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Abstract

We consider a problem where a group of agents is interested in some goods provided by a supplier with multiple sources. To be served, each agent should be connected directly or indirectly to all sources of the supplier for a safety reason. This problem generalizes the classical minimum cost spanning problem with one source by allowing the possibility of multiple sources. In this paper, we extend the definitions of the folk rule to be suitable for minimal cost spanning tree problems with multiple sources and present its axiomatic characterizations.

Keywords: minimum cost spanning tree problems, multiple sources, folk rule, axiomatic characterizations.

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1 Introduction

A group of agents is interested in a service provided by a supplier with multiple service stations, also called sources. Agents will be served through costly connections. They do not care whether they are connected directly or indirectly to the sources, but they want to be connected to all of them. This may occur for a safety reason. Agents have greater assurances of the service in the sense that they can still enjoy the service even if one or more sources cease to provide it. Also, there could be a situation where several suppliers offer different services by using the same network (Internet, cable TV, etc.) and agents are interested in all of them. These situations generalize classical minimum cost spanning tree problems with one source by allowing the possibility of multiple sources.

Given a cost spanning tree problem with multiple sources, the least costly way of connecting all agents to all sources (or minimum cost spanning tree) must be sought. This tree can be obtained, in polynomial time, by using the same algorithms as in the classical minimum cost spanning tree problem, for instance, Prim (1956) algorithm or Kruskal (1957) algorithm. Nevertheless, some variants of this problem are not so easy from a computational point of view: the fixed cost spanning forest problem studied in Granot and Granot (1992), where there are potential sites to construct facilities with fixed construction costs; the multi-source spanning tree problem studied in Farley et al. (2000), where the objective is to compute the spanning tree that minimizes the sum of the distances from each source to every other node; and the hop constrained Steiner trees with multiple root nodes studied in Gouveia et al. (2014).

Once it is known how to construct the minimum cost spanning tree, another interesting issue that usually arises is how to allocate that cost to the agents. Our paper studies this issue in minimum cost spanning tree problems with multiple sources. Even though many papers in the literature on Operations Research or Economics study how to allocate the minimum cost to agents in the classical setting with a single source, there are only a few devoted to this issue in the setting of multiple sources. Rosenthal (1987) introduces the minimum cost spanning forest game where there are several sources that offer the same service and agents want to be connected to at least one source. He associates a cooperative game with this problem and shows that its core is non-empty. Kuipers (1997) studies a problem where there are multiple sources, each of them offering a different service, and each agent specifies the set of sources that she wants to be connected to. He associates a cooperative game with this problem and seeks to determine the conditions under which the core is non-empty.

Our approach is different because we want all agents to be connected to all sources. From this perspective our problem can be seen as a particular case of Kuipers (1997) where all agents demand to be connected to all sources. Nevertheless, the cooperative game that we set up to study this problem is different. In the two papers mentioned above, the cost of a coalition S is the minimum cost of connecting all members in S to some sources under the assumption that S is allowed to use nodes outside of S. We follow the standard approach (as in the classical minimum cost spanning tree problem) and assume that agents in S can not use the locations of agents outside of S.

In the classical minimum cost spanning tree problem, the most popular rule is the so called "folk rule", which is studied in many papers. The folk rule has been proved to satisfy very appealing properties. It chooses an allocation in the core and is monotonic in the population and in the cost matrix. It is also additive in the cost matrix, which makes it easy to compute. Our first aim is to extend the definition of the folk rule to our setting by using the following four approaches:

1. as the Shapley value of the irreducible game (Bergantiños and Vidal-Puga 2007),
2. as an obligation rule (Tijs et al. 2006; Bergantiños and Kar 2010),
3. as a partition rule (Bergantiños et al. 2010 and 2011),
4. through a cone-wise decomposition (Branzei et al. 2004; Bergantiños and Vidal-Puga 2009).

We show that all four approaches make the same recommendation, the folk rule. We also provide its axiomatic characterizations.

The paper is structured as follows. Section 2 introduces minimum cost spanning tree problems with multiple sources. Section 3 extends the four definitions of the folk rule to our setting and show that they coincide in our setting. Section 4 presents its axiomatic characterizations.

2 The model

Let $N = \{1, \ldots, |N|\}$ be a set of agents and $M = \{s_1, ..., s_{|M|}\}$ be a set of sources. We are interested in a network whose nodes are elements of $N \cup M$. We denote by $|N|$ and $|M|$ the cardinalities of N and M, respectively. For each N and M, a *cost matrix* $C = (c_{ij})_{i,j \in N \cup M}$ represents the cost of a direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each i, $j \in N \cup M$ and $c_{ii} = 0$ for each $i \in N \cup M$. Since $c_{ij} = c_{ji}$ for each i, $j \in N \cup M$, we will work with undirected arcs $\{i, j\}$. We denote the set of all cost matrices over $N \cup M$ as $\mathcal{C}^{N \cup M}$. Given C, $C' \in \mathcal{C}^{N \cup M}$, $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all i, $j \in N \cup M$. Similarly, given x, $y \in \mathbb{R}^N$, $x \leq y$ if $x_i \leq y_i$ for each $i \in N$.

A *minimum cost spanning tree problem with multiple sources*, or a *problem*, is characterized by a triple (N, M, C) where N is the set of agents, M is the set of sources, and C is the cost matrix in $\mathcal{C}^{N \cup M}$. Given a subset $S \subset N$, we denote by (S, M, C) the restriction of the problem to the subset of agents S. The classical minimum cost spanning tree problem, or the *classical problem* for short, corresponds to the case where M has a single element, which is denoted by 0.

For each network g and each pair of distinct nodes i and $j \in N \cup M$, a *path from i to j in g* is a sequence of distinct arcs $g_{ij} = \{\{i_{s-1}, i_s\}\}_{s=1}^{p}$ such that $\{i_{s-1}, i_s\} \in g$ for each $s \in \{1, 2, \ldots, p\}$, $i = i_0$, and $j = i_p$. A *cycle* is a path from i to i. For each

i, $j \in N \cup M$, *i and j are connected in g* if there is a path from i to j. A *tree* is a connected network without any cycle.

For each network g, $S \subset N \cup M$ is a *connected component* if (1) for each i, $j \in S$, i and j are connected in g and (2) S is maximal, i.e., for each $i \in S$ and each $j \notin S$, i and j are not connected in g. Let $P(g) = \{S_k(g)\}_{k=1}^{n(g)}$ be the partition of $N \cup M$ into *connected components* induced by g. For each network g, let $S(P(g), i)$ be the element of $P(g)$ to which i belongs. Let $P(N \cup M)$ denote the set of all partitions of $N \cup M$ and $P = \{S_1, \ldots, S_{|P|}\}$ be a generic element of $P(N \cup M)$. For each P, $P' \in P(N \cup M)$, P is *finer* than P' if for each $S \in P$ there is $T \in P'$ such that $S \subset T$. Given a finite set S, $\Delta(S) = \{x \in \mathbb{R}$ such that $x_i \in [0,1]$ for each $i \in S$ and $\sum\limits_{i \in S} x_i = 1\}$ is the simplex over S.

For each problem (N, M, C) and each network g, the cost associated with g is defined as $c(N, M, C, g) = \sum\limits_{\{i,j\} \in g} c_{ij}$. When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N, M, C, g)$. Our first objective is to construct a network which minimizes the cost of connecting all agents to all sources, which can be achieved by a *minimal tree*. Formally, a tree t is a minimal tree if $c(t) = \min\{c(g) : \ g \text{ is a tree}\}$. A minimal tree always exists but it does not necessarily have to be unique. Kruskal algorithm (1956) computes a minimal tree. It constructs such a tree by sequentially adding the cheapest arc avoiding cycles.

Formally, let $A^0(C) = \{\{i, j\} : \ i, j \in N \cup M \text{ and } i \neq j\}$ and $g^0(C) = \emptyset$.

Step 1: Take an arc $\{i, j\} \in A^0(C)$ such that $c_{ij} = \min\limits_{\{k,\ell\} \in A^0(C)} \{c_{k\ell}\}$. If there are more than one arcs satisfying this condition, select just one. Let $\{i^1(C), j^1(C)\} = \{i, j\}$, $A^1(C) = A^0(C) \setminus \{i, j\}$ and $g^1(C) = \{i^1(C), j^1(C)\}$.

Step $p+1$ ($p = 1, \ldots, |N| + |M| - 2$): Take an arc $\{i, j\} \in A^p(C)$ such that $c_{ij} = \min\limits_{\{k,\ell\} \in A^p(C)} \{c_{k\ell}\}$. If there are more than one arcs satisfying this condition, select just one. Two cases are possible:

1. If $g^p(C) \cup \{i, j\}$ has a cycle, then go to the beginning of Step $p+1$ with new $A^p(C)$ obtained from $A^p(C)$ by deleting $\{i, j\}$, that is, $A^p(C) = A^p(C) \setminus \{i, j\}$, and $g^p(C)$ the same.

2. If $g^p(C) \cup \{i, j\}$ has no cycle, then take $\{i^{p+1}(C), j^{p+1}(C)\} = \{i, j\}$, $A^{p+1}(C) = A^p(C) \setminus \{i, j\}$, and $g^{p+1}(C) = g^p(C) \cup \{i^{p+1}(C), j^{p+1}(C)\}$, and go to Step $p+2$.

This process is completed in $|N| + |M| - 1$ steps, exactly the minimum number of arcs that are needed in order to connect all agents with all sources. $g^{|N|+|M|-1}(C)$ is a tree obtained from the Kruskal algorithm (the algorithm leads to a tree which is not always unique). When there is no ambiguity, we write A^p, g^p, and $\{i^p, j^p\}$ instead of $A^p(C)$, $g^p(C)$, and $\{i^p(C), j^p(C)\}$ respectively. We denote by $m(N, M, C)$ the cost of a minimal tree in (N, M, C).

Once the minimal tree is obtained, an interesting issue is how to divide its cost among the agents. A *cost allocation rule,* or a *rule,* is a map f that associates with each problem (N, M, C) a vector of cost shares $f(N, M, C) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(N, M, C) = m(N, M, C)$.

Example 1 *Let* (N, M, C) *be such that* $N = \{1, 2, 3\}$, $M = \{a, b\}$, $c_{1a} = 7$, $c_{12} = 8$, $c_{3b} = 9$, $c_{1b} = 10$, *and* $c_{ij} = 20$ *otherwise. The unique minimal tree is* $\{\{1, a\}, \{1, 2\}, \{1, b\}, \{3, b\}\}$ *and* $m(N, M, C) = 34$.

3 The folk rule in minimum cost spanning tree problems with multiple sources

In this section, we extend four definitions of the folk rule to our setting and show that they make the same recommendation. The first one is defined as the Shapley value of the irreducible game, the second as an obligation rule, the third as a partition rule, and the fourth through simple problems.

3.1 The Shapley value of the irreducible game

In the classical problem, Bergantiños and Vidal-Puga (2007) define the folk rule as the Shapley value of the irreducible game. We now extend this definition to our problem. Let (N, M, C) be a problem and t a minimal tree in (N, M, C). We define the *minimal network* (N, M, C^t) associated with t where $c^t_{ij} = \max_{\{k,\ell\} \in g_{ij}} \{c_{k\ell}\}$ and g_{ij} denotes the unique path in t from i to j. It is well known that C^t does not depend on the choice of the minimal tree. Following Bird (1976), the *irreducible problem* (N, M, C^*) of (N, M, C) can thus be defined as the minimal network (N, M, C^t) associated with any minimal tree t. C^* is referred to as the *irreducible matrix*.

A *game with transferable utility*, briefly a *game*, is a pair (N, v), where v is a real-valued function defined on all coalitions $S \subseteq N$ satisfying $v(\emptyset) = 0$. The *irreducible game* is a pair (N, v_{C^*}) such that for each $S \subset N$, $v_{C^*}(S) = m(S, M, C^*)$, which means that the value of a coalition is the minimum cost in C^* of connecting the agents in S to all sources using only the locations of the members in S.

Let Π_N be the set of all permutations over the finite set N. For each $\pi \in \Pi_N$, let $Pre(i, \pi)$ be the set of agents of N preceding i in the order π, i.e., $Pre(i, \pi) = \{j \in N \text{ such that } \pi(j) < \pi(i)\}$. For each $i \in N$, the Shapley value of a game (N, v) (Shapley 1953) is the average of her marginal contributions:

$$Sh_i(N, v) = \frac{1}{|N|!} \sum_{\pi \in \Pi_N} \big\{ v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi)) \big\}.$$

Definition 1 *For each problem* (N, M, C), *the rule* f^{Sh} *is defined as the Shapley value of the irreducible game associated with* (N, M, C). *Namely,* $f^{Sh}(N, M, C) = Sh(N, v_{C^*})$.

We now compute f^{Sh} in Example 1. Since the unique minimal tree is $\{\{1,a\},\{1,2\},\{1,b\},\{3,b\}\}$, $c^*_{1a}=7$, $c^*_{12}=8$, $c^*_{1b}=10$, and $c^*_{3b}=9$. Besides, $c^*_{2a}=8$, and $c^*_{ij}=10$ otherwise. The irreducible game is as follows:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_{C^*}(S)$	17	18	19	25	26	27	34

Thus,

$$f^{Sh}(N,M,C)=\left(\frac{62}{6},\frac{68}{6},\frac{74}{6}\right)=(10.33,11.33,12.33).$$

3.2 Obligation rules

Tijs et al. (2006) define the family of obligation rules for the classical problem by introducing obligation functions. Let $N_0=N\cup\{0\}$ be a set of nodes where 0 is the unique source in the classical problem. An obligation function is a map o assigning to each $S\in 2^{N_0}\setminus\{\emptyset\}$ a vector $o(S)$ meeting the requirement that $o(S)\in\Delta(S)$ if $0\notin S$, $o_i(S)=0$ for each $i\in S$ if $0\in S$, and for each $S,T\in 2^{N_0}\setminus\{\emptyset\}$ such that $S\subset T$ and $i\in S$, $o_i(S)\geq o_i(T)$. An obligation function can be interpreted as follows. Assume that agents in S are connected with one another. Now, they need to construct an arc from any agent in S to the source so that they are all connected. Thus, $o_i(S)$ represents the proportion of the cost of the arc that each agent $i\in S$ must pay. If the agents in S are already connected to the source, then they do not need to construct any additional arc so that their obligation is zero, $o_i(S)=0$ for each $i\in S$.

The obligation rule associated with an obligation function o is defined through the Kruskal algorithm as follows. The cost of each arc that is constructed at each step of the Kruskal algorithm is divided among the agents who benefit from its construction. Each agent pays the difference between her obligation to the component to which she belongs before the arc is added and the one afterwards. Tijs et al. (2006) prove that f^o is well-defined, namely, it is independent of the choice of the minimal tree by the Kruskal algorithm. The folk rule corresponds to the obligation function where for each $S\subset N$ and each $i\in S$, $o^*_i(S)=\dfrac{1}{|S|}$.

We now extend this definition to our problem. Let $P=\left\{S_1,...,S_{|P|}\right\}\in P(N\cup M)$. Note that in the classical problem, if $i\in S_k$, then the obligation of agent i depends only on S_k (the element of the partition to which i belongs). However, in our problem, it depends on the whole structure of the partition in connected components. We assume that for each $S_k\in P$, agents in S_k are connected with one another. The obligation of each $i\in N$ in P, $o_i(P)$, is defined as follows.

(1) A link that joins two components of P with sources: Since all agents in N are interested in such a link, all agents have an equal obligation over that link.

(2) A link that joins a component S_k with no source ($S_k\cap M=\emptyset$) to a component $S_{k'}$ with a source ($S_{k'}\cap M\neq\emptyset$): Since only agents in S_k are interested in such a link, only agents in S_k have obligations over it.

Formally, for each $i \in S_k \cap N$, the obligation function o^* is defined as

$$o_i^*(P) = \begin{cases} \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} & \text{if } S_k \cap M \neq \emptyset, \\ \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \dfrac{1}{|S_k|} & \text{if } S_k \cap M = \emptyset. \end{cases} \quad (1)$$

It is straightforward to see that when there is a single source ($|M| = 1$), o^* coincides with the obligation function associated with the folk rule in the classical problem.

The obligation rule f^{o^*} associated with the obligation function o^* is defined in the same way as in the classical problem.

Definition 2 *For each problem (N, M, C) and each $i \in N$, the rule f^{o^*} is defined as*

$$f_i^{o^*}(N, M, C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p}[o_i^*(P(g^{p-1})) - o_i^*(P(g^p))].$$

In Proposition 1, we show that f^{o^*} is well-defined, namely, for each (N, M, C), f^{o^*} divides $m(N, M, C)$ among the agents and is independent of the minimal tree selected by the Kruskal algorithm.

We now compute f^{o^*} in Example 1.

Arc	$P(g)$	$o_1^*(P(g))$	$o_2^*(P(g))$	$o_3^*(P(g))$
$\emptyset$	$\{1, 2, 3, a, b\}$	$\frac{2-1}{3} + \frac{1}{1} = 1 + \frac{1}{3}$	$\frac{2-1}{3} + \frac{1}{1} = 1 + \frac{1}{3}$	$\frac{2-1}{3} + \frac{1}{1} = 1 + \frac{1}{3}$
$\{1, a\}$	$\{1a, 2, 3, b\}$	$\frac{2-1}{3} = \frac{1}{3}$	$\frac{2-1}{3} + 1 = 1 + \frac{1}{3}$	$\frac{2-1}{3} + 1 = 1 + \frac{1}{3}$
$\{1, 2\}$	$\{12a, 3, b\}$	$\frac{2-1}{3} = \frac{1}{3}$	$\frac{2-1}{3} = \frac{1}{3}$	$\frac{2-1}{3} + 1 = 1 + \frac{1}{3}$
$\{3, b\}$	$\{12a, 3b\}$	$\frac{2-1}{3} = \frac{1}{3}$	$\frac{2-1}{3} = \frac{1}{3}$	$\frac{2-1}{3} = \frac{1}{3}$
$\{1, b\}$	$\{123ab\}$	0	0	0

Thus,

$$\begin{aligned} f_1^{o^*}(N, M, C) &= c_{1a} + \frac{1}{3}c_{1b} = 7 + \frac{10}{3} = 10.33, \\ f_2^{o^*}(N, M, C) &= c_{12} + \frac{1}{3}c_{1b} = 8 + \frac{10}{3} = 11.33, \\ f_3^{o^*}(N, M, C) &= c_{3b} + \frac{1}{3}c_{1b} = 9 + \frac{10}{3} = 12.33. \end{aligned}$$

3.3 Partition rules

Bergantiños et al. (2010, 2011) introduce a family of rules using the Kruskal algorithm. At each step of the algorithm, the cost of the selected arc is divided among the agents by using *sharing functions*. A sharing function ϱ is a map that specifies the part of the cost paid by each agent at each step of the Kruskal algorithm.

We now explain the sharing function inducing the folk rule. Assume that when an arc is added, components S_k and S_ℓ are joined. The sharing function is defined through the following principles.

1. When a component with no source is joined to one with a source, only agents in the component with no source obtain benefits. Thus, the full cost of the arc is paid by the agents in the component with no source.

2. When two components with no source are joined, agents in both components benefit. We assume that the total amount paid by one component is proportional to the number of agents in the other. We further assume that all agents in the same component pay the same amount.

For each $i \in S_k$, the proportion of the cost paid by agent i is:

$$\begin{cases} 0 & \text{if } 0 \in S_k, \\ \dfrac{1}{|S_k|} & \text{if } 0 \in S_\ell, \\ \dfrac{|S_\ell|}{|S_k \cup S_\ell||S_k|} & \text{if } 0 \notin S_k \cup S_\ell. \end{cases}$$

Next we extend the definition of the sharing function to our problem. Let $P = \left\{S_1, ..., S_{|P|}\right\} \in P(N \cup M)$. We assume that for each $S_k \in P$, agents in S_k are connected to one another. Let P' be a partition obtained from P after components S_k and S_ℓ are joined. We define the sharing function ϱ as follows: Cases 1 and 2 are similar to the ones in the classical problem, but Case 3 is new.

1. When we join a component with no source to one with a source, only agents in the component with no source benefit. Thus, the full cost of the arc is paid by the agents in the component with no source.

2. When we join two components with no source, agents of both components benefit. We assume that the total amount paid by one component is proportional to the number of agents in the other. We further assume that all agents in the same component pay the same amount.

3. When we join two components with sources, all agents in the problem benefit. Thus, the cost of that arc is divided equally among all agents in the problem.

Formally, for each $i \in N$, the sharing function ϱ^* is defined as

$$\varrho_i^*(P,P') = \begin{cases} \dfrac{1}{|N|} & \text{if } S_k \cap M \neq \emptyset, S_\ell \cap M \neq \emptyset, \\ \dfrac{1}{|S_k|} & \text{if } S_k \subseteq N,\ S_\ell \cap M \neq \emptyset, \text{ and } i \in S_k, \\ \dfrac{|S_\ell|}{|S_k \cup S_\ell||S_k|} & \text{if } S_k \cup S_\ell \subseteq N \text{ and } i \in S_k, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

It is clear that $\varrho^*(P,P') \in \Delta(N)$.

Definition 3 *For each problem (N,M,C) and each $i \in N$, the rule f^{ϱ^*} is defined as*

$$f_i^{\varrho^*}(N,M,C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p}[\varrho_i^*(P(g^{p-1}),P(g^p))].$$

In Proposition 1, we show that f^{ϱ^*} is well-defined, namely, it does not depend on the choice of the minimal tree by the Kruskal algorithm.

We now compute f^{ϱ^*} in Example 1.

Arc	$P(g^{p-1}), P(g^p)$	$\varrho_1^*(P(g^{p-1}),P(g^p))$	$\varrho_2^*(P(g^{p-1}),P(g^p))$	$\varrho_3^*(P(g^{p-1}),P(g^p))$
$\{1,a\}$	$\{1,a,2,3,b\}$ $\{1a,2,3,b\}$	1	0	0
$\{1,2\}$	$\{1a,2,3,b\}$ $\{12a,3,b\}$	0	1	0
$\{3,b\}$	$\{12a,3,b\}$ $\{12a,3b\}$	0	0	1
$\{1,b\}$	$\{12a,3b\}$ $\{123ab\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Thus,

$$\begin{aligned} f_1^{\varrho^*}(N,M,C) &= c_{1a} + \frac{1}{3}c_{1b} = 7 + \frac{10}{3} = 10.33, \\ f_2^{\varrho^*}(N,M,C) &= c_{12} + \frac{1}{3}c_{1b} = 8 + \frac{10}{3} = 11.33, \\ f_3^{\varrho^*}(N,M,C) &= c_{3b} + \frac{1}{3}c_{1b} = 9 + \frac{10}{3} = 12.33. \end{aligned}$$

3.4 The cone-wise decomposition

Norde et al. (2004) prove that each classical problem can be written as a non-negative linear combination of classical *simple problems* where the costs of the arcs are either 0 or 1. Branzei et al. (2004) define the folk rule first in the classical simple problem as follows. Agents connected to the source through a 0 cost path pay nothing. Agents connected with one another through a 1 cost path pay the cost of connecting to the source equally. Then they extend this definition to the general problem in a linear way following the result by Norde et al. (2004).

We first introduce the folk rule in the classical simple problem following Branzei et al. (2004). For each simple problem (N_0, C) and each $S \subset N$, two agents i, $j \in N$, $i \neq j$ are (C, S)-*connected* if there exists a path g_{ij} from i to j satisfying that for all $\{k, \ell\} \in g_{ij}$, $c_{k\ell} = 0$ and $\{k, \ell\} \subset S$. Also, $S \subset N$ is a C-*component* if two conditions hold: First, for all i, $j \in S$, i and j are (C, S)-*connected*. Second, S is maximal, i.e., if $S \subsetneq T$, then there exist i, $j \in T$, $i \neq j$, such that i and j are not (C, T)-*connected*. It is obvious that the set of C-components is a partition of N.

For each simple problem (N_0, C), the folk rule is defined as follows. For each $i \in N$, let S_i be the C-component to which i belongs. Then,

$$f_i(N_0, C) = \begin{cases} \dfrac{1}{|S_i|} & \text{if } c_{0j} = 1 \text{ for each } j \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

Namely, agents in a C-component who are connected to the source at 0 cost pay nothing, whereas agents in a C-component who are connected to the source at 1 cost divide this cost equally among the members.

Next lemma adapts the results of Norde et al. (2004) to our setting.

Lemma 1 *For each problem (N, M, C), there exist a positive number $m(C) \in \mathbb{N}$, a sequence $\{C^q\}_{q=1}^{m(C)}$ of cost matrices, and a sequence $\{x^q\}_{q=1}^{m(C)}$ of non-negative real numbers satisfying three conditions:*

(1) $C = \sum_{q=1}^{m(C)} x^q C^q$.

(2) *For each $q \in \{1, \ldots, m(C)\}$, there exists a network g^q such that $c^q_{ij} = 1$ if $\{i, j\} \in g^q$ and $c^q_{ij} = 0$ otherwise.*

(3) *For each $q \in \{1, \ldots, m(C)\}$ and each $\{i, j, k, \ell\} \subset N_0$, if $c_{ij} \leq c_{k\ell}$, then $c^q_{ij} \leq c^q_{k\ell}$.*

Branzei et al. (2004) extend the definition of the folk rule to a classical problem (N_0, C) using Lemma 1, so that the folk rule is defined as

$$\sum_{q=1}^{m(C)} x^q f\left(N_0, C^q\right)$$

where $f\left(N_0, C^q\right)$ denotes the folk rule in the simple problem (N_0, C^q).

We now apply this approach to our problem. Since we have multiple sources, we need to adapt the procedure. First, we need to modify the definition of C-component. Instead of considering each component as a subset of N, we now consider a C-component as a subset of $N \cup M$.

Let (N, M, C) be a simple problem. Denote by $P = \{S_1, \ldots, S_{|P|}\}$ the set of C-components. The rule f^{CW} for simple problems is defined as follows. We first connect each component with no source to a component with sources and divide the cost equally among the agents in the component. Then we connect the components with sources with one another and divide the cost equally among all agents. Formally, for each $i \in N$, let $S(P, i)$ be the C-component to which i belongs. Then,

$$f_i^{CW}(N, M, C) = \begin{cases} \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} & \text{if } S(P,i) \cap M \neq \emptyset, \\ \dfrac{1}{|S(P,i)|} + \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} & \text{if } S(P,i) \cap M = \emptyset. \end{cases}$$

Definition 4 *For each problem (N, M, C) and each $i \in N$, the rule f^{CW} is defined as*

$$f_i^{CW}(N, M, C) = \sum_{q=1}^{m(C)} x^q f_i^{CW}(N, M, C^q).$$

We now compute f^{CW} in Example 1. Note that $C = \sum_{q=1}^{5} x^q C^q$ where $x^1 = 7$, $x^2 = x^3 = x^4 = 1$, $x^5 = 10$, and

Arcs	C^1	C^2	C^3	C^4	C^5
$\{a, 1\}$	1	0	0	0	0
$\{1, 2\}$	1	1	0	0	0
$\{b, 3\}$	1	1	1	0	0
$\{b, 1\}$	1	1	1	1	0
$\{a, b\}$	1	1	1	1	1
$\{a, 2\}$	1	1	1	1	1
$\{a, 3\}$	1	1	1	1	1
$\{b, 2\}$	1	1	1	1	1
$\{1, 3\}$	1	1	1	1	1
$\{2, 3\}$	1	1	1	1	1

We compute $f^{CW}(N, M, C^q)$ for each $q = 1, ..., 5$.

1. C^1-components are $\{1, 2, 3, a, b\}$.

$$f^{CW}(N, M, C^1) = \left(1 + \frac{1}{3}, 1 + \frac{1}{3}, 1 + \frac{1}{3}\right).$$

2. C^2-components are $\{a1, 2, 3, b\}$.

$$f^{CW}(N, M, C^2) = \left(\frac{1}{3}, 1 + \frac{1}{3}, 1 + \frac{1}{3}\right).$$

3. C^3-components are $\{a12, 3, b\}$.

$$f^{CW}(N, M, C^3) = \left(\frac{1}{3}, \frac{1}{3}, 1 + \frac{1}{3}\right).$$

4. C^4-components are $\{a12, b3\}$.

$$f^{CW}(N, M, C^4) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

5. C^5-components are $\{ab123\}$.

$$f^{CW}(N, M, C^5) = (0, 0, 0).$$

Then,

$$\begin{aligned} f^{CW}(N, M, C) &= \sum_{q=1}^{5} x^q f^{CW}(N, M, C^q) \\ &= 7\left(1 + \frac{1}{3}, 1 + \frac{1}{3}, 1 + \frac{1}{3}\right) + \left(\frac{1}{3}, 1 + \frac{1}{3}, 1 + \frac{1}{3}\right) \\ &\quad + \left(\frac{1}{3}, \frac{1}{3}, 1 + \frac{1}{3}\right) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + 10\,(0, 0, 0) \\ &= (10.33, 11.33, 12.33)\,. \end{aligned}$$

3.5 Equivalence of four approaches

In Proposition 1, we show that the obligation rule f^{o^*} and the Kruskal sharing rule f^{ϱ^*} are well-defined. Also, in Theorem 1, we prove that all four approaches make the same recommendation. The proofs of Proposition 1 and Theorem 1 are given in Appendix.

Proposition 1 *f^{o^*} and f^{ϱ^*} are well-defined.*

Theorem 1 *For each problem (N, M, C),*

$$f^{Sh}(N, M, C) = f^{o^*}(N, M, C) = f^{\varrho^*}(N, M, C) = f^{CW}(N, M, C).$$

4 Axiomatic characterizations of the folk rule

Here, we provide axiomatic characterizations of the folk rule. We begin with an extension of the axioms discussed in the classical problem. Our first axiom, *independence of irrelevant trees,* requires that the cost allocation chosen by a rule should depend only on the edges that belong to a minimal tree. This axiom is introduced in Bergantiños and Vidal-Puga (2007) and also used in Bogomolnaia and Moulin (2010) under the name of reductionism.

Independence of irrelevant trees (IIT). For each (N, M, C) and (N, M, C'), if they have a common minimal tree t such that $c_{ij} = c'_{ij}$ for each $\{i, j\} \in t$, then $f(N, M, C) = f(N, M, C')$.

Equivalently, IIT can be stated as for each (N, M, C), $f(N, M, C) = f(N, M, C^*)$, where C^* is an irreducible matrix associated with (N, M, C).

Cost monotonicity requires that if some cost increases, then no agent ends up better off. This axiom has been widely discussed in the literature: Dutta and Kar (2004); Tijs et al. (2006); Bergantiños and Vidal-Puga (2007); Lorenzo and Lorenzo-Freire (2009); and Bergantiños and Kar (2010).

Cost monotonicity (CM). For each (N, M, C) and (N, M, C'), if $C \leq C'$, then $f(N, M, C) \leq f(N, M, C')$.

It is easy to check that CM implies IIT.

Additivity requires that a cost allocation be an additive function of problems, that is, for each (N, M, C) and (N, M, C'), $f(N, M, C + C') = f(N, M, C) + f(N, M, C')$. However, there is no rule satisfying additivity. Therefore, as in the classical problem, we formulate a weaker version of additivity, *cone-wise additivity* (Norde et al. 2004; Bergantiños and Kar 2010; Bogomolnaia and Moulin 2010) which requires the additivity property to hold only for a pair of problems where the orders of all arcs (in which their costs are increasing) are the same in two problems.

Cone-wise additivity (CA). For each (N, M, C) and (N, M, C') and each order $\sigma : \{\{i, j\}\}_{i,j \in N \cup M, i<j} \to \left\{1, 2, ..., \frac{|N \cup M|(|N \cup M|+1)}{2}\right\}$, if for each $i, j, k, \ell \in N \cup M$ such that $\sigma\{i, j\} \leq \sigma\{k, \ell\}$, $c_{ij} \leq c_{k\ell}$ and $c'_{ij} \leq c'_{k\ell}$, then $f(N, M, C + C') = f(N, M, C) + f(N, M, C')$.

We now introduce a monotonicity property concerned with the changes in the set of agents. *Population monotonicity* requires that if new agents join the problem, then no agent in the initial problem should be worse off. PM has been widely discussed in the literature: Dutta and Kar (2004); Tijs et al. (2006); Bergantiños and Vidal-Puga (2007, 2008); Lorenzo and Lorenzo-Freire (2009); Bergantiños and Kar (2010); and Bogomolnaia and Moulin (2010).

Population monotonicity (PM). For each (N, M, C), each $S \subset T \subseteq N$, and each $i \in S$, $f_i(S, M, C) \geq f_i(T, M, C)$.

Core selection requires that no coalition of agents has an incentive to deviate from the grand coalition and to build their own minimal tree.

Core selection (CS). For each (N, M, C) and each $S \subset N$, $\sum_{i \in S} f_i(N, M, C) \leq m(S, M, C)$.

It is straightforward to show that PM implies CS. For each $S \subset N$ and each $i \in S$, PM implies that $f_i(N, M, C) \leq f_i(S, M, C)$, so that $\sum\limits_{i \in S} f_i(N, M, C) \leq \sum\limits_{i \in S} f_i(S, M, C)$. Since $\sum\limits_{i \in S} f_i(S, M, C) = m(S, M, C)$, PM implies CS.

Suppose that two subsets, S and $N \backslash S$, can connect to all sources separately or jointly. *Separability* (Bergantiños and Vidal-Puga 2007 and 2009; Bergantiños et al. 2011) requires that if the minimal costs in two situations are the same, then the same assignment should be made to all agents in two circumstances.

Separability (SEP). For each (N, M, C) and each $S \subset N$, if $m(N, M, C) = m(S, M, C) + m(N \backslash S, M, C)$, then

$$f_i(N, M, C) = \begin{cases} f_i(S, M, C) & \text{if } i \in S, \\ f_i(N \backslash S, M, C) & \text{if } i \in N \backslash S. \end{cases}$$

Note that PM also implies SEP. By PM, for each $i \in S$, $f_i(N, M, C) \leq f_i(S, M, C)$ and for each $i \in N \backslash S$, $f_i(N, M, C) \leq f_i(N \backslash S, M, C)$. If $m(N, M, C) = m(S, M, C) + m(N \backslash S, M, C)$, then from the definition of a rule, we have the desired conclusion.

Symmetry requires that if two agents have the same costs for all connections with nodes, then their cost assignments should be the same.

Symmetry (SYM). For each (N, M, C) and each i, $j \in N$, if $c_{ik} = c_{jk}$ for each $k \in N \cup M \backslash \{i, j\}$, then $f_i(N, M, C) = f_j(N, M, C)$.

We now introduce a property specifically designed for our problem, which requires that if the cost between two sources increases, then all agents should be affected by the same amount.

Equal treatment of source costs (ETSC). For each (N, M, C) and (N, M, C') and each a, $b \in M$, if for each k, $\ell \in M \cup N$ such that $\{k, \ell\} \neq \{a, b\}$, $c_{k\ell} = c'_{k\ell}$, then for each i, $j \in N$, $f_i(N, M, C') - f_i(N, M, C) = f_j(N, M, C') - f_j(N, M, C)$.

In the context of the classical problem, this axiom is related to *constant share of extra costs* (Bergantiños and Kar 2010), which requires that if the connection cost to the source increases by the same amount for all agents, then all agents should share this extra cost by the same amount. However, *constant share of extra costs* is concerned with the cost change in the arc between agents and the source, but ETSC is concerned with the cost change in the arc between two sources.

We are ready to present axiomatic characterizations of the folk rule. First, we show that the folk rule satisfies all axioms introduced in the above.

Proposition 2 *The folk rule satisfies IIT, CM, CA, PM, CS, SEP, SYM, and ETSC.*

The proof is given in Appendix.

We now provide axiomatic characterizations of the folk rule.

Theorem 2 (a) *A rule satisfies IIT, CA, CS, SYM, and ETSC if and only if it is the folk rule.*

(b) *A rule satisfies IIT, CA ,SEP, SYM,and ETSC if and only if it is the folk rule.*

The proof is given in Appendix. Also, in Appendix, we show that all axioms in Theorem 2 are independent.

Remark 1 *In the classical problem, Bergantiños et al. (2011) characterizes the folk rule by imposing the axioms of CM, CA, CS (or SEP), and SYM. Since CM implies IIT and the folk rule satisfies CM, the folk rule can alternatively be characterized by imposing CM instead of IIT. By adding ETSC to the list, we obtain characterizations of the folk rule in our problem. This axiom is important since we need to specify how a rule should respond to cost changes between sources differently from the classical problem.*

Appendix:

In the appendix, we present the proofs of the results. We also show that all axioms of Theorem 2 are independent.

Proof of Proposition 1. We need to prove two statements. First, f^{o^*} and f^{ϱ^*} divide the cost of the minimal tree $m(N, M, C)$ among the agents. Second, the definition of f^{o^*} and f^{ϱ^*} does not depend on the choice of the minimal tree by the Kruskal algorithm.

We start with f^{o^*}. In order to prove that f^{o^*} divides $m(N, M, C)$ among the agents, it suffices to prove that for each $p = 1, \ldots, |N| + |M| - 1$, the cost of arc $\{i^p, j^p\}$ is allocated in full among the agents in N.

Given $P = \left\{S_1, ..., S_{|P|}\right\} \in P(N \cup M)$ it is trivial to see that $\sum\limits_{i \in N} o_i^*(P) = |P| - 1$. Then,

$$\begin{aligned}
\sum_{i \in N}[o_i^*(P(g^{p-1})) - o_i^*(P(g^p))] &= \sum_{i \in N} o_i^*(P(g^{p-1})) - \sum_{i \in N} o_i^*(P(g^p)) \\
&= |P(g^{p-1})| - 1 - (|P(g^p)| - 1) \\
&= |P(g^{p-1})| - |P(g^p)| \\
&= 1
\end{aligned}$$

Next we prove that f^{o^*} does not depend on the choice of the minimal tree by the Kruskal algorithm. Given a tree $t = \{\{i^p, j^p\}\}_{p=1}^{|N|+|M|-1}$ obtained by the Kruskal algorithm, we define the followings:

- $B^0(t) = \emptyset$, $c^0(t) = c^0 = 0$.
- $c^1(t) = \min\limits_{\{k,\ell\} \in t \setminus B^0(t)} \{c_{k\ell}\}$, $c^1 = \min\limits_{\{k,\ell\} \subset N \cup M, c_{k\ell} > c^0} \{c_{k\ell}\}$, and
  $B^1(t) = \{\{i,j\} \in t : c_{ij} = c^1(t)\}$.
- In general, $c^q(t) = \min\limits_{\{k,\ell\} \in t \setminus \cup_{r=0}^{q-1} B^r(t)} \{c_{k\ell}\}$, $c^q = \min\limits_{\{k,\ell\} \subset N \cup M, c_{k\ell} > c^{q-1}} \{c_{k\ell}\}$, and $B^q(t) = \{\{i,j\} \in t : c_{ij} = c^q(t)\}$.

This process ends when we find $m(t) \leq |N| + |M| - 1$ such that $\cup_{r=0}^{m(t)-1} B^r(t) \subsetneq t = \cup_{r=0}^{m(t)} B^r(t)$. Note that $m(t)$ denotes the number of arcs in t with different costs.

By the Kruskal algorithm, for all $q = 1, ..., m(t)$, $c^q(t) = c^q$. Next, we prove that $P(B^1(t)) = P(\{\{i,j\} : \ c_{ij} \leq c^1\})$. Since $B^1(t) \subset \{\{i,j\} : \ c_{ij} \leq c^1\}$, $P(B^1(t))$ is finer than $P(\{\{i,j\} : \ c_{ij} \leq c^1\})$. Suppose that $P(B^1(t)) \neq P(\{\{i,j\} : \ c_{ij} \leq c^1\})$. Then, there exist S, $S' \in P(B^1(t))$, $S \neq S'$, $k \in S$, and $\ell \in S'$ such that $c_{k\ell} \leq c^1$. Thus, $B^1(t) \cup \{\{k,\ell\}\}$ has no cycle and $\{k,\ell\} \notin t$, which contradicts the construction of t by the Kruskal algorithm. Then, $P(B^1(t)) = P(\{\{i,j\} : \ c_{ij} \leq c^1\})$.

Suppose now that for all $q < q_0$,

$$P\left(\cup_{r=0}^{q} B^r(t)\right) = P(\{\{k,\ell\} : \ c_{k\ell} \leq c^q\}).$$

Using arguments similar to those used in the case $q = 1$, we can prove that

$$P(\cup_{r=0}^{q_0} B^r(t)) = P(\{\{i,j\} : \ c_{ij} \leq c^{q_0}\}).$$

Since $t = \cup_{r=1}^{m(t)} B^r(t)$ and $c_{ij} = c^r$ for all $\{i,j\} \in B^r(t)$ and all $r = 0, \ldots, m(t)$,

$$\begin{aligned}
f_i^o(N,M,C) &= \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \left[o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) \right] \\
&= \sum_{q=1}^{m(t)} \left(\sum_{p=|\cup_{r=0}^{q-1} B^r(t)|+1}^{|\cup_{r=0}^{q} B^r(t)|} c_{i^p j^p} \left[o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) \right] \right) \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(g^{|\cup_{r=0}^{q-1} B^r(t)|})) - o_i^*(P(g^{|\cup_{r=0}^{q} B^r(t)|})) \right] \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(\cup_{r=0}^{q-1} B^r(t))) - o_i^*(P(\cup_{r=0}^{q} B^r(t))) \right] \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(\{\{i,j\} : c_{ij} \leq c^{q-1}\})) - o_i^*(P(\{\{i,j\} : c_{ij} \leq c^q\})) \right]. \quad (3)
\end{aligned}$$

Thus, f^{o^*} does not depend on the minimal tree t.

To prove that f^{ϱ^*} is well-defined, it is enough to show that at each step p of the Kruskal algorithm and for each $i \in N$,

$$\varrho_i^*(P(g^{p-1}), P(g^p)) = o_i^*(P(g^{p-1})) - o_i^*(P(g^p)).$$

Assume without loss of generality that $g^p = g^{p-1} \cup \{k, \ell\}$, $P(g^{p-1}) = \{S_1, \ldots, S_r\}$, $k \in S_1$, $\ell \in S_2$, and $P(g^p) = \{S'_2, \ldots, S'_r\}$ where $S'_2 = S_1 \cup S_2$ and $S'_j = S_j$ for each $j = 3, \ldots, r$. We consider four cases:

1. $S_1 \cup S_2 \subset N$:

   (a) $i \notin S'_2$. Since $S'_i = S_i$, it is trivial to see that

   $$o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) = 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)).$$

   (b) $i \in S'_2$. Assume that $i \in S_1$ (since the other case is similar, we omit it). Then,

   $$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{1}{|S_1|} - \frac{1}{|S_1 \cup S_2|} = \frac{|S_2|}{|S_1 \cup S_2||S_1|} \\ &= \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

2. $S_1 \cap M \neq \emptyset$ and $S_2 \cap M \neq \emptyset$:

   (a) $i \notin S'_2$ and $S_i \subset N$.

   $$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \frac{1}{|S_i|} \\ &\quad - \frac{|\{S'_j \in P(g^p) : S'_j \cap M \neq \emptyset\}| - 1}{|N|} - \frac{1}{|S'_i|} \\ &= \frac{1}{|N|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

   (b) $i \notin S'_2$ and $S_i \cap M \neq \emptyset$.

   $$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S'_j \in P(g^p) : S'_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &= \frac{1}{|N|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

(c) $i \in S_2'$. Suppose that $i \in S_1$ (since the other case is analogous, we omit it). Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= \frac{1}{|N|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

3. $S_1 \subset N$ and $S_2 \cap M \neq \emptyset$ (since the case $S_1 \cap M \neq \emptyset$ and $S_2 \subset N$ is similar, we omit it):

(a) $i \notin S_2'$ and $S_i \subset N$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{1}{|S_i|} - \frac{1}{|S_i'|} \\ &= 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

(b) $i \notin S_2'$ and $S_i \cap M \neq \emptyset$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

(c) $i \in S_2' \cap S_1$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1}) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \frac{1}{|S_1|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= \frac{1}{|S_1|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

(d) $i \in S_2' \cap S_2$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1}) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \qquad \blacksquare \end{aligned}$$

Proof of Theorem 1. From the proof of Proposition 1, $f^{o^*} = f^{\varrho^*}$. We now prove that $f^{Sh} = f^{CW}$ and $f^{\varrho^*} = f^{CW}$.

We first prove that f^{CW} and f^{Sh} coincide in simple problems. Let (N, M, C) be a simple problem. Let $P = \{S_1, \ldots, S_{|P|}\}$ be the set of C-components. For each $i \in N \cup M$, let $S(P, i)$ be the C-component to which i belongs. Assume that t is a minimal tree. It is easy to prove that all the elements inside a component are connected at zero cost in t, while the components connect to one another through arcs of cost 1. Note that in the irreducible problem (N, M, C^*) we have that $c^*_{ij} = 0$ when $S(P, i) = S(P, j)$ while $c^*_{ij} = 1$ when $S(P, i) \neq S(P, j)$. Thus, the set of C-components and C^*-components coincide. Recall that for each $i \in N$,

$$f_i^{CW}(N, M, C) = \begin{cases} \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} & \text{if } S(P, i) \cap M \neq \emptyset, \\ \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \dfrac{1}{|S(P, i)|} & \text{otherwise.} \end{cases}$$

$$f_i^{Sh}(N, M, C) = Sh_i(N, v_{C^*}) = \frac{1}{|N|!} \sum_{\pi \in \Pi} (v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi))).$$

We consider two cases:

1. $S(P, i) \cap M \neq \emptyset$. For each order $\pi \in \Pi$, if $\pi(i) = 1$, agent i has to pay the cost of connecting its component to all sources. Thus, $v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi)) = |\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1$. If $\pi(i) > 1$, this means that when this agent arrives all the components with sources are already connected. Thus, $v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi)) = 0$. Therefore,

$$\begin{aligned} f_i^{Sh}(N, M, C) &= \frac{1}{|N|!} \sum_{\pi \in \Pi} (v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi))) \\ &= \frac{1}{|N|!} \sum_{\pi \in \Pi : \pi(i) = 1} (|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1) \\ &= \frac{1}{|N|!} (|N| - 1)! \, (|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1) \\ &= \frac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &= f_i^{CW}(N, M, C). \end{aligned}$$

2. $S(P, i) \cap M = \emptyset$. For each order $\pi \in \Pi$, we compute $v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi))$ distinguishing several cases.

    (a) $Pre(i, \pi) \cap S(P, i) \neq \emptyset$. Thus, $v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi)) = 0$.

    (b) $Pre(i, \pi) \cap S(P, i) = \emptyset = Pre(i, \pi)$. Then $\pi(i) = 1$. Thus, $v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi)) = |\{S_j \in P : S_j \cap M \neq \emptyset\}|$.

(c) $Pre(i,\pi) \cap S(P,i) = \emptyset \neq Pre(i,\pi)$. In this case, $\pi(i) > 1$. Thus, $v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = 1$.

Let $\Pi^* = \{\pi \in \Pi : Pre(i,\pi) \cap S(P,i) = \emptyset \text{ and } \pi(i) > 1\}$. Taking into account the computations above, we have that

$$f_i^{Sh}(N,M,C) = \frac{1}{|N|}|\{S_j \in P : S_j \cap M \neq \emptyset\}| + \frac{1}{|N|!}|\Pi^*|.$$

Note that

$$\frac{1}{|N|!}|\Pi^*| = \frac{1}{|N|!}\sum_{k=1}^{|N|-|S(P,i)|} \frac{(|N|-|S(P,i)|)!}{(|N|-|S(P,i)|-k)!}(|N|-k-1)!.$$

We consider $|S(P,i)| = m+1$. Then,

$$\begin{aligned}
\frac{1}{|N|!}|\Pi^*| &= \sum_{k=1}^{|N|-m-1} \frac{(|N|-m-1)!(|N|-k-1)!}{(|N|-m-k-1)!|N|!} \\
&= \frac{(|N|-m-1)!m!}{|N|!}\sum_{k=1}^{|N|-m-1} \binom{|N|-k-1}{m}.
\end{aligned}$$

Since

$$\begin{aligned}
\binom{x+1}{y+1} - \binom{x}{y+1} &= \frac{(x+1)!}{(y+1)!\,(x-y)!} - \frac{x!}{(y+1)!\,(x-y-1)!} \\
&= \frac{[(x+1)-(x-y)]\,x!}{(y+1)!\,(x-y)!} \\
&= \frac{x!}{y!\,(x-y)!} \\
&= \binom{x}{y}
\end{aligned}$$

we have that

$$\begin{aligned}
\sum_{k=1}^{|N|-m-1} \binom{|N|-k-1}{m} &= \sum_{k=1}^{|N|-m-2} \binom{|N|-k-1}{m} + \binom{m}{m} \\
&= \sum_{k=1}^{|N|-m-2} \left[\binom{|N|-k}{m+1} - \binom{|N|-k-1}{m+1}\right] + \binom{m}{m} \\
&= \binom{|N|-1}{m+1} - \binom{m+1}{m+1} + \binom{m}{m} \\
&= \binom{|N|-1}{m+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{|N|!}|\Pi^*| &= \frac{(|N|-m-1)!m!}{|N|!}\begin{pmatrix} |N|-1 \\ m+1 \end{pmatrix} \\
&= \frac{(|N|-m-1)!m!}{|N|!}\frac{(|N|-1)!}{(m+1)!\,(|N|-m-2)!} \\
&= \frac{|N|-m-1}{|N|(m+1)} \\
&= \frac{1}{m+1}-\frac{1}{|N|} \\
&= \frac{1}{|S(P,i)|}-\frac{1}{|N|}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_i^{Sh}(N,M,C) &= \frac{|\{S_j \in P : S_j \cap M \neq \emptyset\}|}{|N|}+\frac{1}{|S(P,i)|}-\frac{1}{|N|} \\
&= \frac{|\{S_j \in P : S_j \cap M \neq \emptyset\}|-1}{|N|}+\frac{1}{|S(P,i)|} \\
&= f_i^{CW}(N,M,C).
\end{aligned}$$

Now we consider a general problem (N,M,C) and $i \in N$. Thus,

$$f_i^{CW}(N,M,C)=\sum_{q=1}^{m(C)} x^q f_i^{CW}(N,M,C^q)=\sum_{q=1}^{m(C)} x^q Sh_i(N,v_{(C^q)^*}).$$

Since the Shapley value satisfies additivity on v,

$$\sum_{q=1}^{m(C)} x^q Sh_i(N,v_{(C^q)^*}) = Sh_i\left(N, v_{\sum_{q=1}^{m(C)} x^q (C^q)^*}\right).$$

It only remains to prove that $C^* = \sum_{q=1}^{m(C)} x^q (C^q)^*$. Let t be a minimal tree and g_{ij} the unique path in t from i to j. We know that $c_{ij}^* = \max_{\{k,\ell\}\in g_{ij}} \{c_{k\ell}\} = c_{i'j'}$. By Lemma 1, we know that the order of the arcs according to its cost is preserved in each C^q. So t is also a minimal tree for each simple problem C^q. Thus, $c_{ij}^{q*} = \max_{\{k,\ell\}\in g_{ij}} \{c_{k\ell}^q\} = c_{i'j'}^q$ and hence

$$c_{ij}^* = c_{i'j'} = \sum_{q=1}^{m(C)} x^q c_{i'j'}^q = \sum_{q=1}^{m(C)} x^q c_{ij}^{q*}.$$

We now prove that f^{o^*} coincides with f^{CW}. Let (N,M,C) be a problem and t, $m(t)$, and c^k $(k=1,...,m(t))$ be as in the proof of Proposition 1 when we proved that

f^{o^*} does not depend on the minimal tree chosen by the Kruskal algorithm. By Lemma 1, $C = \sum_{q=1}^{m(C)} x^q C^q$. Besides, by Norde et al. (2004), we have that $c^1 = \min\{c_{ij} : c_{ij} > 0\}$ and

$$c_{ij}^1 = \begin{cases} 0 & \text{when } c_{ij} < c^1, \\ 1 & \text{when } c_{ij} \geq c^1. \end{cases}$$

In general, for each $q = 2, \ldots, m(C)$,

$$c^q = \min\{c_{ij} : c_{ij} > c^{q-1}\},$$

$$c_{ij}^q = \begin{cases} 0 & \text{when } c_{ij} < c^q, \\ 1 & \text{when } c_{ij} \geq c^q, \end{cases}$$

and

$$x^q = \begin{cases} c^1 & \text{when } q = 1, \\ c^q - c^{q-1} & \text{when } q > 1. \end{cases}$$

For each $q = 1, ..., m(C)$, the set of C^q-components coincides with $P(\{\{i,j\} : c_{ij} \leq c^{q-1}\})$. Obviously, $m(t) \leq m(C)$ and t is a minimal tree in C^q for each $q = 1, ..., m(C)$. Besides, for each $q > m(t)$ and each $\{i,j\} \in t$, $c_{ij}^q = 0$. By definition of f^{o^*}, for each $i \in N$ and each $q = m(t)+1, ..., m(C)$, $f_i^{CW}(N, M, C^q) = 0$. Then,

$$f^{CW}(N, M, C) = \sum_{q=1}^{m(C)} x^q f^{CW}(N, M, C^q) = \sum_{q=1}^{m(t)} x^q f^{CW}(N, M, C^q).$$

By definition of o^* and f^{CW}, for each $i \in N$ and each $q = 1, ..., m(t)$,

$$f_i^{CW}(N, M, C^q) = o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{q-1}\})),$$

where we denote $c^0 = 0$.

Therefore,

$$\begin{aligned} f_i^{CW}(N, M, C) &= \sum_{q=1}^{m(t)} x^q f_i^{CW}(N, M, C^q) \\ &= \sum_{q=1}^{m(t)} x^q o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{q-1}\})) \\ &= c^1 o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^0\})) + \sum_{q=2}^{m(t)} (c^q - c^{q-1}) o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{q-1}\})) \\ &= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{q-1}\})) - o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^q\})) \right] \\ &\quad + c^{m(t)} o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{m(t)}\})). \end{aligned}$$

Since $P(\{\{i,j\} : c_{ij} \leq c^{m(t)}\}) = \{N \cup M\}$, for each $i \in N$, $o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{m(t)}\})) = 0$. Therefore,

$$f_i^{CW}(N,M,C) = \sum_{q=1}^{m(C)} c^q \left[o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^{q-1}\})) - o_i^*(P(\{\{i,j\} : \ c_{ij} \leq c^q\}))\right].$$

By (3), we deduce that $f_i^{CW}(N,M,C) = f_i^{o^*}(N,M,C)$. ■

Proof of Proposition 2.

(1) *The folk rule satisfies IIT*: By Theorem 1 the folk rule can be defined as the Shapley value of the irreducible game. Thus, the folk rule satisfies IIT.

(2) *The folk rule satisfies CM*: Let (N,M,C) and (N,M,C') be such that $C \leq C'$. We will prove that $f^{o^*}(N,M,C) \leq f^{o^*}(N,M,C')$. It is enough to prove it when there exists $a,b \in N \cup M$ such that $c_{ab} < c'_{ab}$ and $c_{ij} = c'_{ij}$ when $\{i,j\} \neq \{a,b\}$.

Suppose that there is a minimal tree t in (N,M,C) such that $\{a,b\} \notin t$. This means that t is also a minimal tree in the problem (N,M,C') with exactly the same costs. Since the folk rule satisfies IIT, $f^{o^*}(N,M,C) = f^{o^*}(N,M,C')$. Now suppose that $\{a,b\} \in t$ for each minimal tree t in (N,M,C). Let T be the set of trees in (N,M,C) that do not contain the arc $\{a,b\}$ and $x = \min_{t \in T} c(N,M,C,t) - m(N,M,C)$.

We distinguish several cases:

Case 1. $c'_{ab} - c_{ab} \leq x$. Given a minimal tree t in (N,M,C), we have that t is also a minimal tree in (N,M,C'). Consider the set

$$A = \{\{i,j\} \in t : c_{ab} < c_{ij} < c'_{ab}\}.$$

We have two subcases:

Subcase 1.a. $A = \emptyset$. We can apply the Kruskal algorithm to problems (N,M,C) and (N,M,C') in such a way that we select the arcs of t in the same order. Therefore, for each $i \in N$,

$$f_i^{o^*}(N,M,C') - f_i^{o^*}(N,M,C) = (c'_{ab} - c_{ab})\left(o_i^*(P) - o_i^*(P^{ab})\right)$$

where P is the partition in connected components before arc $\{a,b\}$ is selected by the Kruskal algorithm and P^{ab} is the partition obtained after arc $\{a,b\}$ is selected. Note that $P^{ab} = P \setminus \{S(P,a), S(P,b)\} \cup (S(P,a) \cup S(P,b))$. Let $i \in N$.

Subcase 1.a.i. $S(P,a) \cap M \neq \emptyset$ and $S(P,b) \cap M \neq \emptyset$. Then,

$$\begin{aligned}
& (c'_{ab} - c_{ab})(o_i^*(P) - o_i^*(P^{ab})) \\
= \ & (c'_{ab} - c_{ab}) \left(\frac{|S_k \in P : S_k \cap M \neq \emptyset| - 1}{|N|} - \frac{|S_k \in P : S_k \cap M \neq \emptyset| - 2}{|N|} \right) \\
= \ & \frac{c'_{ab} - c_{ab}}{|N|} \\
\geq \ & 0.
\end{aligned}$$

Subcase 1.a.ii. $S(P,a)\cap M\neq\emptyset$ and $S(P,b)\cap M=\emptyset$. Since the case $S(P,a)\cap M=\emptyset$ and $S(P,b)\cap M\neq\emptyset$ is similar, we omit it.
(1) If $i\notin S(P,a)\cup S(P,b)$, then

$$\left(c'_{ab}-c_{ab}\right)\left(o_i^*(P)-o_i^*(P^{ab})\right)=0.$$

(2) If $i\in S(P,a)$, then

$$\left(c'_{ab}-c_{ab}\right)\left(o_i^*(P)-o_i^*(P^{ab})\right)=0.$$

(3) If $i\in S(P,b)$, then

$$\left(c'_{ab}-c_{ab}\right)\left(o_i^*(P)-o_i^*(P^{ab})\right)=\frac{\left(c'_{ab}-c_{ab}\right)}{|S(P,b)|}\geq 0.$$

Subcase 1.a.iii. $S(P,a)\cap M=\emptyset$ and $S(P,b)\cap M=\emptyset$.
(1) If $i\notin S(P,a)\cup S(P,b)$, then

$$\left(c'_{ab}-c_{ab}\right)\left(o_i^*(P)-o_i^*(P^{ab})\right)=0.$$

(2) If $i\in S(P,a)$ (since the case $i\in S(P,b)$ is similar, we omit it), then

$$\left(c'_{ab}-c_{ab}\right)\left(o_i^*(P)-o_i^*(P^{ab})\right)=\left(c'_{ab}-c_{ab}\right)\left(\frac{1}{|S(P,a)|}-\frac{1}{|S(P,a)\cup S(P,b)|}\right)\geq 0.$$

Subcase 1.b. $A\neq\emptyset$. When we apply the Kruskal algorithm to problems (N,M,C) and (N,M,C'), the arc $\{a,b\}$ is selected later in (N,M,C'). Let

$$c^0=c_{ab}\text{ and }c^0_{ij}=\begin{cases}c^0 & \text{if }\{i,j\}=\{a,b\},\\ c_{ij} & \text{otherwise.}\end{cases}$$

For each $k\geq 1$, let

$$\begin{aligned}c^k &= \min\{c_{ij}:\{i,j\}\in A, c_{ij}>c^{k-1}\}\text{ and}\\ c^k_{ij} &= \begin{cases}c^k & \text{if }\{i,j\}=\{a,b\},\\ c_{ij} & \text{otherwise.}\end{cases}\end{aligned}$$

We apply this procedure until we find r such that $c^r_{ab}=\max\{c_{ij}:\{i,j\}\in A\}$. By setting $C^{r+1}=C'$, we have a sequence of problems $\{(N,M,C^k)\}_{k\in\{0,\ldots,r+1\}}$ such that $C^0=C$ and $C^{r+1}=C'$. Note that t is a minimal tree in each of those problems. Besides, for each pair of problems (N,M,C^k) and (N,M,C^{k+1}) we can select the arcs of t in the same order following the Kruskal algorithm.

Thus, using arguments similar to those used in subcase 1.a, for each $k=0,...,r$ and each $i\in N$,

$$f_i^{o^*}(N,M,C^{r+1-k})-f_i^{o^*}(N,M,C^{r-k})\geq 0.$$

Then, for each $i \in N$,

$$f_i^{o^*}(N,M,C') - f_i^{o^*}(N,M,C) = \sum_{k=0}^{r} \left[f_i^{o^*}(N,M,C^{r+1-k}) - f_i^{o^*}(N,M,C^{r-k}) \right] \geq 0.$$

Case 2. $c'_{ab} - c_{ab} > x$. Let the problem (N, M, C'') be such that $c''_{ab} = c_{ab} + x$ and $c''_{ij} = c_{ij}$ otherwise. Let t' be a minimal tree in (N, M, C'). Obviously $\{a, b\} \notin t'$ and t' is also a minimal tree in (N, M, C''). Since the folk rule f^{o^*} satisfies IIT, for each $i \in N$,

$$f_i^{o^*}(N,M,C') - f_i^{o^*}(N,M,C) = f_i^{o^*}(N,M,C'') - f_i^{o^*}(N,M,C).$$

Since (N, M, C'') satisfies the condition of Case 1, for each $i \in N$,

$$f_i^{o^*}(N,M,C'') - f_i^{o^*}(N,M,C) \geq 0.$$

(3) *The folk rule satisfies CA*: By Theorem 1 the folk rule can be defined as f^{CW}, the cone-wise decomposition. Thus, it is obvious that it satisfies CA.

(4) *The folk rule satisfies PM*: It is enough to show that for each $k \in N$ and each $i \in N \backslash \{k\}$, $f_i^{o^*}(N,M,C) \leq f_i^{o^*}(N\backslash\{k\},M,C)$. Without loss of generality, let $k = |N| = n$.

First, we claim that if $c_{ns} = \alpha$ for each $s \in M$, $c_{ni} = \beta$ for each $i \in N\backslash\{n\}$, and $\beta > \alpha > \max_{i,j \in N \cup M \backslash \{n\}} \{c_{ij}\}$, then for each $i \in N\backslash\{n\}$, $f_i^{o^*}(N,M,C) \leq f_i^{o^*}(N\backslash\{n\},M,C)$.

Let $t = \{\{i^p(N,M,C), j^p(N,M,C)\}\}_{p=1}^{|N|+|M|-1}$ be a minimal tree chosen by the Kruskal algorithm. Then, (i) $\{i^{|N|+|M|-1}(N,M,C), j^{|N|+|M|-1}(N,M,C)\} = \{n, s\}$ for some $s \in M$, (ii) $\{n, s\}$ is the only arc that agent n is linked in the tree t, and (iii) $N\backslash\{n\}$ and M are already connected under $g^{|N|+|M|-2}(N,M,C)$. Also, the subtree $\{\{i^p(N,M,C), j^p(N,M,C)\}\}_{p=1}^{|N|+|M|-2}$ is a minimal tree in $(N\backslash\{n\}, M, C)$ and for each $p = 1, \ldots, |N| + |M| - 2$, $\{i^p(N,M,C), j^p(N,M,C)\} = \{i^p(N \setminus \{n\}, M, C), j^p(N \setminus \{n\}, M, C)\}$. Then, for each $i \in N\backslash\{n\}$,

$$\begin{aligned} f_i^{o^*}(N,M,C) &= \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \left[o_i^* \left(P\left(g^{p-1}(N,M,C) \right) \right) - o_i^* \left(P\left(g^{p}(N,M,C) \right) \right) \right] \\ &= \sum_{p=1}^{|N|+|M|-2} c_{i^p j^p} \left[o_i^* \left(P\left(g^{p-1}(N,M,C) \right) \right) - o_i^* \left(P\left(g^{p}(N,M,C) \right) \right) \right] \end{aligned}$$

where the last equality comes from the fact that for each $i \in N\backslash\{n\}$,

$$o_i^* \left(P\left(g^{|N|+|M|-2}(N,M,C) \right) \right) = o_i^* \left(P\left(g^{|N|+|M|-1}(N,M,C) \right) \right) = 0.$$

Note that for each $p = 1, \ldots, |N|+|M|-2$, $P\left(g^p(N,M,C)\right)\backslash\{n\} = P\left(g^p(N \setminus \{n\}, M, C)\right)$, for each $i \in N \setminus \{n\}$, $S\left(P\left(g^p(N,M,C)\right), i\right) = S\left(P\left(g^p(N \setminus \{n\}, M, C)\right), i\right)$, and $\{S_j \in P\left(g^p(N,M,C)\right) : S_j \cap M \neq \emptyset\} = \{S_j \in P\left(g^p(N \setminus \{n\}, M, C)\right) : S_j \cap M \neq \emptyset\}$.

Let $i \in N \backslash \{n\}$. For each $p = 1, \ldots, |N| + |M| - 2$, let

$$q^p = |\{S_j \in P\left(g^p(N, M, C)\right) : S_j \cap M \neq \emptyset\}| = |\{S_j \in P\left(g^p(N \setminus \{n\}, M, C)\right) : S_j \cap M \neq \emptyset\}|$$

and

$$s^p = |S(P(g^p(N, M, C)), i)| = |S(P(g^p(N \setminus \{n\}, M, C)), i)|.$$

We consider several cases:

Case 1. $S\left(P\left(g^{p-1}(N, M, C)\right), i\right) \cap M \neq \emptyset$. Then, $S\left(P\left(g^p(N, M, C)\right), i\right) \cap M \neq \emptyset$. Now

$$\begin{aligned} & o_i^*\left(P\left(g^{p-1}(N, M, C)\right)\right) - o_i^*\left(P\left(g^p(N, M, C)\right)\right) \\ = \; & \frac{q^{p-1}}{|N|} - \frac{q^p}{|N|} \leq \frac{q^{p-1}}{|N \setminus \{n\}|} - \frac{q^p}{|N \setminus \{n\}|} \\ = \; & o_i^*\left(P\left(g^{p-1}(N \setminus \{n\}, M, C)\right)\right) - o_i^*\left(P\left(g^p(N \setminus \{n\}, M, C)\right)\right). \end{aligned}$$

where the last inequality comes from the fact that $q^{p-1} \geq q^p$.

Case 2. $S\left(P\left(g^{p-1}(N, M, C)\right), i\right) \cap M = \emptyset$ and $S\left(P\left(g^p(N, M, C)\right), i\right) \cap M \neq \emptyset$. Now,

$$\begin{aligned} & o_i^*\left(P\left(g^{p-1}(N, M, C)\right)\right) - o_i^*\left(P\left(g^p(N, M, C)\right)\right) \\ = \; & \frac{q^{p-1}}{|N|} + \frac{1}{s^{p-1}} - \frac{q^p}{|N|} \leq \frac{q^{p-1}}{|N \setminus \{n\}|} + \frac{1}{s^{p-1}} - \frac{q^p}{|N \setminus \{n\}|} \\ = \; & o_i^*\left(P\left(g^{p-1}(N \setminus \{n\}, M, C)\right)\right) - o_i^*\left(P\left(g^p(N \setminus \{n\}, M, C)\right)\right). \end{aligned}$$

Case 3. $S\left(P\left(g^{p-1}(N, M, C)\right), i\right) \cap M = \emptyset$ and $S\left(P\left(g^p(N, M, C)\right), i\right) \cap M = \emptyset$. Now,

$$\begin{aligned} & o_i^*\left(P\left(g^{p-1}(N, M, C)\right)\right) - o_i^*\left(P\left(g^p(N, M, C)\right)\right) \\ = \; & \frac{q^{p-1}}{|N|} + \frac{1}{s^{p-1}} - \frac{q^p}{|N|} - \frac{1}{s^p} \leq \frac{q^{p-1}}{|N \setminus \{n\}|} + \frac{1}{s^{p-1}} - \frac{q^p}{|N \setminus \{n\}|} - \frac{1}{s^p} \\ = \; & o_i^*\left(P\left(g^{p-1}(N \setminus \{n\}, M, C)\right)\right) - o_i^*\left(P\left(g^p(N \setminus \{n\}, M, C)\right)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} f_i^{o^*}(N, M, C) &= \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p}\left[o_i^*\left(P\left(g^{p-1}(N, M, C)\right)\right) - o_i^*\left(P\left(g^p(N, M, C)\right)\right)\right] \\ &\leq \sum_{p=1}^{|N|+|M|-2} c_{i^p j^p}\left[o_i^*\left(P\left(g^{p-1}(N \setminus \{n\}, M, C)\right)\right) - o_i^*\left(P\left(g^p(N \setminus \{n\}, M, C)\right)\right)\right] \\ &= f_i^{o^*}(N \setminus \{n\}, M, C), \qquad (4) \end{aligned}$$

as desired.

Let $\alpha = \max_{i,j \in N \cup M} \{c_{ij}\} + 1$ and $\beta = \alpha + 1$. Let $C^0 \in \mathcal{C}^{N \cup M}$ be such that $c_{ns}^0 = \alpha$ for each $s \in M$ and $c_{ij}^0 = c_{ij}$ otherwise. For each $r = 1, ..., |N| - 1$, let $C^r \in \mathcal{C}^{N \cup M}$ be such that $c^r_{nr} = \beta$ and for each $\{i,j\} \neq \{n,r\}$, $c^r_{ij} = c^{r-1}_{ij}$. Let $i \in N \setminus \{n\}$. Since f^{o^*} satisfies CM,

$$f_i^{o^*}(N,M,C) \leq f_i^{o^*}(N,M,C^0) \leq f_i^{o^*}(N,M,C^1) \leq \cdots \leq f_i^{o^*}(N,M,C^{|N|-1}).$$

Applying (4) to $C^{|N|-1}$,

$$f_i^{o^*}(N,M,C^{|N|-1}) \leq f_i^{o^*}(N\backslash\{n\},M,C^{|N|-1}).$$

Since $C^{|N|-1} = C$, we conclude that f^{o^*} satisfies PM.

(5) *The folk rule satisfies CS and SEP*: Since PM implies CS and SEP, the result holds.

(6) *The folk rule satisfies SYM*: By Theorem 1 the folk rule can be obtained as the Shapley value of the game associated with the irreducible problem. It is trivial to prove that if two agents are symmetric in the problem (N,M,C), then they will also be symmetric in the irreducible problem (N,M,C^*) and hence, in the game associated with the irreducible problem. Since the Shapley value satisfies SYM, the folk rule also does.

(7) *The folk rule satisfies ETSC*: Let (N,M,C) and (N,M,C') be two problems satisfying the conditions in the statement of ETSC. Suppose that there is a minimal tree in (N,M,C) such that $\{a,b\} \notin t$. Thus, t is also a minimal tree in (N,M,C') with the same costs. Since the folk rule satisfies IIT, we have that $f^{o^*}(N,M,C) = f^{o^*}(N,M,C')$. Assume that $\{a,b\} \in t$ for each minimal tree t in (N,M,C). Let T be the set of all trees in (N,M,C) that do not contain $\{a,b\}$. Let $x = \min_{t\in T} c(N,M,C,t) - m(N,M,C)$. We consider several cases.

Case 1. $c'_{ab} - c_{ab} \leq x$. Note that a minimal tree t in (N,M,C) is also a minimal tree in (N,M,C'). Now consider the set $A = \{\{i,j\} \in t : c_{ab} < c_{ij} < c'_{ab}\}$. The proof is divided into two subcases:

Subcase 1.a. $A = \emptyset$. We can apply the Kruskal algorithm to (N,M,C) and (N,M,C') in such a way that the arcs of t are selected in the same order. Then, for each $i \in N$,

$$f_i^{o^*}(N,M,C') - f_i^{o^*}(N,M,C) = (c'_{ab} - c_{ab})(o_i^*(P) - o_i^*(P^{ab}))$$

where P is the partition in connected components before arc $\{a,b\}$ is selected by the Kruskal algorithm and P^{ab} is the partition obtained after arc $\{a,b\}$ is selected. Note that $P^{ab} = P\backslash\{S(P,a),S(P,b)\} \cup \{S(P,a) \cup S(P,b)\}$. By the definition of o^*, for each $i \in N$,

$$\begin{aligned}
&(c'_{ab} - c_{ab})(o_i^*(P) - o_i^*(P^{ab})) \\
&= (c'_{ab} - c_{ab})\Big(\frac{|\{S_k \in P : S_k \cap M \neq \emptyset\}| - 1}{|N|} - \frac{|\{S_k \in P : S_k \cap M \neq \emptyset\}| - 2}{|N|}\Big) \\
&= \frac{c'_{ab} - c_{ab}}{|N|}.
\end{aligned}$$

Subcase 1.b. $A \neq \emptyset$. When we apply the Kruskal algorithm to (N, M, C) and (N, M, C'), the arc $\{a, b\}$ is selected later in (N, M, C') than in (N, M, C). Let

$$c^0 = c_{ab} \; and c^0_{ij} = \begin{cases} c^0 & \text{if } \{i,j\} = \{a,b\}, \\ c_{ij} & \text{otherwise.} \end{cases}$$

For each $r \geq 1$, let $c^r = \min\{c_{ij} : \{i, j\} \in A, c_{ij} > c^{r-1}\}$ and

$$c^r_{ij} = \begin{cases} c^r & \text{if } \{i,j\} = \{a,b\}, \\ c_{ij} & \text{otherwise.} \end{cases}$$

We apply this procedure until we find $\bar{r}$ such that $c^{\bar{r}}_{ab} = \max\{c_{ij} : \{i, j\} \in A\}$. By setting $C^{\bar{r}+1} = C'$, we have a sequence of problems $\{(N, M, C^r)\}_{r \in \{0,\ldots,\bar{r}+1\}}$ such that $C^0 = C$ and $C^{\bar{r}+1} = C'$. Note that t is a minimal tree in each of those problems. In addition, for each pair of problems (N, M, C^r) and (N, M, C^{r+1}), $r \in \{0, ..., \bar{r}\}$, we can select the arcs of t in the same order by following the Kruskal algorithm. Therefore, by using arguments similar to Subcase 1.a,

$$\begin{aligned} f^{o^*}_i(N, M, C') - f^{o^*}_i(N, M, C) &= \sum_{r=0}^{\bar{r}} \left[f^{o^*}_i(N, M, C^{\bar{r}+1-r}) - f^{o^*}_i(N, M, C^{\bar{r}-r}) \right] \\ &= \sum_{r=0}^{\bar{r}} \frac{c^{\bar{r}+1-r} - c^{\bar{r}-r}}{|N|} = \frac{c^{\bar{r}+1} - c^0}{|N|} = \frac{c'_{ab} - c_{ab}}{|N|}. \end{aligned}$$

Case 2. $c'_{ab} - c_{ab} > x$. Let (N, M, C'') be such that $c''_{ab} = c_{ab} + x$ and $c''_{ij} = c_{ij}$ otherwise. Let t' be a minimal tree in (N, M, C'). Obviously, $\{a, b\} \notin t'$ and t' is also a minimal tree in (N, M, C''). Since f^{o^*} satisfies IIT, for each $i \in N$,

$$f^{o^*}_i(N, M, C') - f^{o^*}_i(N, M, C) = f^{o^*}_i(N, M, C'') - f^{o^*}_i(N, M, C).$$

Since (N, M, C'') satisfies the condition of Case 1, for each $i \in N$,

$$f^{o^*}_i(N, M, C'') - f^{o^*}_i(N, M, C) = \frac{c'_{ab} - c_{ab}}{|N|} = \frac{x}{|N|},$$

as desired. ■

Proof of Theorem 2.

(a) By Proposition 2, the folk rule satisfies the five axioms. Conversely, let f be a rule satisfying the five axioms. For each partition $P = \{S_1, S_2, ..., S_{|P|}\} \in P(N \cup M)$, we define the function $o(P) = f(N, M, C^P)$ where $c^P_{ij} = 0$ if $i, j \in S_k$ for some $k \in \{1, ..., |P|\}$ and $c^P_{ij} = 1$ otherwise. Note that

$$\sum_{i \in N} o_i(P) = \sum_{i \in N} f_i(N, M, C^P) = m(N, M, C^P) = |P| - 1.$$

We claim that $f = f^o$ where for each (N, M, C) and each $i \in N$,

$$f_i^o(N, M, C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \Big[o_i\big(P(g^{p-1})\big) - o_i\big(P(g^p)\big) \Big].$$

Since f and f^o satisfy CA, by Lemma 1, $f(N, M, C) = \sum_{q=1}^{m(C)} f(N, M, x^q C^q)$ and $f^o(N, M, C) = \sum_{q=1}^{m(C)} f^o(N, M, x^q C^q)$. Therefore, it is enough to prove that f coincides with f^o in problems (N, M, C) where there exists a network g such that $c_{ij} = x$ if $\{i, j\} \in g$ and $c_{ij} = 0$ otherwise. Let $P(g) = \{S_1, ..., S_r\}$ be the partition induced by g over $N \cup M$.

When we use the Kruskal algorithm in this problem, we first connect the nodes within the same component with zero cost until step $(|N| + |M| - r)$. Then, we connect the nodes from different components with the constant cost x. Thus, for each $i \in N$,

$$\begin{aligned} f_i^o(N, M, C) &= \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \Big[o_i\Big(P(g^{p-1})\Big) - o_i\Big(P(g^p)\Big) \Big] \\ &= \sum_{p=|N|+|M|-r+1}^{|N|+|M|-1} x \Big[o_i\Big(P(g^{p-1})\Big) - o_i\Big(P(g^p)\Big) \Big] \\ &= x \Big[o_i\Big(P(g^{|N|+|M|-r+1})\Big) - o_i\Big(P(g^{|N|+|M|-1})\Big) \Big] \\ &= x \Big[f_i(N, M, C^{P(g^{|N|+|M|-r+1})}) - f_i(N, M, C^{P(g^{|N|+|M|-1})}) \Big]. \end{aligned}$$

Note that $P(g^{|N|+|M|-r+1}) = P(g)$ and $P(g^{|N|+|M|-1}) = N \cup M$.

Since $c_{ij}^{N \cup M} = 0$ for each i, $j \in N \cup M$ and f satisfies CA, for each $i \in N$, $f_i(N, M, C^{N \cup M}) = 0$, which implies that $f_i^o(N, M, C) = x f_i(N, M, C^{P(g)})$. Now, consider C' such that $c'_{ij} = \frac{1}{x} c_{ij}$ for each i, $j \in N \cup M$. Note that $C'^* = C^{P(g)}$. By IIT, $f(N, M, C^{P(g)}) = f(N, M, C')$. By CA, $f^o(N, M, C) = x f(N, M, C')$.

Using similar arguments as in Bergantiños et al. (2010, p.708), we can prove that $x f(N, M, C') = f(N, M, xC')$, which implies that $f^o(N, M, C) = f(N, M, xC')$. Since $xC' = C$, we conclude that $f = f^o$, as desired.

It remains to prove that $o = o^*$. Now, let $P = \{S_1, ..., S_q, ..., S_{|P|}\}$ be a partition such that $S_k \cap M \neq \emptyset$ when $k \leq q$ and $S_k \subset N$ when $k > q$. Note that $|\{S_k \in P : S_k \cap M \neq \emptyset\}| = q$. We introduce a sequence of problems $\{(N, M, C^r)\}_{r=1,2,...,q}$ where $C^1 = C^P$ and for each $r > 1$, C^r is obtained from C^{r-1} such that $c^r_{a^{r-1} a^r} = 0$ if $a^{r-1} \in S_{r-1} \cap M$ and $a^r \in S_r \cap M$, and $c^r_{ij} = c^{r-1}_{ij}$ otherwise. By ETSC, for each $r = 2, \ldots, q$ and each i, $j \in N$,

$$f_i(N, M, C^{r-1}) - f_i(N, M, C^r) = f_j(N, M, C^{r-1}) - f_j(N, M, C^r).$$

Since

$$\sum_{i\in N}\big[f_i(N,M,C^{r-1})-f_i(N,M,C^r)\big]=\sum_{i\in N}f_i(N,M,C^{r-1})-\sum_{i\in N}f_i(N,M,C^r)$$
$$=m(N,M,C^{r-1})-m(N,M,C^r)$$
$$=1,$$

for each $i\in N$,

$$f_i(N,M,C^{r-1})-f_i(N,M,C^r)=\frac{1}{|N|}.$$

Therefore, for each $i\in N$,

$$f_i(N,M,C^1)-f_i(N,M,C^q)=\sum_{r=2}^{q}\big[f_i(N,M,C^{r-1})-f_i(N,M,C^r)\big]$$
$$=\frac{q-1}{|N|}.$$

Thus,

$$o(P)=f_i(N,M,C^1)=\frac{q-1}{|N|}+f_i(N,M,C^q). \tag{5}$$

By CS, for each $k=q+1,\dots,|P|$, $\sum\limits_{i\in S_k\cap N}f_i(N,M,C^q)\leq m(S_k\cap N,M,C^q)=1$ and for each $i\in(\cup_{k=1}^q S_k)\cap N$, $f_i(N,M,C^q)\leq m(\{i\},M,C^q)=0$. Since $\sum\limits_{i\in N}f_i(N,M,C^q)=m(N,M,C^q)=|P|-q$, for each $k=q+1,\dots,|P|$, $\sum\limits_{i\in S_k\cap N}f_i(N,M,C^q)=1$ and for each $i\in(\cup_{k=1}^q S_k)\cap N$, $f_i(N,M,C^q)=0$. By (1) and (5), for each $i\in(\cup_{k=1}^q S_k)\cap N$,

$$o_i(P)=\frac{q-1}{|N|}=o_i^*(P)$$

For each $k=q+1,...,|P|$ and each i, $j\in S_k$, i and j are symmetric, so that by SYM, for each $i\in S_k$ $(k>q)$, $f_i(N,M,C^q)=\frac{1}{|S_k|}$. From (1) and (5), we have that

$$o_i(P)=\frac{q-1}{|N|}+\frac{1}{|S_k|}=o_i^*(P).$$

(b) By Proposition 2, the folk rule satisfies the five axioms. Conversely, let f be a rule satisfying the five axioms. From the same argument as in (a), we obtain (5). Note that $m(N,M,C^q)=|P|-q$, for each $k=q+1,\dots,|P|$, $m(S_k,M,C^q)=1$, and for each $i\in(\cup_{k=1}^q S_k)\cap N$, $m(\{i\},M,C^q)=0$, which together imply that $m(N,M,C^q)=\sum\limits_{k=1}^{q}\big(\sum\limits_{i\in S_k\cap N}m(\{i\},$

$M, C^q)\big) + \sum_{k=q+1}^{|P|} m(S_k, M, C^q)$. By SEP, for each $k = q+1, \ldots, |P|$, $f_i(N, M, C^q) = f_i(S_k, M, C^q)$, which implies that $\sum_{i \in S_k \cap N} f_i(N, M, C^q) = \sum_{i \in S_k \cap N} f_i(S_k, M, C^q) = m(S_k, M, C^q) = 1$ and for each $i \in (\cup_{k=1}^{q} S_k) \cap N$, $f_i(N, M, C^q) = f_i(\{i\}, M, C^q) = m(\{i\}, M, C^q) = 0$. Once again, by using the same argument as in the proof of (a), we conclude that f coincides with the folk rule. ■

Next, we show that all axioms are independent in Theorem 2.

(1) Dropping *Independence of irrelevant trees*: Let f^w be a rule defined for simple problems. For each simple problem (N, M, C), we consider $g = \{\{i, j\} \subset N \cup M : c_{ij} = 0\}$. For each $i \in N$, let

$$w_i = \begin{cases} \frac{1}{|\{\{i,j\}: j \in S(P(g),i) \ and c_{ij}=0\}|} & \text{if } S(P(g), i) \neq \{i\}, \\ 1 & \text{otherwise.} \end{cases}$$

For each $i \in N$, let f^w be

$$f_i^w(N, M, C) = \begin{cases} \frac{|\{S_k \in P(g): S_k \cap M \neq \emptyset\}| - 1}{|N|} & \text{if } S(P(g), i) \cap M \neq \emptyset, \\ \frac{|\{S_k \in P(g): S_k \cap M \neq \emptyset\}| - 1}{|N|} + \frac{w_i}{\sum\limits_{j \in S(P(g),i)} w_j} & \text{otherwise.} \end{cases}$$

This rule is extended to general problems using Lemma 1. The rule f^w satisfies CA, CS, SEP, SYM, and ETSC, but not IIT.

(2) Dropping *Cone-wise additivity*: We first introduce some notion in the classical problem following Bergantiños and Vidal-Puga (2015). For each classical problem (N_0, C) and each $S \subset N$, let

$$\delta_S = \begin{cases} \min\limits_{j \in N_0 \setminus \{i\}} c_{ij} & \text{if } S = \{i\}, \\ \min\limits_{i \in S, j \in N_0 \setminus S} c_{ij} - \max\limits_{\{i,j\} \in \tau(S)} c_{ij} & \text{if } |S| > 1, \end{cases}$$

where $N_0 = N \cup \{0\}$ and $\tau(S)$ is a minimal tree in (S, C_S) connecting all agents in S. Let $Ne(N_0, C)$ be a set of all coalitions $S \subset N$ and $|S| > 1$ such that $\delta_S > 0$. Let $\hat{o} = \{\hat{o}^x\}_{x \in \mathbb{R}_+}$ be a parametric family of functions defined as

$$\hat{o}_i^x(N) = \begin{cases} \frac{1}{|N|} & \text{if } |N| \neq 2, \\ \frac{1}{2} & \text{if } |N| = 2 \text{ and } x > 1, \\ \max\{\frac{1}{3}, \min\{\frac{c_{0i}}{c_{01}+c_{02}}, \frac{2}{3}\}\} & \text{if } |N| = 2 \text{ and } x \leq 1. \end{cases}$$

Let C^* be the irreducible cost matrix of C. For each (C^*, x) and each $i \in N$, let

$$e_i(C^*, x) = \int_0^x \hat{o}_i^t(N) dt.$$

Now, we define the rule f^e such that for each classical problem (N_0, C) and each $i \in N$,

$$f_i^e(N_0, C) = c_{0i}^* - \sum_{S \in Ne(N_0,C), i \in S} (\delta_S - e_i((S, C_S^*), \delta_S)).$$

Next, we extend this rule to our problem. For all (N, M, C), let t be a minimal tree in the irreducible problem (N, M, C^*) where all sources are connected among themselves. Let t_M be the restriction of t to M. We now consider the classical problem $(N_0, \bar{C})$ such that $\bar{t} = \{\{i,j\} \in t : i, j \in N\} \cup \{\{0,i\} : i \in N \text{ and } \{i,j\} \in t \text{ for some } j \in M\}$. It is easy to see that $\bar{t}$ is a tree that connects all agents in N to 0. Let $\bar{c_{ij}} = c_{ij}^*$ if $i, j \in N$ and $\{i,j\} \in \bar{t}$; $\bar{c_{0i}} = \max\{c_{k\ell}^* : \{k,\ell\} \in g_{ii^M}^t\}$, where i^M is the first source in the unique path connecting agent i to each source in t; and $\bar{c_{ij}} = \max\{\bar{c_{k\ell}} : \{k,\ell\} \in g_{ij}^{\bar{t}}\}$ if $i, j \in N$ and $\{i,j\} \notin t$. For each problem (N, M, C), let

$$f^e(N, M, C) = \frac{c(t_M)}{|N|} + f^e(N_0, \bar{C}).$$

The rule f^e satisfies CM (thus, IIT), PM (thus, CS and SEP), SYM and ETSC, but not CA.

(3) Dropping *Core selection* or *Separability*: The egalitarian rule f^E, defined as for each (N, M, C) and each $i \in N$, $f_i^E(N, M, C) = \frac{m(N,M,C)}{|N|}$, satisfies CA, IIT, SYM, and ETSC, but not CS or SEP.

(4) Dropping *Symmetry*: Let $\tilde{o}$ be a function such that for each $P \in P(N \cup M)$ and each $i \neq n$,

$$\tilde{o}_i(P) = \begin{cases} \frac{|\{S \in P: S \cap M \neq \emptyset\}|-1}{|N|} & \text{if } i \in S_k,\ S_k \cap M = \emptyset \text{ and } n \in S_k, \\ \frac{|\{S \in P: S \cap M \neq \emptyset\}|-1}{|N|} + \frac{1}{|S_k|} & \text{if } i \in S_k,\ S_k \cap M = \emptyset \text{ and } n \notin S_k, \\ \frac{|\{S \in P: S \cap M \neq \emptyset\}|-1}{|N|} & \text{if } i \in S_k \text{ and } S_k \cap M \neq \emptyset, \end{cases}$$

and

$$\tilde{o}_n(P) = \begin{cases} \frac{|\{S \in P: S \cap M \neq \emptyset\}|-1}{|N|} + 1 & \text{if } n \in S_k \text{ and } S_k \cap M = \emptyset, \\ \frac{|\{S \in P: S \cap M \neq \emptyset\}|-1}{|N|} & \text{if } n \in S_k \text{ and } S_k \cap M \neq \emptyset. \end{cases}$$

Let $f^{\tilde{o}}$ be a rule such that for each (N, M, C) and each $i \in N$,

$$f_i^{\tilde{o}}(N, M, C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \Big[\tilde{o}_i\big(P(g^{p-1})\big) - \tilde{o}_i\big(P(g^p)\big)\Big].$$

The rule $f^{\tilde{o}}$ satisfies CA, IIT, CS, SEP, and ETSC, but not SYM.

(5) Dropping *Equal treatment of source costs*: Let $P = \{S_1, ..., S_q, ..., S_{|P|}\}$ be a partition in $P(N \cup M)$, where $S_k \cap M \neq \emptyset$ if $k \leq q$ and $S_k \cap M = \emptyset$ if $k > q$. Let t be a number

of agents in an element in P containing no source, i.e., $t = |\{i \in N : i \in S_k \ (S_k \in P)$ and $S_k \cap M = \emptyset\}|$. Let ϵ be an arbitrarily small number such that $\epsilon \in (0, \frac{1}{|N||M|})$. Let o^ϵ be a function such that for each $P \in P(N \cup M)$ and each $i \in N$, if $0 < t < |N|$,

$$o_i^\epsilon(P) = \begin{cases} \frac{1-\epsilon}{|N|}(|\{S \in P : S \cap M \neq \emptyset\}| - 1) + \frac{1}{|S_k|} & \text{if } i \in S_k \text{ and } S_k \cap M = \emptyset, \\ \frac{|N|-t(1-\epsilon)}{|N|(|N|-t)}(|\{S \in P : S \cap M \neq \emptyset\}| - 1) & \text{if } i \in S_k \text{ and } S_k \cap M \neq \emptyset, \end{cases}$$

and if $t = 0$ or $t = |N|$, $o_i^\epsilon(P) = o^*(P)$. Let f^{o^ϵ} be a rule such that for each (N, M, C) and each $i \in N$,

$$f_i^{o^\epsilon}(N, M, C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \Big[o_i^\epsilon\big(P(g^{p-1})\big) - o_i^\epsilon\big(P(g^p)\big) \Big].$$

The rule f^{o^ϵ} satisfies CA, IIT, CS, SEP, and SYM, but not ETSC.

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