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Abstract. In this paper, we consider a one-to-one matching model where the population expands with the arrival of a man and a woman. Individuals in this population are matched, before and after the expansion, according to a version of the deferred acceptance algorithm (Gale and Shapley, 1962) where men propose and women reject or (tentatively or permanently) accept. Using computer simulations of this model, we study how the percentage of matches disrupted (undisrupted) with the expansion of the population is affected when the initial size of the population and the size of correlation in the preferences of individuals change.

Key words: One-to-one matching; deferred acceptance; stability; external stability.

1 Introduction

It is well known that for every two-sided (heterosexual) population in which individuals' preferences satisfy some mild assumptions there always exists a one-to-one (monogamic) matching that is stable. Under such a matching no pair of individuals would prefer each other to their mates and no individual would prefer staying single to his/her match. This celebrated result is due to Gale and Shapley (1962) who provide the proof by showing that a dynamic procedure —called the deferred acceptance (DA) algorithm— always produces, after some finite iterations, a stable matching. According to this algorithm, individuals in one side of the population propose for matching and individuals in the other side reject or (tentatively or permanently) accept these proposals. Because of its simplicity and computational power, this algorithm has been very popular and useful —since it was proposed (and in some markets even before that)— to solve many theoretical and practical matching problems (e.g. Roth, 1984, 2002, 2008; Balinski and Sönmez, 1999; Abdulkadiroğlu, Pathak, and Roth, 2005; and Pathak and Sönmez, 2008).

In this paper, we aim to investigate —with the help of computer simulations— how the stable matching generated by the DA algorithm is affected when the population expands with the arrival of a man and a woman. To measure the investigated effect, we count the number of matches disrupted (undisrupted) after the expansion of the population, and in more detail, the number of individuals who change/retain their marital status and the number of individuals who change/retain their partners. Then, we explore how all these numbers change with the size of population and the size of correlation in the preferences of individuals over potential partners.

While our research question and our findings are novel to the best of our knowledge, the destabilizing effects caused by a change in the matching population are already known in the matching literature since the works of Blum et al. (1997) and Cantala (2004), dealing with one-to-one markets and many-to-one markets respectively. In this literature, the closest work to ours is due Gabszewicz et al (2008), who show by example that entry to the matching population can heavily destabilize one-to-one marriages and suggest a new stability concept, called k-external stability, to account for the disruptive effects of a change in the matching population. In more detail, Gabszewicz et al (2008) say that the matching population (the marriage market) is k-externally stable if at any stable matching at least k of the matches are not disrupted whenever the matching population expands with the entry of a man and a woman. Using this definition, they show that a matching population with n men and n women would become 0-externally stable (the worst possible case) if all individuals had common (homogenous) preferences over potential mates whereas the degree of external stability would be $n-2$ (the best possible case) if all individuals had peak load preferences, i.e., a structure of preferences requiring that on a common ranking of individuals the further distant a potential mate is from the rank of an individual, the least preferred it is. We should note that in our study we do not consider all possible stable matchings associated with the matching population; we focus on the stable matching generated by the DA algorithm, only. Hence, when we expand the matching population, we measure the extent of the survival of the matches under this particular matching only, instead of calculating the degree of external stability of the whole matching population. In addition, unlike in Gabszewicz et al (2008), we consider heterogenous preferences, using a setup we borrow from Saglam (2019a, 2019b). This setup allows us to vary, in our simulations, the degree of correlation in the preferences from zero to one and to measure the possible effects of this variation on the percentage of disrupted/undisrupted matches.[1] Moreover, we measure these effects by varying the initial size of the matching population, as well.

The rest of our paper is organized as follows. Section 2 introduces our model and Section 3 presents our results. Finally, Section 4 concludes.

2 Model

Consider a population involving a finite set of men M and a finite set of women W, where each individual can be matched to another individual only if they are from opposite sexes. Let $\mathcal{N} = M \cup W$ denote this population. Individual $i \in \mathcal{N}$ derives the utility $U_i(j)$ when he/she is matched with individual j, and derives the utility $U_i(i)$ when he/she is unmatched to any individual (and remains single). Given any $X, Y \in \{M, W\}$ with $X \neq Y$, any $i \in X$, and any $j, k \in Y \cup \{i\}$, we say that i prefers j to k if $U_i(j) > U_i(k)$. We assume that each individual in the population knows all

[1] Using this setup, Saglam (2019b) extends the Todd and Miller's (1999) results in mutual sequential mate search obtained under homogenous preferences to the case of heterogenous preferences whereas Saglam (2019a) studies how the heterogeneity in individuals' preferences and the intensity of their learning about their own aspirations (before a matching takes place) can affect the likelihood of marriage and divorce as well as the balancedness and the speed of matching under the stable outcome of the DA algorithm.

relevant utilities for himself/herself before a matching takes place, and these utilities do not change during the matching process.

A matching is a one-to-one function $\mu : \mathcal{N} \to \mathcal{N}$ such that for each $m \in M$ and $w \in W$, $\mu(w) = m$ if and only if $\mu(w) = m$. In addition, $\mu(m) \notin W$ if and only if $\mu(m) = m$, and $\mu(w) \notin M$ if and only if $\mu(w) = w$. Individuals m and w are matched to each other if $\mu(m) = w$, and individual i is single if $\mu(i) = i$.

A matching μ is said to be *acceptable* for individual i if $U_i(\mu(i)) \geq U_i(i)$; i.e., the utility of individual i from the match $\mu(i)$ is not below the utility from being single. Also, given any matching μ, a man m and a woman w are together called a *blocking pair* for μ if $\mu(m) \neq w$ and $U_i(j) > U_i(\mu(i))$ for $i, j \in \{m, w\}$ with $i \neq j$; i.e., m and w are not matched under μ and they prefer each other to their matches at μ. Given these definitions, a matching μ is said to be *stable* if μ is acceptable for each individual and there exists no blocking pair for μ.

A celebrated result in matching theory, due to Gale and Shapley (1962), shows that there exists a stable matching for every matching population provided that individuals' preferences satisfy completeness and transitivity.[2] Moreover, under these assumptions, a stable matching can always be obtained as the outcome of a procedure, called the deferred acceptance (DA) algorithm, which has two versions depending upon the roles of men and women. In one of these algorithms, men propose to women and women give rejections or (tentative or permanent) acceptances, while in the other algorithm proposals are given by women and rejections/acceptances by men. The matching outcomes of these two algorithms, even though they are both stable, have different welfare implications. Gale and Shapley (1962) shows that if all individuals have strict preferences, the DA algorithm, when men (women) propose, always produces the men-optimal (women-optimal) stable matching; i.e. a stable matching which is preferred by all men (women) to any other stable matching.

[2]Given two alternatives x and y, an individual is said to weakly prefer x to y if he/she strictly prefers x to y or is indifferent between them. The preference ordering of this individual over the set of possible alternatives satisfies *completeness* if for any two alternatives in this set it is true that he/she weakly prefers one of them to the other. Also, his/her preference ordering satisfies *transitivity* if for any three alternatives x, y, z it is true that he/she weakly prefers x to z whenever he/she weakly prefers x to y and weakly prefers y to z.

We will denote by DA-MP (DA-WP) the deferred acceptance with men proposing (women proposing). Below, we describe the DA-MP algorithm with $k \geq 1$ steps. Initially, all individuals are single in this algorithm. (Interchanging the roles of men and women, one can simply obtain the DA-WP algorithm from below.)

Step $k \geq 1$: Every man who is in step 1, or who was rejected in step $k-1$ when k is at least 2, proposes to his most preferred woman in his updated list of acceptable women (if any). A man makes no proposal if his list is empty. Each woman holds the most preferred acceptable proposal she has received until now and rejects all other proposals. Then, each man rejected in this step deletes the woman who rejected him from his list of acceptable women. (Above, if any individual is indifferent between any two potential mates, he/she is allowed to break the tie arbitrarily.) The algorithm terminates when no further proposal is made by any man, and at this step each woman is matched to the man (if any) whose proposal she is holding.

Gale and Shapley (1962) proved that the above algorithm must always yield, after finite steps, a stable matching.

3 Results

Below, we study through computer simulations how the external stability of the matching outcome of the DA-MP algorithm is affected by a change in the size of heterogeneity in the preferences of individuals when the initial size of the matching population is also varied. We conduct all simulations using GAUSS Software Version 3.2.34 (Aptech Systems, 1998). (The program code of the simulations and the resulting data are available upon request.)

We assume that the number of men and the number of women in the initial population are the same and denoted by n; and we vary in our simulations the number n between the integers 2 and 100. For each n, we denote the corresponding population by $\mathcal{N}_n$. We consider the addition of a man and a women to this population, and call the expanded population by $\mathcal{N}^e_{n+1}$. We denote the set of men and women in the initial and extended populations by M_n, W_n and M^e_{n+1}, W^e_{n+1}.

We model the preferences of individuals in the population $\mathcal{N}_{n+1}^e$ (which contains $\mathcal{N}_n$ by construction) using the preference structure in Saglam (2018, 2019). Formally, we assume that for any n, any $X, Y \in \{M_{n+1}^e, W_{n+1}^e\}$ with $X \neq Y$, any $i \in X$, and any $j \in Y$, the match utility $U_i(j)$ of individual i derived from individual j satisfies

$$U_i(j) = \omega\, U^c(j) + (1-\omega)\, U_i^p(j),$$

where $U^c(j)$ denotes the common part of the utility any individual in X can derive from individual j, $U_i^p(j)$ denotes the private part of the utility only individual i can derive from individual j, and ω denotes a weight parameter that is, for simplicity, assumed to be common for all individuals in $X \cup Y$. In our simulations, the utilities $U^c(j)$ and $U_i^p(j)$ in the above expression are always randomly drawn from a uniform distribution of values in $[0, 100]$. On the other hand, the weight parameter ω is varied inside the set of values $\{0,\ 1/3,\ 2/3,\ 1\}$. It is clear that the higher the value of ω, the less heterogenous the preferences of individuals. In particular, $\omega = 0$ and $\omega = 1$ induce completely heterogenous (perfectly uncorrelated) and completely homogenous (perfectly correlated) preferences, while $\omega = 1/3$ and $\omega = 2/3$ induce partially heterogenous (imperfectly correlated) preferences.

Our simulation results are reported in the six panels of Figure 1. Panel (a) illustrates the percentage of matches disrupted with the expansion of the population by the arrival of a man and a woman. Our results show that the higher the correlation (the lower the heterogeneity) in the preferences of individuals, the lower the percentage of disrupted matches. Moreover, the size of the population, unless it is extremely small, has no remarkable effect on the percentage of disrupted matches, irrespective of the size of heterogeneity in the preferences. Here, the calculation of disrupted matches include an individual if the marital status of this individual changes (from single to married or from married to single) or if this individual is married with a different partner before and after the expansion of the population. We report these three types of disruptions separately in panels (b)-(d).

Panel (b) shows that as the population size becomes higher, the percentage of individuals who change —after the expansion of the population— their marital status from married to single becomes smaller and smaller, always tending towards zero.

The size of heterogeneity in the preferences has negligible effects on our results. Nevertheless, we always obtain the lowest percentage values when the preferences are homogenous. All these observations are also valid for panel (c), where we report the percentage of individuals who change —after the expansion of the population— their marital status from married to single. The only remarkable difference between panels (b) and (c) is about the levels. We observe that when the population expands, there are more individuals who change their marital status from single to married (panel c) than from married to single (panel b), especially when the initial size of the population is sufficiently small. In panel (d), we report the percentage of individuals who are married with different partners before and after the expansion of the population. We should immediately see that this percentage is monotonically decreasing in the size of correlation (ω), irrespective of the initial size of the population. However, the population size can have significantly large effects when it is extremely small. In that case, we observe that the higher the size of the population, the higher the percentage of married individuals who become divorced and marry with another partner after the expansion of the population.

In the last two panels of Figure 1, we analyze undisrupted matches. In panel (e), we illustrate the percentage of individuals who are married with the same partner before and after the expansion of the population. We observe that this percentage is much lower —irrespective of the population size— when the preferences of individuals are perfectly correlated than when they are uncorrelated. However, a change in the correlation parameter is found to be much more effective when it is smaller. The same observation also holds for the population size. Irrespective of the level of correlation in the preferences, the initial size of the population, when sufficiently small, has a positive impact on the percentage of individuals who do not change their partners after the expansion of the population. Finally, in panel (f) we report the percentage of individuals who are single both before and after the expansion of the population. We observe that this percentage is always increasing in the size of correlation in the preferences, while it is generally decreasing in the population size unless the preferences of individuals are very highly correlated.

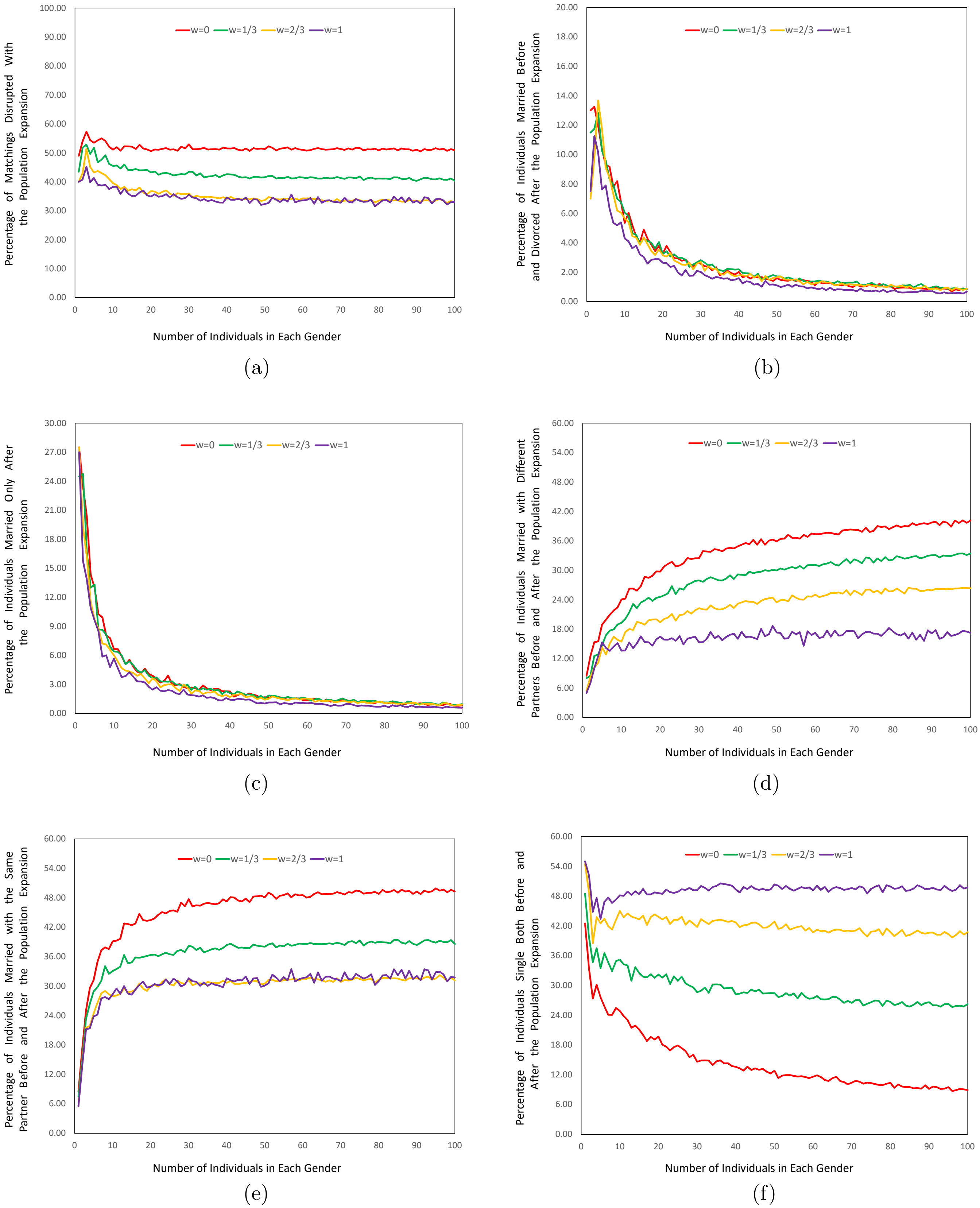

Figure 1. Various Effects of the Population Expansion on the Outcome of the DA algorithm.

4 Conclusion

In this paper, we have studied how the stable matching outcome generated by the DA algorithm is affected when the matching population expands with the arrival of a man and a woman. Using Monte Carlo simulations, we have showed that the size of correlation in the preferences of individuals has, in general, a negative effect on the overall percentage of disrupted matches, irrespective of the initial size of the population. While this negative effect is observed for both the set of individuals who change their marital status (from married to single or from single to married) and the set of married individuals who change their partners after the expansion of the population, the magnitude of the effect is incomparably larger in the latter case. On the other hand, the initial size of the population is found to have no remarkable effects on the overall percentage of disrupted matches, unless it is extremely small. However, this last result becomes no longer valid when we identify the type of disruption in our simulations. We find that the percentage of individuals who change —after the expansion of the population— their marital status is hyperbolically decreasing with the population size, whereas the percentage of individuals who are married to a different partner before and after the expansion of the population is mildly increasing unless the size of correlation in the preferences is extremely high.

Our results on disrupted matches imply opposite effects on undisrupted matches. That is, the population size should have no remarkable effect on the overall percentage of undisrupted matches, unless it is extremely small. In addition, the size of correlation in the preferences should, in general, have a positive effect on the overall percentage of undisrupted matches. We find that these overall effects can partially hold when we consider, in isolation, the percentage of individuals who are married with the same partner before and after the expansion of the population. In that case, the size of correlation, when it is not too high, has a negative effect whereas the size of population, when it is extremely small, can have a positive effect. On the other hand, when we take into consideration the percentage of individuals who are single both before and after the expansion of the population, the size of correlation in the preferences has always a positive, and remarkably large, effect whereas the size of population has a negative effect (only) if the size of correlation is sufficiently small.

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