Mixed rules in multi-issue allocation situations

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Abstract

Multi-issue allocation situations study problems where we have to divide an estate among a group of agents. The claim of each agent is a vector specifying the amount claimed by each agent on each issue. We present a two-stage rule. First we divide the estate among the issues following the constrained equal awards rule. Second, the amount assigned to each issue is divided among the agents proportionally to their demands on this issue. We apply such rule to two real problems: the distribution of natural resources between countries and the distribution of budget for education and research between universities.

Keywords: multi-issue allocation situations, proportional rule, constrained equal awards rule.

1 Introduction

Bankruptcy situations study problems where an estate must be divided among several claimants. The problem arises when the estate is not enough to cover all claims. A typical example is when a firm goes bankrupt. The objective is to identify well-behaved rules for dividing the estate among the agents. This literature originates in O'Neill (1982) and Aumann and Maschler (1985). Two of the most popular rules are the proportional rule and the Constrained Equal Awards (CEA) rule. See Thomson (2003, 2015) for a survey.

In bankruptcy situations each agent's claim is a number. In many real situations we have to divide an estate among a group of agents, as in bankruptcy for instance, but the claim of each agent is a vector. In many Spanish Universities, once they have decided on the total annual operating budget for each department, the procedure is as follows. The university decides the money that will be assigned to the departments. The departments then submit a quantified request for each issue, typically research and teaching. Finally, the university decides the amount each department receives. More examples are: The European Community divides the budget

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among several issues (agriculture, roads, research, ...) and each state (Spain, France, ...) has claims over the different issues. The government of Spain divides the budget among several issues (health, education, ...) and each "Comunidad Autónoma" (Galicia, Madrid, Catalonia, ...) has claims over the different issues.

A Multi-Issue Allocation (MIA) is an extension of the basic bankruptcy model described above. In a MIA situation an estate must be divided among agents that make claims for several issues and the estate does not cover all claims. These situations were introduced in Calleja *et al* (2005). An approach used in the literature to solve these situations consists in considering a two-stage procedure. First, the estate is divided among issues. Second, the amount assigned to each issue is divided among agents. With this approach a rule is a matrix f where f_{ki} denotes the amount received by agent i on issue k. Several papers have provided axiomatic characterizations of rules. Lorenzo-Freire *et al* (2010) and Bergantiños *et al* (2011) study the two stage CEA rule. In the first stage, the estate is divided among the issues following the CEA bankruptcy rule. Second, the amount assigned to each issue is divided among the agents following also the CEA bankruptcy rule. Moreno-Ternero (2009) and Bergantiños *et al* (2010) study the two stage proportional rule.

We then study the two-stage rule that divides the estate among the issues following the CEA bankruptcy rule and inside each issue following the proportional bankruptcy rule. We give two axiomatic characterizations of this rule. Both are obtained by combining characterizations of the CEA bankruptcy rule and the proportional bankruptcy rule. The first one combines the results of Yeh (2006) for the CEA bankruptcy rule and Chun (1988) for the proportional bankruptcy rule. The second one combines the results of Herrero and Villar (2002) for the CEA bankruptcy rule and Chun (1988) for the proportional bankruptcy rule. It is very important to characterize rules axiomatically because we need fair allocation procedures. For example, these methods can be used to solve monetary problems where an asset must be divided among several creditors or resource allocation problems where a natural resource such water, oil or natural gas, must be divided between several territories claiming for the exploitation of these resources.

In addition to the two axiomatic characterizations of this sharing rule, in the last section of the paper we apply the rule to two real examples: the distribution of natural resources between countries and the distribution of budget for education and research between universities. We also analyze the consequences on the properties used in the axiomatic characterizations in the real problems we consider.

Natural resource problems can be solved by using this type of models. The Caspian Sea is a region that has a large reserve of oil and natural gas. Even the reserves have not yet been well quantified, the first estimations suggest that we would be ahead of the world's largest oil and natural gas reserves. Currently, the Caspian Sea shares borders with Azerbaijan, Iran, Kazakhstan, Russia and Turkmenistan. International organizations have proposed different legal status for the sharing of the Caspian Sea among the five countries, but there is no definitive agreement yet. For this reason, these countries are in continuous dispute over the ownership of the resources located in certain areas of the region. In the negotiations we have two countries claiming the ownership of the same oil or natural gas wells. Thus, some authors as Sheikhmohammady and Madani (2008) propose to resolve the dispute through a bankruptcy situation. In particular, they compare the results obtained using different bankruptcy rules as the proportional and the constrained equal awards. In this paper, we take the data from Sheikhmohammady and Madani (2008) and apply the MIA rule introduced in this paper to such data.

In European countries, universities are mostly funded with public money. They provide teaching and research. The regional or central government must divide the budget between universities to cover the spent on these two issues. Although universities are non-market based structures, politics aims to rectors the use of business concepts in the budget manage. Therefore, we observe that the budget is divided among universities using competitive rules. As well, universities give the same importance to teaching as to research. Several papers have studied this problem from different perspectives. See, for instance, Del Rey (2001), Ball and Butler (2004), and Gautier and Wauthy (2007). Faden and Gal (2001) study the redistribution of funds for teaching and research among universities in the North Rhine-Westphalia region in Germany. We study the same problem but our proposal is to solve it using the MIA rule we have introduced.

The paper is organized as follows. In Section 2 we introduce MIA situations. In Section 3 we introduce and study the two stage MIA rule. In Section 4 we apply such rule to the two real cases.

2 Multi-issue allocation situations

In this section we introduce bankruptcy situations, multi-issue allocation situations, and some rules.

We now introduce several concepts in bankruptcy situations. See, for instance, Thomson (2003, 2015) for a survey on this topic.

A *bankruptcy situation* is a triple (N, E, c). $N = \{1, \ldots, n\}$ is the set of agents. The estate $E \geq 0$ represents the amount to be divided among the agents. Finally $c = (c_i)_{i \in N} \in \mathbb{R}^N_+$ and c_i denotes the claim of player i. It is assumed that $0 \leq E \leq \sum\limits_{i \in N} c_i$. When no confusion arises we use (E, c) instead of (N, E, c).

A *bankruptcy rule* is a map ψ that associates with each bankruptcy situation (E, c) a vector $\psi(E, c) \in \mathbb{R}^N$ satisfying that $\sum\limits_{i \in N} \psi_i(E, c) = E$ and $0 \leq \psi_i(E, c) \leq c_i$ for each $i \in N$.

The *proportional rule* (P) is defined for each $i \in N$ as $P_i(E, c) = \lambda c_i$ where λ satisfies $\sum\limits_{i \in N} \lambda c_i = E$.

The *constrained equal awards rule* (CEA) is defined for each $i \in N$ as $CEA_i(E, c) = \min\{\lambda, c_i\}$ where λ satisfies $\sum\limits_{i \in N} \min\{\lambda, c_i\} = E$.

We now introduce several properties of bankruptcy rules that are relevant for this paper.

Conditional full compensation (FC). For each bankruptcy situation (E, c) and each $i \in N$, if $\sum\limits_{j \in N} \min\{c_j, c_i\} \leq E$, then $\psi_i(E, c) = c_i$.

Claims monotonicity (CM). For each bankruptcy situation (E, c) and each $i, j \in N$, if $c_j \geq c_i$, then $\psi_j(E, c) \geq \psi_i(E, c)$.

No advantageous transfer (NAT). For each (E, c), (E, c'), and $M \subset N$ such that $c_i = c'_i$ when $i \in N \setminus M$ and $\sum\limits_{i \in M} c'_i = \sum\limits_{i \in M} c_i$, then $\sum\limits_{i \in M} \psi_i(E, c) = \sum\limits_{i \in M} \psi_i(E, c')$. This property can be defined in a strong way by claiming, $\psi_i(E, c) = \psi_i(E, c')$ for all $i \in N \setminus M$. The idea is that no group of agents should receive more by transferring claims among themselves.

Composition down (CD). For each (E, c) and each $E' \in \mathbb{R}$ such that $E' \geq E$, $f(E, c) = f(E, f(E', c))$.

Calleja *et al* (2005) introduce multi-issue allocation situations generalizing bankruptcy situations.

A *multi-issue allocation (MIA) situation* is a 4-tuple (R, N, E, C), where $R = \{1, \ldots, r\}$ is the set of issues. $N = \{1, \ldots, n\}$ is the set of agents. $E \geq 0$ is the estate to be divided. $C = (c_{ki})_{k \in R, i \in N} \in \mathbb{R}_+^{R \times N}$ and c_{ki} represents the amount claimed by agent $i \in N$ on issue $k \in R$. We assume $0 \leq E \leq \sum\limits_{k \in R} \sum\limits_{i \in N} c_{ki}$.

Note that a bankruptcy situation is a MIA situation with $|R| = 1$.

For defining a rule we have two possibilities. First, to assign to each agent a part of the estate, see Calleja *et al* (2005), González-Alcón *et al* (2007), and Ju *et al* (2007). Second, to specify the amount each agent receives in each issue, see Moreno-Ternero (2009), Lorenzo-Freire *et al* (2010), and Bergantiños *et al* (2010, 2011). In this paper we follow the second approach. Thus,

A *multi-issue allocation (MIA) rule* f is a map that associates with each *MIA* situation (R, N, E, C) a matrix $f(R, N, E, C) \in \mathbb{R}^{R \times N}$ satisfying:

- $0 \leq f_{ki}(R, N, E, C) \leq c_{ki}$ for each $k \in R$ and each $i \in N$.
- $\sum\limits_{k \in R} \sum\limits_{i \in N} f_{ki}(R, N, E, C) = E$.

Lorenzo-Freire *et al* (2010) define a two-stage procedure to define *MIA* rules from bankruptcy rules. They first apply a bankruptcy rule for dividing the estate among the issues. Later, the amount assigned to each issue is divided among the agents claiming on this issue. We follow the same idea considering that the bankruptcy rule used at each step can be different.

Let ψ and ϕ be two bankruptcy rules and let (R, N, E, C) be a MIA situation. The *two-stage rule* $f^{\psi,\phi}(R, N, E, C)$ is the MIA rule obtained from the following two-stage procedure:

1. First stage. Consider the so-called bankruptcy situation among the issues (R, E, c^R), where $c^R = (c_1^R, \ldots, c_r^R) \in \mathbb{R}^R$ denotes the vector of total claims in the issues, *i.e.* for each $k \in R$, $c_k^R = \sum\limits_{i \in N} c_{ki}$. The amount E is divided among the issues using the bankruptcy rule ψ.
2. Second stage. For each $k \in R$, consider the bankruptcy situation given by $(N, \psi_k(R, E, c^R), (c_{ki})_{i \in N})$ and apply the bankruptcy rule ϕ.

Thus, for each $k \in R$ and each $i \in N$ we define

$$f_{ki}^{\psi,\phi}(R, N, E, C) = \phi_i(N, \psi_k(R, E, c^R), (c_{ki})_{i \in N}).$$

3 The mixed rule

In many Spanish universities, when deciding the annual operating budget that each department will receive, the procedure is as follows. The university decides the money that will be assigned to the departments. The departments submit a quantified amount for each issue, typically research and teaching. Finally, the university decides the amount each department will receive for each issue.

The university authorities argue that research and teaching are two important issues of the university and no one is more important than the other. In our model this idea can be applied by claiming that the amount devoted to research and teaching should be as equal as possible. Nevertheless, within each issue (research or teaching) the authorities argue that agents are, in general, different. Then, the amount they receive should recognize these differences. For instance, in the university of Vigo, the amount each department receives for research is proportional to the number of points the department obtains. The points are computed taking into account several issues as publications, books, and so on. In our model this idea could be applied by claiming that the amount each agent receives in each issue is proportional to his claim on the issue.

We then study the two stage rule which divides among the issues following the CEA bankruptcy rule and inside each issue following the proportional bankruptcy rule. We give two axiomatic characterizations which are obtained by combining characterizations of the CEA bankruptcy rule and the proportional bankruptcy rule.

The *two-stage constrained equal awards (CEA) - proportional (PRO) rule* is the two-stage MIA rule $f^{\psi,\phi}$ where $\psi = CEA$ and $\phi = P$. Then, $f^{CEA,P}$ is defined as follows: for each (R, N, E, C), each $k \in R$, and each $i \in N$,

$$f_{ki}^{CEA,P}(R, N, E, C) = P_i \left(N, CEA_k(R, E, c^R), (c_{ki})_{i \in N}\right).$$

It is obvious that $f^{CEA,P}$ can be rewritten as $\frac{c_{ki}}{c_k^R} \min\{\lambda, c_k^R\}$ where λ satisfies $\sum\limits_{k \in R} \min\{\lambda, c_k^R\} = E$.

Theorem 4 in Thomson (2003) presents several characterizations of the CEA bankruptcy rule obtained, among others, by Herrero and Villar (2002) and Yeh (2006). In Theorem 8, Thomson (2003) presents several characterizations of the proportional bankruptcy rule obtained, among others, by Chun (1988) and Young (1988).

Our axiomatic characterizations are obtained by combining these characterizations. This is not a trivial issue. For instance, Young (1988) characterizes the proportional bankruptcy rule with self duality and composition down. Self duality is defined in MIA situations as in bankruptcy situations (Bergantiños *et al* 2011). Since $f^{CEA,P}$ does not satisfy self duality, we cannot use this result for obtaining a characterization of $f^{CEA,P}$.

We give two axiomatic characterizations of $f^{CEA,P}$. The first one combines the results of Yeh (2006) and Chun (1988). The second one combines the results of Herrero and Villar (2002) and Chun (1988).

We now extend bankruptcy properties to MIA properties. In some cases the definition is the same, as in the case of composition down. In other cases the properties are adapted by claiming them among the issues or within each issue.

No advantageous transfer inside the issues ($NATI$). Let (R,N,E,C), (R,N,E,C'), $k \in R$, and $M \subset N$ be such that $\sum_{i\in M} c'_{ki} = \sum_{i\in M} c_{ki}$ and $c'_{li} = c_{li}$ when $l \in R \setminus \{k\}$ or $i \in N \setminus M$. Then for each $i \in N \setminus M$,

$$f_{ki}(R,N,E,C) = f_{ki}(R,N,E,C').$$

$NATI$ says that if a group of agents redistribute their claims inside an issue, the amount assigned to the other agents in this issue does not change.

Conditional full compensation among the issues (FCA). For each (R,N,E,C) and each $k \in R$ such that $\sum_{l\in R} \min\left\{\sum_{i\in N} c_{li}, \sum_{i\in N} c_{ki}\right\} \leq E$,

$$\sum_{i\in N} f_{ki}(R,N,E,C) = \sum_{i\in N} c_{ki}.$$

The idea behind this property is that if the total claim in an issue is small when comparing with the total claim in the other issues, then the total amount allocated to issue k must coincide with its total claim. Note that FCA can be rewritten as $f_{ki}(R,N,E,C) = c_{ki}$ for all $i \in N$ instead of $\sum_{i\in N} f_{ki}(R,N,E,C) = \sum_{i\in N} c_{ki}$.

Claims monotonicity among the issues (CMA). Let (R,N,E,C), (R,N,E,C'), and $k \in R$ be such that $\sum_{i\in N} c_{ki} \leq \sum_{i\in N} c'_{ki}$ and $c_{li} = c'_{li}$ for each $l \in R \setminus \{k\}$. Then for each $i \in N$,

$$\sum_{i\in N} f_{ki}(R,N,E,C) \leq \sum_{i\in N} f_{ki}(R,N,E,C').$$

CMA says that if the total claim of an issue increases, then the total allocation assigned to this issue cannot decrease.

Composition down is defined as in bankruptcy situations.
Composition down (CD). For each (R,N,E,C) and each $E' \in \mathbb{R}$ such that $E' \geq E$,

$$f(R,N,E,C) = f(R,N,E,f(R,N,E',C)).$$

Theorem 1 (a) *If* $|N| \geq 3$*,* $f^{CEA,P}$ *is the unique rule satisfying* FCA*,* CMA*, and* $NATI$*.*
(b) $f^{CEA,P}$ *is the unique rule satisfying* CD*,* FCA*, and* $NATI$*.*

Proof of Theorem 1.
Using similar arguments to those used in Bergantiños *et al* (2010, 2011) we can prove that $f^{CEA,P}$ satisfies FCA, CMA, CD, and $NATI$.

(a) We now prove that $f^{CEA,P}$ is the only one. Let f be a rule satisfying FCA, CMA, and $NATI$. We first prove that for each $k \in R$,

$$\sum_{i\in N} f_{ki}\left(R,N,E,C\right) = \sum_{i\in N} f_{ki}^{CEA,P}\left(R,N,E,C\right). \tag{1}$$

Let $q = (q_{ki})_{k\in R, i\in N}$ be such that for each $k \in R$, $(q_{ki})_{i\in N}$ belongs to the simplex in $\mathbb{R}^N$. For each bankruptcy situation $(R, E, (x_k)_{k\in R})$, we define the bankruptcy rule f^q such that for each $k \in R$,

$$f_k^q(R, E, (x_k)_{k\in R}) = \sum_{i\in N} f_{ki}(R, N, E, (q_{ki}x_k)_{k\in R, i\in N}).$$

Since f is a MIA rule, f^q is a bankruptcy rule. We now prove that f^q satisfies FC and CM in bankruptcy situations.

- FC. Let $k \in R$ be such that $\sum_{l\in R} \min\{x_l, x_k\} \leq E$. Thus,

$$\sum_{l\in R} \min\left\{\sum_{i\in N} q_{li}x_l, \sum_{i\in N} q_{ki}x_k\right\} \leq E.$$

  Since f satisfies FCA, $f_{ki}(R, N, E, (q_{li}x_l)_{l\in R, i\in N}) = q_{ki}x_k$ for all $i \in N$. Hence, $f_k^q(R, E, (x_k)_{k\in R}) = \sum_{i\in N} q_{ki}x_k = x_k$.

- CM. Let $k, k' \in R$ be such that $x_k \geq x_{k'}$. Thus, $\sum_{i\in N} q_{ki}x_k \geq \sum_{i\in N} q_{k'i}x_{k'}$. Since f satisfies CMA,

$$\begin{aligned} f_k^q(R, E, (x_l)_{l\in R}) &= \sum_{i\in N} f_{ki}(R, N, E, (q_{li}x_l)_{l\in R, i\in N}) \\ &\geq \sum_{i\in N} f_{k'i}(R, N, E, (q_{li}x_l)_{l\in R, i\in N}) \\ &= f_{k'}^q(R, E, (x_l)_{l\in R}). \end{aligned}$$

Yeh (2006) proves that CEA is the unique bankruptcy rule satisfying FC and CM. Thus, $f^q = CEA$.

Consider $(x_k)_{k\in R}$ and q such that $x_k = \sum_{i\in N} c_{ki}$ and $q_{ki} = \frac{c_{ki}}{x_k}$ for each $(k, i) \in R \times N$. Now, it is trivial to deduce that (1) holds.

We now prove that for each $k \in R$ and $i \in N$,

$$f_{ki}(R, N, E, (c_{lj})_{l\in R, j\in N}) = f_{ki}^{CEA,P}(R, N, E, (c_{lj})_{l\in R, j\in N}).$$

Let $d^k = (d_{lj})_{l\in R\setminus\{k\}, j\in N} \in \mathbb{R}_+^{R\setminus\{k\}\times N}$. For each bankruptcy situation $(N, E, (y_i)_{i\in N})$ we define the bankruptcy rule f^{d^k} such that for each $i \in N$

$$f_i^{d^k}(N, E, (y_j)_{j\in N}) = f_{ki}(R, N, E^d, (d_{lj})_{l\in R, j\in N})$$

where $d_{kj} = y_j$ and E^d satisfies $\min\left\{\lambda^d, \sum_{i\in N} d_{ki}\right\} = E$ and $\sum_{l\in R} \min\left\{\lambda^d, \sum_{i\in N} d_{li}\right\} = E^d$. We prove that $f^{d^k} = P$.

We first prove that f^{d^k} is well defined, namely, that it is a bankruptcy rule.

- Since f is a MIA rule, for each $i \in N$,

$$0 \leq f_{ki}(R, N, E^d, (d_{lj})_{l \in R, j \in N}) \leq d_{ki} = y_i.$$

- By (1),

$$\begin{aligned}
\sum_{i \in N} f_i^{d^k}(N, E, (y_j)_{j \in N}) &= \sum_{i \in N} f_{ki}(R, N, E^d, (d_{lj})_{l \in R, j \in N}) \\
&= \sum_{i \in N} f_{ki}^{CEA,P}(R, N, E^d, (d_{lj})_{l \in R, j \in N}) \\
&= \min\left\{\lambda, \sum_{i \in N} d_{ki}\right\}
\end{aligned}$$

where $\sum\limits_{l \in R} \min\left\{\lambda, \sum\limits_{i \in N} d_{li}\right\} = E^d$. Thus, $\lambda = \lambda^d$ and hence,

$$\sum_{i \in N} f_i^{d^k}(N, E, (y_j)_{j \in N}) = \min\left\{\lambda^d, \sum_{i \in N} d_{ki}\right\} = E.$$

We now prove that f^{d^k} satisfies NAT in bankruptcy situations. Let (N, E, y), (N, E, y'), and $M \subset N$ such that $y_i = y'_i$ when $i \in N \setminus M$ and $\sum\limits_{i \in M} y'_i = \sum\limits_{i \in M} y_i$. Thus, $d'_{li} = d_{li}$ when $l \in R \setminus \{k\}$ or $i \in N \setminus M$ and $\sum\limits_{i \in M} d'_{ki} = \sum\limits_{i \in M} y'_i = \sum\limits_{i \in M} y_i = \sum\limits_{i \in M} d_{ki}$.

For each $i \in N \setminus M$,

$$f_i^{d^k}(N, E, y') = f_{ki}(R, N, E^{d'}, (d'_{lj})_{l \in R, j \in N}).$$

Since f satisfies $NATI$,

$$f_{ki}(R, N, E^{d'}, (d'_{lj})_{l \in R, j \in N}) = f_{ki}(R, N, E^{d'}, (d_{lj})_{l \in R, j \in N}).$$

By (1), it is easy to deduce that $E^d = E^{d'}$. Then,

$$f_{ki}(R, N, E^{d'}, (d_{lj})_{l \in R, j \in N}) = f_i^{d^k}(N, E, y).$$

Hence f^{d^k} satisfies NAT. Since P is the unique bankruptcy rule satisfying NAT (Chun, 1988), $f^{d^k} = P$.

We define d^k such that for each $l \in R \setminus \{k\}$ and each $j \in N$, $d_{lj} = c_{lj}$. Besides, we take $y_j = d_{kj} = c_{kj}$ for all $j \in N$. Because of the definition of CEA it is easy to conclude that $CEA_k(R, E, c^R)^d = E$. Now

$$\begin{aligned}
f_{ki}(R, N, E, c) &= f_i^{d^k}(N, CEA_k(R, E, c^R), (c_{kj})_{j \in N}) \\
&= P_i(N, CEA_k(R, E, c^R), (c_{kj})_{j \in N}) \\
&= f_{ki}^{CEA,P}(R, N, E, (c_{lj})_{l \in R, j \in N}).
\end{aligned}$$

(b) The proof of the uniqueness is similar to case (a) and we omit it. ■

We prove that the properties used in Theorem 1 are independent. Let α be the bankruptcy rule in which we fulfill the claims by increasing order. Namely, given (E,c) such that $c_1 \leq c_2 \leq .... \leq c_n$ we can find $j \in N$ satisfying $\sum\limits_{i \leq j} c_i \leq E < \sum\limits_{i \leq j+1} c_i$. Thus, $\alpha_i(E,c) = c_i$ when $i \leq j$, $\alpha_{j+1}(E,c) = E - \sum\limits_{i \leq j} c_i$ and $\alpha_i(E,c) = 0$ when $i > j+1$.

Let δ be the bankruptcy rule defined by Herrero and Villar (2002).

Next tables show that the properties used in Theorem 1 are independent. The proofs are left to the reader.

Independence of the properties of Theorem 1 (a):

Properties / Rules	$f^{P,P}$	$f^{\alpha,P}$	$f^{CEA,CEA}$
FCA	No	Yes	Yes
CMA	Yes	No	Yes
$NATI$	Yes	Yes	No

Independence of the properties of Theorem 1 (b):

Properties / Rules	$f^{P,P}$	$f^{\delta,P}$	$f^{CEA,CEA}$
FCA	No	Yes	Yes
CD	Yes	No	Yes
$NATI$	Yes	Yes	No

4 Applications to real problems

In this section we apply the rule introduced in the previous section to two real problems. The sharing of natural resources among countries and the sharing of funds among universities.

4.1 Sharing oil and natural gas resources of the Caspian Sea among countries

The Caspian Sea is a region that has a large reserve of oil and natural gas. The proven and probable reserves have not yet been well quantified. The US Energy Information Administration (EIA) estimates "that there were 48 billion barrels of oil and 292 trillion cubic feet (Tcf) of natural gas in proved and probable reserves within the basins that make up the Caspian Sea and surrounding area in 2012" (EIA report on the Caspian Sea, 2013). If these measurements were true, we would be ahead of the world's largest oil and natural gas reserves. That is why the exploitation of these resources is one of the main tasks of the countries located in this region.

Currently, the Caspian Sea shares borders with Azerbaijan, Iran, Kazakhstan, Russia and Turkmenistan. International organizations have proposed different legal status for the sharing of the Caspian Sea among the five countries, but there is still no a definitive agreement. For this reason, these countries are in continuous dispute over the ownership of the resources located in certain areas of the region. Newer countries do not have the technology to exploit oil and natural gas, and therefore they have to resort to agreements with oil companies. Obviously, the greater the reserves managed, the easier it will be to sign agreements with these oil companies for the joint exploitation of resources. Zimnitskaya and Geldern (2011) deals in detail with the problem of the sharing of the Caspian Sea among the five countries in dispute.

The legal proposals for the division of the Caspian Sea have been numerous. Before the dissolution of the Soviet Union (USSR), the Caspian Sea was divided between the USSR and Iran from a series of treaties. However, after the demise of the USSR, three new Border States appear in the Caspian Sea that also seek to exploit these resources. These countries are Russia, Azerbaijan, Kazakhstan and Turkmenistan. However, these new countries do not agree to continue with the old treaties signed between the USSR and Iran.

The UN proposes a sharing based on the Law of the Sea (UNCLOS). Article 15 of UNCLOS requires that the sea of the countries with opposite or adjacent coasts may not be extended "beyond the median line every point of which is equidistant from the nearest points on the baselines form which the breadth of the territorial seas of each of the [two] States is measured". Other possible distribution rules would be the equal division of the Caspian Sea or the equal distribution of oil and natural gas reserves (Condominium).

In the negotiations countries propose different rules for solving the sharing problem. This position has resulted in two countries claiming the ownership of the same oil or natural gas wells. Thus, some authors as Sheikhmohammady and Madani (2008) propose to resolve the dispute in the Caspian Sea through a bankruptcy situation. In particular, they compare the results using some bankruptcy rules as the proportional and the constrained equal awards. In this paper, we take the data from Sheikhmohammady and Madani (2008) and apply the MIA rule analyzed in this paper to such data.

The MIA situation (R, N, E, C) associated to the Caspian Sea is the following:

1. R has two issues: oil and natural gas.
2. N has 5 agents, the five countries we have mentioned.
3. The estate E is the value of the total reserves of oil and natural gas. According to Sheikhmohammady and Madani (2008) it is 13,512 billion of dollars. We should mention that these values could change as new oil and natural gas prospecting are carried out in the region.
4. The vector of claims C is also obtained from Sheikhmohammady and Madani (2008). Table 1 below shows the values in billions of dollars of oil and gas claims made by the five countries. The total value of the all the claims is 17,791 billion dollars.

Table 1. Values of oil and natural gas claimed by countries

Countries	Oil (billion \$)	Natural gas (billion \$)
Azerbaijan	2,060	521
Iran	2,016	686
Kazakhstan	5,050	918
Russia	2,016	686
Turkmenistan	2,190	1,648
Total claims	13,332	4,459

Now we apply the MIA rule $f^{CEA,P}$ to the previous MIA situation. Remember that we first divide the estate between the two issues following the CEA rule, and secondly, we divide the money allocated to each issue among the five countries following the P rule. Theorem 1 shows that $f^{CEA,P}$ is the only rule that satisfies the CD, FCA, and $NATI$ properties. The implications of these properties in the Caspian Sea problem are very interesting. CD implies that if new prospecting actions change the amounts of oil or natural gas (very common in the Caspian Sea), and the current value is smaller than the expected value, it is indifferent to solve the new problem or to solve a problem with the reduced estate and where the claims are the allocations in the overestimated estate. Thus, renegotiations will be easier. FCA tells us that if the claim of one of the issues (oil of natural gas) is less than the equal distribution of the state between both issues, that issue should be fully compensated. $NATI$ implies that if two countries reach a new agreement on ownership of an oil well, this change does not affect the allocation of other countries.

Table 2 below shows the results obtained after applying $f^{CEA,P}$.

Table 2. Allocation based on $f^{CEA,P}$.

Countries	Oil (billion \$)	Natural gas (billion \$)	Total allocation
Azerbaijan	1,399	521	1,920
Iran	1,369	686	2,055
Kazakhstan	3,429	918	4,347
Russia	1,369	686	2,055
Turkmenistan	1,487	1,648	3,135
Sum	9,053	4,459	13,512

Notice that in this case countries receive the claimed amount of natural gas and the rest of the total value of the reserves (9,053 billion of dollars) is distributed among the five countries proportionally to their claims of oil.

Table 3 below compares the results obtained by $f^{CEA,P}$ with the ones discussed in Sheikhmohammady and Madani (2008). Namely, the P rule, the CEA rule and the AP rule.

Table 3. Comparing the different allocations

Countries	P rule	CEA rule	AP rule	$f^{CEA,P}$
Azerbaijan	1,960	2,580	1,895	1,920
Iran	2,052	2,702	1,984	2,055
Kazakhstan	4,533	2,764	4,831	4,347
Russia	2,052	2,702	1,984	2,055
Turkmenistan	2,915	2,764	2,818	3,135
Sum	13,512	13,512	13,512	13,512

Source: Sheikhmohammady and Madani (2008).

Sheikhmohammady and Madani (2008) conclude that the CEA rule is the best one when we consider social choice but the P rule is better when we try to maximize the minimum satisfaction. The $f^{CEA,P}$ is closer to the P rule than to the CEA rule.

More natural resource problems can be solved by using this type of models. An example is the resolution of conflict in water resources. Nowadays there is not an accepted international agreement about a sharing rule of water resources. There exists a normative principle of "equitable and reasonable utilization". Since there is not a consensus about how to interpret equitable and reasonable utilization, conflicts frequently appear when, for example, a river flows between different borders or several countries share borders in a lake or a sea. For example, Mianabadi *et al* (2014) and Oftaldeh *et al* (2016) propose a bankruptcy approach to solve some water conflicts.

4.2 Sharing funds for teaching and research among universities: the case of North Rhine-Westphalia

Funding of European universities is done mostly through public money. Universities and regional governments meet together to discuss a funding plan for a specific period (usually for 5 or 10 years). Generally, each university presents to the government some objectives, to be reached in that period, and the estimated cost to achieve them. The government also presents the total budget for universities.

Traditionally, university funding depended primarily on the number of students enrolled. Currently, this funding seeks to enhance the two main functions of a university: teaching and research. Governments are increasingly aware that the decision to enroll in a particular university depends on the quality of education as well as the prestige of the university in the field of research.

Thus, in these meetings, universities and government decide how much money is destined for teaching and research. Finally, this money is distributed among the different universities attending to both the objectives set by them and by the government. In this distribution, the government seeks to encourage competition between universities by rewarding with greater budget those institutions with greater achievements in the two issues.

The problem is to agree on how to evaluate the achievements in teaching and research (its weight in funding). Universities are very heterogeneous (some universities are older than others, for example), and usually do not agree on the sharing rule to be applied.

In this context, a MIA procedure can be a good proposal for the allocation of the budget. Each university makes a claim for money to the government for teaching and a claim for research, based on valuations of achievements obtained in the past. Since the valuations each university assigns to teaching and research do not coincide among different universities, the final sum of the claims usually exceeds the total amount the public administration is willing to invest in financing the universities. The MIA procedure, analyzed in this work, proposes a fair solution to this problem since it satisfies desirable axiomatic properties in this type of sharing rules.

As an example, we have applied our rule to a real case that has already been studied in the literature. The redistribution of funds for teaching and research among universities in the North Rhine-Westphalia region in Germany (Fanden and Gal, 2001).

The North Rhine-Westphalia government brings together the 15 universities in the region to distribute public funds for universities. The government proposes to allocate funds according to three teaching criteria and two research criteria. The teaching criteria are the percentages of academic staff, number of students and number of graduates of each university over the total of the region. The research criteria correspond to the percentages of external funds obtained for research and the number of theses approved on the regional total. From these variables, universities and government have to agree on the weight that each one of these variables will have in the distribution of the budget.

Negotiations between universities and government were formally described as a Multiple Criteria Decision Making (MCDM) problem. The solution to this problem consisted of the five weights that are assigned to each of the teaching and research variables. This weighting vector transforms the percentages of each university with respect to the five different variables into a total share of the university budget. Universities and authority valued seven situations resulting from the resolution of seven different MCDMs (see Table 4). Each of them gave different budgets to the 15 universities involved. Universities and government agreed on the sharing rule shown in column 7 of Table 4.

Table 4. Weighting vectors for the seven proposed sharing rules.

	1	2	3	4	5	6	7
Teaching							
Acad. personnel	0.2	0	0	0.0503	0.1546	0	0.2
Students	0.2	0	0.2614	0.2647	0.1348	0.2598	0.2
Graduates	0.2	0.1942	0.5120	0.4648	0.5051	0.5806	0.35
Research							
Outside funds	0.2	0.4305	0.1746	0.1720	0.2045	0.1596	0.2
Ph. Ds	0.2	0.3752	0.0520	0.0482	0	0	0.05

Source: Fandel and Gal (2001).

However, what happens when no agreement is reached? At this point, this problem could be considered as a MIA situation (R, N, E, C) as follows.

1. R has two issues: teaching and research.
2. N has 15 agents, the 15 universities.
3. The estate E is the budget devoted by government to the universities. According with Fandel and Gal (2001) 148.58 millions of DM of the year 1996, when the authors realized the study of redistribution of the budget among the universities.

4. We compute the vector of claims C as follows. The claim of each university in teaching and research is defined as the highest allocation in teaching and research that each university will receive among the seven sharing rules mentioned above. In Table 5 below we compute such claims.

Table 5. Claims based on the best rule for each university.

Universities	Best rule	Teaching	Research	Total
Aachen	2	4.42	21.19	25.61
Bielefeld	2	1.04	9.21	10.26
Bochum	2	2.55	13.25	15.80
Bonn	2	3.43	17.18	20.61
Dortmund	4	9.31	1.84	11.15
Dusseldorf	2	0.83	6.32	7.15
Cologne	6	13.78	2.07	15.84
Munster	6	14.77	2.17	16.94
DSH Cologne	6	2.44	0.28	2.71
Duisburg	5	4.8	0.66	5.46
Essen	6	6.85	0.77	7.61
Paderborn	6	7.59	0.87	8.46
Siegen	5	4.65	1.23	5.88
Wuppertal	7	5.58	1.47	7.04
FU Hagen	7	2.10	1.21	3.31
Total claims		84.13	79.71	163.84

Source: Calculations obtained from data in Fandel and Gal (2001).

Notice that the total claim of the universities is 163.48, greater than the estate (148.58).

If we solve this problem as a MIA situation we proceed as follows. At the first stage, the total budget is distributed between teaching and research following the CEA rule. Thus, half of the total budget is devoted to teaching and half to research. Later, the part of the budget devoted to teaching (respectively research) is divided among the universities following the proportional rule. As we demonstrated above, this rule is the unique that would meet the properties of CD, FCA, and $NATI$. The same good implications of such properties, as explained in the previous example, can be found here.

Table 6 below shows the allocation obtained with $f^{CEA,P}$ and the allocation used in practice.

Table 6. Allocation based on $f^{CEA,P}$.

Universities	$f^{CEA,P}$ Teaching	Practice Teaching	$f^{CEA,P}$ Research	Practice Research	$f^{CEA,P}$ Total	Practice Total
Aachen	3.91	15.03	19.75	6.09	23.66	21.12
Bielefeld	0.92	5.09	8.59	3.41	9.51	8.51
Bochum	2.25	10.36	12.35	4.13	14.60	14.49
Bonn	3.03	11.85	16.01	5.04	19.04	16.89
Dortmund	8.22	9.09	1.71	2.03	9.93	11.12
Dusseldorf	0.73	4.45	5.89	1.84	6.62	6.29
Cologne	12.17	11.51	1.93	3.39	14.09	14.90
Munster	13.04	12.36	2.02	3.50	15.06	15.86
DSH Cologne	2.15	1.89	0.26	0.38	2.41	2.27
Duisburg	4.24	4.58	0.61	0.85	4.85	5.44
Essen	6.04	6.32	0.71	1.19	6.76	7.52
Paderborn	6.70	6.72	0.81	1.27	7.51	7.99
Siegen	4.11	4.49	1.14	1.35	5.25	5.84
Wuppertal	4.93	5.58	1.37	1.47	6.29	7.04
FU Hagen	1.86	2.10	1.13	1.21	2.99	3.31
Total	74.29	111.44	74.29	37.14	148.58	148.58

Source: Fandel and Gal (2001) and calculations from such data.

Looking at Table 5 we realize that for five universities the best sharing rule is 2, for five universities the best sharing rule is 6, for two universities is 5, for two universities is 7, and for one university is 4. Looking at Table 4 we realize that in sharing rule 2 research has a greater weight than teaching (80% research versus 20% teaching) while for sharing rules 4, 5, 6 and 7 teaching has a greater weight than research. Notice that in the selected sharing rule (7) the weight of teaching is 75% and the weight of research is 25%. Since five universities are better in sharing rule 2 (where research is more important than teaching), 10 universities are better under sharing rules where teaching is more important than research, and the sharing rule selected is 7, we can guess that the sharing rule selected represents the interest of the majority of the universities, but not all of them. Looking at Table 6 we realize that the five universities for which the best sharing rule is 2, receive more with the MIA rule. The other ten universities but DSH Cologne, receive less with the MIA rule.

During the negotiations between the universities and the government, universities know that the final allocation they will receive depends on the sharing rule used. Thus, each university has strong incentives to support the sharing rules in which receives a higher budget, instead of supporting the sharing rules which is more fair for the whole university system.

In practice is quite difficult to avoid this problem. If we want to mitigate it, then we have several possibilities. One, is to select the sharing rule supported by the majority of the universities. This seems to be behind the allocation used in this case. The problem is the asymmetric treatment of the universities. For some of them we select the sharing rule that gives them a large allocation, but for others we select a sharing rule that gives them a small allocation.

With the MIA rule we avoid this asymmetry because for each university we select as claims the allocation that gives such university the largest amount (among the seven sharing rules). Since this allocation is above the budget, we apply the MIA rule for obtaining an allocation dividing the real budget.

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