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On Societies Choosing Social Outcomes, and their Memberships: Internal Stability and Consistency*

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Abstract: We consider a society whose members have to choose not only an outcome from a given set of outcomes but also the subset of agents that will remain members of the society. We study the extensions of approval voting, plurality voting, Borda methods and Condorcet winners to our setting from the point of view of their consistency and internal stability properties.

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1 Introduction

Classical social choice studies problems where a fixed set of agents have to choose an outcome from a given set of social outcomes, and agents have preferences only on the set of outcomes. However, there are settings where, depending on the chosen outcome, some agents may not want to remain in the society. For instance, membership of a political party may depend on the positions that the party takes on issues like the death penalty, abortion, or the possibility of allowing the independence of a region of the country. A professor in a department may consider to look for a position in another university if he considers that the recruitment of the department has not being satisfactory to his standards. Hence, to be able to deal with such situations the classical social choice model has to be modified to include explicitly the possibility that members may leave the society as the consequence of the chosen outcome.

There is a large literature that has already considered explicitly, in specific settings, the dependence of the final society on the choices made by the initial society.[1] Those papers study alternative models in terms of the voting methods under which members choose the social outcome and the timing under which members reconsider their membership. In this note (as we also do in the companion paper Bergantiños, Massó, and Neme (2016)) we look at the general setting without being specific about the two issues. We do that by considering that the set of alternatives are all pairs formed by a subset of the original society (an element in 2^N, the subset of agents that will remain in the society) and an outcome in X. Then, we assume that agents' preferences are defined over the set of alternatives $2^N \times X$ and satisfy two natural requirements. First, each agent has strict preferences between any two alternatives, provided the agent belongs to at least one of the two corresponding societies. Second, each agent is indifferent between two alternatives, provided the agent is not a member of any of the two corresponding societies; namely, agents do not care about the outcome chosen by societies they do not belong to.

We consider rules that operate on this restricted domain of preference profiles by selecting, for each profile, an alternative (a final society and an outcome). In Bergantiños, Massó, and Neme (2016) we characterize the class of strategy-proof, unanimous and non-bossy rules as the family of all serial dictator rules.

[1] See for instance Barberà, Maschler and Shalev (2001), Barberà and Perea (2002), and Berga, Bergantiños, Massó and Neme (2004, 2006).

For applications where the profile is common knowledge (and hence, the revelation of agents' preferences is not an strategic issue) we focus on internal stable and consistent rules (see Thomson (1994, 2007) and Bergantiños, Massó and Neme (2015) for the study of consistent rules in other social choice settings). Internal stability is a minimal requirement of individual rationality, and it is a desirable property whenever membership is voluntary (*i.e.*, nobody can force an agent to remain in the society if the agent does not want to do so). A rule is consistent if the following property holds. Apply the rule to a given profile and consider the new problem where the new society is formed by the chosen subset of agents at the original profile. A consistent rule chooses at the subprofile of preferences of the agents that remain members of the society the same alternative. Thus, a consistent rule does not require to reapply the rule after an alternative has been chosen.

We adapt well-known voting methods to our setting, with the goal of making them either internally stable or consistent, or both. We show that plurality voting and the Borda method do not satisfy consistency. However, approval voting not only satisfies internal stability and consistency but it also satisfies efficiency, neutrality and participation (which requires individual rationality for all agents, not only for those agents remaining in the society). Finally, we show that the Condorcet winner is internal stable, consistent, efficient, anonymous, neutral and satisfies participation at those profiles where an alternative beats all other alternatives by majority voting.

The paper is organized as follows. In Section 2 we describe the model. Section 3 contains the definitions of the properties of rules that we will be interested in. Section 4 contains the analysis of well-known rules from the point of view of their internal stability and consistency properties.

2 Preliminaries

This section follows closely Bergantiños, Massó, and Neme (2016). Let $N = \{1, ..., n\}$ be the set of *agents*, where $n \geq 2$, and let X be the set of possible *outcomes*. We are interested in situations where some agents may not be part of the final society, perhaps as the consequence of the chosen outcome. To model such situations, let $A = 2^N \times X$ be the set of (social and final) *alternatives* and assume that each agent $i \in N$ has preferences over the set of possible alternatives A. We will often use the notation a for a generic alternative $(S, x) \in A$; *i.e.,* $a \equiv (S, x)$, $a' \equiv (S', x')$, and so on. Let R_i denote agent i's (weak) *preference* over A, where for any pair of alternatives $a, a' \in A$, aR_ia' means that agent i considers alternative a to be at least as good as alternative a'. Let P_i and I_i denote the strict and indifference relations induced by R_i over A, respectively; namely, for any pair of alternatives $a, a' \in A$, aP_ia' if and only if aR_ia' and $\neg a'R_ia$, and aI_ia' if and only if aR_ia' and $a'R_ia$. We assume that agents do not care about alternatives whenever they do not belong to their corresponding final societies and they are not indifferent between pairs of alternatives whenever they do belong to at least one of the two corresponding final societies. Namely, we assume that agent i's preferences R_i over A satisfy the following two properties: for all $x, y \in X$ and $S, T \in 2^N$,

(P.1) if $i \notin S \cup T$ then $(S, x)\, I_i\, (T, y)$; and

(P.2) if $i \in S \cup T$ and $(S, x) \neq (T, y)$ then either $(S, x)\, P_i\, (T, y)$ or $(T, y)\, P_i\, (S, x)$.

Let $\mathcal{R}_i$ be the set of preferences of agent $i \in N$ over A satisfying (P.1) and (P.2), and let $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$ be the set of (preference) *profiles*.

We denote the subset of alternatives that agent i is indifferent to any alternative for which i is not a member of the corresponding final society by

$$[\varnothing]_i = \{a \in A \mid aI_i\,(\varnothing, x) \text{ for some } x \in X\}\,.$$

By (P.1), $(\varnothing, x)I_i(\varnothing, y)$ for all $x, y \in X$ and $[\varnothing]_i$ is the indifference class generated by the empty society. Observe that $[\varnothing]_i$ may be at the top of i's preferences. With an abuse of notation we often treat, when listing a preference ordering, the indifference class $[\varnothing]_i$ as if it were an alternative; for instance, given R_i and $a \in A$ we write $aR_i[\varnothing]_i$ to represent that aR_ia' for all $a' \in [\varnothing]_i$.

The *top* of R_i, denoted by $\tau(R_i)$, is the set of all best alternatives according to R_i; namely,

$$\tau(R_i) = \{(S, x) \in A \mid (S, x)\, R_i\, (T, y) \text{ for all } (T, y) \in A\}\,.$$

A *rule* is a social choice function $f : \mathcal{R} \to A$ selecting, for each profile $R \in \mathcal{R}$, an alternative $f(R) \in A$. To be explicit about the two components of the alternative chosen by f at R, we will often write $f(R)$ as $(f_N(R), f_X(R))$, where $f_N(R) \in 2^N$ and $f_X(R) \in X$.

To clarify the model, we relate it with the two examples used in the introduction. The initial members of the political party correspond to the agents, the set of outcomes to the set of choices and the set S, if the chosen alternative is (S, x), to the set of final members of the party that still want to stay after it supports outcome x. Similarly, all professors in the department correspond to the agents, the set of outcomes to all subsets of hired candidates and the set S, if the chosen alternative is (S, x), to the set of professors who remain in the department after x has been selected.

3 Properties of rules

In this section we present several properties that a rule may satisfy. The first three impose conditions on f at each profile.

A rule $f : \mathcal{R} \to A$ is efficient if it always selects a Pareto optimal allocation.

Efficiency For each $R \in \mathcal{R}$ there is no $a \in A$ with the property that $aR_i f(R)$ for all $i \in N$ and $aP_j f(R)$ for some $j \in N$.

The next property is related to the stability of a rule $f : \mathcal{R} \to A$, and captures the idea that agents are able to exit a society at their free will. Internal stability says that no agent belonging to the final society would prefer to leave it.

Internal stability For all $R \in \mathcal{R}$ and all $i \in f_N(R)$, $f(R)\, P_i\, [\varnothing]_i$.

A rule satisfies the property of participation if all agents prefer to be involved in the election of the social alternative rather than to exclude themselves by not submitting their preferences; namely, participation guarantees that the procedure to choose the social alternative is individually rational.

Participation For all $R \in \mathcal{R}$ and all $i \in N$, $f(R)\, R_i [\varnothing]_i$.

Although by definition, participation seems stronger than internal stability, they are equivalent indeed.

Lemma 1 *A rule satisfies participation if and only if is internally stable.*

Proof It is evident that participation implies internal stability. Assume f satisfies internal stability and let $R \in \mathcal{R}$ and $i \in N$. Since $f_{N\setminus\{i\}}\left(R_{|N\setminus\{i\}}\right) \subset N\setminus\{i\}$, $(f_{N\setminus\{i\}}\left(R_{|N\setminus\{i\}}\right), f_X(R_{|N\setminus\{i\}})) \in [\varnothing]_i$. We distinguish between two cases. First, $i \in f_N(R)$. Since f satisfies internal stability, $f(R)\, P_i\, [\varnothing]_i$. Second, $i \notin f_N(R)$. Hence, $f(R) \in [\varnothing]_i$. Thus, $f(R) R_i [\varnothing]_i$.

The next three properties impose conditions on a rule by comparing the alternatives chosen by the rule at two different profiles. A rule is anonymous if the names of the agents are not relevant to select the alternative. To define it formally, let $\pi : N \to N$ be a permutation (one-to-one mapping) of the set of agents. Given $i \in N$, $\pi(i)$ is the agent assigned to i after applying the permutation π to N. The set of all permutations $\pi : N \to N$ will be denoted by Π. For $\pi \in \Pi$ and $1 \leq k \leq n$, we write π_k to denote the agent $\pi^{-1}(k)$. Let $S \in 2^N$ be a subset of agents and π be a permutation of N. Denote by $\pi(S)$ the subset of agents associated to S by π; namely, $\pi(S) = \{i \in N \mid \pi(j) = i \text{ for some } j \in S\}$. Let $R \in \mathcal{R}$ be a profile and $\pi \in \Pi$ be a permutation of the set of agents N. Denote by R^π the new profile where for all $i \in N$, agent $\pi(i)$ has the preference R_i after replacing in the preference R_i each alternative (S, x) by $(\pi(S), x)$.

ANONYMITY For all $R \in \mathcal{R}$ and all permutation $\pi \in \Pi$ of the set of agents, $f_N(R^\pi) = \pi(f_N(R))$ and $f_X(R^\pi) = f_X(R)$.

Remark 2 In our setting anonymity and efficiency are incompatible (no rule satisfies both properties). To see that consider the case where $N = \{1, 2\}$, $X = \{x\}$, and R_1 and R_2 are as follows: $(\{1\}, x) \, P_1 \, [\varnothing]_1 \, P_1 \, (N, x)$ and $(\{2\}, x) \, P_2 \, [\varnothing]_2 \, P_2 \, (N, x)$. If f is efficient, $f(R) \in \{(\{1\}, x), (\{2\}, x)\}$. Suppose $f(R) = (\{1\}, x)$; *i.e.*, $f_N(R) = \{1\}$ (the other case proceeds similarly, and hence we omit it). Consider the permutation π where $\pi(1) = 2$ and $\pi(2) = 1$. Since $\pi(\{1\}) = \{2\}$, $R^\pi = (R_2, R_1)$ and the sets of efficient alternatives at R and at R^π coincide, $f_N(R^\pi) = \{1\} \neq \{2\} = \pi(\{1\}) = \pi(f_N(R))$. Hence, f is not anonymous.

A rule is neutral if the name of the outcome does not play any role in selecting the social alternative. To define it formally, let $\sigma : X \to X$ be a permutation of the set of outcomes. Given $x \in X$, $\sigma(x)$ is the outcome assigned to x after applying the permutation σ to X. The set of all permutations $\sigma : X \to X$ will be denoted by Σ. For $\sigma \in \Sigma$ and $x \in X$, we write σ_x to denote the outcome $\sigma^{-1}(x)$. Let $Y \subseteq X$ be a non-empty subset of outcomes and σ be a permutation of X. Denote by $\sigma(Y)$ the subset of outcomes associated to Y by σ; namely, $\sigma(Y) = \{x \in X \mid \sigma(y) = x \text{ for some } y \in Y\}$. Let $R \in \mathcal{R}$ be a profile and $\sigma \in \Sigma$ be a permutation of the set of outcomes X. Denote by R^σ the new profile where for all $i \in N$ the preference R_i^σ is obtained from R_i after replacing each pair (S, x) by $(S, \sigma(x))$.

NEUTRALITY For all $R \in \mathcal{R}$ and all permutation $\sigma \in \Sigma$ of X, $f(R^\sigma) = (f_N(R), \sigma(f_X(R)))$.

A rule is consistent if the following requirement holds. Apply the rule to a given profile and consider the subset of agents that are members of the chosen society. Construct the new subprofile of preferences restricted to this new set of chosen agents. Then, the rule does not require to modify the chosen alternative because when applied to the new subprofile the new alternative coincides with the alternative chosen at the original profile. To define the property formally, we first need an additional notation. Given a profile $R = (R_i)_{i \in N} \in \mathcal{R}$ and a subset of agents $S \subset N$ we denote by $R_{|S}$ the restriction of R to 2^S. Namely, given $i \in T \cap T'$, $T \cup T' \subset S$ and $x, y \in X$, $(T, x) \left(R_{|S}\right)_i (T', y)$ if and only if $(T, x) R_i (T', y)$. Second, we will have to specify how a given rule can be applied to a subprofile. One way of doing so it is to see a rule $f : \mathcal{R} \to A$ as it were a family of rules. Given a nonempty subset $S \in 2^N \backslash \{\varnothing\}$, denote by $\mathcal{R}^S$ the set of subprofiles $R_{|S} = (R_{|i})_{i \in S}$ where each $R_{|i}$, $i \in S$, is defined over pairs in $2^S \times X$ and it is obtained by restricting R_i only to alternatives in $2^S \times X$. Thus, a rule f can be identified with the collection $\{f^S\}_{S \in 2^N \backslash \{\varnothing\}}$ of rules where for each $S \in 2^N \backslash \{\varnothing\}$, $f^S : \mathcal{R}^S \to 2^S \times X$. If no confusion can arise, we often omit the superscript S and write $f(R_{|S})$.

Consistency For all $R \in \mathcal{R}$, $f(R) = f\left(R_{|f_N(R)}\right)$.

4 Consistent and internally stable rules

In Bergantiños, Massó and Neme (2016) we characterize the class of all strategy-proof, unanimous and non-bossy rules as the family of serial dictatorial rules. Here, we consider situations where the strategic manipulation in the preference revelation game is not an issue and will look for rules satisfying two meaningful properties in our setting, assuming agents report truthfully their preferences. Internal stability (no agent, member of the chosen society, wants to leave it) is a specially interesting property because in most societies, agents are not obliged to stay in the society if they want to leave it. The second property is consistency. Assume that the rule f has selected the alternative (S, x) at $R \in \mathcal{R}$. Thus, agents in S might want to reconsider again the choice of alternative (S, x). Consistency says that if f is applied to $R_{|S}$, the pair (S, x) would be selected again. Hence, members of the new society S do not need to reconsider the choice (S, x) made by the former society N.

To look for consistent rules satisfying also internal stability we ask whether three of the most prominent rules in classical social choice satisfy them. Recall that in the classical setting the goal is to select an outcome, from a given set X, taking into account the strict preferences of agents over X. The rules we consider are:

1. Approval voting. Each $i \in N$ votes for a subset X_i of X. For each outcome $x \in X$, compute the number of received votes; namely, $|\{i \in N : x \in X_i\}|$. The outcome

with more votes is selected. A tie-breaking rule should be applied whenever several outcomes obtain the largest number of votes.

2. Plurality voting. Each $i \in N$ votes for an outcome $x_i \in X$. The outcome with more votes is selected. A tie-breaking rule should be applied whenever several outcomes obtain the largest number of votes.

3. Borda method. Each $i \in N$ ranks all outcomes. Assign a preestablished number of points to each outcome depending on its position in the order. For each outcome, compute the sum, over all agents, of the points obtained by such outcome. Select the outcome with more points. A tie-breaking rule should be applied whenever several outcomes obtain the largest number of points.

We adapt the three voting methods to our setting, where the set of alternatives is $2^N \times X$. In addition, we will have to deal with the indifferences arising from property (P.1) of preference relations.

1. Approval voting. Each $i \in N$ votes for all $a \in A$ such that $aP_i [\varnothing]_i$ (if any).

2. Plurality voting. Each $i \in N$ votes for his top alternative $\tau(R_i)$. If $\tau(R_i) = [\varnothing]_i$, assume that i votes for all $a \in [\varnothing]_i$.

3. Borda method. For each $i \in N$, consider $[\varnothing]_i$ as a single alternative in i' rank. For each $(S, x) \in A$ and each $i \in N \backslash S$, assign to (S, x) the score obtained by $[\varnothing]_i$.

Example 1 below shows that none of these extensions satisfy internal stability.

Example 1 Let $R \in \mathcal{R}$ and $x \in X$ be such that for all $i \in N \backslash \{1\}$, $\tau(R_i) = (N, x)$, $\tau(R_1) = [\varnothing]_1$ and for each (S, y) with $1 \in S$, $(N, x) R_1 (S, y)$. Then, the three adapted voting methods choose (N, x) at R. Nevertheless, (N, x) is not internal stable because agent 1 prefers to leave the society. ■

Since we are interested in identifying rules satisfying internal stability, we modify the previous methods by considering only pairs (S, x) that are internally stable for each $i \in S$ according to R_i; namely, $(S, x) P_i [\varnothing]_i$ for each $i \in S$. In approval voting agents vote only for pairs that are internally stable. In plurality voting each agent votes for his best internally stable pair. In a Borda method we consider only the rank, given by the preference, among the internally stable pairs. With these modifications the three methods satisfy internal stability by definition. Denote by f^{AV}, f^{P} and f^{B} Approval voting, the Plurality voting, and the Borda method, respectively.

Our first result is negative: plurality voting and Borda method do not satisfy consistency. To see that, consider Example 2 below.

Example 2 Let $N = \{1,2,3,4,5,6\}$ and $X = \{y_1, y_2, y_3, y_4, y_5\}$ and consider the following profile $R \in \mathcal{R}$. For each $i \in N$, $(S,x)\, P_i\, [\varnothing]_i$ whenever $i \in S$ (namely, all pairs are internally stable). In addition, R is one among all those profiles satisfying the following properties, where the first column indicates the rank of each of the six preference relations.

Rank	R_1	R_2	R_3	R_4	R_5	R_6
First	(N, y_1)	(N, y_2)	(N, y_3)	$(N\backslash\{6\}, y_4)$	$(N\backslash\{6\}, y_4)$	(N, y_5)
Second	$(N\backslash\{6\}, y_1)$	$(N\backslash\{6\}, y_1)$	$(N\backslash\{6\}, y_1)$	$(N\backslash\{1\}, y_4)$	$(N\backslash\{1\}, y_4)$	
Third	$(N\backslash\{6\}, y_4)$	$(N\backslash\{6\}, y_4)$	$(N\backslash\{6\}, y_4)$	$(N\backslash\{2\}, y_4)$	$(N\backslash\{2\}, y_4)$	
Fourth				$(N\backslash\{3\}, y_4)$	$(N\backslash\{3\}, y_4)$	
Fifth				$(N\backslash\{6\}, y_1)$	$(N\backslash\{6\}, y_1)$	

First, plurality voting does not satisfy consistency since $f^{P}(R) = (N\backslash\{6\}, y_4)$ but at the same time $f^{P}\left(R_{|N\backslash\{6\}}\right) = (N\backslash\{6\}, y_1)$. Consider now the classical definition of the Borda method where the scores from the worst to the best alternative are given by $0, 1, 2, ..., k-2, k-1$, where k is the number of available alternatives. It is possible to select a profile R' satisfying the above rankings in such a way that $f^{B}(R') = (N\backslash\{6\}, y_4)$. Besides, $f^{B}\left(R'_{|N\backslash\{6\}}\right) = (N\backslash\{6\}, y_1)$. Hence, this Borda method is not consistent. ■

Fortunately, approval voting satisfies not only consistency but also other desirable properties. Before stating this result formally we propose a tie-breaking rule, to be used whenever more than one alternative obtains the highest number of votes. Let ρ be a monotonic order over the family of subsets of 2^N. Namely, given $S, T \in 2^N$ such that $S \subset T$, $T \rho S$. Observe that $N \rho S$ for all $S \neq N$.

Fix a monotonic order ρ over 2^N. Denote by $f^{AV,\rho}$ the approval voting that uses ρ to break ties. Formally, let $A' = \{(S_k, x_k)\}_{k=1}^{K}$ be the set of alternatives that have received the largest number of votes according to approval voting at profile R. First select the society $S \in \{S_1, ..., S_K\}$ ranked higher by ρ and consider the subset of alternatives $\{(S_{k'}, x_{k'}) \in A' \mid S_{k'} = S\}$. Select now agent $i \in S$ ranked higher by ρ (as a singleton set) and choose finally at R the alternative that is most preferred by i among those in the family $\{(S_{k'}, x_{k'}) \in A' \mid S_{k'} = S\}$.

Proposition 1 below states that any Approval voting $f^{AV,\rho}$ is consistent and satisfies participation and, by Remark 1, internal stability, together with other desirable properties.

Proposition 1 *Let ρ be a monotonic order over 2^N. Then, the Approval voting $f^{AV,\rho}$ satisfies consistency, efficiency, neutrality and participation. Moreover, in the subdomain of profiles where no tie-breaking rule is needed, $f^{A,\rho}$ also satisfies anonymity.*

Proof of Proposition 1 Observe that if (S,x) is approved by agent $i \in N$, then $i \in S$. This fact will be repeatedly used in the proof to show that $f^{AV,\rho}$ satisfies the properties, which we consider separately.

- Consistency. Let $R \in \mathcal{R}$ be an arbitrary profile and let $(S,x) \in A$ be such that $S \subset f_N^{AV,\rho}(R)$. The set of agents approving (S,x) at R coincides with the set of agents approving (S,x) at $R_{|f_N^{A,\rho}(R)}$. Hence, it follows that $f^{AV,\rho}(R_{|f_N^{A,\rho}(R)}) = f^{AV,\rho}(R)$. Thus, $f^{AV,\rho}$ satisfies consistency.

- Efficiency. Suppose otherwise; namely, there exist $R \in \mathcal{R}$ and $(S,x) \in A$ such that $(S,x)\, R_i f^{AV,\rho}(R)$ for all $i \in N$ and $(S,x) \neq f^{AV,\rho}(R)$. Let $i \in f_N^{AV,\rho}(R)$. Since $f^{AV,\rho}$ satisfies internal stability, $f^{AV,\rho}(R)\, P_i\, [\varnothing]_i$. Hence, $i \in S$ and $(S,x)\, P_i f^{AV,\rho}(R)$. We consider two cases. First, assume $f_N^{AV,\rho}(R) \subsetneq S$. Since for each $i \in S \backslash f_N^{AV,\rho}(R)$, $f^{AV,\rho}(R) = [\varnothing]_i$ and $(S,x)\, R_i f^{AV,\rho}(R)$ it follows that $(S,x)\, P_i\, [\varnothing]_i$. Thus, all agents in S approve (S,x), which contradicts the definition of $f^{AV,\rho}(R)$. Second, assume $f_N^{AV,\rho}(R) = S$. Thus, $f^{A,V\rho}(R) = (S,y)$ with $y \neq x$ and all agents in S approve both, (S,x) and (S,y). Hence, the tie-breaking rule ρ has been used to select $f^{AV,\rho}(R)$. Thus, there exists $i \in S$ such that $f^{AV,\rho}(R)\, P_i\, (S,x)$ which is a contradiction.

- Neutrality. Let $R \in \mathcal{R}$ be a profile and σ a permutation of X. Observe that for any $(S,x) \in A$ the number of agents approving (S,x) at R coincides with the number of agents approving (S,σ_x) at R^σ. We consider two cases. First, assume it is not necessary to apply ρ to select $f^{AV,\rho}(R)$. Namely, $f^{AV,\rho}(R)$ has been approved at R by more agents that any other alternative (S,x). Thus, $(f_N^{AV,\rho}(R), \sigma_{f_X^{AV,\rho}(R)})$ has been approved at R^σ by more agents that any other alternative (S,x). Hence, $f^{AV,\rho}(R^\sigma) = (f_N^{AV,\rho}(R), \sigma_{f_X^{AV,\rho}(R)})$. Second, assume it is necessary to apply ρ to select $f^{AV,\rho}(R)$. Let $\{(S_k,x_k)\}_{k=1}^K$ be the set of alternatives receiving the largest number of votes at R. Thus, $\{(S_k,\sigma_{x_k})\}_{k=1}^K$ is the set of alternatives receiving the largest number of votes at R^σ. Hence, $f_N^{AV,\rho}(R) = f_N^{AV,\rho}(R^\sigma)$. Now, let $i \in f_N^{AV,\rho}(R)$ be the agent with the highest ranking according to ρ (as a singleton set) and let $i' \in f_N^{AV,\rho}(R^\sigma)$ be the agent with the highest ranking according to ρ. Obviously, $i' = i$. Thus, $f_X^{AV,\rho}(R^\sigma) = \sigma_{f_X^{AV,\rho}(R)}$.

- Participation. By definition, $f^{A,\rho}$ satisfies internal stability. Hence, by Lemma 1, $f^{AV,\rho}$ satisfies participation.

- Anonymity in the profiles where no tie-breaking rule is needed. Assume that to select the alternative at profile R the tie-breaking ρ is not used. Then, $f^{AV,\rho}(R)$ has been approved at R by more agents than any other alternative (S,x). Hence, the number of agents approving (S,x) at R coincides with the number of agents approving $(\pi(S),x)$ at R^π. Thus, $(\pi(f_N^{AV,\rho}(R)), f_X^{AV,\rho}(R))$ has been approved at R^π by more agents that any other alternative (S,x). Hence, $f^{AV,\rho}(R^\pi) = \left(\pi(f^{AV,\rho}(R)), f_X^{AV,\rho}(R)\right)$, which means that $f^{AV,\rho}$ satisfies anonymity at profile R. ■

We end the paper by applying the Condorcet winner to our setting. First, we recall the definition of the Condorcet winner at a profile (over the set of outcomes) in the classical setting. Fix a profile over X and $x, y \in X$. We say that x beats y if the number of agents preferring x to y is larger that the number of agents preferring y to x. We say that x is a Condorcet winner (at a profile over X) if there is no y such that y beats x. It could be the case that no Condorcet winners exists or that there are several Condorcet winners (at a profile over X). Thus, the Condorcet winner is not a rule according to our definition.

We adapt the notion of a Condorcet winner to our setting as we have already did for the previous three rules. Fix $R \in \mathcal{R}$ and two different alternatives (S,x) and (T,y). All agents in the set $S \cup T$ strictly prefer one alternative to the other one while all agents in the set $N \backslash (S \cup T)$ are indifferent between (S,x) and (T,y). Thus, (S,x) beats (T,y) at R if the number of agents strictly preferring (S,x) to (T,y) is larger than the number of agents strictly preferring (T,y) to (S,x). In order to ensure that the chosen alternative satisfies internal stability at R we only consider alternatives (S,x) satisfying internal stability at R (namely, for all $i \in S$, $(S,x)P_i[\varnothing]_i$). When several Condorcet winners exist we apply the same tie-breaking rule ρ as in approval voting.

We say that a profile $R \in \mathcal{R}$ is *resolute* if there is $a \in A$ such that a beats a' for all $a' \neq a$. Thus, the Condorcet winner selects a at R. Let $f^{C,\rho}(R)$ denote the Condorcet winner (if any) at R. If $R \in \mathcal{R}$ is resolute, then $f^{C,\rho}(R)$ is independent of ρ and $\left|f^{C,\rho}(R)\right| = 1$. Proposition

2 states that the Condorcet winner at resolute profiles satisfies the same properties as approval voting, at such profiles.

Proposition 2 *Let R be a resolute profile. Then, $f^{C,\rho}(R)$ satisfies consistency, efficiency, anonymity, neutrality, and participation at R.*

Proof of Proposition 2 Fix a resolute profile R and set $f^{C,\rho}(R) = (S,x)$. We show that $f^{C,\rho}(R)$ satisfies the properties at R.

- Consistency. We prove that $f^{C,\rho}\left(R_{|S}\right) = (S,x)$ by showing that at $R_{|S}$, (S,x) beats (T,y) for all $(T,y) \neq (S,x)$ with $T \subset S$. Let (T,y) be an alternative with the above properties. Since (S,x) beats (T,y) at R, the number of agents in N preferring (S,x) to (T,y) is larger than the number of agents in N preferring (T,y) to (S,x). Moreover, each agent in $N \backslash S$ is indifferent between (S,x) and (T,y). Thus the number of agents in S preferring (S,x) to (T,y) (or (T,y) to (S,x)) coincides with the number of agents in N preferring (S,x) to (T,y) (or (T,y) to (S,x)). Hence, (S,x) beats (T,y) at $R_{|S}$.

- Efficiency. Suppose otherwise; namely, there exists (T,y) such that $(T,y)\, R_i\, (S,x)$ for all $i \in N$ and $(S,x) \neq (T,y)$. Let $i \in S$. Since (S,x) satisfies internal stability, $(S,x)\, P_i\, [\varnothing]_i$. Hence, $i \in T$ and $(T,y)\, P_i\, (S,x)$. Each agent in $N \backslash T$ is indifferent between (S,x) and (T,y). Thus (T,y) beats (S,x), which is a contradiction.

- Anonymity. Observe that $(\pi(S),x)$ beats $(\pi(T),y)$ at R^{π}, for each $(T,y) \neq (S,x)$. Hence, $f^{C,\rho}(R^{\pi}) = (\pi(S),x)$, which means that $f^{C,\rho}$ satisfies anonymity at profile R.

- Neutrality. Observe that (S,σ_x) beats (T,σ_y), at R^{σ}, for each $(T,y) \neq (S,x)$. Hence, $f^{C,\rho}(R^{\sigma}) = (S,\sigma_x)$, which means that $f^{C,\rho}$ satisfies neutrality at R.

- Participation. By definition, $f^{C,\rho}(R)$ satisfies internal stability at R. By Lemma 1, $f^{C,\rho}(R)$ satisfies participation. ■

Nevertheless, for profiles R that are not resolute the Condorcet winner $f^{C,\rho}(R)$, even when it is unique, may not satisfy consistency. To see that, consider the following example.

Example 3 Let $N = \{1,2,3,4,5\}$, $X = \{y_1, y_2\}$, and let ρ be any monotonic order satisfying $\{1\}\,\rho\,\{2\}\,\rho\,\{3\}\,\rho\,\{4\}$. In addition, take any profile R from all those satisfying the following properties, where the first column indicates the rank of each of the five preference relations.

Rank	R_1	R_2	R_3	R_4	R_5
First	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_2)$	$(N\backslash\{5\},y_2)$	(N,y_1)
Second	$(N\backslash\{5\},y_2)$	$(N\backslash\{5\},y_2)$	(N,y_1)	(N,y_1)	$[\varnothing]_5$
Third	(N,y_1)	(N,y_1)	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_1)$	
Fourth	$[\varnothing]_1$	$[\varnothing]_2$	$[\varnothing]_3$	$[\varnothing]_4$	

The only internally stable alternatives are $(N\backslash\{5\},y_1)$, $(N\backslash\{5\},y_2)$, and (N,y_1). Notice that, at R, $(N\backslash\{5\},y_1)$ is tied with $(N\backslash\{5\},y_2)$, $(N\backslash\{5\},y_2)$ beats (N,y_1) and (N,y_1) beats $(N\backslash\{5\},y_1)$. Since there exists a unique Condorcet winner, $(N\backslash\{5\},y_2)$, it must be the case that $f^{C,\rho}(R)=(N\backslash\{5\},y_2)$. The subprofile $R_{|N\backslash\{5\}}$ is given by

	$R_{\vert N\backslash\{5\}_1}$	$R_{\vert N\backslash\{5\}_2}$	$R_{\vert N\backslash\{5\}_3}$	$R_{\vert N\backslash\{5\}_4}$
First	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_2)$	$(N\backslash\{5\},y_2)$
Second	$(N\backslash\{5\},y_2)$	$(N\backslash\{5\},y_2)$	$(N\backslash\{5\},y_1)$	$(N\backslash\{5\},y_1)$
Third	$[\varnothing]_1$	$[\varnothing]_2$	$[\varnothing]_3$	$[\varnothing]_4$

At $R_{|N\backslash\{5\}}$, $(N\backslash\{5\},y_1)$ is tied with $(N\backslash\{5\},y_2)$. Thus, applying the tie-breaking rule ρ, and since agent 1 prefers $(N\backslash\{5\},y_1)$ to $(N\backslash\{5\},y_2)$, we have that $f^{C,\rho}(R)=(N\backslash\{5\},y_1)$, which means that $f^{C,\rho}$ does not satisfy consistency. ■

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