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Abstract

We study the relationship between two widely used risk measures, the spectral measures and the distortion risk measures. In both cases, the risk measure can be thought of as a re-weighting of some initial distribution. We prove that spectral risk measures are equivalent to distorted risk pricing measures, or equivalently, spectral risk functions are related to distortion functions. Besides that we prove that distorted measures are absolutely continuous with respect to the original measure.

Key words. Coherent risk measure, distortion function, Spectral measures, Risk Aversion Function.

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1 Introduction

The quantification of market risk for derivative pricing, for portfolio optimization and pricing risk for insurance purposes has generated a large amount of theoretical and practical work, with a variety of interconnections.

Two lines of research in these areas are based upon, and depart from, foundational desiderata consisting of establishing axioms that both, the market risk measure and the risk pricing measure, have to satisfy.

Value at Risk (VaR) is one of most popular risk measures, due to its simplicity. VaR indicates the minimal loss incurred in the worse outcomes of the portfolios. But this risk measure is not always sub-additive, nor convex. So, Artzner, Delbaen, Ebner and Heath (1999) proposed the main properties that a risk measures must satisfy, thus establishing the notion of coherent risk measure.

After coherent risk measures and their properties were established, other classes of measures have been proposed, each with distinctive properties: convex (Föllmer and Shied, 2002), spectral (Acerbi, 2002) or deviation measures (Rockafellar et al. 2006).

The coherent risk measures were used for capital allocation and portfolio optimization as in Rockafellar, Uryasev and Zabarankin (2002), as well as to price options in incomplete markets, as in Cherny (2006).

The spectral risk measures are coherent risk measure that satisfies two additional conditions. These measures have been applied to futures clearinghouse margin requirements in Cotter and Down (2006). Acerbi and Simonetti (2002) extend the results of Pflug-Rockafellar-Uryasev methodology to spectral risk measures.

A description of the axioms of risk pricing measures with many applications to insurance can be found in Wang, Young and Panjer (1997), in Wang (1998) and in the monograph by Kass, Goovaerts, Dhaene and Denuit (2001). From this line of work has evolved the concept of distorted risk measure, which ties up with the older notion of capacity. Capacities are non-additive, monotone set functions which extend the notion of integral in a peculiar way. The evolution of this concept, from Choquet's work in the 1950's until the 1990's can be traced back from the review by Denneberg (1997).

Interestingly enough, there have been some natural points of contact between actuarial and financial risk theory. One one hand, concepts in actuarial risk theory can be used to solve problems in derivative pricing, and vice versa. A few papers along these lines are the ones by Embrechts (1996) Gerber and Shiu (2001), Schweitzer (2001), Goovaerts and Laeven (2006) and Madan and Unal (2004).

So, in the literature there are a lot set of risk measures, the difference is the properties that satisfies. It is very interesting study the equivalence between this risk measures. We shall establish an equivalence between spectral risk measures, a special class of risk measures and distorted risk pricing measures. Then we shall examine some other way of computing distorted measures.

This paper is organized as follows: in the Section 2 we introduce the concept of coherent and spectral risk measure and the idea of a distortion measure. We present different examples of these measures. In the Section 3 we present that exist a relationship between the spectral risk measure and the distortion risk measure, so we proof that all spectral risk measure is defined by a concave distortion risk measure. We show that the inverse relationship is verified, all risk coherent distortion measure is a spectral risk measure. Moreover, we obtained the form of the distortion function and the Risk Aversion function in both cases. In the Section 4 exploring the nature of the distorted distribution function and the relationship the distorted distributions between different investors. Finally, the Section 5 conclude the paper.

2 Preliminaries

We shall consider a one period market model $(\Omega, \mathcal{F}, P)$. The information about the market, that is the σ-algebra $\mathcal{F}$, can be assumed to be generated by a finite collection of random variables, i.e., $\mathcal{F} = \sigma(S_0, S_1, ...S_N)$, where the $\{S_j \,|\, j = 0, ..., N\}$ are the basic assets traded in the market. We shall model the present worth of our position by $X \in \mathcal{L}_\infty(P)$ (as Delbaen (2003)), that is, essentially all bounded random variables. This somewhat restrictive framework greatly simplifies the proofs.

Definition 2.1 *A coherent risk measure is defined to be a function $\rho : \mathcal{L}_\infty(P) \to \mathbb{R}$ that satisfies the following axioms:*

1. *Translation Invariance: For any $X \in \mathcal{L}_\infty(P)$ and $a \in \mathbb{R}$, we have $\rho(X+a) = \rho(X) - a$.*

2. *Positive homogeneity: For any* $X \in \mathcal{L}_\infty(P)$ *and* $\lambda \geq 0$*, we have* $\rho(\lambda X) = \lambda\rho(X)$.

3. *Monotonicity: For any* X *and* $Y \in \mathcal{L}_\infty(P)$*, such that* $X \leq Y$ *then* $\rho(X) \geq \rho(Y)$.

4. *Subadditivity: For any* X *and* $Y \in \mathcal{L}_\infty(P)$*,* $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

These properties insure that the diversification reduce the risk of the portfolio and if position size directly increase risk (consequences of lack of liquidity) it is been computed in the future net worth of the position.

One example of coherent risk measures is the Conditional Value at Risk (CVaR). This measure indicate the expected loss incurred in the worse cases of the position. It is the most popular alternative to the Value at Risk.

$$CVaR_\alpha(X) = -E_P[X \mid X \leq -VaR_\alpha(X)]\ .$$

where $VaR_\alpha(X) = -inf\{x : P[X \leq x] > \alpha\}$

The spectral risk measures are defined by a general convex combination of Conditional Value at Risk.

Definition 2.2 *An element* $\phi \in \mathcal{L}_1([0,1])$ *is called an admissible risk spectrum if*

1. $\phi \geq 0$

2. ϕ *is decreasing*

3. $\|\phi\| = \int_0^1 |\phi(t)|dt = 1$.

Definition 2.3 *Let, an admissible risk spectrum* $\phi \in \mathcal{L}_1([0,1])$ *the risk measure*

$$\rho_\phi(X) = -\int_0^1 q_X(u)\phi(u)du$$

is called the spectral risk measure generated by ϕ.

ϕ is called the Risk Aversion Function and assigns, in fact, different weights to different p-confidence level of the left tail. Any rational investor can express her subjective risk aversion by drawing a different profile for the weight function ϕ. The spectral risk measures are a subset of coherent risk measures as Acerbi proves. Specifically, a spectral measure can be associated with a coherent risk measures that has two additional properties, law invariance and comonotone additivity. Law invariance in particular is a important property for applications since it is a necessary property for a risk measure to be estimable from empirical data.

Theorem 2.1 *The risk measure $\rho_\phi(X)$ be defined by*

$$\rho_\phi(X) = -\int_0^1 q_X(u)\phi(u)du \tag{1}$$

is a coherent risk measure. Here, for $u \in (0,1)$, $q(u) = \inf\{x \,|\, F(x) \geq u\}$ is the left continuous inverse of $F(x) = P(X \leq x)$

Comment 2.1 *Note that if $X \geq 0$, then $q(u) \geq 0$ and $\rho(X) < 0$, that is, positive worth entails no risk.*

Example 2.1 *The Conditional Value at Risk is a spectral risk measure defined by the Risk Aversion Function:*

$$\phi(p) = \frac{1}{\alpha} 1_{\{0 \geq p \geq \alpha\}} \tag{2}$$

Example 2.2 *Other example of Risk Aversion Function is defined by Cotter and Dowd (2006)*

$$\phi(u) = \frac{Re^{-R(1-u)}}{1-e^{-R}}$$

where R is the user's coefficient of absolute risk aversion.

The Value at Risk is not a spectral risk measure because it is not a coherent risk measure and it not satisfies the comonotone additive property.

On the other hand, Wang (1996) defines a family of risk measures by the concept of distortion function as introduced in Yaaris dual theory of choice under risk. So, the distortion risk measures are defined by a distortion function.

Definition 2.4 *We shall say that $g : [0,1] \to [0,1]$ is a distortion function if*

1. $g(0) = 0$ *and* $g(1) = 1$.
2. g *is non-decreasing function.*

For applications to insurance risk pricing it is convenient to think of the liabilities as positive variables, we restrict ourselves to $X \in \mathcal{L}_\infty^+(P)$, i.e., to positive random variables, which we think about as losses or liabilities. If we were to relate this to the previous interpretation, we would say that our position is $-X$. The companion theorem characterizing the distorted risk measure induced by g is the following.

Theorem 2.2 *Define the distorted risk measure $D_g(X)$ induced by g on the class $\mathcal{L}_\infty(P)$ by*

$$D_g(X) = \int_0^\infty g(S(x))dx + \int_{-\infty}^0 [g(S(x)) - 1]dx. \tag{3}$$

where $S(x) = 1 - F_X(x)$. Then $D_g(X)$ has the following properties:

1. $X \leq Y$ *implies* $D_g(X) \leq D_g(Y)$.
2. $D_g(\lambda X) = \lambda D_g(X)$ *for all positive* λ. $D_g(c) = c$ *whenever* c *is a constant risk.*
3. *If the risks X and Y are comonotone, then* $D_g(X + Y) = D_g(X) + D_g(Y)$.
4. *If g is concave then* $D_g(X + Y) \leq D_g(X) + D_g(Y)$.
5. *If g is convex then* $D_g(X + Y) \leq D_g(X) + D_g(Y)$.

Hardy and Wirch (2001) have shown that a risk measure based on a distortion function is coherent if and only if the distortion function is concave. So, it can be shown that if g is concave the generated risk measure is spectral.

A distortion risk measure is the expectation of a new variable, with changed probabilities, re-weighting the initial distribution.

Example 2.3 *The VaR can be defined by the distortion function:*

$$g(x) = \begin{cases} 0 & \text{if } x < \alpha \\ 1 & \text{if } x \geq \alpha \end{cases}, \tag{4}$$

Example 2.4 *The CVaR is a distortion risk measure with respect to the following distortion function:*

$$g(x) = \begin{cases} \frac{x}{\alpha} & \text{if } x \leq \alpha \\ 1 & \text{if } x \geq \alpha \end{cases} . \tag{5}$$

Example 2.5 *Some distortion risk functions are used for insurance risk pricing.*

1. *Dual-power functions:* $g(u) = 1 - (1-u)^{\nu}$ *with* $\nu \geq 1$.
2. *Proportional Hazard transform:* $g(u) = u^{\frac{1}{\gamma}}$ *with* $\gamma \geq 1$.
3. *Wang's distortion function:* $g_{\alpha}(u) = \Phi[\Phi^{-1}(u) + \alpha], \quad u \in (0,1)$ *where* Φ *is the standard Normal distribution.*

More examples are quadratic function or Denneberg's absolute deviation principle (see Wang 1996 for more details).

The Wang's distortion function is defined to pricing financial and insurance risks. Wang transform risk measure uses the whole distribution and that it accounts for extreme low-frequency and high severity losses.

Let us now recall a couple of results about quantiles. The following are taken from the nice expose by Laurent(2003), where the results we spell out in section 2 are hinted at. First of all we need the notion of set of quantiles.

Definition 2.5 *Given a probability space* $(\Omega, , P)$ *as above and a random variable* X *and* $\alpha \in (0,1)$, *the* α*-quantile set of* X *is defined to be*

$$Q_X(\alpha) = \{x \in \mathbb{R} \,|\, P(X < x) \leq \alpha \leq P(X \leq x)\}.$$

Theorem 2.3 *With the notations introduced above*

$$Q_X(\alpha) = [q_X(\alpha), q_X^{+}(\alpha)],$$

where, as above

$$q_X(\alpha) = \inf\{x \,|\, P(X \leq x) \geq \alpha\} = \sup\{x \,|\, P(X < x) < \alpha\},$$

and

$$q_X^+(\alpha) = \sup\{x \mid P(X < x) \geq \alpha\} = \inf\{x \mid P(X \leq x) > \alpha\}.$$

The following characterization is important: for $u \in (0,1)$ *and* $x \in \mathbb{R}$ *we have*

$$q_X^+(u) \geq x \Leftrightarrow P(X < x) \geq u.$$

Also, for $\alpha \in (0,1)$,

$$Q_X(\alpha) = [q_X(\alpha), q_X^+(\alpha)],$$

which can be used to establish that

$$Q_{-X}(\alpha) = -Q_X(1-\alpha),$$

and in particular that $q_{-X}(\alpha) = -q_X(1-\alpha)$.

For the proof of the first theorem of section 3, we shall need the following version of the transference theorem (see section 6.5 in Kingman and Taylor(1966)). Set $G(x) = P(X < x) = F(x-)$. Then clearly $G(x)$ is increasing and left continuous. We have

Theorem 2.4 *(Transference theorem) For every positive, measurable* $h : (0,1) \to \mathbb{R}$ *we have*

$$\int_0^1 h(u)dq^+(u) = \int_{\mathbb{R}} h(G(x))dx,$$

where q^+ *denotes the right quantile of* $F(x) = P(X \leq x)$.

Proof It suffices to prove the result for $h(u) = I_{(a,b]}(u)$ with $0 < a < b \leq 1$. In this case, involving the characterization mentioned in theorem 2.3, we have that

$$\int_0^1 I_{(a,b]}(u)dq^+(u) = q^+(b) - q^+(a) = \int_{\mathbb{R}} I_{(q^+(a),q^+(b)]}(x)dx = \int_{\mathbb{R}} I_{(a,b]}(G(x))dx$$

which concludes our proof. □

Corollary 2.1 *Under the assumptions of the theorem we have*

$$\int_0^1 h(u)dq^+(u) = \int_{\mathbb{R}} h(F(x))dx.$$

Proof Just recall that G(x) differs from $F(x)$ at a countable set of points □

3 Equivalence between spectral and distortion risk measures

In this section, we prove the relationship between the spectral measures of risk and the distorted measures.

Theorem 3.1 *Let ϕ be a piecewise continuous, admissible spectral function and let $\rho_\phi(X)$ the spectral risk measure be defined on the class of positive and bounded risks. Then for every X, $D_g(X) = \rho_\phi(-X)$ is a coherent distortion risk measure with concave distortion function satisfying $g'(u) = \phi(u)$*

Proof Consider $\rho_\phi(-X) = -\int_0^1 q_{-X}(u)\phi(u)du$ and note that $q_{-X}(u) = -q_X^+(1-u)$. The fact that X is bounded, say, $m \leq X \leq M$ is used to assert that $q_X(0) = m$ and $q_X(1) = M$. Consider now

$$\begin{aligned}\rho_\phi(-X) &= -\textstyle\int_0^1 q_{-X}(u)\phi(u)du = \int_0^1 q_X^+(1-u)\phi(u)du \\ &= \textstyle\int_0^1 q_X^+(u)\phi(1-u)du = \int_0^1 q_X^+(u)d\psi(u),\end{aligned}$$

where $\psi(u) = \int_{(1-u)}^1 \phi(s)ds$. Clearly, the assumptions about ϕ yield that $\psi(0) = 0$ and $\psi(1) = 1$. Invoke now integration by parts to obtain

$$\int_0^1 q_X^+(u)d\psi(u) = \psi(1)q_X^+(1) - \psi(0)q_X^+(0) - \int_0^1 \psi(u)dq_X(u) = \int_0^1 (1-\psi(u))dq_X^+(u).$$

Now, bring in the definition of ψ and a simple change of variables formula to recompose the two chains into

$$\rho_\phi(-X) = \int_0^1 (1-\psi(u))dq_X^+(u) = \int_0^1 g(1-u)dq_X^+(u) = \int_0^1 g(1-F_X(x))dx = D_g(X),$$

after invoquing the transference theorem and where we did set $g(u) = \int_0^u \phi(s)ds$ thus concluding the proof. □

In the next result we show how to use the previous theorem to take care of the case in which our position is described by a bounded, non necessary positive random variable.

Theorem 3.2 *Let ϕ be an admissible spectral function and let $\rho_\phi(X)$ be defined on the class $\mathcal{L}_\infty(P)$. Then the identity $D_g(-X) = \rho_\phi(X)$ holds with distortion function satisfying $g'(u) = \phi(u)$.*

Proof Assume $m \leq X \leq M$ for some $m < 0 < M$. Let us consider the shifted loss position $M - X$. Then $0 \leq M - X \leq M - m$. According to theorem 3.1, $D_g(M - X) = \rho_\phi(X - M)$ with ρ and ϕ as in the statement. On the other hand, a two short computations allow us to verify that $\rho_\phi(X - M) = \rho_\phi(X) + M$ as well as $D_g(M - X) = D_g(-X) + M$. Thus concludes our proof. $\square$

Comment 3.1 *By appropriate truncation we could extend to position (risks) in $\mathcal{L}_2(P)$, provided the needed continuity is established.*

Example 3.1 *The risk measure CVaR is a spectral risk measure (see Example 2.1). If we apply the last Theorem to (2) we have that the Conditional Value at Risk is a distortion risk measure defined by:*

$$g(u) = \int_0^u \phi(s)ds = \int_0^u \frac{1}{\alpha} 1_{\{0 \geq s \geq \alpha\}} = \begin{cases} \frac{u}{\alpha} & if\, u \leq \alpha \\ 1 & if\, u \geq \alpha \end{cases}$$

We have obtained the same result that in Example 2.4

Example 3.2 *We can calculate the Risk Aversion Function for the distortion risk function in Example 2.5.*

1. *Dual-power measure: $\phi(u) = (1 - u)^\nu$ with $\nu \geq 1$.*
2. *Proportional Hazard measure: $\phi(u) = \frac{1}{\gamma} u^{\frac{1}{\gamma} - 1}$ with $\gamma \geq 1$.*
3. *Wang's measure: $\phi_\alpha(u) = e^{[\alpha\Phi^{-1}(u) - \frac{\alpha^2}{2}]}$.*

Observe that for Proportional Hazard and Wang's measures, the Risk Aversion function is not bounded at zero. Moreover, the Risk Aversion function of the Wang's measure decreases more quickly than that of the Proportional Hazard. So, the investor using the Wang's risk measure is more risk averse than other investor that measure the risk by the Proportional Hazard distortion because the first investor give more importance to the higher loss than the last one.

The previous theorem admits the following reciprocal, the proof of which can follows reversing the steps of the proof of the previous theorem.

Theorem 3.3 *Let g a concave distortion function, and let D_g be the associated distorted risk measure. Then $\phi(u) = g'(u)$ defines a spectral measure ρ_ϕ such that $\rho(X) = D_g(-X)$.*

As simple example, it is easy to check that the derivative of the distortion risk associated to Value at Risk (5) is the Risk Aversion Function of CVaR (2)

We have thus established that both methods to construct risk measures, either by means of distortion risk functions or by admissible spectral functions, are equivalent. In both, the risk measure can be thought of as a re-weighting of the initial distribution. Moreover, the derivative of the distortion risk function indicate the way of this re-weighting, as Balbas et.al (2006) have indicated.

Comment 3.2 *These correspondences also provide an indirect proof of the fact that for a concave distortion function g, the risk measure defined by 3 is a coherent risk measure.*

4 A representation theorem

The following is a formalization of an idea implicit in Reesor and McLeish's (2002) work. It involves exploring the nature of the distorted distribution function $F_X^*(x) = g(F_X(x))$. One such study was undertaken by Hurlimann in several papers, but it goes in a different direction than the one we follow here. Reesor and McLeish establish a link between the risk measures defined by a relative entropy and a distortion risk measure.

We begin with a result in measure theory.

Theorem 4.1 *Let $dm^* = dF^*$ and $dm = dF$ be two measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $m^* << m$ having continuous density ψ. Then there exists a distortion function g such that $F^*(x) = g(F(x))$.*

Proof Define $g(u) = \int_0^u \psi(q(s))ds$, where for $0 < u < 1$ we denote by $q(u)$ the left continuous inverse of F. Clearly, g is increasing, continuous, with $g(0) = 0$ and $g(1) = 1$.

Let us now verify that $g(F(x)) = F^*(x)$. An application of the transference theorem (or a variation on the change of variables theme, see [KT]) yields that

$$F^*(x) = \int_{-\infty}^{x} \psi(t)dF(t) = \int_{\mathbb{R}} I_{(-\infty,x])}(t)\psi(t)dF(t)$$

$$= \int_0^1 I_{(-\infty,x])}(F^{-1}(u))\psi(F^{-1}(u))du = \int_0^{F(x)} \psi(F^{-1}(u))du = g(F(x))$$

since $u \leq F(x) \Leftrightarrow F^{-1}(u) \leq x$, and we are through. $\square$

Theorem 4.2 *Let g be a piecewise continuously differentiable distortion function as above. Then the measure $dm^* = dF^*_X$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ induced by the distorted distribution function F^*_X is absolutely continuous with respect to $dm = dF_X$ having density $\psi(x) = g'(F_X(x))$.*

Proof It boils down to noticing that $F^*_X(x) = g(F_X(x))$ implies that $dF^*_X(x) = g'(F_X(x))dF_X(x)$, which stochastic calculus buffs may want to think about as the finite variation version of Ito's formula. $\square$

Comment 4.1 *To relate our result with the special case considered in* [RM], it is sufficient to assume that $\psi(x) = \frac{1}{Z(\lambda)}\exp(\sum_{i=1}^{N} \lambda_i h_i(x))$, where $h_i(x)$ is some finite collection of independent functions, which are assumed to be such that all integrals displayed converge. Let $Z(\lambda) = \int_{-\infty}^{\infty} \exp(\sum_{i=1}^{N} \lambda_i x^i)dF(x)$ a normalization factor, and the λ_j are to be chosen so that $\int x^j dF^*(x) = m_j$, and the m_j are known moments. Then according to the Theorem 4.1, $F^*(x) = g(F(x))$ for an appropriate g ask Reesor and McLeish showed.

We have a the following simple observation: If $\int h_i(x)dF(x) = \mu_i$ are known generalized moments, and g is as in theorem 4.1, then $F^*(x) = g(F(x))$ has moments $\int \hat{h}_i(x)dF^*(x) = \mu_i$ with $\hat{h}_i(x) = h_i(x)/g'(F(x))$.

Comment 4.2 *Consider two agents that assign different physical measures to their market models. Let $F^*(x)$ and $F(x)$ be the distribution function describing the statistical nature of some asset to each of them. Intuitively we may expect that $dF^* \sim dF$. What theorem 4.1 asserts that upon some conditions on the density of F^* with respect to F, each may conclude that the other has a distorted view of reality with respect to him/herself.*

Comment 4.3 *Two agents may have the same point of view of reality, that is, both agents have the same market model, but may have different risk aversion functions, for example the first agent measure his level of risk by the distortion function g_1 and the agent two by g_2. If the agents have the same opinion about what losses are important, that is, the loss which they assign a new probability positive, or in the same sense the percentiles that they consider to measure the lever of their risk.*

In this case, $F_1^*(x) = g_1(x)$ and $F_2^*(x) = g_2(x)$ are absolutely continuous one respect the other. And applying the Theorem 4.1 we have $F_1^*(x) = h(F_2^*(x))$. If both distortion functions are strictly increasing and continuous, the difference of the agent's risk aversion is given by $h = g_1 \circ g_2^{-1}$.

5 Conclusions

The paper prove that the spectral risk measures are related to distorted risk pricing measures. Thus we have two representations at hand for a given measure, and may choose which representation is more convenient for the application at hand. Also, distorted risk pricing measures are absolutely continuous with respect to the measure that they distort. This allows us, for example, to interpret different physical probabilities (or different generalized scenarios) as distorted views of reality, one with respect to the other.

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