+k+\Delta)) - \text{cov}(x(t+\Delta), x(t+k)) - \text{cov}(x(t), x(t+k+\Delta)) + \text{cov}(x(t), x(t+k))
\]

(A 4.3.1.1)

In general holds for 0<s<t<T:

\[
\text{var}(x(t) - x(s)) = \text{var}(x(t)) - 2\text{cov}(x(t), x(s)) + \text{var}(x(s))
\]

\[
\text{cov}(x(t), x(s)) = \frac{1}{2} \left\{ \text{var}(x(t)) + \text{var}(x(s)) - \text{var}(x(t) - x(s)) \right\}
\]

\[
\text{cov}(x(t), x(s)) = \frac{1}{2} \left\{ t^{2H} \text{var}(x(0)) + s^{2H} \text{var}(x(0)) - (t-s)^{2H} \text{var}(x(0)) \right\}
\]

Set H = ½, it follows:

\[
\text{cov}(x(t), x(s)) = \text{var}(x(s)) \quad (A 4.3.1.2)
\]

(A 4.3.1.1) together with (A 4.3.1.2) yields:

\[
\text{cov}(dx(t), dx(t+k)) = \text{var}(x(t+\Delta)) - \text{var}(x(t+\Delta)) - \text{var}(x(t)) + \text{var}(x(t)) = 0
\]

Part II.)

\[
\text{cov}(dx(t), dx(t+k)) = 0 \Rightarrow
\]

\[
\Rightarrow \text{cov}(x(t+\Delta), x(t+k+\Delta)) + \text{cov}(x(t), x(t+k)) = \text{cov}(x(t+\Delta), x(t+k)) + \text{cov}(x(t), x(t+k+\Delta))
\]
\[\begin{align*}
&\Leftrightarrow \frac{1}{2} \left\{ \text{var}(x(t+\Delta+k)) + \text{var}(x(t+\Delta)) - \text{var}(x(t+k+\Delta) - x(t+\Delta)) \right\} \\
&+ \frac{1}{2} \left\{ \text{var}(x(t+k)) + \text{var}(x(t)) - \text{var}(x(t+k) - x(t)) \right\} \\
&= \frac{1}{2} \left\{ \text{var}(x(t+k)) + \text{var}(x(t+\Delta)) - \text{var}(x(t+k) - x(t+\Delta)) \right\} \\
&+ \frac{1}{2} \left\{ \text{var}(x(t+k+\Delta)) + \text{var}(x(t)) - \text{var}(x(t+k+\Delta) - x(t)) \right\} \\
&\Leftrightarrow \frac{1}{2} \left\{ \text{var}(x(t+k+\Delta) - x(t+\Delta)) \right\} + \frac{1}{2} \left\{ \text{var}(x(t+k) - x(t)) \right\} \\
&= \frac{1}{2} \left\{ \text{var}(x(t+k) - x(t+\Delta)) \right\} + \frac{1}{2} \left\{ \text{var}(x(t+k+\Delta) - x(t)) \right\} \\
&\Leftrightarrow \frac{1}{2} \left\{ k^{2H} \text{var}(x(0)) \right\} + \frac{1}{2} \left\{ k^{2H} \text{var}(x(0)) \right\} \\
&= \frac{1}{2} \left\{ (k-\Delta)^{2H} \text{var}(x(0)) \right\} + \frac{1}{2} \left\{ (k+\Delta)^{2H} \text{var}(x(0)) \right\} \\
&\Leftrightarrow k^{2H} = (k-\Delta)^{2H} + (k+\Delta)^{2H}
\end{align*}\]

this only holds for \( H = \frac{1}{2} \)

Part III.)

Finally we need to prove that only if the increments \( dx(t) \) are independent, they are uncorrelated.

For any increment \( t, t+\nabla \):

\[ x(t+\Delta) - x(t) = dx(t) \cong x(0) \left( [t+\Delta]^H - t^H \right) \]

So the increments are equal in distribution but the scaling factor \( s = ([t+\Delta]^H - t^H) \). For any two random variables \( X, Y \) that are identical but two scaling factors \( s_X, s_Y \) it holds:

\[ E[XY] = s_X s_Y E \left[ \tilde{X} \tilde{Y} \right] ; \quad \tilde{X} = \frac{X}{s_X} ; \tilde{Y} = \frac{Y}{s_Y} \]

This means that only if \( X \) and \( Y \) are independent they are uncorrelated.

Here from follows, that only if \( H = \frac{1}{2} \) all \( dx(s) \) are independent.

(ii)

Suppose the covariance of two increments: \( dX(t) = X_{t+h} - X_t \) and \( dX(s) = X_{s+h} - X_s \) with \( s+h \leq t \) and \( t-s = nh \) then:
\[ \text{cov}_N(n) = \frac{1}{2} h^{2H} \left( (n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \right) \approx h^{2H} H (2H - 1) n^{2H-2} \]

It follows:

\[ H > 0.5 \Rightarrow \text{cov}_N(n) > 0 \]
\[ H < 0.5 \Rightarrow \text{cov}_N(n) < 0 \]

This proves (iii).

QED.

**C 1 Proof of Theorem 5.2.1.1.2:**

Theorem 1.9 from REF014 guaranties that all stable distributions are continuous with an infinitely differentiable density. This allows for numerical integration and thus proofs the Theorem.

QED.
Guido Venier: A new Model for Stock Price Movements

References


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REF019 T.Liu, Clive W.J. Granger, Walter P: Heller (1992) “Using the correlation exponent to decide whether an economic time series is chaotic.”


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