Subsistence, saturation and irrelevance in preferences

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Abstract

We present a microeconomic analysis of subsistence consumption in a standard consumer theory setting. We define *subsistence and saturation induced irrelevance* (SSI) preferences for an individual who makes choices over two goods: a basic good and a non-basic good. The basic good has two key features: subsistence and saturation. We axiomatize SSI preferences using two key concepts: (i) 'irrelevance' of a good in a consumption bundle (increasing its amount does not make the consumer better off) and (ii) an 'unhappy set' (any bundle outside such a set is preferred to all bundles inside). SSI preferences more adequately represent the decision problem of the poor and give certain new insights on consumer behavior that are not captured by widely used utility functions such as Stone-Geary. We also axiomatize a generalized version of Leontief (GL) preferences, for which irrelevance is solely driven by complementarity.

Keywords: subsistence; saturation; irrelevance; unhappy sets; generalized Leontief

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"Of the nonpossession of the matter of subsistence in such quantity as is necessary to the support of life, death is the consequence: and such natural death is preceded by a course of suffering much greater than what is attendant on the most afflictive violent deaths employed for the purpose of punishment."
—Jeremy Bentham, *Pannomial Fragments* (1843).

1 Introduction

Subsistence is the minimum amount of basic necessities essential for a person's survival. It can be expressed alternatively in terms of income (e.g., $1.25 per day) or in terms of nutrition such as a certain daily calorie requirement. As extreme poverty and hunger continue to pose a major global challenge, subsistence remains a useful concept for policymakers. Poverty lines are based on estimates of the cost of a consumption basket considered minimal for the survival of a family (World Bank, 2013). Effective policies to end hunger require knowledge of *food deficit*, which is measured by comparing the average dietary energy of undernourished people with the minimum amount of dietary energy needed to maintain body weight and carry out light activity (Food and Agriculture Organization of the United Nations, 2014).

Jeremy Bentham (1843), one of the founding fathers of utility theory, considered "securing the existence of, and sufficiency of, the matter of subsistence for all the members of the community" an important milestone towards achieving "the all embracing end—the greatest happiness of the greatest number of the individuals belonging to the community in question." Yet the treatment of subsistence consumption is far from adequate in a standard utility maximization setting. This was pointed out long back by Stigler (1950),[1] but the lacuna still remains. Stone-Geary utility functions are widely used to model subsistence.[2] Under these functions it is implicitly assumed that utility is negative below the subsistence level, so when an individual cannot purchase the good above that level, there is no choice problem of interest. This essentially sidesteps the problem of the poor, because for a poor person the decision problem precisely occurs in the region below the subsistence, where the individual makes choices to get as close to the subsistence as possible. The Stone-Geary approach is also inadequate to understand the problems of individuals who have only marginally exceeded the subsistence threshold, as a rise in the prices of basic goods such as food can take such individuals below the subsistence.

Seeking to address these inadequacies, this paper presents a microeconomic analysis of subsistence consumption in a standard consumer theory framework. We axiomatize subsistence consumption in a utility maximization setting where an individual makes

[1]To quote: "Occasionally it was stated that the marginal utility of a necessity falls rapidly as its quantity increases and the like; and there were some mystical references to the infinite utility of subsistence. These were *ad hoc* remarks, however, and were not explicitly developed parts of the formal theory."

[2]See, e.g., Rebelo (1992) and Steger (2000), who use Stone-Geary functions to study the role of subsistence in economic growth. Sharif (1986) provides a survey of measurement issues of subsistence. Subsistence consumption has also been associated with Giffen behavior, i.e., upward sloping demand curve (Jensen and Miller, 2008).

choices over two goods: a basic good which is a necessity such as food[3] and a non-basic good.[4] In developing our theory we appeal to two distinct aspects of a basic necessity. First, the individual requires a minimum critical level of this good. This is the subsistence requirement. If this requirement is not met, the non-basic good is not useful. The second aspect is saturation, which is in line with the concept of 'abundance' proposed by Bentham (1843). Saturation implies that once the individual has consumed sufficiently large amounts of the basic good, consuming more of it may not be beneficial.

Subsistence and saturation generate *irrelevance* of one of the goods. A good is irrelevant at a consumption bundle if increasing its amount without changing the amount of the other good keeps the consumer indifferent. The non-basic good is irrelevant in the subsistence zone (i.e., when the subsistence requirement is not met), while the basic good becomes irrelevant when its saturation is reached. Incorporating these features, we define *subsistence and saturation induced irrelevance* (SSI) preferences. For such preferences there are potentially three zones in the commodity space. Apart from the two zones where one of the goods is irrelevant, there can be an intermediate region (where the consumption of the basic good has exceeded the subsistence level but not yet reached saturation) in which none of the goods is irrelevant. In this region the individual has a standard consumer preference where two goods can be imperfect substitutes. SSI preferences thus enrich consumer theory by allowing for the existence of poverty and prosperity in different regions of the commodity space. This formalizes Bentham's concepts of subsistence and abundance in terms of individual preference. Theorem 1 axiomatizes SSI preferences.

SSI preferences are nonhomothetic. Their nonhomotheticity is driven by the irrelevance of the non-basic good in the subsistence zone. Since this is a key aspect of the consumption choice problem of the poor, SSI preferences can help us to have a better understanding of some empirically observed consumption patterns of the poor. For example, the global consumption database of the World Bank (2010), which looks at sector-wise shares of consumption for different income groups in developing countries, indicates the presence of nonhomotheticities.[5] Stone-Geary utility functions are also nonhomothetic, but SSI preferences give certain new insights on consumer behavior that are not captured by Stone-Geary functions. Under SSI preferences, an increase in the price of the basic good may take a consumer from non-subsistence to subsistence zone. Whether a specific instance of poverty is temporary (driven by fluctuations in food prices) or chronic (due to low levels of income) can thus be better understood in terms of SSI preferences. SSI preferences may also give rise to what we call 'subsistence inertia' where it is optimal for a consumer to buy only the basic good even though he is outside the subsistence zone, implying that the absence of the non-basic good in a

[3]The Sanskrit word for bare subsistence *grāsācchādana* makes the components of subsistence particularly clear. It is a compound consisting of two words: *grāsa* (food) and *ācchādana* (clothing).

[4]Jensen and Miller (2010) also consider a two-good setting to study subsistence behavior. However, both goods in their model are basic goods (food items that contribute calories) and there is substitutability between them. Substitutability across different basic goods as an optimization problem was first analyzed by Stigler (1945).

[5]World Bank (2010) `http://datatopics.worldbank.org/consumption/home`

consumption basket does not reveal that a consumer is in his subsistence zone.[6]

The concept of irrelevance that is used to define SSI preferences also applies when the two goods are complements, as in Leontief preferences. If an individual prefers two spoons of sugar with every cup of tea and has one cup of tea, then sugar becomes irrelevant after two spoons. For such preferences, complementarity between the goods implies that at any consumption bundle one of the two goods is saturated and therefore rendered irrelevant. Theorem 2 axiomatizes a generalized version of Leontief (GL) preferences.

Apart from the notion of irrelevance, the other key concept that is central for our axiomatizations is an *unhappy set*. Given a preference relation a set of consumption bundles is said to be an unhappy set if every bundle outside this set is preferred to all bundles inside. This is meant to capture the state of a poor person who has extreme urge to come out of poverty. For SSI preferences, the subsistence zone is the largest unhappy set that has the property that the non-basic good is irrelevant at *every* bundle of the set. But for GL preferences if there is a set of consumption bundles such that a certain good is irrelevant at every bundle of the set, it can never be an unhappy set. Thus roughly speaking, SSI and GL preferences are characterized by the presence or the absence of unhappiness in irrelevance. It is a case of too little versus too much. Irrelevance of the non-basic good in SSI preferences stems from the fact that there is too little of the basic good, but for GL preferences, irrelevance of a good is driven by too much of that good in relation to the other good.

SSI preferences share some properties with lexicographic preferences, but there are important differences.[7] For a consumer who is in the subsistence zone under SSI, the preference ordering over all bundles having different amounts of the basic good follows lexicographic order. But this order breaks down for any two bundles in the subsistence zone with the same amount of the basic good. In contrast to a lexicographic preference, such bundles lie on the same indifferent curve under SSI. This is why unlike a lexicographic preference, SSI is continuous.

A closely related paper is by Basu and Van (1998), who study a model of child labour by using subsistence and lexicographic ordering to define the preference of a household over two goods: a consumption good and a binary choice on whether or not to send the child to work. Under their preference a household sends its child to work only if its consumption without child labor income drops below the subsistence level. In contrast to SSI, this preference is discontinuous and it induces lexicographic order in the non-subsistence zone. Our paper contributes to the literature of consumer behavior by providing a theoretical framework that can be useful to reexamine different issues of consumption choice in the specific context of the decision making problem of the poor.

The paper is organized as follows. We present the analytical framework in Section 2. We axiomatize SSI preferences in Section 3. In Section 4 we present an axiomatization

[6]See Section 2.2 for examples of SSI that illustrate aspects not captured by Stone-Geary functions.

[7]See Fishburn (1975) for an axiomatization of lexicographic preferences. Axiomatizations of other different consumer preferences include Milnor (1974), Maskin (1979), Segal and Sobel (2002). The main difference of our approach from this literature is that our axioms are on the regions of irrelevance embedded in SSI and GL preferences.

of GL preferences. We conclude in Section 5 where we discuss the key insights provided by SSI preferences. Proofs of the results are provided in the appendix.

2 The analytical framework

Consider the problem of an individual in a two-good setting where the set of goods is $\{1, 2\}$. The individual has a consumption set $X = X_1 \times X_2$ where $X_i = \mathbb{R}_+$ for $i \in \{1, 2\}$, and $X = \mathbb{R}^2_+$. A consumption bundle is $x = (x_1, x_2) \in X$ where x_i stands for the amount of good i. Generic points in X will be denoted by x, y, z. If for all $i \in \{1, 2\}$: (a) $x_i > y_i$, then we say $x > y$ and (b) $x_i \geq y_i$, then $x \geq y$.

The individual's preference on X is defined using the binary relation $\succsim$ where "$x \succsim y$" stands for "the individual prefers x to y". The strict preference is defined as $x \succ y \Leftrightarrow [x \succsim y]$ and [not $y \succsim x$]. The indifference relation is defined as $x \sim y \Leftrightarrow [x \succsim y]$ and $[y \succsim x]$. Throughout we consider preference relations on X that are rational, continuous and monotone (monotone refers to weak monotonicity, that is, for any $x, y \in X$ with $x > y$, we have $x \succ y$).

2.1 SSI preferences

A basic necessity such as food has two key features. The first feature is the subsistence requirement: the individual requires a minimum critical level of the necessity. If this requirement is not met, other goods are not useful. The second feature is saturation. Beyond a point, consuming more of it may not be beneficial. For a preference relation in a two-good setting, the common aspect of these two features is 'irrelevance' in one of the two goods.

Definition 1 Good 2 is *irrelevant at a bundle* x if $x \sim (x_1, y_2)$ for all $y_2 > x_2$. Similarly good 1 is *irrelevant at a bundle* x if $x \sim (y_1, x_2)$ for all $y_1 > x_1$. For $i \in \{1, 2\}$, good i is *relevant at a bundle* x if it is not irrelevant there.

Let $i, j \in \{1, 2\}$ and $i \neq j$. We say that *a bundle* y *involves* x_i if $y_i = x_i$. Thus, the set of all bundles involving x_i is $\{y \in X | y_i = x_i\}$.

Definition 2 Consider a preference relation $\succsim$ on X which is rational, continuous and monotone. It is *subsistence and saturation induced irrelevance preference* (or an SSI preference) with respect to good 1 if it satisfies the following properties.

(I) *Subsistence*: $\exists\ \underline{Q} \in (0, \infty)$ such that

- (a) *Subsistence zone* $[0, \underline{Q}]$: for every $x_1 \in [0, \underline{Q}]$, good 2 is irrelevant at all bundles involving x_1;
- (b) *Weak non-subsistence zone* $(\underline{Q}, \infty)$: for every $x_1 > \underline{Q}$, $\exists\ y_1 \in (\underline{Q}, x_1)$ such that good 2 is relevant at some bundle involving y_1.

(II) *Weak saturation*: $\exists\ x_2 \in X_2$ and $\overline{Q}(x_2) \in \mathbb{R}_+$ such that good 1 is irrelevant at x if $x_1 \geq \overline{Q}(x_2)$ and it is relevant at x if $x_1 < \overline{Q}(x_2)$.

Definition 2 has zones of subsistence, weak non-subsistence and weak saturation in preferences. Here good 1 is the basic good and $\underline{Q}$ stands for the subsistence threshold. good 2 is the non-basic good.[8] For instance, if good 1 represents food, then $\underline{Q}$ stands for the critical amount of food that corresponds to the minimum calorie requirements of the individual. The subsistence zone specifies that if the consumption of good 1 is below this critical level, then good 2 does not have any benefit (Property I(a)).

Property I(b) says that once the amount of good 1 exceeds $\underline{Q}$ we can always find a bundle with lower amount of good 1 at which good 2 is relevant. In other words, for any $x_1 = \underline{Q} + \varepsilon$ (where $\varepsilon > 0$, no matter how small), there is $y_1 \in (\underline{Q}, \underline{Q} + \varepsilon)$ such that good 2 is relevant at some bundle involving y_1. Once $x_1 > \underline{Q}$, we are in the weak non-subsistence zone in that the total irrelevance of good 2 disappears there. As we shall see, the properties of SSI preference ensure the existence of a subset of the weak non-subsistence zone that is a non-subsistence zone in a stronger sense.

Property (II) of the definition says that there is at least one $x_2 \in X_2$ and a corresponding threshold $\overline{Q}(x_2)$ such that for consumption bundles involving x_2, any unit of good 1 beyond $\overline{Q}(x_2)$ has no benefit. This captures the saturation aspect of a basic good in a weak sense.[9]

An SSI preference has two implications that are stated in Observation 1. First, there is a natural order between the threshold of subsistence and any threshold of weak saturation: for any $x_2 \in X_2$ where weak saturation holds, we have $\underline{Q} \leq \overline{Q}(x_2)$. Second, for any consumption bundle where the amount of the basic good exceeds the weak saturation level (that is, $x_1 > \overline{Q}(x_2)$), the non-basic good is necessarily beneficial. Formally, call an interval $(a, \infty) \subseteq X_1$ a *strong non-subsistence zone* if for every $x_1 \in (a, \infty)$ there is a bundle involving x_1 at which good 2 is relevant. Observation 1 shows that if weak saturation holds for $x_2 \in X_2$, then the interval $(\overline{Q}(x_2), \infty)$ is a strong non-subsistence zone. That is, for every $x_1 > \overline{Q}(x_2)$, there is a bundle involving x_1 at which good 2 is relevant.

Observation 1 *For an SSI preference consider any $x_2 \in X_2$ at which weak saturation (Property (II)) holds. Then $\underline{Q} \leq \overline{Q}(x_2)$, with strict inequality if $x_2 > 0$. Moreover, the interval $(\overline{Q}(x_2), \infty)$ is a strong non-subsistence zone.*

Before presenting the axiomatization of SSI preferences, we look at certain aspects of consumer behavior under this preference that distinguishes it from related preferences.

2.2 Consumer behavior under SSI preferences

2.2.1 SSI preferences and Stone-Geary utility functions

Stony-Geary utility functions are often used to model subsistence. The first two examples point out how certain implications of subsistence are better captured by SSI preferences in comparison to Stone-Geary utility functions. For both examples, $u : \mathbb{R}^2_+ \to \mathbb{R}$

[8]Henceforth when we refer to an SSI preference, it will be implicit that it is with respect to good 1, that is, good 1 is the basic good and good 2 is the non-basic good.

[9]A stronger notion of saturation requires that such a property holds *for every* $x_2 \in X_2$. Formally, there is *strong saturation* with respect to good 1 if for every $x_2 \in X_2$, $\exists\, \overline{Q}(x_2) \in \mathbb{R}_+$ such that good 1 is irrelevant at x if $x_1 \geq \overline{Q}(x_2)$ and it is relevant at x if $x_1 < \overline{Q}(x_2)$.

is a utility function that represents an SSI preference. The prices of goods $1, 2$ are denoted by $p_1, p_2 > 0$ and the income of the consumer is $w > 0$, so that the utility maximization problem for the consumer is

$$\text{choose } x \in \mathbb{R}^2_+ \text{ to maximize } u(x) \text{ subject to } p_1x_1 + p_2x_2 \leq w \tag{1}$$

Example 1 Let $0 < \underline{Q} < \overline{Q} < \infty$. Define the *net-usefulness function* $g : X_1 \to \mathbb{R}_+$ as

$$g(x_1) := \begin{cases} 0 & \text{if } 0 \leq x_1 \leq \underline{Q}, \\ x_1 - \underline{Q} & \text{if } \underline{Q} < x_1 < \overline{Q}, \\ \overline{Q} - \underline{Q} & \text{if } x_1 \geq \overline{Q}. \end{cases} \tag{2}$$

That is, $g(x_1) = \max\{x_1 - \underline{Q}, 0\} + \min\{\overline{Q} - x_1, 0\}$. The net-usefulness function $g(.)$ is continuous, non-decreasing and piecewise linear. This function captures the usefulness of the basic good beyond subsistence requirement. Let $0 < \alpha < 1$. Using the net-usefulness function, consider the utility function $u(x) = \min\{x_1, \underline{Q}\} + [g(x_1)]^\alpha x_2^{1-\alpha}$, that is,

$$u(x) = \begin{cases} x_1 & \text{if } 0 \leq x_1 \leq \underline{Q}, \\ \underline{Q} + \left(x_1 - \underline{Q}\right)^\alpha x_2^{1-\alpha} & \text{if } \underline{Q} < x_1 < \overline{Q}, \\ \underline{Q} + \left(\overline{Q} - \underline{Q}\right)^\alpha x_2^{1-\alpha} & \text{if } x_1 \geq \overline{Q}. \end{cases} \tag{3}$$

The preference represented by (3) is an SSI preference with strong saturation (where saturation threshold $\overline{Q}(x_2) = \overline{Q}$ for all $x_2 \in X_2$). Denote $\underline{w}(p_1) = p_1\underline{Q}$, $\overline{w}(p_1) = p_1\overline{Q}$ and $\widehat{w}(p_1) = \overline{w}(p_1) + (1 - \alpha)(\overline{w}(p_1) - \underline{w}(p_1))/\alpha$. When $u(x)$ is given by (3), the unique solution $x^* = (x_1^*, x_2^*)$ to the utility maximization problem (1) is

$$x^* = \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if } w \in (0, \underline{w}(p_1)], \\ \left(\underline{Q} + \frac{\alpha(w - \underline{w}(p_1))}{p_1}, \frac{(1-\alpha)(w - \underline{w}(p_1))}{p_2}\right) & \text{if } w \in \left(\underline{w}(p_1), \widehat{w}(p_1)\right), \\ \left(\overline{Q}, w - p_1\overline{Q}\right) & \text{if } w \geq \widehat{w}(p_1). \end{cases} \tag{4}$$

Observe that the utility function of (3) is different from a Stone-Geary utility function. For $\underline{Q} < x_1 < \overline{Q}$ in (3), $u(x)$ resembles a Stone-Geary function, but there is a qualitative difference. The minimum income required to achieve the subsistence consumption $\underline{Q}$ is a function of p_1: $\underline{w}(p_1)$. This important aspect, intrinsic to subsistence, is missing from Stone-Geary utility functions as it implicitly assumes that any consumer *always* has enough wealth to stay outside the subsistence zone without any reference to the price of the basic good. However, an increase in the price of certain basic good such as foodgrains may very well push a consumer from non-subsistence to subsistence zone.

Also note that if $w \geq \overline{w}(p_1)$, the consumer can afford the saturation level $\overline{Q}$ of the basic good. However, for the interval $[\overline{w}(p_1), \widehat{w}(p_1))$, it is optimal to buy less than saturation level of the basic good and more of the non-basic good.

2.2.2 Subsistence inertia

By *subsistence inertia* we mean a situation where a consumer continues to buy only the basic good even outside the subsistence zone. That is, even if a consumer has adequate income to buy more than subsistence level of the basic good ($w > p_1\underline{Q}$) so that the non-basic good is beneficial, it might still be optimal to not to buy the non-basic good at all.

Example 2 Let $0 < \underline{Q} < \overline{Q} < \infty$. Using the net-usefulness function $g(.)$ defined in Example 1, consider the utility function $u(x) = \min\{x_1, \overline{Q}\} + g(x_1)x_2$, that is,

$$u(x) = \begin{cases} x_1 & \text{if } 0 \leq x_1 \leq \underline{Q}, \\ x_1 + (x_1 - \underline{Q})x_2 & \text{if } \underline{Q} < x_1 < \overline{Q}, \\ \overline{Q} + (\overline{Q} - \underline{Q})x_2 & \text{if } x_1 \geq \overline{Q}. \end{cases} \tag{5}$$

This utility function also represents an SSI preference (with strong saturation). For $u(x)$ given by (5), the unique solution $x^* = (x_1^*, x_2^*)$ to the utility maximization problem (1) is as follows where as before $\underline{w}(p_1) = p_1\underline{Q}$ and $\overline{w}(p_1) = p_1\overline{Q}$.

For $p_2 < \overline{w}(p_1) - \underline{w}(p_1)$, we have

$$x^* = \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if } w \in (0, \underline{w}(p_1) + p_2], \\ \left(\frac{w+\underline{w}(p_1)+p_2}{2p_1}, \frac{w-\underline{w}(p_1)-p_2}{2p_2}\right) & \text{if } w \in (\underline{w}(p_1) + p_2, 2\overline{w}(p_1) - \underline{w}(p_1) - p_2), \\ \left(\overline{Q}, w - \overline{w}(p_1)\right) & \text{if } w \geq 2\overline{w}(p_1) - \underline{w}(p_1) - p_2. \end{cases} \tag{6}$$

For $p_2 \geq \overline{w}(p_1) - \underline{w}(p_1)$, we have

$$x^* = \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if } w \in (0, \overline{w}(p_1)], \\ \left(\overline{Q}, w - \overline{w}(p_1)\right) & \text{if } w \geq \overline{w}(p_1). \end{cases} \tag{7}$$

If $w > \underline{w}(p_1)$, the consumer can afford more than the subsistence level $\underline{Q}$ of good 1 and good 2 can be beneficial. Yet it may be the case in (6) and (7) where it is optimal not to buy good 2 at all. For (7), buying good 2 is optimal only when $w > \overline{w}(p_1)$. However, for (6), although the consumer can afford the saturation level $\overline{Q}$ of good 1 if $w > \overline{w}(p_1)$, it may still be optimal to buy less than $\overline{Q}$ of good 1 and more of good 2.

2.2.3 SSI preference with weak saturation

Both the examples considered so far have SSI preferences with strong saturation. This is an example of an SSI preference with weak rather than strong saturation.

Example 3 For $x = (x_1, x_2) \in \mathbb{R}_+^2$ consider the following continuous utility function

$$u(x) = \begin{cases} x_1 \text{ if } x_1 \leq 1, \\ 1 + \min\{\sqrt{(x_1-1)x_2}, x_2\} \text{ if } (1 < x_1 \leq 2) \text{ or } (x_1 > 2 \text{ and } x_2 \leq 1), \\ 1 + \min\left\{\sqrt{(x_1-1)x_2}, \frac{1+\sqrt{1+4(x_1-1)(x_2-1)}}{2}\right\} \text{ if } x_1 > 2 \text{ and } x_2 > 1. \end{cases}$$

Some indifference curves of this preference are drawn in Figure 1. This utility function represents an SSI preference in which the subsistence zone is $[0, 1]$, that is, $\underline{Q} = 1$. Note that $(1, \infty) \subset X_1$ is a strong (and hence weak) non-subsistence zone. The weak

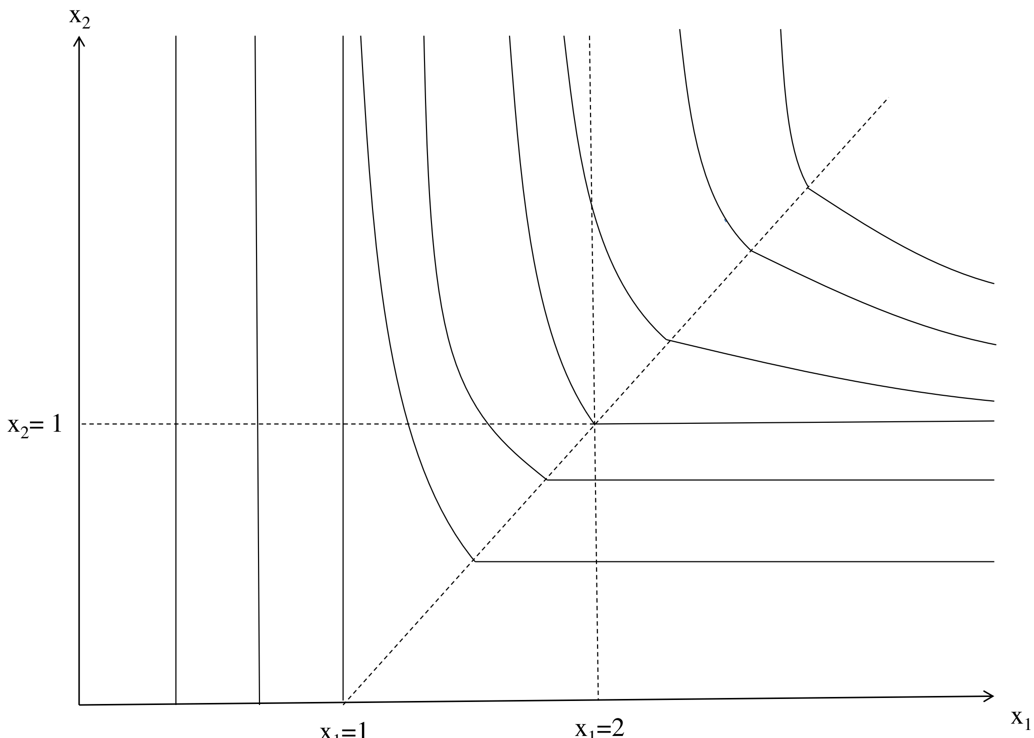

Figure 1

saturation property holds for $x_2 \in [0,1]$. For any such x_2, there is $\overline{Q}(x_2) = x_2 + 1$ (see Figure 1) such that good 1 is irrelevant at x if $x_1 \geq \overline{Q}(x_2)$ and relevant if $x_1 < \overline{Q}(x_2)$. Note that $\overline{Q}(0) = \underline{Q}$ and $\overline{Q}(x_2) > \underline{Q}$ for $x_2 \in (0,1]$. As shown in Observation 1, $(x_2 + 1, \infty) \subset X_1$ is a strong non-subsistence zone for every $x_2 \in [0,1]$. Finally observe that at any bundle with $x_1 > 1$ and $x_2 > 1$, both goods are relevant.

To see a particular context in which the properties of Example 3 fits well, suppose good 1 is food and good 2 is physical activity, broadly construed. For instance, good 2 can stand for labor which, when productive, brings monetary benefits. In the subsistence zone the individual does not have sufficient nutrition, so he is too weak to have any benefit from physical activity. Above this zone, for any $x_2 \leq 1$, there is a threshold (given by $x_2 + 1$) beyond which good 1 is not useful. This is because to sustain low levels of physical activity the individual does not require an ever increasing amount of food and saturation is reached after a while. In particular, when $x_2 = 0$ (no physical activity), saturation is reached at the subsistence level (see Figure 1). However, there is no point of saturation once $x_2 > 1$. For any such x_2, a higher amount of good 1 makes the individual better off. This is because for jobs that require higher amount of labor, better nutrition makes labor more productive.

These examples demonstrate a glimpse of the intricacies of consumer behavior that can be associated with SSI preferences. Whether saturation in an SSI preference is weak or strong depends on how the non-basic good relates with the basic good. In the first two examples, good 2 is considered to be a non-basic consumption good, which does not share the same relation that physical activity has with food as in the last example.

2.2.4 SSI and lexicographic preferences

The preference relation $\succsim$ on X is a *lexicographic preference with linear order* $1 <_0 2$ on the two goods if the following hold: $x \succsim y$ if either $x_1 > y_1$ or $[x_1 = y_1$ and $x_2 \geq y_2]$.

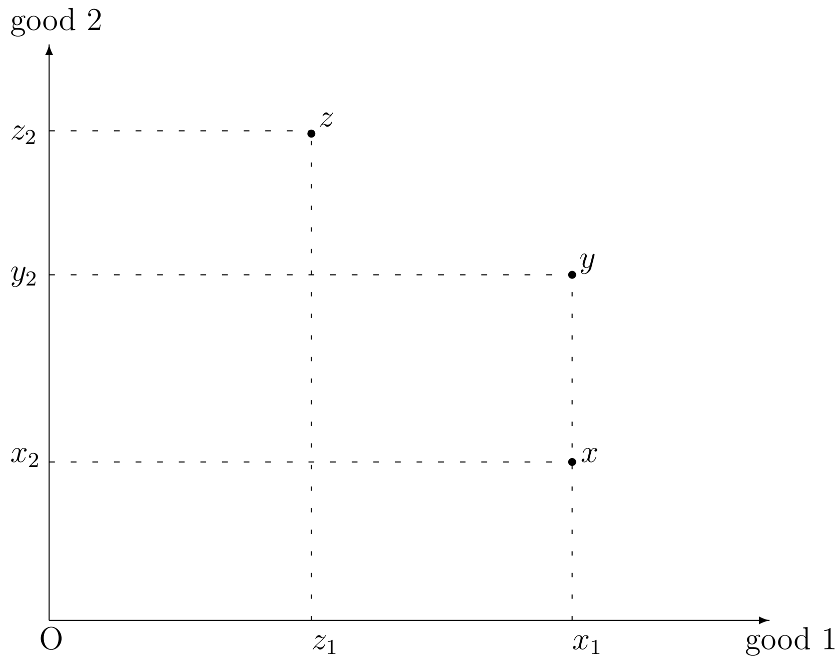

Figure 2

Consider two bundles that are in the subsistence zone of an SSI preference. If they have different amounts of good 1 (like points $x = (x_1, x_2)$ and $z = (z_1, z_2)$ in Figure 2), then their preference ordering in the SSI is same as lexicographic. However, if the bundles have the same amount of good 1 (like points $x = (x_1, x_2)$ and $y = (x_1, y_2)$ in Figure 2), the orderings of SSI and lexicographic are very different. Such bundles lie on the same indifference curve for SSI, while for lexicographic, they are strictly ordered in terms of the amount of good 2. Indeed, SSI preference is continuous while lexicographic is not. On the other hand, lexicographic is strong monotone, while SSI is not.

2.3 Unhappy sets

We introduce the notion of unhappy sets which will be used in our axiomatizations.

Definition 3 For a preference relation $\succsim$ on X, a set $S \subseteq X$ is an *unhappy set* if for any $y \notin S$, $y \succ x$ for every $x \in S$.

Remarks For any preference relation $\succsim$ on X, the empty set and the set X are both unhappy sets. Any lower contour set or strict lower contour set is an unhappy set. For any rational and continuous preference relation on $\mathbb{R}^n_+$, if S is an unhappy set which is a non empty proper subset of $\mathbb{R}^n_+$, then S is either open or closed, but not both. If S is closed, it is a lower contour set and if S is open, it is a strict lower contour set.

3 Axiomatization of SSI preferences

We characterize SSI preferences using Axiom 1 and Axiom 2. Axiom 1 requires that irrelevance of the non-basic good is at least partially driven by inadequacy of the basic good. Axiom 2 requires that there exists at least one bundle where the basic good is irrelevant. Thus for each of the two goods there is a structural transition in preference. Theorem 1 shows that this requirement uniquely characterizes SSI preferences.

To state the axioms, for $i, j \in \{1, 2\}$ and $i \neq j$, denote by B_i the set of all consumption bundles at which good j is irrelevant. That is,

$$B_1 := \{x \in X | \text{good 2 is irrelevant at } x\} \text{ and } B_2 := \{x \in X | \text{good 1 is irrelevant at } x\}$$

Axiom 1 *Unhappiness driven irrelevance:* B_1 has an unhappy subset of positive area.

Axiom 2 B_2 is non-empty.

Theorem 1 *Consider a preference relation* $\succsim$ *on* $X = \mathbb{R}^2_+$ *which is rational, continuous and monotone. The following statements are equivalent.*

(SSI1) *The preference relation* $\succsim$ *on* X *satisfies Axiom* 1 *and Axiom* 2.

(SSI2) *The preference relation* $\succsim$ *on* X *is a SSI preference with respect to good* 1.

Idea of the proof: To see that an SSI preference satisfies Axiom 1 and Axiom 2, first observe that the set $S_0 = \{x \in X \mid x_1 \in [0, \underline{Q})\}$ is a subset of B_1 since good 2 is irrelevant for any consumption bundle in S_0. Moreover, by monotonicity of preference any bundle in $X \setminus S_0$ is strictly preferred to any bundle in S_0. Hence S_0 is an unhappy set of positive area, so Axiom 1 holds. Property (II) of SSI implies Axiom 2.

To prove that Axiom 1 and Axiom 2 imply SSI preference we use the following chain of arguments. First, by Axiom 1 there is a set $S \subseteq B_1$ which is an unhappy set of positive area. Therefore, there exists $x \in S$ such that $x_1 > 0$ (see Figure 3).

By monotonicity of preference, x is weakly preferred to any point in the rectangle $OExA$. Since S is an unhappy set, any point in this rectangle must be in S and hence in B_1. In particular for all points lying on the line OE, good 2 is irrelevant and hence all indifference curves are parallel vertical lines in this region. This indicates the existence of a subsistence zone; in particular, the interval OE is a subset of the subsistence zone. To fully characterize the subsistence zone we have to use Axiom 2.

Axiom 2 requires non-emptiness of B_2 and hence there exists $y = (y_1, y_2) \in B_2$. This gives rise to weak saturation. The point $(y_1, 0)$ must be to the right of OE, otherwise two indifference curves will intersect. Finally the subsistence zone must be bounded since $(y_1, 0)$ and anything to the right of it cannot belong to the subsistence zone. Existence of weak non-subsistence zone follows.

One implication of Axiom 1 is B_2 cannot have an unhappy set of positive area, for if it does, then there will be a region where indifference curves are parallel horizontal lines each meeting the horizontal axis. But then they will intersect with the indifference curves of the interval OE (see IC(y)in Figure 3). Given this, an alternative formulation of the axioms (modulo relabeling of the two goods) would be: (i) at least one of B_1, B_2 has an unhappy subset of positive area and (ii) B_1, B_2 are non-empty.

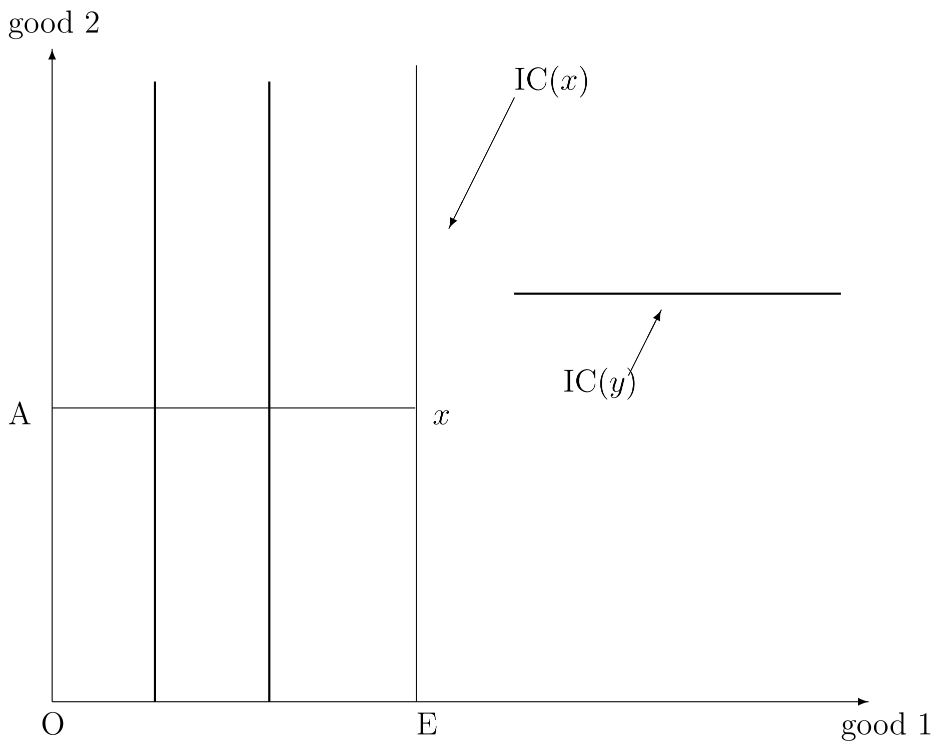

Figure 3

3.1 Robustness of the axioms of SSI preference

A Leontief preference satisfies Axiom 2 but not Axiom 1. Let us now check the robustness of Axiom 2. This axiom is useful not only to generate weak saturation, but it is also necessary for the existence of a non-subsistence zone. Without it, a non-subsistence zone might not exist. Without a reference to a situation of non-subsistence, the notion of subsistence may not be meaningful.

Corollary 1 *Consider a preference relation $\succsim$ on $X = \mathbb{R}^2_+$ which is rational, continuous and monotone. The following statements are equivalent.*

(S1) *The preference relation $\succsim$ on X satisfies Axiom* 1.

(S2) *For the preference relation $\succsim$ on X, either property* (I) *of Definition* 2 *holds, or good* 2 *is irrelevant at all bundles.*

Recall that in property (I) of Definition 2, the subsistence zone is $[0, \underline{Q}]$ for $0 < \underline{Q} < \infty$, which results in a weak non-subsistence zone $(\underline{Q}, \infty)$. The preference in (S2) of Corollary 1 includes the case where $\underline{Q} = \infty$, in which case there is no non-subsistence zone at all, rendering good 2 to be irrelevant at all bundles.

4 Generalized Leontief preferences

In the previous section we characterized an SSI preference that has the feature that each good had stretches of irrelevance. The only well-known preference where irrelevance in both goods exists is a Leontief preference. For this preference at least one good is irrelevant at any bundle $x \in X$. Thus irrelevance spans the entire domain of preferences. Formally, a preference relation $\succsim$ on X is a *Leontief preference* if there exists $a > 0$ such that for any $x, y \in X$, $x \succsim y$ if and only if $\min\{ax_1, x_2\} \geq \min\{ay_1, y_2\}$. In that case there is a linear function $F(x_1) = ax_1$ such that given any $x_1 \in X_1$, both goods are irrelevant at $(x_1, F(x_1))$, good 1 is irrelevant at $(y_1, F(x_1))$ for any $y_1 > x_1$ and good 2 is irrelevant at (x_1, y_2) for any $y_2 > F(x_1)$. The ratio $1/a$ is the fixed coefficient of substitutability between the two goods. However, there is no apparent pressing need to keep the substitution fixed across the two goods. For example, with one cup of tea a day, an individual may want two spoons of sugar, but if the same individual drinks ten cups of tea a day, he may take less than twenty spoons of sugar if he is diabetic. Thus for Leontief preferences the proportion of substitutability may well vary as we vary the amount of any one good. Incorporating this generality of variable substitutability, ceteris paribus, we define the 'generalized Leontief' preference as follows.

Definition 4 The preference relation $\succsim$ on X is a *generalized Leontief preference* (or a GL preference) if there exists surjective[10] and increasing function $F : X_1 \to X_2$ with $F(0) = 0$ such that for any $x_1 \in X_1$:

(i) at any bundle $(x_1, F(x_1))$, both goods $1, 2$ are irrelevant;

(ii) good 1 is irrelevant at any bundle $(y_1, F(x_1))$ for $y_1 > x_1$;

(iii) good 2 is irrelevant at any bundle (x_1, y_2) for $y_2 > F(x_1)$.

Observe that since F is surjective and increasing, it is also one-to-one and continuous. The domain of the inverse function of F is X_2. Recall that

$$B_1 = \{x \in X | \text{good 2 is irrelevant at } x\} \text{ and } B_2 = \{x \in X | \text{good 1 is irrelevant at } x\}$$

To axiomatize GL preferences, it will be useful to define

$$A_1 := \{x_1 \in X_1 | \exists x_2 \in X_2 \text{ such that } x = (x_1, x_2) \in B_1\} \text{ and}$$

$$A_2 := \{x_2 \in X_2 | \exists x_1 \in X_1 \text{ such that } x = (x_1, x_2) \in B_2\}.$$

That is, for $i \neq j$, $A_i \subseteq X_i$ is the set of all *elements* x_i for which there exists a bundle involving x_i at which good j is irrelevant.

Axiom 3 *Irrelevance without unhappiness:* Neither B_1 nor B_2 has an unhappy subset of positive area.

Axiom 4 *Spanning axiom:* $A_1 = X_1$, $A_2 = X_2$ and $B_1 \cup B_2 = X$.

We axiomatize GL preferences using these two axioms. Monotonicity of preference ensures that B_1 and B_2 both cannot have an unhappy set of positive area (see Lemma 1 of the Appendix), implying that Axiom 3 is the compliment of Axiom 1 (modulo relabeling of the goods).

[10] A function $F : X_1 \to X_2$ is a *surjective* function if for any $x_2 \in X_2$, $\exists\ x_1 \in X_1$ such that $F(x_1) = x_2$.

Theorem 2 *Consider a preference relation $\succsim$ on $X = \mathbb{R}^2_+$ which is rational, continuous and monotone. The following statements are equivalent.*

(GL1) *The preference relation $\succsim$ on X satisfies Axiom* 3 *and Axiom* 4.

(GL2) *The preference relation $\succsim$ on X is a generalized Leontief preference.*

Idea of the proof: For GL preferences, $A_i = X_i$ for $i = 1, 2$. Moreover, $B_1 = \{x \in X \mid x_2 \geq F(x_1)\}$, $B_2 = \{x \in X \mid x_2 \leq F(x_1)\}$, implying that $B_1 \cup B_2 = X$, so Axiom 4 holds. For any $x \in B_i$, there exists $y \notin B_i$ such that $x \sim y$, so there does not exist an unhappy subset of B_i. Thus Axiom 3 also holds.

To see the converse consider any $x = (x_1, x_2) > (0, 0)$ and, given Axiom 4, assume without loss of generality that $x \in B_1$. If $(x_1, 0) \in B_1$, then $S = \{y \in X \mid y_1 \in [0, x_1)\}$ is an unhappy set of positive area. Since $B_1 \cup B_2 = X$ (Axiom 4) and indifference curves cannot intersect it follows that $S \subset B_1$, contradicting Axiom 3. So we must have $(x_1, 0) \in B_2$. Then the indifference curve containing x cannot meet the horizontal axis. Hence there exists $y_2 \in (0, x_2]$ such that $(x_1, y_2) \in B_2$ and $(x_1, z_2) \in B_1$ for all $z_2 \geq y_2$. Taking $F(x_1) = y_2$ and using $A_i = X_i$ (for $i = 1, 2$) it can be shown that $F(.)$ is an surjective and increasing function with $F(0) = 0$.

4.1 Robustness of axioms

Axiom 3 and Axiom 4 have three requirements: (i) $A_i = X_i$ for $i = 1, 2$, (ii) $B_1 \cup B_2 = X$ and (iii) none of B_1 and B_2 has an unhappy subset of positive area. In each of the following examples, only one of requirements (i)-(iii) is violated, and we see that we do not get a generalized Leontief preference.

Example 4 Consider the preference represented by utility function u where $k > 0$.

$$u(x_1, x_2) = \begin{cases} \min\{x_1/(k - x_1), x_2\} & \text{if } x_1 < k, \\ x_2 & \text{if } x_1 \geq k. \end{cases}$$

Some indifference curves of this preference are drawn in Figure 4. For this example, $B_1 \cup B_2 = X$ and none of B_1 and B_2 has an unhappy set of positive area. However, $A_1 = [0, k)$ although $A_2 = X_2$. This preference is not GL but is "locally Leontief" (for $x_1 < k$) with saturation of good 1 at $x_1 = k$.

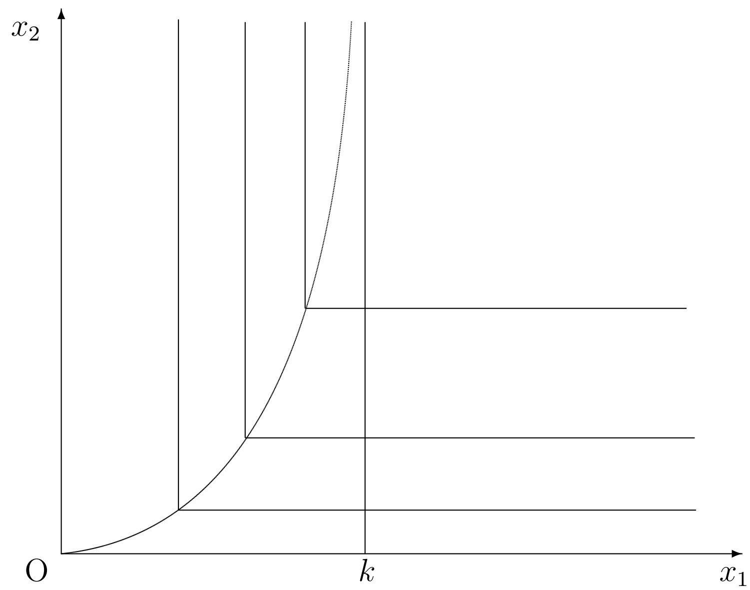

Figure 4

Example 5 Consider the preference represented by the utility function

$$u(x_1, x_2) = \begin{cases} x_2 & \text{if } x_2 \leq x_1/2, \\ (x_1 + x_2)/3 & \text{if } x_1/2 < x_2 < 2x_1, \\ x_1 & \text{if } x_2 \geq 2x_1. \end{cases}$$

Some indifference curves of this preference are drawn in Figure 5. For this example, Axiom 3 holds and $A_i = X_i$ for $i = 1, 2$. However, $B_1 \cup B_2 \neq X$.

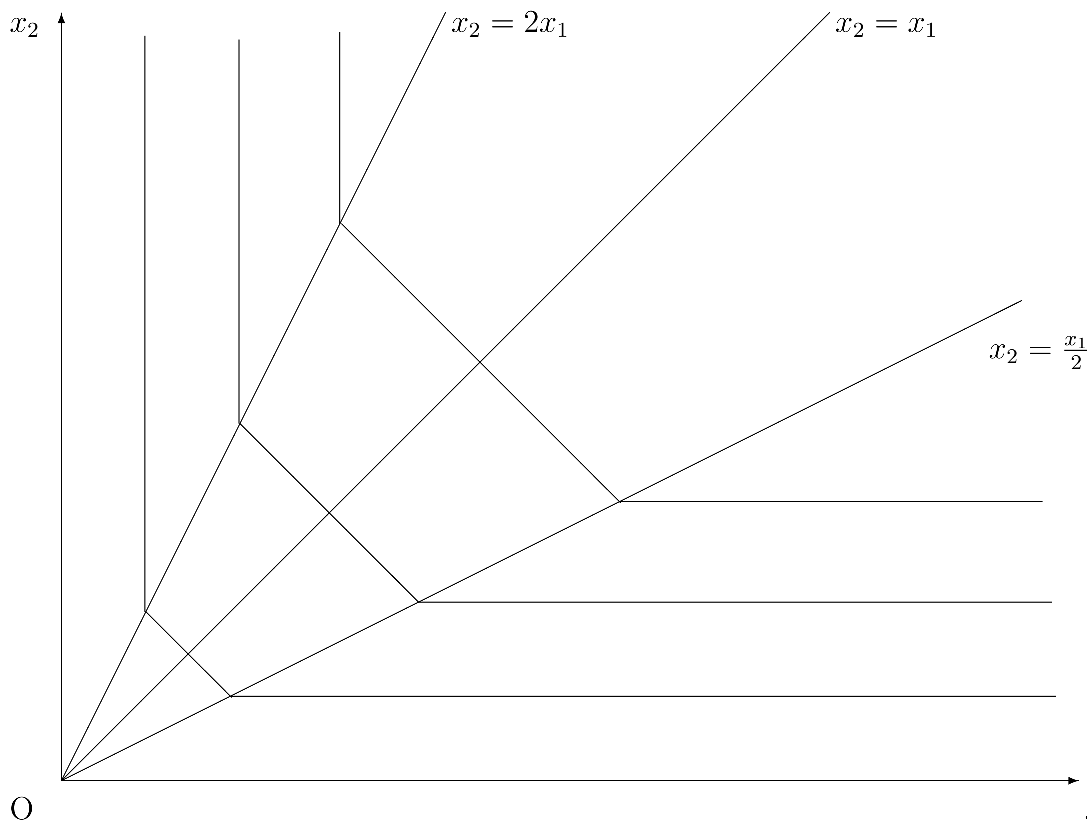

Figure 5

Example 6 Consider the preference represented by utility function u where $k > 0$.

$$u(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 \leq k, \\ k + \min\{x_1 - k, x_2\} & \text{if } x_1 > k. \end{cases}$$

Some indifference curves of this preference are drawn in Figure 6.

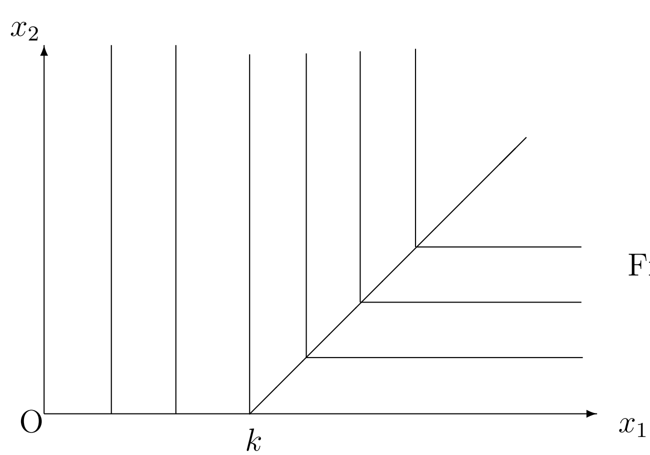

Figure 6

For Example 6, Axiom 4 holds. But Axiom 3 does not hold since B_1 has an unhappy subset of positive area. The set $\{(x_1, x_2) | x_1 \in [0, k], x_2 \in X_2\} \subset B_1$ is an unhappy set. We get "locally Leontief" (for $x_1 > k$) and subsistence with respect to good 1 for $x_1 \leq k$.

5 Concluding remarks

As poverty and hunger continue to pose major challenges for development, subsistence remains a useful concept for policymakers. However, in spite of its wide use, the treatment of subsistence in the literature of consumer theory is far from adequate. Seeking to fill this gap we introduce subsistence and saturation in preferences (SSI) in a standard consumer theory framework with two goods: a basic good and a non-basic good. In so doing, we formalize Bentham's notions of subsistence and abundance. The modeling of subsistence and the axiomatization of these preferences are based on two key concepts: (i) irrelevance of a good in certain consumption bundles and (ii) unhappy sets. These concepts are also used subsequently to axiomatize a generalized version of Leontief (GL) preferences. For SSI preferences, irrelevance together with the presence of unhappy sets generates subsistence. By contrast, for GL preferences, if there is a set of consumption bundles such that a certain good is irrelevant at every bundle of the set, it can never be an unhappy set.

As illustrated in Section 2.2, SSI preferences bring out certain aspects of consumer behavior that are ignored by Stone-Geary utility functions. These utility functions implicitly assume that consumers always have enough income to stay outside the subsistence zone regardless of the price of the basic good. By contrast, SSI preferences demonstrate that a price rise of the basic good may well take a consumer from non-subsistence to the subsistence zone. SSI preferences can thus provide better insights for poverty analysis. For instance, whether a specific instance of poverty is temporary (due to a fluctuation of food prices) or chronic (driven by persistently low levels of income) can be better understood from these preferences.

SSI preferences also bring out another aspect of consumer behavior which we call subsistence inertia. It may be the case that for moderate levels of income, it is optimal for a consumer to buy only the basic good even though he is outside the subsistence zone. Therefore the absence of the non-basic good in the consumption basket does not necessarily imply that the individual is in his subsistence zone. This insight can be useful in avoiding misallocation of resources while implementing policies that seek to specifically target the poor.

In conclusion, in this paper we have presented a microeconomic analysis of subsistence consumption. The theoretical framework of this paper can be useful to better understand decision making behavior of the poor.

6 Appendix

Proof of Observation 1: Let $x_2 \in X_2$ be such that property (II) holds there. For the first part, suppose on the contrary that $\underline{Q} > \overline{Q}(x_2)$. Then $(\overline{Q}(x_2), x_2) \sim (\underline{Q}, x_2)$ (by property (II)) and $(\underline{Q}, x_2) \sim (\underline{Q}, y_2)$ for any $y_2 > x_2$ (by property (I)(a)). By transitivity, $(\overline{Q}(x_2), x_2) \sim (\underline{Q}, y_2)$ for any $y_2 > x_2$ which violates monotonicity. So we must have $\underline{Q} \leq \overline{Q}(x_2)$.

Let $x_2 > 0$. If $\underline{Q} = \overline{Q}(x_2) = Q$, then $(Q, 0) \sim (Q, x_2)$ (by (I)(a)) and $(Q, x_2) \sim x$ for any $x_1 > Q$ (by (II)), implying $(Q, 0) \sim x$ for any $x_1 > Q$ which violates monotonicity. So we must have $\underline{Q} < \overline{Q}(x_2)$ if $x_2 > 0$.

To prove that $(\overline{Q}(x_2), \infty)$ is a strong non-subsistence zone with respect to good 1, we have to show that for any $x_1 > \overline{Q}(x_2)$, good 2 is relevant at some bundle involving x_1. Suppose, on the contrary, $\exists\, x_1 > \overline{Q}(x_2)$ such that good 2 is irrelevant at all bundles involving x_1. Then $x \sim (x_1, y_2)$ for any $y_2 > x_2$. But since $x_1 > \overline{Q}(x_2)$, by property (II) we have $x \sim (\overline{Q}(x_2), x_2)$. By transitivity, $(x_1, y_2) \sim (\overline{Q}(x_2), x_2)$, which violates monotonicity, a contradiction. ■

6.1 Irrelevance: some implications

We define two functions $f_1, f_2 : X \to \{0, 1\}$ that captures the notion of irrelevance.

$$f_1(x) \equiv \begin{cases} 0 \text{ if } x \sim (y_1, x_2) \text{ for all } y_1 \geq x_1, \\ 1 \text{ otherwise.} \end{cases}$$

$$f_2(x) \equiv \begin{cases} 0 \text{ if } x \sim (x_1, y_2) \text{ for all } y_2 \geq x_2, \\ 1 \text{ otherwise.} \end{cases}$$

The function $f_1(x)$ captures irrelevance of good 1 at bundle x. Similarly, the function $f_2(x)$ captures irrelevance of good 2 at bundle x. Observation 2 shows that if a good is irrelevant at a bundle, then it continues to remain so for all bundles where its quantity is increased keeping the quantity of the other good unchanged. Observation 2 also shows that the converse is true, which is proved using continuity of the preference relation.

Observation 2 (i) $f_2(x) = 0 \Leftrightarrow f_2(x_1, y_2) = 0$ *for all* $y_2 > x_2$ *and* (ii) $f_1(x) = 0 \Leftrightarrow f_1(y_1, x_2) = 0$ *for all* $y_1 > x_1$.

Proof: We prove (i), proof of (ii) is similar. Let $f_2(x) = 0$. Then $x \sim (x_1, y_2)$ for any $y_2 > x_2$. Hence $(x_1, y_2) \sim (x_1, z_2)$ for any $z_2 > y_2 > x_2$, implying that $f_2(x_1, y_2) = 0$.

Conversely, let $f_2(x_1, y_2) = 0$ for all $y_2 > x_2$. Then $(x_1, y_2) \sim (x_1, z_2)$ for all $z_2 > y_2 > x_2$. Let $x^n = (x_1, x_2 + 1/n)$ and $y^n = (x_1, y_2 + 1/n)$ for $n = 1, 2, \ldots$. Then $x^n \sim y^n$, and hence $x^n \succsim y^n$ for $n = 1, 2, \ldots$. Since $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = (x_1, y_2)$, by continuity we have $x \succsim (x_1, y_2)$. Since $y_2 > x_2$, by monotonicity we have $(x_1, y_2) \succsim x$. We then conclude that $x \sim (x_1, y_2)$ for any $y_2 > x_2$, proving that $f_2(x) = 0$. ■

We conclude from Observation 2 that for every $x_i \in A_i$, $\exists\, \alpha_i(x_i) \in X_j = \mathbb{R}_+$ such that

$$f_j(x) = \begin{cases} 0 \text{ if } x_j \geq \alpha_i(x_i), \\ 1 \text{ otherwise.} \end{cases} \tag{8}$$

It follows from (8) that $B_i = \{x \in X | x_i \in A_i, x_j \geq \alpha_i(x_i)\}$. For $x_i \in A_i$, let $B_i(x_i)$ be the set of all bundles involving x_i at which good j is irrelevant, that is, $B_i(x_i) :=$

$\{y \in X | y_i = x_i, y_j \geq \alpha_i(x_i)\}$. It is immediate that $B_i = \cup_{x_i \in A_i} B_i(x_i)$. For any $x_i \in X_i$, define the set of all bundles involving x_i as $M_i(x_i) := \{y \in X | y_i = x_i\}$. Observe that for any $x_i \in A_i$, $B_i(x_i) \subseteq M_i(x_i)$. Moreover $B_i(x_i) = M_i(x_i)$ if and only if $\alpha_i(x_i) = 0$. The last equality implies that good j is irrelevant at *all* bundles involving x_i.

Observation 3

(i) *Let $x_i, y_i \in A_i$ and $y_i < x_i$. Then $x \succ y$ for any $x \in B_i(x_i)$ and $y \in B_i(y_i)$.*

(ii) *Let $x_i > 0$. If $B_i(y_i) = M_i(y_i)$ for all $y_i \in [0, x_i)$, then $x_i \in A_i$ and $B_i(x_i) = M_i(x_i)$.*

Proof: Without loss of generality (w.l.o.g.), let $i = 1$.

(i) Let $y \in B_1(y_1)$. Consider any $z_2 > \max\{y_2, \alpha_1(x_1)\}$. Then $(x_1, z_2) \in B_1(x_1)$. Since $x_1 > y_1$ and $z_2 > y_2$, by monotonicity $(x_1, z_2) \succ y$. Since $(x_1, z_2) \sim x$ for any $x \in B_1(x_1)$ the result follows from transitivity.

(ii) Consider two sequences $x^n = (x_1 - 1/n, x_2)$, $y^n = (x_1 - 1/n, 0)$ where $x_2 > 0$ and $n > 1/x_1$. Since $y_1 \in A_1$ and $\alpha_1(y_1) = 0$ for $y_1 \in [0, x_1)$, we have $x^n, y^n \in M_1(x_1 - 1/n) = B_1(x_1 - 1/n)$. Hence $x^n \sim y^n$ and in particular, $y^n \succsim x^n$. Since $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = (x_1, 0)$, by continuity we have $(x_1, 0) \succsim x$. Since $x_2 > 0$, by monotonicity we have $x \succsim (x_1, 0)$, implying that $x \sim (x_1, 0)$ for any $x_2 > 0$. This proves the result. ■

Consider any two arbitrary bundles at both of which good j is irrelevant. The first part of Observation 3 shows that the preference ordering of these two bundles is completely determined by amounts of good i. The second part shows that if for any $y_i < x_i$, good j is irrelevant at all bundles involving y_i, then good j is also irrelevant at all bundles involving x_i.

6.2 SSI preferences

For $i = 1, 2$, a set $S \subseteq B_i$ is a *maximal unhappy subset* of B_i if (a) S is an unhappy set and (b) $\nexists\, T \subseteq B_i$ such that T is an unhappy set and $S \subset T$. Lemma 1 (that follows) will be used to prove Theorem 1. Part (I) of Lemma 1 shows that if for some $x_1 > 0$, good 2 is irrelevant at all bundles involving any $y_1 \in [0, x_1]$ then Axiom 1 holds. Part (II) shows that the converse is also true. Moreover, if Axiom 1 holds, then B_1 has a unique maximal unhappy subset $\overline{S}$ which has the property that if $x = (x_1, x_2) \in \overline{S}$, then $(x_1, 0) \in \overline{S}$ and consequently good 2 is irrelevant at all bundles involving x_1. Finally if Axiom 1 holds, then B_2 cannot have an unhappy subset of positive area.

Given Axiom 1, an immediate consequence of Lemma 1(I) is that the set $T(x_1) = \{y \in X \mid y_1 \in [0, x_1]\} \subseteq B_1$ is an unhappy set and the indifference curves in $T(x_1)$ are all parallel to the X_2 axis.

Lemma 1 (I) *If $x_1 > 0$, $[0, x_1] \subseteq A_1$ and $B_1(y_1) = M_1(y_1)$ for all $y_1 \in [0, x_1]$, then Axiom 1 holds.*

(II) *Suppose Axiom 1 holds.*

(i) *Let* $S \subseteq B_1$ *be an unhappy set of positive area. If* $x \in S$*, then* $\alpha_1(y_1) = 0$ *for all* $y_1 \in [0, x_1]$ *and* $\cup_{y_1 \in [0,x_1]} B_1(y_1) = \cup_{y_1 \in [0,x_1]} M_1(y_1) \subseteq S$*.*

(ii) B_1 *has a unique maximal unhappy subset* $\overline{S}$*, which has the following properties: Either* (a) $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1]} M_1(y_1)$ *or* (b) $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1)} M_1(y_1)$ *for some* $\overline{x}_1 \in (0, \infty)$*, or* (c) $\overline{S} = \cup_{y_1 \in \mathbb{R}_+} M_1(y_1) = \mathbb{R}^2_+$*.*

(iii) *Suppose* (a) *or* (b) *of* (ii) *holds. Then for every* $x_1 > \overline{x}_1$*,* $\exists\ y_1 \in (\overline{x}_1, x_1)$ *such that either* $y_1 \notin A_1$*, or* $y_1 \in A_1$ *and* $\alpha_1(y_1) > 0$*.*

(iv) B_2 *cannot have an unhappy subset of positive area.*

Proof of Lemma 1: (I) Let $y_1 \in [0, x_1]$. Let $T := \cup_{y_1 \in [0,x_1)} B_1(y_1) = \cup_{y_1 \in [0,x_1)} M_1(y_1) \subseteq B_1$. To prove that T is an unhappy set, first we show that $x \succ y$ for any $y \in T$. Observe that $x \in M_1(x_1) = B_1(x_1)$. Let $y \in T$. Then $y \in M_1(y_1) = B_1(y_1)$ for some $y_1 < x_1$.By Observation 3(i), we conclude that $x \succ y$.

To complete the proof we show that $z \succ y$ for any z such that $z_1 > x_1$. Monotonicity of preference implies that $z \succsim (x_1, 0)$ for any such z. From the preceding paragraph, we have $(x_1, 0) \succ y$ for any $y \in T$. By transitivity, $z \succ y$ for any $y \in T$. This proves that T is an unhappy set. As $x_1 > 0$, the area of T is positive. So Axiom 1 holds.

(II) (i) Let $S \subseteq B_1$ has positive area. Then $\exists\ x \in S$ where $x_1 > 0$. Consider such $x \in S$. Since $y \sim x$ for all $y \in B_1(x_1)$ and S is an unhappy set, we must have $B_1(x_1) \subseteq S$.

Next observe that if $\alpha_1(x_1) > 0$ for some $x \in S$, we can find y such that $y_1 = x_1$ and $y_2 \in [0, \alpha_1(x_1))$. Then $y \notin B_1$, so we have $y \notin S$. But $x \succsim y$ (by continuity and monotonicity of $\succsim$), which contradicts that S is an unhappy set. Hence for any $x \in S$, we must have $\alpha_1(x_1) = 0$, implying that $B_1(x_1) = M_1(x_1) \subseteq S$.

Now we show that if $x \in S$, then $y \in S$ for any y such that $y_1 < x_1$. To see this, consider z such that $z_1 = x_1$ and $z_2 > y_2$. Since $B_1(x_1) = M_1(x_1) \subseteq S$, we have $z \in S$. By monotonicity, $z \succ y$. As S is an unhappy set, we must have $y \in S$.

From the preceding paragraphs we conclude that if $x \in S$, then $\alpha_1(y_1) = 0$ for all $y_1 \in [0, x_1]$ and $\cup_{y_1 \in [0,x_1]} B_1(y_1) = \cup_{y_1 \in [0,x_1]} M_1(y_1) \subseteq S$. This proves (i).

(ii) First observe that if S, T are two subsets of B_1 that are both unhappy sets, then either $S \subseteq T$ or $T \subseteq S$. If neither holds, then $\exists\ x \in S, y \in T$ such that $x \notin T, y \notin S$. If $x_1 = y_1$, then $y \in M_1(x_1) \subseteq S$, a contradiction. So $x_1 \neq y_1$. W.l.o.g., let $y_1 < x_1$. But then from the last paragraph, we have $y \in M_1(y_1) \subseteq S$, again a contradiction.

Therefore, if Axiom 1 holds, then it has a unique maximal unhappy subset $\overline{S}$ and this set has positive area. From part (i) we conclude that either $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1]} M_1(y_1)$ or $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1)} M_1(y_1)$ for some $0 < \overline{x}_1 < \infty$, or $\overline{S} = \cup_{y_1 \in \mathbb{R}_+} M_1(y_1) = \mathbb{R}^2_+$.

(iii) If (a) or (b) of (ii) holds, then $y_1 \in A_1$ and $\alpha_1(y_1) = 0$ for all $y_1 \in [0, \overline{x}_1]$ (for (b), the result for $y_1 = \overline{x}_1$ follows from Observation 3(ii)). Suppose, on the contrary $\exists$ $x_1 > \overline{x}_1$ where the assertion (iii) does not hold. Then for every $y_1 \in (\overline{x}_1, x_1)$, we have $y_1 \in A_1$ and $\alpha_1(y_1) = 0$, so that $B_1(y_1) = M_1(y_1)$. Let $\widetilde{S}^* := \cup_{y_1 \in [0,x_1)} M_1(y_1)$. Then $\overline{S} \subset \widetilde{S}^* \subseteq B_1$. By part (I), $\widetilde{S}^*$ is an unhappy set, which contradicts (II)(ii).

(iv) Suppose on the contrary both B_1, B_2 have unhappy subsets of positive area. Then by part (II)(i), for $i = 1, 2$, $\exists\ x_i > 0$ such that $x_i \in A_i$ and $\alpha_i(x_i) = 0$. Then

$(x_1, 0) \sim x$ (since $\alpha_1(x_1) = 0$) and $(0, x_2) \sim x \sim (y_1, x_2)$ for any $y_1 > x_1$ (since $\alpha_2(x_2) = 0$). This implies $(x_1, 0) \sim (y_1, x_2)$. But since $y_1 > x_1$ and $x_2 > 0$, by monotonicity we must have $(y_1, x_2) \succ (x_1, 0)$, a contradiction. This proves (iv). ■

Proof of Theorem 1: We first prove (SSI1)⇒ (SSI2).

Proof of property (I) (subsistence): Since Axiom 1 holds, by Lemma 1(II)(ii), B_1 has a unique maximal unhappy subset $\overline{S}$.

Now we show that $\overline{S} \neq \mathbb{R}^2_+$. To see this, first note that since Axiom 1 holds, by Lemma 1(II)(iv), B_2 cannot have an unhappy subset of positive area. Moreover, by Axiom 2, B_2 is non-empty and so is A_2. Let $x_2 \in A_2$, $y_1 > x_1 \geq \alpha_2(x_2)$ and $y_2 = x_2$. Then $x, y \in B_2(x_2)$, so that $x \sim y$. If $\overline{S} = \mathbb{R}^2_+$, then $x, y \in \overline{S} \subseteq B_1$. As $x \in M_1(x_1) = B_1(x_1)$, $y \in M_1(y_1) = B_1(y_1)$ and $y_1 > x_1$, by Observation 3(i) we have $y \succ x$, a contradiction. So we must have $\overline{S} \neq \mathbb{R}^2_+$.

From the preceding paragraph and by Lemma 1(II)(ii) we conclude that either $\overline{S} = \cup_{y_1 \in [0, \overline{x}_1)} M_1(y_1)$ or $\overline{S} = \cup_{y_1 \in [0, \overline{x}_1]} M_1(y_1)$ for some $\overline{x}_1 \in (0, \infty)$. In either case, by Observation 3(ii) we have $\alpha_1(y_1) = 0$ for all $y_1 \in [0, \overline{x}_1]$. Taking $\underline{Q} = \overline{x}_1$ proves property (I)(a) of SSI preference holds. Property (I)(b) of SSI preference with respect to good 1 follows from Lemma 1(II)(iii).

Proof of property (II) (weak saturation): Since B_2 is non-empty, $\exists\ x_2 \in X_2$ and $\alpha_2(x_2) \geq 0$ such that good 1 is relevant at x if $x_1 < \alpha_2(x_2)$ and it is irrelevant at x if $x_1 \geq \alpha_2(x_2)$. Taking $\overline{Q}(x_2) = \alpha_2(x_2)$ proves property (II) of SSI preference holds. From continuity and monotonicity of preference it also follows that $\underline{Q} = \overline{x}_1 \leq \overline{Q}(x_2) = \alpha_2(x_2)$ and the inequality is strict if $x_2 > 0$.

We now prove (SSI2)⇒(SSI1). We consider the SSI preference with respect to good 1 and show that it satisfies Axiom 1. Observe from the subsistence property that $[0, \underline{Q}] \subseteq A_1$ and $B_1(x_1) = M_1(x_1)$ for all $x_1 \in [0, \underline{Q}]$. Then by Lemma 1(I), it follows that Axiom 1 holds. To show that Axiom 2 holds, observe from the weak saturation property that $\{x \in X | x_1 \geq \overline{Q}\} \subseteq B_2$ so that B_2 is non-empty. ■

Proof of Corollary 1: We first prove (S1)⇒ (S2). Since Axiom 1 holds, by Lemma 1(II)(ii), B_1 has a unique maximal unhappy subset $\overline{S}$. If either (a) or (b) of Lemma 1(II)(ii) holds, then property (I) of Definition 2 holds. So suppose (c) of Lemma 1(II)(ii) holds, i.e., $\overline{S} = \mathbb{R}^2_+$. Then $A_1 = \mathbb{R}_+$ and $\alpha_1(x_1) = 0$ for all $x_1 \in \mathbb{R}_+$, implying that good 2 is irrelevant at all bundles.

To prove (S2)⇒ (S1), if property (I) of Definition 2 holds, then from the proof of Theorem 1 it follows that Axiom 1 holds. Otherwise, $B_1 = \mathbb{R}^2_+$, which is itself an unhappy set of positive area. ■

6.3 GL preferences

To prove Theorem 2 we will use the following lemmas. Given Axiom 4, Lemma 2 shows that if a good is irrelevant (relevant) at a bundle and its amount is decreased (increased), then it continues to be irrelevant (relevant) at the new bundle.

Lemma 2 *Suppose $\succsim$ satisfies Axiom 4.*

(I) *Let $i, j \in \{1, 2\}$ and $i \neq j$. For any $x_i \in X_i$, $f_i(x)$ is non-decreasing in x_j.*

(II) *If $x_i \in A_i$, then $y_i \in A_i$ and $\alpha_i(y_i) \leq \alpha_i(x_i)$ for all $y_i \in [0, x_i)$.*

Proof: W.l.o.g. take $i = 1$ and $j = 2$.

(I) We have to show that $f_2(y_1, x_2) \leq f_2(x)$ for all $y_1 < x_1$ and $f_2(y_1, x_2) \geq f_2(x)$ for all $y_1 > x_1$. Since $f_2(.)$ equals 0 or 1, it is sufficient to show: (a) if $f_2(x) = 0$, then $f_2(y_1, x_2) = 0$ for all $y_1 < x_1$ and (b) if $f_2(x) = 1$, then $f_2(y_1, x_2) = 1$ for all $y_1 > x_1$. If (a) does not hold, then $\exists$ x and $y_1 < x_1$ such that $f_2(x) = 0$ and $f_2(y_1, x_2) = 1$, i.e., $(y_1, x_2) \notin B_1$. By Axiom 4, we must have $(y_1, x_2) \in B_2$, so that $\alpha_2(x_2) \leq y_1 < x_1$. Hence $(y_1, x_2), x \in B_2(x_2)$, implying $(y_1, x_2) \sim x$. Since $f_2(x) = 0$, we have $x \sim (x_1, z_2)$ for any $z_2 > x_2$. By transitivity, $(y_1, x_2) \sim (x_1, z_2)$ which violates monotonicity, so (a) must hold. If (b) does not hold, then $\exists$ z and $\widetilde{z}_1 > z_1$ such that $f_2(z) = 1$ and $f_2(\widetilde{z}_1, z_2) = 0$. Taking $x_1 = \widetilde{z}_1$, $x_2 = z_2$ and $y_1 = z_1$ contradicts (a). Hence (b) must hold.

(II) If $x_1 \in A_1$, then $\exists$ $\alpha_1(x_1) = x_2$ such that $f_2(x_1, y_2) = 0$ $\forall$ $y_2 \geq x_2$. By Lemma 2(I), for any $y_1 \in [0, x_1)$, we have $f_2(y_1, x_2) = 0$. By definition of $\alpha_1(.)$, we have $\alpha_1(y_1) \leq x_2 = \alpha_1(x_1)$ for all $y_1 \in [0, x_1)$. ■

Since $A_i = X_i$ (by Axiom 4), $\alpha_i(.)$ is defined for any $x_i \in X_i$. Lemma 3 derives properties of this function and as a consequence we get the function $F(.)$ specified in the definition of GL preference.

Lemma 3 *Suppose the preference relation $\succsim$ on X satisfies Axiom 3 and Axiom 4. The following hold for $i, j \in \{1, 2\}, i \neq j$.*

(I) *$\alpha_i(x_i) > 0$ for any $x_i > 0$.*

(II) *$\alpha_i(0) = 0$.*

(III) *$\alpha_j(\alpha_i(x_i)) = x_i$.*

(IV) *$\alpha_i(x_i)$ is increasing for all $x_i \geq 0$.*

(V) *$\alpha_i(x_i)$ is an surjective function from X_i to X_j, i.e., for every $x_j \in X_j$, $\exists$ $x_i \in X_i$ such that $\alpha_i(x_i) = x_j$.*

Proof: W.l.o.g., take $i = 1, j = 2$.

(I) Suppose on the contrary $\alpha_1(x_1) = 0$ for some $x_1 > 0$. Then by Lemma 2(II), $\alpha_1(y_1) = 0$ for all $y_1 \in [0, x_1]$. Then by Lemma 1(I), Axiom 1 holds, contradicting Axiom 3.

(II) Suppose on the contrary $\alpha_1(0) = x_2 > 0$. Let $y_2 \in (0, x_2)$. Then $(0, y_2) \notin B_1$ (since $y_2 < \alpha_1(0)$) and $(0, y_2) \notin B_2$ (since $0 < \alpha_2(y_2)$, part (I)), i.e., $y_2 \notin B_1 \cup B_2$, which contradicts Axiom 4.

(III) By (II), the result clearly hold for $x_1 = 0$, so let $x_1 > 0$. Then $\alpha_1(x_1) > 0$ (by (I)). Let $x_2 < \alpha_1(x_1)$. Then $x \notin B_1$, so by Axiom 4 we must have $x \in B_2$, implying that $\alpha_2(x_2) \leq x_1$ for all $x_2 < \alpha_1(x_1)$.

Now we show $\alpha_2(\alpha_1(x_1)) \leq x_1$. Denote $x_2 = \alpha_1(x_1)$. Suppose on the contrary $\alpha_2(x_2) = y_1 > x_1$ and let $y_2 = x_2$. Then $y \succ x$. For any neighborhoods N_y, N_x around y, x we can find $z \in N_y, \widetilde{z} \in N_x$ such that $z_2 = \widetilde{z}_2 < x_2 = \alpha_1(x_1)$ and $z_1 > \widetilde{z}_1 \geq x_1$.

Since $x_1 \geq \alpha_2(z_2)$, we have $z, \widetilde{z} \in B_2(z_2)$, so that $z \sim \widetilde{z}$. This contradicts continuity of $\succsim$ (see Definition C1, page 16, Rubinstein (2006)), proving that $\alpha_2(\alpha_1(x_1)) \leq x_1$.

Denote $\alpha_1(x_1) = y_2$ and $\alpha_2(y_2) = y_1$. If $y_1 < x_1$, then $y, (x_1, y_2) \in B_2(y_2)$, so that $y \sim (x_1, y_2)$. Let $z_2 > y_2 = \alpha_1(x_1)$. Then $(x_1, z_2), (x_1, y_2) \in B_1(x_1)$, implying $(x_1, z_2) \sim (x_1, y_2)$. By transitivity, $y \sim (x_1, z_2)$, a contradiction (since $x_1 > y_1$ and $z_2 > y_2$). Hence we must have $y_1 \geq x_1$, i.e., $\alpha_2(\alpha_1(x_1)) \geq x_1$. From the conclusion of the previous paragraph, we conclude that $\alpha_2(\alpha_1(x_1)) = x_1$.

(IV) Since $\alpha_1(0) = 0$ and $\alpha_1(x_1) > 0$ for any $x_1 > 0$, $\alpha_1(x_1)$ is increasing at $x_1 = 0$. By Lemma 2(II), $\alpha_1(x_1)$ is non-decreasing. If it is not increasing for all $x_1 > 0$, $\exists\ x_1 > y_1 > 0$ such that $\alpha_1(x_1) = \alpha_1(y_1) = x_2 > 0$. By part (III), we then have $\alpha_2(x_2) = \alpha_2(\alpha_1(x_1)) = x_1$ and $\alpha_2(x_2) = \alpha_2(\alpha_1(y_1)) = y_1 < x_1$, a contradiction.

(V) By (II), the result holds for $x_2 = 0$. Suppose $\exists\ x_2 > 0$ such that $\alpha_1(x_1) \neq x_2$ $\forall\ x_1 \in X_1$. Since $\alpha_1(.)$ is continuous and $\alpha_1(0) = 0$, we must have $\alpha_1(x_1) < x_2$ for all $x_1 \in X_1$. By Axiom 4, $A_2 = X_2$. Hence $x_2 \in A_2$ and $\alpha_2(x_2)$ is well defined. Taking $x_1 = \alpha_2(x_2)$ above, we have $\alpha_1(\alpha_2(x_2)) < x_2$, which contradicts (III). ■

Proof of Theorem 2: $(L1) \Rightarrow (L2)$ By Axiom 4, for $i = 1, 2$, $A_i = X_i$ and $\alpha_i(x_i)$ is well defined for all $x_i \in X_i$. Note from Lemma 3 that $\alpha_1(.) : X_1 \to X_2$ is an increasing and surjective function with $\alpha_1(0) = 0$ (the same property holds for $\alpha_2(.) : X_2 \to X_1$ and $\alpha_2(.)$ is the inverse function of $\alpha_1(.)$). Taking $F(x_1) = \alpha_1(x_1)$, by Lemma 3(III) it follows that (i)-(iii) of Definition 4 hold.

$(L2) \Rightarrow (L1)$ Suppose the preference is generalized Leontief. Then for $i = 1, 2$, $A_i = X_i = \mathbb{R}_+$. For any $x_1 \in X_1$, we have $\alpha_1(x_1) = F(x_1)$ and for any $x_2 \in A_2$, we have $\alpha_2(x_2) = F^{-1}(x_2)$, and $F(0) = 0$. Hence $B_1(x_1) = \{(x_1, x_2) | x_2 \geq F(x_1)\}$ and $B_2(x_2) = \{(x_1, x_2) | x_1 \geq F^{-1}(x_2)\}$. So we have $B_i = \cup_{x_i \in X_i} B_i(x_i)$ for $i = 1, 2$, and $B_1 \cup B_2 = X$. Therefore, Axiom 4 holds.

It remains to show that Axiom 3 holds. If for some $i = 1, 2$, $\exists\ S \subseteq B_i$ such that S is an unhappy set of positive area, then $\exists\ x \in S$ such that $x_i > 0$. By Lemma 1(II)(i), this will imply that $\alpha_i(x_i) = 0$ for all $y_i \in [0, x_i]$, a contradiction since $\alpha_i(y_i) > 0$ for all $y_i > 0$. ■

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