How to apply penalties to avoid delays in projects

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Abstract

A planner wants to carry out a project involving several firms. In many cases the planner, for instance the Spanish Administration, includes in the contract a penalty clause that imposes a payment per day if the firms do not complete their activities or the project on time. We discuss two ways of including such penalty clauses in contracts. In the first the penalty applies only when the whole project is delayed. In the second the penalty applies to each firm that incurs a delay even if the project is completed on time. We compare the two penalty systems and find that the optimal penalty (for the planner) is larger in the second method, the utility of the planner is always at least as large or larger in the second case and the utility of the firms is always at least as large or larger in the first. Surprisingly, the final delay in the project is unrelated to which penalty system is chosen.

Keywords: game theory; PERT; delays; penalties

1 Introduction

Assume that an agent, which we will call the planner, wants to carry out a project. The planner could be a public or private organization and the project could be the construction of some kind of infrastructure such as a bridge or a building. Typically, the project involves different activities that might be performed by different firms. Thus, the planner allocates each activity to a different firm. Each firm becomes responsible for performing its activity in a specified time. The planner wants to carry out the project by a deadline and will suffer a cost if the deadline is not met.

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We consider that the firms responsible for the activities are not affected directly by the cost suffered by the planner if the project is delayed. For instance, consider that the planner is a private firm that wants to adapt a warehouse to install a new data server and one of the activities of the project is the electrical installation. If that installation is delayed the server cannot work. This delay in the set-up of the server directly affects the planner, but it does not affect the firm responsible for the electrical installation. Thus, the planner should do something to encourage firms to complete their activities in the agreed time. In practice, the usual way is to impose a penalty on firms that cause a delay. This point is included in the contract between the planner and the firm in such a way that when there is a delay caused by the firm the amount received by the firm decreases in proportion to the delay in the activity.

When the planner is a private organization contracts between the planner and firms are private. Thus, it is difficult to know how penalty clauses (if any) are described in them. But when the planner is a public organization it is possible to learn that information. For instance, Spanish law sets a general framework for contracts in the public sector[1]. One part of that general legislation[2] describes how penalties should be applied to firms that cause a delay. The main issues in this area of Spanish law are outlined below.

First, any project must have a deadline. Firms must complete the project by that deadline. The administration can also stipulate interim deadlines in some contracts, in which case the firms must also meet those interim deadlines.

Second, if firms suffer delays (in meeting the project or interim deadlines) for which they are responsible, then the administration can cancel the contract or impose penalties on them. The general penalty is applied on a daily basis in a proportion of 0.2€ per 1000€ of the total cost estimate for the project. However, the administration can include different penalties in some contracts. Whenever the penalty reaches a multiple of 5% of the total cost estimated of the project, the administration must decide between canceling the project or continuing to apply the penalties.

Third, penalties must be applied by deducting them from the total amount that the firm is to receive from the administration.

In this paper we present a formal model for analyzing situations of this kind.

[1] We are talking about the "Real Decreto Legislativo 3/2011, de 14 de noviembre, por el que se aprueba el texto refundido de la Ley de Contratos del Sector Público". Ministerio de Economía y Hacienda.

BOE number 276. Date: November 16th, 2011. Reference BOE-A-2011-17887

[2] LIBRO IV: Efectos, cumplimiento y extinción de los contratos administrativos

TÍTULO I: Normas Generales

CAPÍTULO III: Ejecución de los contratos

Artículo 212: Ejecución defectuosa y demora

Our model is inspired by Spanish law, but it could be applied to a wide class of situations including administrations in other countries. For example, in February 2015 the U.S. Department of Housing and Urban Development updated the General Conditions for Construction Contracts -Public Housing Programs (Chicago Housing Authority, 2015). Among these conditions there is a liquidated damages clause that states: "If the Contractor fails to complete the work within the time specified in the contract, or any extension, [...] the Contractor shall pay to the Public Housing Agency as liquidated damages, the sum of $........[Contracting Officer insert amount] for each day of delay. If different completion dates are specified in the contract for separate parts or stages of the work, the amount of liquidated damages shall be assessed on those parts or stages which are delayed." There are also other clauses concerning the right of the Public Housing Agency to terminate the Contractor's right to proceed with the work due to an excessive delay. This issue has also been studied in other countries such as Italy, Kuwait and the Republic of the Congo (see, D'Alpaos *et al* (2009), Al-Tabtabai *et al* (1998), and Louzolo-Kimbembe and Mbani (2013)).

This model could also be applied to the private sector. For example, in its Guide for Supplementary Conditions the American Institute of Architects suggests including a liquidated damages clause in construction contracts in the following terms: "9.11 The Contractor and the Contractor's surety, if any, shall be liable for and shall pay the Owner the sums hereinafter stipulated as liquidated damages, and not as a penalty, for each calendar day of delay after the date established for Substantial Completion in the Contract Documents until the Work is substantially complete" (AIA Document A503-2007).

In this paper the PERT (Project Evaluation Review Technique) is used to model projects. Non-cooperative games in extensive form are used for modeling the situations faced by the different agents involved in the project.

In our model we have two kinds of agent: the planner, who wants to complete a project, and the firms, which are hired by the planner to complete the various activities involved in the project. Typically, the planner is a public or a private institution. Our non-cooperative game has three stages.

Stage 1. The planner decides what penalty will be applied to firms in case of delay. We assume that the penalty is proportional to the delay.

Stage 2. Following the structure of the project, firms decide how much effort they will devote to their assigned activities. Since we are studying situations in which firms are responsible for their delays, we assume that they can complete their allocated activities within the specified time if they devote the resources at their disposal. Thus, a delay in activity results from devoting fewer resources than required. We also assume that firms can obtain profit from devoting part of their resources to activities unrelated to the project. Thus, firms must find a balance between the utility that they obtain by devoting their resources to outside jobs and the penalty that will be imposed on them by the planner.

Stage 3. The planner pays the firms. We assume that the planner receives a utility from the completion of the project. Once the project is completed the planner knows what delays there have been in all activities. Each firm receives the amount agreed for the completion of its activity, minus the penalties applied for delays in that activity (if any). As in Spanish law, we consider two ways of applying penalties: First, only when the whole project is delayed, i.e. if a particular activity is delayed but the whole project is completed on time the planner does not apply the penalty. Second, the planner applies the penalty to each firm whose allocated activity is delayed (regardless of whether or not the whole project is delayed).

Actually, we consider two non-cooperative games in which Stages 1 and 2 are the same but Stage 3 is different. Since the utility of the agents is different in the two games, the equilibria could be different. In this paper we study and compare the equilibria in the two cases. Our main findings are the following.

The optimal penalty for the planner depends on the profit obtained by the firms when devoting their resources to activities other than the project. Thus, under Spanish law it is better to set the penalty depending on the project under consideration than to apply the general penalty. Moreover, the amount of the penalty also depends on how it is applied (when the whole project is delayed or always), which is not the case here.

For each project, the utility of each firm when penalties are applied only when the whole project is delayed is as great or greater than its utility when penalties are always applied. By contrast, the utility of the planner when penalties are always applied is as great or greater than its utility when penalties are applied only when the whole project is delayed.

The delay of the project is unrelated. Sometimes is greater when penalties are applied only when the whole project is delayed. Sometimes is greater when penalties are always applied. Assume that there is a set of agents (other than the planner and the firms) that need the project to be completed. For instance the project could be the construction of a new hospital, highway, etc. Those agents clearly want the project to be completed as soon as possible, but in such a situation it is not clear which penalty system is better.

The paper is organized as follows. The next subsection briefly reviews the literature related to our paper. Section 2 describes in detail the situations that we study. Section 3 analyzes the case where the planner applies the penalties only when the whole project is delayed. Section 4 analyzes the other case, where penalties are always applied. Section 5 compares the results obtained in the two cases. Finally, we present some concluding remarks.

1.1 Literature review

As far as we know there are not many papers studying how to manage delays in projects when several activities are involved. In the literature on civil engineering, a field in which the contract is cancelled in some countries if the total amount of penalties becomes very high, there are some papers that seek to find the limit for penalties so that the cancellation date does not exceed the due date of the project. Cases in point are the papers by Al-Tabtabai *et al* (1998), D'Alpaos *et al* (2009) and Louzolo-Kimbembe and Mbani (2013).

Focusing on economic literature, there is typically assumed to be a delay in the project caused by delays in several activities. This delay generates a cost that has to be paid by those responsible for the activities that cause the delay. The main question addressed in this literature is how this cost should be divided fairly between those responsible for these activities.

Bergantiños and Sánchez (2002) propose two rules: The first is based on cost sharing literature. They associate a cost sharing problem as in Moulin and Shenker (1992) with each project with delays. Then they study the serial cost sharing rule of that cost sharing problem. The second rule is based on cooperative games: They associate a cooperative game with each project with delays, then study the Shapley value (Shapley (1953)) of that game.

Branzêi *et al* (2002) propose several rules following two different approaches. In the first approach they associate a bankruptcy problem with each project with delays (see, for instance Aumann and Maschler (1985) or the survey by Thomson (2003)). Then they compute various bankruptcy rules. In the second approach they introduce more rules defined directly from the project with delays.

Estévez-Fernández *et al* (2007) associate a new cooperative game with each project with delays. The core of that cooperative game is studied.

Estévez-Fernández (2012) considers a more general model than in Estévez-Fernández *et al* (2007). For instance, the cost function of this paper is more general. A cooperative game (different from the one considered in Estévez-Fernández *et al* (2007)) is associated with each project with delays. The core of that new cooperative game is studied. For instance, it is proved that the core is non empty.

Although our paper also studies projects with delays, the approach taken is quite different from the papers mentioned above, which attempt to divide the cost fairly between the activities by means of the Shapley value, the serial cost sharing rule, bankruptcy rules or the core. Here, we seek to study mechanisms incentivizing firms to behave in the "right" way. The literature on implementation addresses the same objective as we do. The idea of that literature is to provide a mechanism (formally, a non-cooperative game) such that when agents which behave only out of self-interest play the mechanism optimally (formally, they play some kind of equilibria) the final outcome is good from a social welfare perspective. Some examples of this literature are Hart and Mas-Colell (1996) and Pérez-Castrillo and Wettstein (2001). These papers study two mechanisms whose equilibria induce the Shapley value. O'Neill (1982) and Dagan *et al* (1997) study two mechanisms whose equilibria induce some bankruptcy rules. Moulin and Shenker (1992) study a mechanism whose equilibria induce the serial cost sharing rule. Perry and Reny (1994) study a mechanism whose equilibria induce elements of the core.

2 The situations studied

There is a planner (denoted by 0), who wishes to carry out a project involving several activities that must be completed in a specific order. Some activities can be performed concurrently while others must be performed sequentially. Each activity is allocated to a different firm, which is responsible for completing it.

Each activity has an estimated duration representing the time needed to complete the activity when the firm devotes all its resources to it. Each firm has agreed with the planner to complete its activity in the allotted time. The planner has agreed with each firm on the amount that each firm will receive for completing its activity on time. The planner obtains a benefit from the completion of the project.

The planner wants to finish the project as soon as possible, given the duration of the activities. If the project is delayed the planner suffers a cost which is linear over the total delay in the project. The firms responsible for the activities are not affected directly if the project is delayed.

The firms can act strategically by assigning some of their resources to other tasks unrelated to the project and thus obtaining extra earnings. In this case they will need more time to complete their activities in the project. For example, assume that a firm is working on the construction of a house with a crew of ten workers and its allocated activity has an eight-week deadline. In this case the workers are the resources of the firm. The roof of a warehouse is broken and needs to be repaired quickly. If the firm does not take this job now it will lose it. Thus, the firm assigns five members of the crew to the job of mending the roof for two weeks. The other five continue to work on the construction of the house. Once the roof is fixed the five workers return to the house. Since the firm needs a crew of ten workers working for 8 weeks to complete its activity on time, it will now finish it in 9 weeks[3].

We assume that the planner cannot observe how much of their resources firms devote to their activities. The planner only observes the real completion time of the activity, denoted by t_i' (we assume that $t_i' \geq t_i$). In the above example the planner observes that the firm completes the activity in nine weeks.

In order to avoid strategic behavior by firms, the planner punishes them if they delay their activities. Before the projects starts, the planner announces a penalty p per unit of delay that firms must pay. We consider two cases.

1. First, the planner will punish firms that delay their activities only when the whole project is delayed.

   If the project is completed on time each firm i receives a_i, even if its activity was delayed.

   If the whole project is delayed, each firm i receives $a_i - p(t_i' - t_i)$.

2. Second, penalties are imposed for all activities that are delayed. In this case each firm i receives $a_i - p(t_i' - t_i)$.

We assume that each firm i has a reserve utility r_i representing the utility per unit of time that firm i obtains from reassigning all its resources to an alternative task unrelated to the project. Thus, if firm i is not punished it obtains $a_i + r_i t_i^*$ where t_i^* is the number of units of time for which its resources have been assigned to an alternative task. If it is punished, it obtains $a_i + r_i t_i^* - p\,(t_i' - t_i)$.

In the above example $t_i^* = 1$ because the firm assigns five workers during two weeks, or equivalently ten workers for 1 week. Notice that, in general, t_i^* coincides with $t_i' - t_i$ (in our case $9 - 8$).

We define the delay d_i of firm i as $t_i' - t_i$. Thus, the previous expressions can be written with d_i instead of t_i^* and $t_i' - t_i$.

[3]Note that only five workers are working on the activity for the first two weeks, which means that they do the same amount of work as 10 workers working for one week. In weeks 3 to 9 the full crew is working on the house.

Remark 1 *In order to simplify the notation of the paper we assume that firms decide how long to delay the activity rather than how much of their resources and time they will devote to other tasks. Moreover, when no confusion arises we identify the firm with its allocated activity.*

We now introduce other notation used in the paper. Given $x = (x_i)_{i \in N}$ and $y = (y_i)_{i \in N}$ we say that x Pareto dominates y if $x_i \geq y_i$ for all $i \in N$ and there exists $j \in N$ such that $x_j > y_j$. Given a subset $M \subset N$ we denote by $x_M = (x_i)_{i \in M}$. Given a subset $S \subset \mathbb{R}^N$ we define the Pareto boundary of S as

$$PB(S) = \{x \in S : \text{there is no } y \in S \text{ such that } y \text{ Pareto dominates } x\}.$$

The procedure described above is modeled as a non-cooperative game in extensive form. We use the PERT for modeling the project considered.

2.1 Modeling the project: The PERT

The PERT (Project Evaluation and Review Technique) is a well known tool of Operations Research for managing complex projects where several activities are involved. A classical reference for PERT is Moder and Phillips (1970). In the PERT model there is a directed graph G where the arcs are the activities and the nodes denote the end or the beginning of one or more activities. We denote the set of arcs by $N = \{1, 2, \ldots, n\}$. For each $i \in N$, b_i and e_i denote the beginning and the ending node of i respectively. There are two special nodes: node origin, which is the unique node such that there is no activity ending at that node, and node end, which is the unique node such that there is no activity beginning at that node. Each activity i has an estimated duration of t_i units of time, planned in an initial schedule.

A path π is a set of consecutive activities from the origin to the end of the project. We denote by Π the set of all paths in G. The duration of a path $\pi \in \Pi$ is the sum of the durations of the activities along this path, i.e. $t_\pi = \sum_{i \in \pi} t_i$. The PERT time T is the minimum time needed to complete the project. Thus, T is the duration of the longest path, namely $T = \max_{\pi \in \Pi} t_\pi$. The slack of a path π, denoted by ps_π, is the amount of time the activities in the path can delay without delaying the project. Thus, $ps_\pi = T - t_\pi$. The slack of an activity i, denoted by as_i, is the maximum time the activity can delay without delaying the project. Thus, $as_i = \min_{\pi \in \Pi : i \in \pi} ps_\pi$. A path or an activity is critical if its slack is 0. This means that any delay will produce a delay in the project.

Given two activities $i, j \in N$ we say that i comes before j, and denote it by $i \prec j$, if activity i needs to be performed before activity j can begin. Given an activity $i \in N$ and a path $\pi \in \Pi$ such that $i \in \pi$, we denote by $Pre(i, \pi)$ the set of activities that precede i in the path π. Analogously, we denote by $Suc(i, \pi)$ the set of activities that follow i in π. Formally:

$$\begin{aligned} Pre(i, \pi) &= \{j \in \pi : j \prec i\} \text{ and} \\ Suc(i, \pi) &= \{j \in \pi : i \prec j\}. \end{aligned}$$

We denote by $Pre(i)$ the set of activities that need to be performed before activity i can begin and $Suc(i)$ denotes the set of activities that need activity i to be performed before they can start. Formally:

$$\begin{aligned} Pre(i) &= \bigcup_{\pi \in \Pi : i \in \pi} Pre(i, \pi), \\ Suc(i) &= \bigcup_{\pi \in \Pi : i \in \pi} Suc(i, \pi). \end{aligned}$$

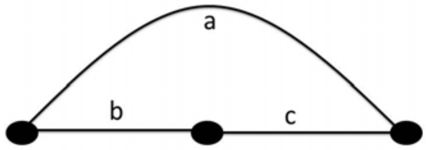
3 First case: penalties only when the project is delayed

We model this situation with a non-cooperative game in extensive form with 3 stages. In the first stage the planner decides the penalty per unit of delay to be paid by firms when the project is delayed. In the second stage the firms decide how much of their resources to put on their activities. The greater the resources the shorter the delay. In the third stage the planner pays the firms and applies the penalties, if any.

As argued in Remark 1, in Stage 2 each firm decides how long to delay its activity. As laid down in Spanish and US law, if a project is delayed the planner (public institution) has the right to cancel the contract or to impose a penalty. Typically, if the delay is short the planner imposes a penalty but if it is long it cancels the contract. Thus, we assume that there is a maximum level of delay, say D_i, such that if the firm delays more than D_i the planner will cancel the contract with the firm. Thus, the firm will not receive the payment a_i. Hence, $d_i \in [0, D_i]$. Note that under this assumption the project will not be canceled, so we focus on the case where the project is always completed.

The firms make their decisions following the structure of the project. This issue can be explained via the following example.

Example 1 *Consider the project given by the following picture.*



where

activities	t_i	D_i	r_i
a	*3*	*5*	*0*
b	*1*	*5*	*2*
e	*1*	*5*	*2*

Stage 1. The planner announces the penalty p to all the firms involved in the project.

Stage 2. Firms a and b decide simultaneously how long to delay their activities. Once firm b has finished, firm e decides its delay. Depending on the decisions of firms a and b firm e could make its decision after a or simultaneously with a. Assume that $d_a = 0$ *and* $d_b = 1$*, so firm e decides at time 2, while firm a is still working. Nevertheless, if* $d_a = 0$ *and* $d_b = 3$*, then firm a finishes at time 3 and firm b at time 4. Hence firm e decides its delay at time 4 when firm a has already finished.*

In general, if activity $j \in Pre(i)$, firm i decides after firm j has finished. If $j \notin Pre(i)$ and $i \notin Pre(j)$ anything is possible: i decides after j has finished, j decides after i has finished, or one of them decides while the other one is still working.

It is important to state what information is available to a firm when the time comes to decide its delay. We assume that each firm knows at time t, when it has to make its decision, which firms have already finished, and what their delays are. Thus, the information set of a firm i is characterized by a triple (t, S, d_S) where t is the time at which it has to start. Thus, all activities preceding i have already finished, i.e.

$$t \in \left[\max_{\pi \in \Pi : i \in \pi} \sum_{j \in Pre(i,\pi)} t_j, \max_{\pi \in \Pi : i \in \pi} \sum_{j \in Pre(i,\pi)} (t_j + D_j) \right],$$

S is the set of activities that have been completed by time t, namely

$$S = \left\{ j \in N : \max_{\pi \in \Pi : j \in \pi} \sum_{k \in Pre(j,\pi) \cup \{j\}} (t_k + d_k) \leq t \right\},$$

and $d_S = (d_j)_{j \in S}$ where d_j is the delay of each activity $j \in S$.

Given a firm $i \in N$, a strategy for firm i is a map d_i that assigns to each information set (t, S, d_S) a delay $d_i(t, S, d_S) \in [0, D_i]$. When no confusion arises we write d_i instead of $d_i(t, S, d_S)$. Besides, we usually write I instead of (t, S, d_S).

Given a strategy profile $d = (d_i)_{i \in N}$ and a path $\pi \in \Pi$, we denote by $t_\pi(d)$ the real duration of the path π when the delays of the activities are given by d. Analogously, we denote by $T(d)$ the real duration of the project. Thus,

$$T(d) = \max_{\pi \in \Pi} t_\pi(d) = \max_{\pi \in \Pi} \sum_{i \in \pi} (t_i + d_i).$$

In this expression d_i stands also for the real delay of activity i when firms play according with with the strategy profile d. When no confusion arises, we will make the same abuse of notation in the rest of the paper.

Remark 2 *We assume that $T(D) > T$. Namely, if all activities are delayed as long as possible then the project will be delayed for sure.*

We now describe Stage 3, where the planner pays the firms.

- The firms. The utility obtained by firm i will be the amount received from the planner, plus the utility the firm obtains from the alternative tasks, minus the penalty caused by its delay (if the whole project is delayed). Formally,

$$u_i(p, d) = \begin{cases} a_i + r_i d_i & \text{if } T(d) \leq T \\ a_i + r_i d_i - p d_i & \text{if } T(d) > T. \end{cases}$$

- Planner. The utility of the planner will be the benefits obtained from the completion of the project, plus (if the project is delayed) the amount obtained from the penalties minus the cost associated with the delay.

$$u_0(p, d) = \begin{cases} a_0 & \text{if } T(d) \leq T \\ a_0 + \sum\limits_{i \in N} p d_i - c(T(d) - T) & \text{if } T(d) > t. \end{cases}$$

Remark 3 *The utility function can be simplified. Since we are assuming that the project is completed, the planner always obtains a_0 and each firm i always receives a_i. Thus, the utility function can be defined in terms of earnings with respect to the status quo $(a_i)_{i \in N \cup 0}$.*

We now present the formal definition of the non-cooperative PERT game taking into account the above comments.

A **PERT problem with delays** is a triple (G, c, r) where G is the graph associated with the PERT problem, c is the cost per unit of time that the planner will incur if the project is delayed, and $r = (r_i)_{i \in N}$ is the vector of reserve utilities of the firms.

We can associate a non-cooperative game $\Gamma(G, c, r)$ with each PERT problem with delays as above, where:

1. **Stage 1**. The planner decides the penalty $p \in [0, +\infty)$.

2. **Stage 2**. The firms, following the structure of the project, decide the vector of delays $d = (d_i)_{i \in N}$.

3. **Stage 3**. The planner pays the firms. Because of Remark 3 the utilities are:

$$u_i(p, d) = \begin{cases} 0 & \text{if } i = 0 \text{ and } T(d) \leq T \\ r_i d_i & \text{if } i \in N \text{ and } T(d) \leq T \\ \sum\limits_{i \in N} p d_i - c(T(d) - T) & \text{if } i = 0 \text{ and } T(d) > T \\ (r_i - p)\, d_i & \text{if } i \in N \text{ and } T(d) > T. \end{cases}$$

Assume that in Example 1, $c = 3$, $r_a = 0$, $r_b = 1$ and $r_e = 2$. Besides, agents play the following strategies $p = 1$, $d_a = 0$, $d_b = 5$ and $d_e = 1$. There are two paths from the origin to the end: $\{a\}$ and $\{b, e\}$. The real duration of the paths when agents play (p, d) are $t_{\{a\}}(d) = 3$ and $t_{\{be\}}(d) = 8$. Thus, the real duration of the project according with (p, d) is $T(d) = \max\{3, 8\} = 8$. Namely, the project has been delayed because $T = 3$. Then,

$$\begin{aligned} u_0(p, d) &= 1\,(5 + 1) - 3\,(8 - 3) = -9 \\ u_a(p, d) &= 0 \\ u_b(p, d) &= (1 - 1)5 = 0 \\ u_e(p, d) &= (2 - 1)\,2 = 2. \end{aligned}$$

Since $p = r_b$, firm b obtains the same utility delaying its activity 5 units than finishing on time.

Remark 4 *Assume that $r_i = p$ for some firm $i \in N$. Assume that the whole project is delayed. This means that firm i will obtain zero whatever its delay. We assume* *that in such cases, firm i prefers not to delay. This can be interpreted as a reputation effect. Planners do not like delays and if a firm does not produce delays its reputation will be better and it will be easier for it to obtain contracts in the future. Thus, we consider that a firm will cause a delay only if it improves its utility by doing so.*

Next we study our model in three different examples: in the first all the activities perform in line; in the second all the activities perform in parallel; and in the third we propose a mixture of the two cases (a slight modification of Example 1).

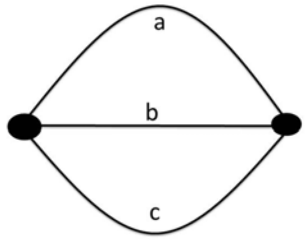
Example 2 *Consider a project where all the activities are in line.*

Since the unique path is the critical one, a delay in one activity will turn into a delay in the whole project. For each p and each $i \in N$, the utility of firm i will be 0 if it has not delayed and $(r_i - p)\, d_i$ if it has delayed d_i. Thus, in any Nash equilibria (NE) of the game if $r_i < p$, firm i chooses $d_i = 0$. If $r_i > p$, firm i chooses $d_i = D_i$. If $r_i = p$, by Remark 4, firm i chooses $d_i = 0$.

Thus, for each p the utility of the planner will be $(p - c) \sum_{i \in N: r_i > p} D_i$. So in an NE the planner chooses the penalty p that maximizes the above expression. If $p \geq c$, the planner will always obtain a non negative utility. If the reserve utilities are not so small (for instance $r_i > c$ for some i) then the planner will choose $p \geq c$.

This example shows that in equilibrium the planner could possibly choose a penalty greater than the cost, which is quite intuitive. Nevertheless, it is also possible that in equilibrium the project could be delayed. This is not counterintuitive. Assume that the cost c of delaying the project is smaller than the reserve utilities of the firms and the planner chooses a penalty p larger than c but smaller that some r_i. Thus, firms will decide to delay their activities because the benefits exceed the penalty. But the planner will also get benefits because the money obtained from the penalties offsets the loss caused by the delay.

Example 3 *Consider a project where the three activities are performed in parallel.*



Besides $c = 2$ *and*

activities	t_i	D_i	r_i
a	*3*	*10*	*1*
b	*3*	*10*	*1*
e	*3*	*10*	*1*

Note that, as in the example above, all activities are critical. Thus, if a firm delays its activity then the whole project will be delayed.

Assume that agents are playing an NE*. If the planner announces a penalty* $p \geq 1$*, all the activities will be completed on time and the payoff for the planner will be* 0*. If the planner announces* $p = 0.9$ *all the firms will choose* $d_i = D_i = 10$*. Thus, the payoff of the planner will be* $3 \cdot 0.9 \cdot 10 - 2 \cdot 10 = 7 > 0$*.*

Next, we consider an example where some activities are performed in parallel and others in sequence.

Example 4 *Consider the project given by the figure in Example 1 where* $c = 5$ *and*

activities	t_i	D_i	r_i
a	*10*	*5*	*0*
b	*4*	*5*	*3*
e	*4*	*5*	*3*

If firms play an NE *in the subgame obtained in Stage 2, depending on* p*, we can obtain several* NE[4]

p	4	2.5	2
(d_a, d_b, d_e)	$(0,2,0)$	$(0,1,1)$	$(0,5,5)$
$T(d) - T$	0	0	8
(u_0, u_a, u_b, u_e)	$(0,0,6,0)$	$(0,0,3,3)$	$(-20,0,5,5)$

[4]This statement is a consequence of our results but could be proved now directly.

In the case $p = 4$ it is easy to see that the NE obtained in Stage 2 is unique. Nevertheless, in the case $p = 2.5$ several NE could exist in the subgame obtained in Stage 2. For instance $(d_a, d_b, d_c) = (0, 0.9, 1.1)$ is also an NE inducing no delay in the project $(T(d) - T = 0)$ and whose payoff vector is $(0, 0, 2.7, 3.3)$. Notice that, even the payoff of the firms is different in both cases, the project is not delayed and so the utility of the planner is the same. Intuitively, what is happening is that firms b and e are dividing the slack of path $\{be\}$ among them in different ways. In the case $p = 2$ the NE obtained in Stage 2 is also unique.

3.1 Equilibria in Stage 2

In this part we characterize the NE of the subgame of $\Gamma(G, c, r)$ obtained in Stage 2, once the penalty p is announced by the planner. First we introduce some concepts used in this characterization.

Given a penalty p and a firm i with $r_i > p$, if firm i delays D_i its utility will be at least $(r_i - p)D_i > 0$. If $r_i \leq p$ and firm i does not delay its activity, its utility will certainly be 0. Thus, we define the **minimal right utility**[5] of firm i as

$$u_i^{mr}(p) = \begin{cases} (r_i - p)D_i & \text{if } r_i > p \\ 0 & \text{if } r_i \leq p. \end{cases}$$

Remark 5 *This minimal right utility can also be interpreted as* max min *utility. Namely, for each penalty p and each firm i we define*

$$u_i^{mm}(p) = \max_{d_i} \min_{d_{-i}} u_i(p, d_i, d_{-i})$$

where d_{-i} stands for the strategies of the agents in $N \setminus \{i\}$.

For each $i \in N$, it is easy to see that when the delay of the other firms is enough to delay the project, namely $T(0, D_{N\setminus\{i\}}) > T$, the minimal right utility and the max min *utility coincide $(u_i^{mr}(p) = u_i^{mm}(p))$.*

[5] The name minimal right is used in reference to the structure of the non-cooperative game, but not the contract between the planner and the firm.

Given a penalty p, let $s_i(p)$ be the **minimal right delay**, i.e. the minimum time by which activity i should be delayed when the project is not delayed to obtain its minimal right utility. Thus,

$$s_i(p) = \begin{cases} \dfrac{(r_i - p)D_i}{r_i} & \text{if } r_i > p \\ 0 & \text{if } r_i \leq p. \end{cases}$$

Notice that $s_i(p) \in [0, D_i]$ always. When no confusion arises we will write s_i instead of $s_i(p)$.

Given a penalty $p \in \mathbb{R}$ such that $\sum\limits_{i\in\pi} s_i(p) \leq ps_\pi$ for all $\pi \in P$, let $F(p)$ be the set of delays of the activities such that each agent delays at least $s_i(p)$ units of time and the project finishes on time. Namely,

$$F(p) = \left\{ x \in \mathbb{R}^N : x_i \in [s_i(p), D_i] \text{ and } \sum_{i\in\pi} x_i \leq ps_\pi \text{ for all } \pi \in P \right\}.$$

In the next proposition we characterize the NE in the subgame obtained in Stage 2.

Proposition 1 *Let $\Gamma(G, c, r)$ be the non-cooperative game induced by the PERT problem with delays (G, c, r). Let p be the penalty chosen by the planner at Stage 1. Assume that firms are playing an NE in the subgame obtained in Stage 2.*

(1) *If there exists a path $\pi' \in \Pi$ such that $\sum\limits_{i\in\pi'} s_i(p) > ps_{\pi'}$, then all NE have the same utility outcome. Besides, the project is delayed under any NE.*

(2) *If $\sum\limits_{i\in\pi} s_i(p) \leq ps_\pi$ for all $\pi \in \Pi$, then any allocation in $\left\{(x_i r_i)_{i\in N} : x \in PB(F(p))\right\}$ can be obtained as the utility vector associated with some NE. Thus, there exist equilibria in which the project is not delayed.*

Proof of Proposition 1.

(1) We prove that there exists an NE with delay. Let us define the strategy profile d where for each $i \in N$ and each information set I,

$$d_i(I) = \begin{cases} 0 & \text{if } r_i \leq p \\ D_i & \text{if } r_i > p. \end{cases}$$

Since $ps_{\pi'} < \sum\limits_{i\in\pi'} s_i(p) \leq \sum\limits_{i\in\pi' : r_i > p} D_i = \sum\limits_{i\in\pi'} d_i$, the project is delayed under d.

Next we prove that d is an NE. Consider $i \in N$ and d'_i a strategy of agent i. We distinguish two cases:

1. $r_i > p$. Assume that $u_i(p, d'_i, d_{-i}) \neq u_i(p, d)$. Then, there exists an information set I of agent i which is achieved under (d'_i, d_{-i}) such that $d'_i(I) < D_i$. So,

$$u_i(p, d'_i, d_{-i}) = \begin{cases} r_i d'_i(I) & \text{if } T(d'_i, d_{-i}) \leq T \\ (r_i - p)d'_i(I) & \text{if } T(d'_i, d_{-i}) > T. \end{cases}$$

   If $T(d'_i, d_{-i}) > T$, then $(r_i - p)d'_i(I) < (r_i - p)D_i = u_i(p, d)$.
   Assume that $T(d'_i, d_{-i}) \leq T$. Since $\sum_{j \in \pi'} s_j(p) > ps_{\pi'}$, $i \in \pi'$ and

$$\sum_{j \in \pi' \setminus \{i\} : r_j > p} D_j + d'_i(I) \leq ps_{\pi'}.$$

   Since $s_j(p) \leq D_j$ for each $j \in \pi' \setminus \{i\}$ with $r_j > p$, $s_j(p) = 0$ for each $j \in \pi' \setminus \{i\}$ with $r_j \leq p$ and $\sum_{j \in \pi'} s_j(p) > ps_{\pi'}$ we deduce that $d'_i(I) < s_i(p)$. So, $r_i d'_i(I) < r_i s_i(p) = (r_i - p)D_i = u_i(p, d)$.

2. $r_i \leq p$. Since $d_i(I) = 0$ for any information set I, the project will be delayed under (d'_i, d_{-i}). Thus,

$$u_i(p, d'_i, d_{-i}) = (r_i - p)d'_i(I) \leq 0 = u_i(p, d).$$

Next we prove that the project is delayed under any NE. Assume there exists an NE $d = (d_i)_{i \in N}$ where the project finishes on time. Thus, $\sum_{i \in \pi} d_i \leq ps_\pi$ for each $\pi \in P$. Since $\sum_{i \in \pi'} s_i(p) > ps_{\pi'}$, there exists an agent $i \in \pi'$ such that $s_i(p) > d_i \geq 0$. Since $s_i(p) > 0$, we have that $r_i > p$. Let d'_i be such that $d'_i(I) = D_i$ for each information set I of agent i. If firm i deviates and plays d'_i instead of d_i, then

$$u_i(p, d'_i, d_{-i}) = \begin{cases} r_i D_i & \text{if } T(d'_i, d_{-i}) \leq T \\ (r_i - p)D_i & \text{if } T(d'_i, d_{-i}) > T. \end{cases}$$

Thus, $u_i(p, d'_i, d_{-i}) \geq (r_i - p)D_i = r_i s_i(p) > r_i d_i$. Since agent i improves by playing d'_i instead of d_i, d is not a NE, which is a contradiction. Thus, under an NE, the project will be delayed.

Finally we prove that the utility outcome associated with any NE is unique. Let d' be an NE. We identify d'_i with the choice of firm i in the information set achieved when all agents play d'. We have proved that the project is delayed. Thus, $u_i(p, d) = (r_i - p) d'_i$. If $r_i < p$, then $d'_i = 0$ (otherwise agent i improves by playing

0 instead of d_i'). If $r_i = 0$ or $r_i = p$, then $u_i(p,d) = 0$ for each d_i'. By Remark 4, $d_i' = 0$. If $r_i > p$, then $d_i' = D_i$ (otherwise agent i improves by playing D_i instead of d_i'). Notice that the outcome associated with d_i' coincides with the outcome of the NE defined at the beginning of the proof. Thus, the utility outcome of the set of NE is unique.

(2) Given $x \in PB(F(p))$, we define d such that for each $i \in N$ and each information set I, $d_i(I) = x_i$. Note that under d, firms allocate the slack in the paths according to x. Besides, the conditions over x guarantee the project finishes on time. Thus, $u_i(p,d) = r_i x_i$ for each $i \in N$.

We prove that d is an NE. Assume firm i changes its strategy to d_i'. We identify d_i' with the choice of firm i in the information set achieved when all agents play according to (d_i', d_{-i}). We distinguish two cases:

1. $d_i' < x_i$. The project finishes on time and

$$u_i\left(p, d_i', d_{-i}\right) = r_i d_i' < r_i x_i = u_i(p,d).$$

2. $d_i' > x_i$. Since $x \in PB(F(p))$ and $d_i' \in [s_i(p), D_i]$, there exist $\pi \in P$ such that $\sum\limits_{j \in \pi \setminus \{i\}} x_j + d_i' > ps_\pi$. Thus, the project will be delayed and $u_i\left(p, d_i', d_{-i}\right) = (r_i - p)d_i'$. We distinguish two cases:

   (a) $r_i \leq p$. In this case

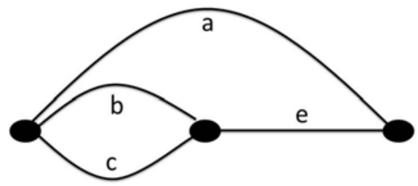
$$u_i\left(p, d_i', d_{-i}\right) = (r_i - p)d_i' \leq 0 \leq u_i(p,d).$$

   (b) $r_i > p$. In this case

$$u_i\left(p, d_i', d_{-i}\right) = (r_i - p)d_i' \leq (r_i - p)D_i = r_i s_i(p) \leq r_i x_i = u_i(p,d). \blacksquare$$

In the first part of Proposition 1 we prove that, basically, we have a unique NE. Nevertheless, in the second part we identify a subset of utility allocations associated with NE. Depending on the case there may be NE where the project is delayed or not. For instance, when $p > \max\limits_{i \in N} r_i$, there is no NE where the project is delayed. However, in the next example, we show there are NE that lead to payoffs that are not associated with any allocation $x \in PB(F(p))$.

Example 5 . *Consider the project given by the following figure*



where

Activities	t_i	D_i	r_i
a	10	3	0
b	2	3	3
e	2	3	3
f	2	8	4

Notice that the value of c *is irrelevant for our analysis. Let* $p = 2$*. We define* d *as follows*

- $d_a(I) = 0$ *for every information set* I*.*
- $d_b(I) = 1$ *for every information set* I*.*
- $d_e(I) = 2$ *for every information set* I*.*
- $d_f(I) = \begin{cases} 4 & \text{if } t = 4, S = \{b, e\}, d_b = 1, \text{ and } d_e = 2 \\ 8 & \text{otherwise.} \end{cases}$

Thus, $u_a(p, d) = 0$*,* $u_b(p, d) = 3$*,* $u_e(p, d) = 6$*, and* $u_f(p, d) = 16$*. Note that under this strategy the slack of path* $\{b, f\}$ *is not completely allocated. It is easy to prove that* d *is a* NE*. For instance, if firm* b *delays 2, instead of* 1*, the project still finishes on time. But if firm* b *does so, firm* f *will delay 8 (instead of 4), the project will be delayed and the utility of firm* b *will be 2.*

Nevertheless, the vector of delays $(0, 1, 2, 4)$ *associated with* d*, does not belong to* $PB(F(2))$ *because* $(0, 2, 2, 4)$ *Pareto dominates* $(0, 1, 2, 4)$ *and* $(0, 2, 2, 4) \in F(2)$*.*

In the next Proposition we characterize the set of NE whose payoffs for the firms are not Pareto dominated by other NE. By part 1 of Proposition 1, when there is a path $\pi' \in \Pi$ such that $\sum_{i \in \pi'} s_i(p) > ps_{\pi'}$ all NE have the same utility outcome. In such a case the set of NE whose payoffs are not Pareto dominated is the set of all NE.

Proposition 2 *Let $\Gamma(G, c, r)$ be the non-cooperative game induced by (G, c, r). Assume that $\sum\limits_{i\in\pi} s_i(p) \leq ps_\pi$ for all $\pi \in \Pi$. Let d be an NE in the subgame obtained in Stage 2. Then, the set of NE whose payoffs are not Pareto dominated is:*

$$\{(r_i x_i)_{i\in N} : x \in PB(F(p))\}.$$

Proof of Proposition 2. Let d be an NE where the project is delayed. By part 2 of Proposition 1 we know that there exist NE where the project is not delayed. Let d' be one of such NE. Let us denote by y and y' the vectors of utilities for the firms associated with both equilibria respectively. Using arguments similar to those used in the proof of part 1 of Proposition 1 we can deduce that for each $i \in N$

$$y_i = \begin{cases} 0 & \text{if } r_i \leq p \\ (r_i - p)D_i & \text{if } r_i > p. \end{cases}$$

Besides, $y' = (x_i r_i)_{i\in N}$ where $x \in PB(F(p))$. We now prove that $y_i \leq y'_i$ for all $i \in N$. If $y_i = 0$, then it is obvious because $r_i \geq 0$ and $x_i \geq s_i \geq 0$. If $y_i = (r_i - p)D_i$, then $y_i = (r_i - p)D_i \leq r_i s_i \leq r_i x_i = y'_i$.

Note that there exists at least one agent j such that $y_j \neq y'_j$. Otherwise both vectors would coincide and, by Remark 4, an agent will choose the strategy that leads to a smaller delay. Thus, y' Pareto dominates y.

Let d be an undominated NE. Since the project ends on time under d, the utilities for the firms under d can be rewritten as $(r_i x_i)_{i\in N}$ where $x_i \in [0, D_i]$ and $\sum\limits_{i\in\pi} x_i \leq ps_\pi$. When we define s_i we have argued that any firm i can obtain, independently of the strategies of the other agents, a utility $r_i s_i$. Thus, $x_i \geq s_i$ for all $i \in N$. Hence $x \in F(p)$. Suppose that $x \notin PB(F(p))$. Then there exists $x' \in F(p)$ such that x' Pareto dominates x. We can assume $x' \in PB(F(p))$ (otherwise we can consider $x'' \in PB(F(p))$ such that x'' Pareto dominates x' and proceed with x'' instead of x'). Under Proposition 1.2, the allocation $(x'_i r_i)_{i\in N}$ can be obtained as the utility vector associated with some NE d'. Thus, the vector of utilities associated with the NE d' Pareto dominates the vector of utilities associated with NE d, contradicting that d is an undominated NE. Thus, we conclude that $x \in PB(F(p))$.

It only remains to prove that if d is an NE whose vector of utilities is given by $(r_i x_i)_{i\in N}$ with $x \in PB(F(p))$, then d is an undominated NE. Suppose not. Then, there exists another NE d' such that the utilities associated with d' Pareto dominates $(x_i r_i)_{i\in N}$. We can assume that d' is an undominated NE (otherwise we take d'' an undominated NE that Pareto dominates d' and we proceed with d''

instead of d'). The utility vector associated with d' can be expressed as $(r_i x'_i)_{i\in N}$ with $x' \in PB(F(p))$. Thus, $r_i x'_i \geq r_i x_i$ for all $i \in N$. If $r_i = 0$, then $x'_i = x_i = 0$ by Remark 4. Then $x'_i \geq x_i$ for each $i \in N$ with $r_i > 0$ and there exists $j \in N$ with $r_j > 0$ such that $x'_j > x_j$. Thus, x' Pareto dominates x, which is a contradiction. ■

We end this subsection with some examples showing that some NE might not be a good prediction of the behavior of firms.

Example 6 *Consider the project given by Example 3 where $c = 5$ and*

activities	t_i	D_i	r_i
a	*3*	*5*	*0*
b	*2*	*5*	*1*
e	*2*	*5*	*1*

Assume that $p = 0.9$. Notice that the three firms take their decisions simultaneously. Thus, they have only one information set. Since $r_a = 0$ and a is a critical activity it is a dominant strategy for firm a to finish on time. Thus, if d is an NE in the subgame in Stage 2 then $d_a = 0$.

There are two NE in the subgame in Stage 2 $d = (0, 1, 1)$ and $d' = (0, 5, 5)$. In the first NE firms b and e delay by 1 unit, the project finishes on time, and each firm (b and e) obtains 1. In the second NE firms b and e delay as long as possible, the project is delayed and each firm (b and e) obtains $(1 - 0.9)\, 5 = 0.5$. Notice that the second NE is Pareto dominated, in terms of the utilities of the firms, by the first one.

The second NE is based on a bad coordination effect. Firm b (or e) thinks that firm e (or b) will delay as long as possible, so its best option is to do likewise. Nevertheless, it is better for both firms to delay by only 1 unit (as the first NE suggests).

This bad coordination effect is intrinsic to the NE when agents play simultaneously. We believe that in this example the first NE predicts the behavior of rational firms better.

Example 7 *Assume that in Example 1 $p = 1.9$. Thus $s_b(p) = s_e(p) = 0.25$. Because of Proposition 2, the set of utilities associated with undominated NE is*

$$\{(0, x, 2 - x) : x \in [0.5, 1.5]\}.$$

Take $(0, 0.6, 1.4)$. *This corresponds to* $d_a = 0$, $d_b = 0.3$, *and* $d_e(I) = 0.7$ *for any information set* I *of firm* e. *We do not believe that this* NE *predicts the behavior of rational firms.*

We now analyze this example in detail. Firms a *and* b *must make their decisions simultaneously and the only information that they have is* $p = 1.9$. *Firm* e *knows the delay of firm* b. *Firm* e *may know the delay of firm* a *(for instance if* $d_b = 4$ *and* $d_a = 0$*) but it also may not (for instance if* $d_b = 1$ *and* $d_a = 0$*). What should firm* e *do in any of its information sets? Of course, if the project is delayed the best decision for firm* e *is to play* $d_e = 5(= D_e)$. *We consider several cases:*

1. *Assume that* $d_b > 1$, *then the project will be delayed. Hence, the best decision for firm* e *is* $d_e = 5$ *and its final payoff will be* $(2 - 1.9)\,5 = 0.5$.

2. *Assume that* $0.75 < d_b \leq 1$. *Two situations are possible depending on the delay of firm* a *(which firm* e *does not know).*

   (a) $d_a > 0$. *Then the project will be delayed and hence the best decision for firm* e *is* $d_e = 5$. *Its final payoff will be* 0.5.

   (b) $d_a = 0$. *If firm* e *chooses* $d_e > 1 - d_b$ *the project will be delayed. In that case it is better to choose* $d_e = 5$. *Hence, the project will be delayed and the utility of firm* e *will be* 0.5.

   If firm e *chooses* $d_e \leq 1 - d_b$ *the project will finish on time. The utility of firm* e *will be* $2\,(1 - d_b) < 2\,(0.25) = 0.5$.

   Thus, the best decision for firm e *is to choose* $d_e = 5$.

3. *Assume that* $d_b \leq 0.75$. *Again, two situations are possible depending on the delay of firm* a *(which firm* e *does not know).*

   (a) $d_a > 0$. *Similarly to Case 2.(a),* $d_e = 5$ *and firm* e *obtains* 0.5.

   (b) $d_a = 0$. *If firm* e *chooses* $d_e > 1 - d_b$ *similarly to case 2.(b),* $d_e = 0.5$ *and firm* e *obtains* 0.5.

   If firm e *chooses* $d_e \leq 1 - d_b$ *the project will finish on time. In that case it is better to choose* $d_e = 1 - d_b$. *Then the utility of firm* e *will be* $2(1 - d_b) \geq 2(0.25) = 0.5$.

   Notice that the best decision for firm e *depends on the decision of firm* a. *If* $d_a > 0$ *then the best decision is* $d_e = 5$, *but if* $d_a = 0$ *the best decision is* $d_e = 1 - d_b$. *Thus, firm* e *should think about what firm* a *will do and make* *its decision accordingly. This case is quite simple because for firm a the payoff from playing* $d_a = 0$ *is larger than the payoff of any* $d_a > 0$*. Hence, firm e will choose* $d_e = 1 - d_b$*.*

We now analyze the behavior of firms a and b. For firm a $d_a = 0$ *dominates any* $d_a > 0$*, so firm a will always choose* $d_a = 0$ *under equilibria.*

Firm b may anticipate that firm a will choose $d_a = 0$ *and firm e will choose* d_e *following the discussion above. If firm b chooses* $d_b > 0.75$*, then firm e will choose* $d_e = 5$*, and firm b will obtain* $(2 - 1.9)d_b \leq 0.5$*. If firm b chooses* $d_b \leq 0.75$*, then firm e will choose* $d_e = 1 - d_b$ *and firm b will obtain* $2d_b$*. Thus, the best decision for firm b is* $d_b = 0.75$*.*

We argue that although there are many NE *in this example, rational firms will play the* NE *where in the equilibrium path,* $d_a = 0$*,* $d_b = 0.75$*, and* $d_e = 0.25$*, whose vector of utilities for the firms is* $(0, 1.5, 0.5)$*.*

Both examples 6 and 7 show that when firms make their decisions they must care about what decisions are made by firms that perform in parallel. In both examples the good prediction is that of the case where firms believe that firms working in parallel will not cause a delay unless it is profitable for them. In the next section we formalize this idea and select the NE that meet this condition.

3.2 Selecting equilibria in Stage 2

In this section we define the optimistic NE. Our idea is to select the NE where each firm behaves rationally in any information set when its beliefs are optimistic, i.e. when each firm thinks that the other firms will not delay the project unless it is profitable for them. Thus, we select the NE that meet two conditions. The first condition is related with the beliefs of the firms. By the time when a firm i must make its decision there are other firms that have already made their decisions (firm b in Example 1 with $i = e$); there are also other firms that will make their decisions after firm i has completed its work (firm e in Example 1 if we consider $i = b$); and still others that make their decisions while firm i is still performing and therefore do not know the final delay of firm i (firm a in Example 1 with $i = b$). In an optimistic NE we assume that firm i believes that the firms in the third group will not cause a delay unnecessarily (namely, these firms are trying to play an NE without delay when possible). The second condition is related to the ideas of the subgame perfect Nash equilibrium ($SPNE$). We assume that each firm behaves rationally in any information set in accordance with its beliefs. In our case this assumption is stronger than saying that agents play an $SPNE$ because, in general, the unique subgame of the subgame obtained in Stage 2 is the whole subgame (see for instance examples 6 and 7).

We now formalize this idea. Consider $i \in N$ and $I = (t, S, (d_j)_{j \in S})$ an information set of firm i. Let $Par(I)$ be the set of activities that perform in parallel with i at time t[6]. Namely,

$$Par(I) = \{k \in N \setminus S : Pre(k) \subset S\}.$$

Note that $i \in Par(I)$.

We now define the **vector of optimistic believes** $o(I) \in \mathbb{R}^N$ for firm i at information set I. We consider several cases:

1. $j \in S$. Firm i knows the exact delay d_j of firm j. Thus,

$$o_j(I) = d_j.$$

2. $j \in Par(I) \setminus \{i\}$. Firm i knows that firm j has started at time $\max\limits_{j \in \pi \in \Pi} \sum\limits_{k \in Pre(j,\pi)} (t_k + d_k)$ and that at time t firm j is still working. Thus, firm j has been working $t - \max\limits_{j \in \pi \in \Pi} \sum\limits_{k \in Pre(j,\pi)} (t_k + d_k)$ units of time.

   Firm i believes firm j will not delay the project unless it is necessary for obtaining its minimal right. Thus, if $t - \max\limits_{j \in \pi \in \Pi} \sum\limits_{k \in Pre(j,\pi)} (t_k + d_k) < t_j + s_j(p)$, then firm i believes the delay of firm j will be $s_j(p)$. Otherwise, if $t - \max\limits_{j \in \pi \in \Pi} \sum\limits_{k \in Pre(j,\pi)} (t_k + d_k) \geq t_j + s_j(p)$, then firm i believes firm j will finish immediately. We model it by saying that j will finish in ε units of time. Thus,

$$o_j(I) = \max \left\{ s_j(p), t - t_j - \max_{j \in \pi \in \Pi} \sum_{k \in Pre(j,\pi)} (t_k + d_k) + \varepsilon \right\}.$$

3. $j \in N \setminus \{S \cup Par(I)\}$. Note that in this case firm i has no information about the delay of j because it has not started yet. So firm i believes firm j will not delay the project at least it is necessary for obtaining its minimal right. Thus,

$$o_j(I) = s_j(p).$$

[6]The set of activities in which firms are still performing at time t.

Let $a(o(I),\varepsilon) = \max_{\pi\in\Pi}\left\{\sum_{j\in\pi} o_j(I) - ps_\pi\right\}$ where by convention $o_i(I) = s_i(p)$. If $a(o(I),\varepsilon) > 0$ for all $\varepsilon > 0$, then the project will be delayed under each NE. If $a(o(I),\varepsilon) \leq 0$ it is possible for firms to play an NE without delays.

Remark 6 *Given the information held by firm i at time t, firm i thinks that the firms that have not finished yet, including itself, will choose a delay equal to its minimal right or will finish immediately. Actually, our results still hold if the beliefs of firm i are modeled by a vector $x(I)$ in which the firms do not delay the project unless it is necessary to do so to obtain their minimal right. Namely, $x_j(I) = o_j(I)$ for all $j \in S$, $x_j(I) \geq o_j(I)$ when $j \notin S$, and $a(x(I),\varepsilon) \leq 0$ when $a(o(I),\varepsilon) \leq 0$.*

Let $\Gamma(G,c,r)$ be the non-cooperative game induced by (G,c,r). Let p be the penalty chosen by the planner at Stage 1. Consider $i \in N$ and I an information set of agent i. Let $x = (x_j)_{j\in N\setminus(Suc(i)\cup\{i\})}$ such that $x_j = d_j$ for each $j \in S$. We define the game $\Gamma^{I,x}(G,c,r)$ as the game induced by $\Gamma(G,c,r)$ at information set I by assuming that the set of firms is $Suc(i) \cup \{i\}$ and the rest of the firms $j \in N \setminus (Suc(i) \cup \{i\})$ have a fixed duration of $t_j + x_j$ units of time (or a fixed delay of x_j units of time). Note that in this game the first decision corresponds to firm i.

Let us denote by $o^i(I)$ the restriction of $o(I)$ to $N \setminus (Suc(i) \cup \{i\})$, namely, $o^i(I) = (o_j(I))_{j\in N\setminus(Suc(i)\cup\{i\})}$.

We say $d = (d_i)_{i\in N}$ is an **optimistic** NE if it is an NE and for any firm i and any information set I we have that $d_{Suc(i)\cup\{i\}}$ induces an NE in $\Gamma^{I,o^i(I)}(G,c,r)$.

We now compute the set of optimistic NE in Example 1 when $p = 1.9$. Let $I = (t, S, d_S)$ be an information set of firm e. We consider several cases.

1. $t > 2$. The project will be delayed for sure. Hence, $d_e(I) = 5$.
2. $t \leq 2$. Thus, $S = \{b\}$, $o_a(I) = 0 = s_a(1.9)$ and $o_b(I) = d_b$. Hence,

$$d_e(I) = \begin{cases} 5 & \text{if } d_b > 0.75 \\ 1 - d_b & \text{if } d_b \leq 0.75. \end{cases}$$

Let I be the information set of firm a. Since $r_a < p$ and a is a critical activity, $d_a(I) = 0$. Note that $o_b(I) = o_e(I) = s_b(1.9) = s_e(1.9) = 0.25$.

Let I be the information set of firm b. For each decision d_b in I, firm b knows the decision of firm e under an optimistic NE. Hence, $d_b(I) = 0.75$.

Thus, the vector of delays when firms play an optimistic NE is $(0, 0.75, 0.25)$ and the utilities are $(0, 1.5, 0.5)$.

Next we prove that there is a unique optimistic NE.

Proposition 3 *Let $\Gamma(G, c, r)$ be the non-cooperative game induced by (G, c, r). For any penalty p there exists a unique optimistic NE in the subgame obtained in Stage 2.*

Proof of Proposition 3. We first prove that there exists at least one optimistic NE. Let $i \in N$ and let I be an information set of firm i. Intuitively, the decision of firm i is as follows: given its optimistic beliefs, if there is enough slack to give to every firm its minimal right and $r_i > 0$, then firm i delays as long as possible, giving the other firms their minimal right. Otherwise firm i delays D_i when $r_i > p$ and 0 when $r_i \leq p$. Formally,

$$d_i(I) = \begin{cases} \min\limits_{i \in \pi \in \Pi} \left\{ D_i, ps_\pi - \sum\limits_{j \in \pi \setminus \{i\}} o_j(I) \right\} & \text{if } r_i > 0 \text{ and } a(o(I), \varepsilon) \leq 0 \text{ for some } \varepsilon > 0 \\ D_i & \text{if } r_i > p \text{ and } a\left(o(I), \varepsilon\right) > 0 \text{ for all } \varepsilon > 0 \\ 0 & \text{if } r_i = 0 \text{ or } r_i \leq p \text{ and } a\left(o\left(I\right), \varepsilon\right) > 0 \text{ for all } \varepsilon > 0. \end{cases} \quad (1)$$

Notice that when $a(o(I), \varepsilon) \leq 0$ for some $\varepsilon > 0$ we have that

$$\min_{i \in \pi \in \Pi} \left\{ D_i, ps_\pi - \sum_{j \in \pi \setminus \{i\}} o_j(I) \right\} \geq 0.$$

Thus, $d_i(I)$ is well defined.

We now prove that $d = (d_i)_{i \in N}$ is an optimistic NE. Let $i \in N$ and I an information set of firm i. We assume that $r_i > 0$ (otherwise, by Remark 4, $d_i(I) = 0$ is the best decision). Let $d_i'(I) \neq d_i(I)$. By simplicity we write d_i (d_i') instead of $d_i(I)$ ($d_i'(I)$). We make an abuse of notation and we identify the utility and strategies in the games $\Gamma(G, c, r)$ and $\Gamma^{I, o^i(I)}(G, c, r)$. We distinguish two cases:

1. $d_i' > d_i$. Thus, $d_i < D_i$. We consider two cases:

(a) $d_i = 0$. Thus, $r_i \leq p$ and $a(o(I), \varepsilon) > 0$ for all $\varepsilon > 0$. This means that the project will be delayed under NE. So

$$u_i(d_{Suc(i)\cup\{i\}} \setminus d_i') = d_i'(r_i - p) \leq 0 = u_i(d_{Suc(i)\cup\{i\}}).$$

(b) $0 < d_i = \min_{i\in\pi\in\Pi}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} o_j(I)\right\} < D_i$. Thus, $r_i > 0$ and $a(o(I), \varepsilon) \leq 0$ for some $\varepsilon > 0$. Then, the project is not delayed under d and $u_i(d_{Suc(i)\cup\{i\}}) = r_i d_i$.

We have that $D_i > d_i \geq o_i(p) = s_i(p)$. By definition of d_i, if firm i increases its delay, the project will be delayed. Thus, $u_i(d_{Suc(i)\cup\{i\}} \setminus d_i') = (r_i - p)d_i'$.

If $r_i \leq p$, then $(r_i - p)d_i' \leq 0 \leq r_i d_i$. If $r_i > p$, then

$$(r_i - p)d_i' \leq (r_i - p)D_i = r_i s_i(p) \leq r_i d_i.$$

2. $d_i' < d_i$. Thus $d_i > 0$. We again consider two cases.

(a) $d_i = D_i$, $r_i > p$ and $a(o(I), \varepsilon) > 0$ for all $\varepsilon > 0$. Then, the project is delayed under d and hence $u_i\left(d_{Suc(i)\cup\{i\}}\right) = (r_i - p)\, D_i$. Two cases are possible.

i. The project is also delayed under $d_{Suc(i)\cup\{i\}} \setminus d_i'$. Thus,

$$u_i\left(d_{Suc(i)\cup\{i\}} \setminus d_i'\right) = (r_i - p)\, d_i' < (r_i - p)\, D_i.$$

ii. The project is not delayed under $d_{Suc(i)\cup\{i\}} \setminus d_i'$. Since $a(o(I), \varepsilon) > 0$ for all $\varepsilon > 0$ we have that $d_i' < o_i(I) = s_i(p)$. Thus,

$$u_i\left(d_{Suc(i)\cup\{i\}} \setminus d_i'\right) = r_i d_i' < r_i s_i(p) = (r_i - p)\, D_i.$$

(b) $d_i = \min_{i\in\pi\in P}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} x_j\right\}$, $r_i > 0$ and $a(o(I), \varepsilon) \leq 0$ for some $\varepsilon > 0$. Then, the project is not delayed under d and $u_i(d_{Suc(i)\cup\{i\}}) = r_i d_i$. Since $d_i' < d_i$ we know that the project is not delayed under $d_{Suc(i)\cup\{i\}} \setminus d_i'$. Thus,

$$u_i\left(d_{Suc(i)\cup\{i\}} \setminus d_i'\right) = r_i d_i' < r_i d_i.$$

We now prove the uniqueness. We will prove that if d is an optimistic NE, then d coincides with formula (1). Let $i \in N$ and I an information set of firm i. If $r_i = 0$, by Remark 4, $d_i(I) = 0$. Thus, we assume $r_i > 0$.

For each $i \in N$ let $l(I)$ denote the maximum number of activities in the largest path from i until the end of the project. Namely,

$$l(i) = \max_{i \in \pi \in \Pi} \{|Suc(i, \pi)|\}.$$

We prove the uniqueness by backward induction on $l(i)$. Assume that i is an activity with $l(i) = 0$. Then, $Suc(i) = \emptyset$. Since i is the unique firm in the game $\Gamma^{I,o^i(I)}(G, c, r)$ and d induces an NE in $\Gamma^{I,o^i(I)}(G, c, r)$, we deduce that $d_i(I)$ is as in formula (1).

Assume now that $l(i) = 1$. Thus, for each $j \in Suc(i)$, $l(j) = 0$, and so these firms behave according with d. We consider several cases:

1. $a(o(I), \varepsilon) > 0$ for all $\varepsilon > 0$. Since $d_{Suc(i) \cup \{i\}}$ is an NE in $\Gamma^{I,o^i(I)}(G, c, r)$, the project is delayed under $d_{Suc(i) \cup \{i\}}$. Thus, $u_i\left(d_{Suc(i) \cup \{i\}}\right) = (r_i - p)\, d_i(I)$. We again consider two cases:

   (a) $r_i \leq p$. If $d_i(I) > 0$, then $u_i\left(d_{Suc(i) \cup \{i\}}\right) < 0$. Since $d_{Suc(i) \cup \{i\}}$ is an NE, we have that $d_i(I) = 0$.

   (b) $r_i > p$. Since $d_{Suc(i) \cup \{i\}}$ is an NE, we have that $d_i(I) = D_i$.

2. $a(o(I), \varepsilon) \leq 0$ for some $\varepsilon > 0$. Then, $d_{Suc(i) \cup \{i\}}$ could be an NE with or without delay in $\Gamma^{I,o^i(I)}(G, c, r)$. Once i has finished, for each $j \in Suc(i)$ it is achieved the information set I_j. We consider several cases.

   (a) $d_i(I) \leq \min_{i \in \pi \in \Pi} \left\{ D_i, ps_\pi - \sum_{j \in \pi \setminus \{i\}} o_j(I) \right\}$. Then, $a(o(I_j), \varepsilon) \leq 0$ for some $\varepsilon > 0$ for each $j \in Suc(i)$. Since $l(j) = 0$, for each $j \in Suc(i)$, we have that $d_j(I_j)$ is as in formula (1). Thus, the project is not delayed and hence, $u_i(d_{Suc(i) \cup \{i\}}) = r_i d_i(I)$.

   In this case $d_i(I) = \min_{i \in \pi \in \Pi} \left\{ D_i, ps_\pi - \sum_{j \in \pi \setminus \{i\}} o_j(I) \right\}$ is strictly better that any $d_i(I) < \min_{i \in \pi \in \Pi} \left\{ D_i, ps_\pi - \sum_{j \in \pi \setminus \{i\}} o_j(I) \right\}$.

(b) $d_i(I) > \min_{i\in\pi\in\Pi}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} o_j(I)\right\}$. Then, there exists π' such that $ps_{\pi'} - \sum_{j\in\pi'\setminus\{i\}} o_j(I) < d_i(I) \le D_i$. Hence, there exists $j \in Suc(i)$ such that $a(o(I_j),\varepsilon) > 0$ for all $\varepsilon > 0$. Since $o_j(I) = s_j(p)$ and $s_j(p) = 0$ when $r_j \le p$ we deduce that $r_j > p$. Since $l(j) = 0$, $d_j(I) = D_j$ (as in formula (1)) and the project is delayed. Hence, $u_i(d_{Suc(i)\cup\{i\}}) = (r_i - p)d_i(I)$. Since d is an NE, $r_i > p$ (otherwise $d_i'(I) = 0$ is better than $d_i(I)$). Then $d_i(I) = D_i$ is strictly better than any $d_i(I) < D_i$.

Since $s_i(p) \le \min_{i\in\pi\in\Pi}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} o_j(I)\right\}$ we have that

$$(r_i - p)\, D_i = r_i s_i(p) \le r_i \min_{i\in\pi\in\Pi}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} o_j(I)\right\}.$$

Now if $s_i(p) < \min_{i\in\pi\in\Pi}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} o_j(I)\right\}$, then firm i obtains more utility delaying $\min_{i\in\pi\in\Pi}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} o_j(I)\right\}$ than delaying D_i. Since $d_{Suc(i)\cup\{i\}}$ is an NE in $\Gamma^{I,o^i(I)}(G,c,r)$, we have that $d_i(I) = \min_{i\in\pi\in\Pi}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} o_j(I)\right\}$.

If $s_i(p) = \min_{i\in\pi\in\Pi}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} o_j(I)\right\}$ then firm i obtains the same utility delaying $\min_{i\in\pi\in\Pi}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} o_j(I)\right\}$ than delaying D_i. Since $d_{Suc(i)\cup\{i\}}$ is an NE in $\Gamma^{I,o^i(I)}(G,c,r)$ and Remark 4, we have that $d_i(I) = \min_{i\in\pi\in\Pi}\left\{D_i, ps_\pi - \sum_{j\in\pi\setminus\{i\}} o_j(I)\right\}$.

Assume now that $l(i) = 2$. If we proceed as in the previous case we obtain than $d_i(I)$ is as in formula (1). By repeating this argument we can prove that $d_i(I)$ is as in formula (1) for each $i \in N$ and each information set I of firm i. ■

3.3 Equilibria in the whole game

Once we know how firms behave under equilibria when the planner announces the penalty, the next step is to focus on the optimal decision for the planner. Our main objective is to calculate the penalty under which the planner maximizes its utility. To that end we assume that the firms play an NE in the subgame of Stage 2. In some cases there is a unique NE but in other cases there can be several NE. Thus, the planner must predict which NE will be played in Stage 2. The results shown in this section hold under the assumption that in Stage 2 firms will play any optimal NE (see Proposition 2). In particular our results also hold when firms play the unique optimistic NE characterized in Proposition 3.

We start with a preliminary result.

Lemma 1 *Let $\Gamma(G, c, r)$ be the non-cooperative game induced by (G, c, r). Assume that for each penalty p firms play an optimal NE in Stage 2. Then, there exists a limit penalty p^*_Γ such that*

1. *If $p < p^*_\Gamma$, then the project will be delayed.*
2. *If $p \geq p^*_\Gamma$, then the project will finish on time.*

Proof of Lemma 1. Let $f(p, \pi)$ denote the sum of the minimal rights of the firms in a path π when the penalty announced by the planner is p. Namely,

$$f(p, \pi) = \sum_{i \in \pi} s_i(p) = \sum_{i \in \pi : r_i > p} \frac{(r_i - p)D_i}{r_i}.$$

Since f is a continuous strictly decreasing piecewise linear function in p from $f(0, \pi) = \sum_{i \in \pi : r_i > 0} D_i$ till $f(p, \pi) = 0$ when $p \geq \max_{i \in N} r_i$, there exists a unique p^1_π such that $f(p^1_\pi, \pi) = ps_\pi$.

By Proposition 1.1, if $p < p^1_\pi$ for some $\pi \in \Pi$, then $\sum_{i \in \pi} s_i(p) > ps_\pi$ and so the project is delayed in any NE at Stage 2. We define $p^*_\Gamma = \max_{\pi \in \Pi} p^1_\pi$. Thus,

1. If $p < p^*_\Gamma$, then the project is delayed in any NE in Stage 2.
2. If $p \geq p^*_\Gamma$, since the firms play an optimal NE in Stage 2, by Proposition 1.2, the project will finish on time. ■

Proposition 4 *Let $\Gamma(G,c,r)$ be the non-cooperative game induced by (G,c,r). Assume that for each penalty p firms play an NE $d(p)$ without delays (when there exists) and the unique NE with delays when there is not an NE without delays in Stage 2. Thus,* $\sup\{u_0(p,d(p)) : p \geq 0\}$ *is achieved for some* $p \in \{p^*_\Gamma, \{r_j\}_{j\in N: r_j < p^*_\Gamma}\}$.

Proof of Proposition 4.

If $p \geq p^*_\Gamma$, by Lemma 1, we know the project finishes on time. Then, the planner gets 0.

If $p < p^*_\Gamma$, by Lemma 1, the project is delayed in any NE $d(p) = (d_i(p))_{i\in N}$ at Stage 2. Let us make an abuse of notation and denote by $d_i(p)$ the delay in the information set of firm i achieved when firms play $d(p)$. By the proof of Proposition 1 the planner will obtain

$$\begin{aligned} u_0\left(p, d(p)\right) &= p\sum_{i\in N} d_i\left(p\right) - c\left(\max_{\pi\in\Pi}\sum_{i\in\pi}(d_i(p)+t_i) - T\right) \\ &= p\sum_{i\in N: r_i > p} D_i - c\left(\max_{\pi\in\Pi}\left(t(\pi) + \sum_{i\in\pi: r_i>p} D_i\right) - T\right). \end{aligned}$$

We decompose this utility in two parts:

$$p\sum_{i\in N: r_i>p} D_i. \tag{2}$$

$$c\left(\max_{\pi}\left(t(\pi) + \sum_{i\in\pi: r_i>p} D_i\right) - T\right). \tag{3}$$

where (2) is the profit obtained from the penalties and (3) is the cost incurred by the planner because of the delay in the project. Note that

- The profit (2) is a right-continuous piecewise linear function on the intervals defined by $\{p^*_\Gamma, \{r_j : r_j < p^*_\Gamma\}\}$. Within each interval this function is strictly increasing in p.

  If $p \geq r_i$ for some $i \in N$, firm i will choose $d_i(p) = 0$ whereas $d_i(p) = D_i$ when $p < r_i$. So (2) decreases when moving from one interval to the next. Thus, the local supremum of (2) is achieved when p left converges to an element in $\{p^*_\Gamma, \{r_j : r_j < p^*_\Gamma\}\}$.

- The cost (3) is a decreasing piecewise constant function on the intervals given by $\{p_\Gamma^*, \{r_j : r_j < p_\Gamma^*\}\}$. So, again, the changes in (3) are given when $p \in \{p_\Gamma^*, \{r_j : r_j < p_\Gamma^*\}\}$.

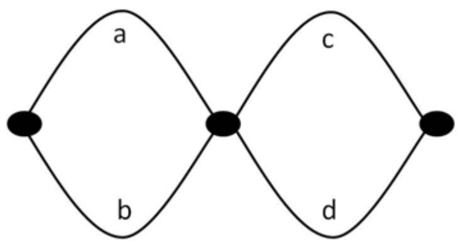
Thus, computing the strategy that will lead to a supremum in the planner's utility is straightforward. When $p \geq p_\Gamma^*$ its utility will be 0 for sure.

When $p < p_\Gamma^*$ we only need to calculate its utility function in $\{p_\Gamma^*, \{r_j : r_j < p_\Gamma^*\}\}$. Note that we talk about the supremum (rather than the maximum) because it is achieved when the penalty left converges to some r_i or to p_Γ^*. ■

Proposition 4 has two immediate consequences. From a theoretical point of view, because of the proof of this proposition, the game $\Gamma(G, c, r)$ may not have NE. This is because in our result we find conditions in order for the planner to obtain the supremum of its utility. From a practical point of view we think that our results can be used for predicting the behavior of rational agents. In our model the penalty p is a real number. In the real world when agents decide about penalties, prices, etc. they typically use natural numbers, which are a multiple of a small amount of money in the relevant currency. For instance, if the currency is the Euros, prices are given as an amount in Euro cents. If we consider a discrete version of our model where p and r are natural numbers (interpreted as amounts in cents) the whole game has NE. This corresponds to the case where the planner chooses $p' - 1$ where p' is (according to Proposition 4) the value at which $\sup\{u_0(p, d(p)) : p \geq 0\}$ is achieved.

We now illustrate the results of Proposition 4 in some examples.

Example 8 *Consider the project given by the following picture*



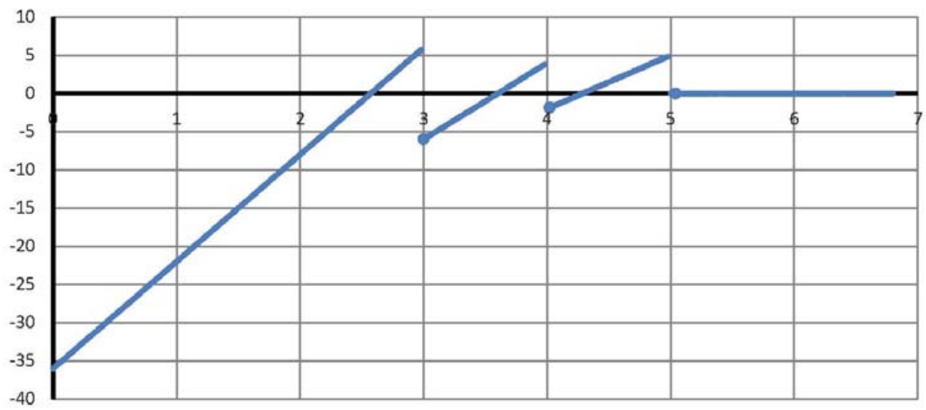
where $c = 6$ and

activities	t_i	D_i	r_i
a	*4*	*4*	*3*
b	*5*	*3*	*5*
e	*8*	*3*	*4*
f	*6*	*4*	*6*

There are four paths: $\pi_1 = \{a, e\}$, $\pi_2 = \{a, f\}$, $\pi_3 = \{b, e\}$, and $\pi_4 = \{b, f\}$. The PERT time is $T = 13$ and the vector of slacks of the paths is given by $ps = (1, 3, 0, 2)$. Besides, the limit penalty for each path is: $p^1_{\pi_1} = 2.88$, $p^1_{\pi_2} = 2.5$, $p^1_{\pi_3} = 5$, and $p^1_{\pi_4} = 3.947$. This means that a penalty as great or greater than $p^*_\Gamma = 5$ is needed if the planner wants the project to finish on time.

The next figure shows the shape of functions (2) *and* (3)*.*

The next figure shows the utility (blue line) of the planner.



Note that the planner must choose a penalty lower than 5 *to obtain a positive payoff. Moreover, the supremum utility is achieved when p left converges to* 3*. Thus, the planner should choose a penalty a little lower than but very close to* 3*.*

We consider Example 3. The utility of the planner is given by

$$u_0(p, d(p)) = \begin{cases} 30p - 20 & \text{if } p < 1 \\ 0 & \text{if } p \geq 1. \end{cases}$$

Thus, the supremum is achieved when p left converges to 1. From a practical point of view the prediction given by our results is that the planner will choose a penalty a little below 1 and firms will delay by 10 units.

We consider Example 4. In this case $p^1_{\{b,e\}} = 2.4$ and $p^1_{\{a\}} = 0$. Thus, the utility of the planner is given by

$$u_0(p, d(p)) = \begin{cases} 10p - 40 & \text{if } p < 2.4 \\ 0 & \text{if } p \geq 2.4. \end{cases}$$

Since $10p - 40 < 0$ when $p < 2.4$, it emerges that the supremum is achieved when $p \geq 2.4$. In particular when $p = 3 = r_b = r_e$. The prediction in this case is that the planner will choose a penalty larger than 2.4 (for instance 3), firms b and e will delay together by two units (the slack of the path $\{b, e\}$) and the project will finish on time. It is important to remark that the planner is indifferent between penalties $p \geq 2.4$,but firms are not. For instance

p	$d_a(p)$	$d_b(p)$	$d_e(p)$	$u_0(p, d(p))$	$u_a(p, d(p))$	$u_b(p, d(p))$	$u_e(p, d(p))$
2.5	0	1.5	0.5	0	0	7.5	2.5
3	0	2	0	0	0	10	0

In this case, the larger p is, the best for firm b and the worst for firm e.

4 Second case: penalties are applied to every delayed firm

We model this situation with a non-cooperative game in extensive form with 3 stages, as in the previous case. The first two stages are the same in both games but the third stage is different. In this case the planner always applies the penalty on paying the firms, i.e. if a firm incurs a delay it will be always punished (no matter whether the project is delayed or not). We conduct an analysis similar to the one for the first case. We first analyze the equilibria in Stage 2 and then we analyze the equilibria of the whole game. In this case most of the theoretical results are quite straightforward.

We first introduce the model formally. With each PERT problem with delays (G, c, r) we associate the non-cooperative game $\Delta(G, c, r)$ where

1. **Stage 1**. The planner decides the penalty $p \in [0, +\infty)$.

2. **Stage 2**. The firms, following the structure of the project, decide the vector of delays d.

3. **Stage 3**. The planner pays the firms.

$$u_i(p,d) = \begin{cases} \sum\limits_{i \in N} pd_i - c(T(d) - T) & \text{if } i = 0. \\ (r_i - p)d_i & \text{if } i \in N. \end{cases}$$

Assume that in Example 1 $c = 3$, $r_a = 0$, $r_b = 2$, and $r_e = 3$. Moreover, agents play the following strategies $p = 1$, $d_a = 0$, $d_b = 0.5$ and $d_e = 0.5$. Then, $T(d) = 3$. The project is not delayed because $T = 3$ but delayed firms are punished anyway:

$$\begin{aligned} u_0(p,d) &= 2\,(0.5 + 0.5) = 2 \\ u_a(p,d) &= 0 \\ u_b(p,d) &= (2-1)0.5 = 0.5 \\ u_e(p,d) &= (3-1)\,0.5 = 1. \end{aligned}$$

4.1 Equilibria in Stage 2

In this section we characterize the set of NE of $\Delta(G,c,r)$ in the subgame obtained in Stage 2. Basically, there is a unique NE where firms with $r_i \leq p$ do not delay and firms with $r_i > p$ delay as long as possible (D_i).

Proposition 5 *Let $\Delta(G,c,r)$ be the non-cooperative game induced by (G,c,r). Let p be the penalty chosen by the planner at Stage 1. Let d be such that for any firm i and any information set I of firm i it holds that*

$$d_i(I) = \begin{cases} 0 & \textit{if } r_i \leq p \\ D_i & \textit{if } r_i > p. \end{cases}$$

Thus, d is an NE in the subgame obtained in Stage 2. Moreover, all NE in the subgame obtained in Stage 2 have the same utility outcome as d.

Proof of Proposition 5. It is obvious that d is an NE.
Let, d' be a profile of strategies. For each $i \in N$, $u_i(d') = (r_i - p)d'_i$.

1. If $r_i \leq p$, then $u_i(d') \leq 0 = u_i(d_i, d'_{-i})$.

2. If $r_i > p$, then $u_i(d') \leq (r_i - p)D_i = u_i(d_i, d'_{-i})$.

Now it is trivial to prove that all NE in the subgame obtained in Stage 2 have the same utility outcome as d. ■

4.2 Equilibria in the whole game

In this section we analyze the whole game under this new penalty system. We obtain similar results to the previous case, but without making any kind of assumption as to the behavior of firms in Stage 2. We only need to assume that firms will play an NE.

We start with a preliminary result.

Lemma 2 *Let $\Delta(G, c, r)$ be the non-cooperative game induced by (G, c, r). Assume that for each penalty p firms play an NE in Stage 2. Then, there exists a limit penalty p^*_Δ such that*

1. *If $p < p^*_\Delta$, then the project will be delayed.*

2. *If $p \geq p^*_\Delta$, then the project will finish on time.*

Besides, this penalty belongs to $\{r_i\}_{i \in N}$ and $p^*_\Delta \geq p^*_\Gamma$.

Proof of Lemma 2. Given a penalty p, the delay of path π when the firms play an NE is given by $\sum\limits_{i \in \pi : r_i > p} D_i$.

For each path π such that $\sum\limits_{i \in \pi : r_i > 0} D_i > ps_\pi$ we take p^2_π such that $\sum\limits_{i \in \pi : r_i > p} D_i > ps_\pi$ for each $p < p^2_\pi$ and $\sum\limits_{i \in \pi : r_i > p} D_i \leq ps_\pi$ for each $p \geq p^2_\pi$. Obviously, $p^2_\pi \in \{r_i\}_{i \in \pi}$.

For each path π such that $\sum\limits_{i \in \pi : r_i > 0} D_i \leq ps_\pi$ we take $p^2_\pi = 0$.

We define $p^*_\Delta = \max\limits_{\pi \in \Pi} \{p^2_\pi\}$.

1. If $p < p^*_\Delta$, then there exists $\pi \in \Pi$ such that $0 \leq p < p^2_\pi$. Thus, $\sum\limits_{i \in \pi : r_i > p} D_i > ps_\pi$ Therefore, the project will be delayed.

2. If $p \geq p^*_\Delta$, then for each $\pi \in \Pi$, $p \geq p^2_\pi$. Thus, for each $\pi \in \Pi$, $\sum_{i \in \pi : r_i > p} D_i \leq p s_\pi$ and so the project will finish on time.

In Lemma 1 we have seen that

$$\sum_{i \in \pi} s_i(p^1_\pi) = \sum_{i \in \pi : r_i > p^1_\pi} s_i(p^1_\pi) = p s_\pi.$$

Since $s_i(p) \leq D_i$ for all $p \in \mathbb{R}^+$, $\sum_{i \in \pi : r_i > p^1_\pi} D_i \geq p s_\pi$. Thus, $p^1_\pi \leq p^2_\pi$ and so$p^*_\Gamma \leq p^*_\Delta$. ■

The main result of this subsection is the following.

Proposition 6 *Let $\Delta(G, c, r)$ be the non-cooperative game induced by (G, c, r). Assume that for each penalty p firms play an NE $d(p)$ in Stage 2. Thus,* $\sup \{u_0 (p, d(p)) : p \geq 0\}$ *is achieved for a $p \in \{r_i\}_{i \in N}$.*

Proof of Proposition 6. By Lemma 2,

1. If $p < p^*_\Delta$ the project will be delayed and the utility of the planner will be

$$u_0(p, d(p)) = p \sum_{i \in N : r_i > p} D_i - c \left(\max_{\pi \in \Pi} \left(t(\pi) + \sum_{i \in \pi : r_i > p} D_i \right) - T \right). \tag{4}$$

2. If $p \geq p^*_\Delta$ the project will finish on time. In this case we need to distinguish two cases:

   (a) If $p \geq \max \{r_i\}_{i \in N}$, all the activities will play 0 and the utility of the planner will be 0.

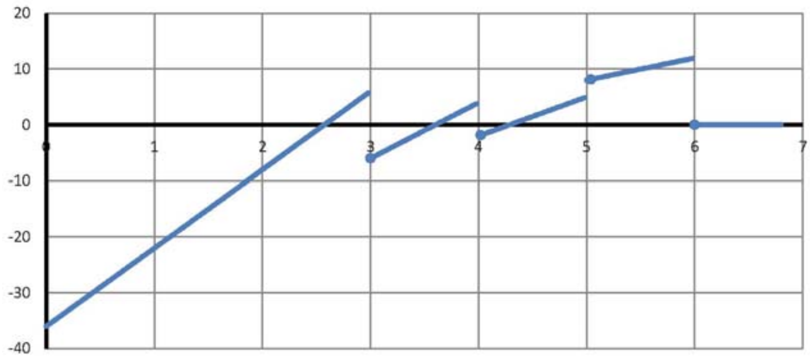
   (b) If $p < \max \{r_i\}_{i \in N}$, despite the project ends on time, the planner will obtain some benefits through the penalties. Thus, $u_0(p, d(p)) = p \sum_{i \in N : r_i > p} D_i$.

Note that the function that computes the benefit obtained from penalties and the function that computes the cost incurred by the planner if the project is delayed are the same as in Proposition 4. So, since the changes in both functions are given when $p \in \{r_i\}_{i \in N}$, the local supremum for the planner is achieved when the penalty p left converges to a $r_j \in \{r_i\}_{i \in N}$.■

The statements of Proposition 4 and Proposition 6 seem to be the same, but the set of values that need to be checked in order to find the supremum differs from one model to the other. Moreover, the supremum may also be different. Note that in the present model the value p^*_Γ plays no role whereas in the first case it is highly important. For instance, given a penalty p, if there is a firm i with $p^*_\Gamma < p < r_i$, in the previous case the project will end on time, while in this case firm i will delay as long as possible and the project could be delayed.

We clarify this issue with an example.

Consider the project in **Example 8**. If we compute the utility of the planner under this second penalty system we obtain the following



Note that in this case the supremum is achieved when $p = 6$, whereas in the previous case it is achieved when $p = 3$.

In the first case the utility of the planner is 6 and the delay in the project is also 6. In this case the utility of the planner is 12 and the delay in the project is 1.

In this example it seems better for the planner to apply the second penalty system because it will obtain greater utility with less delay in the project. This raises the question of whether this happens in general or only in some cases. In the next section we compare the two cases.

5 Comparing both penalty systems

In this section we compare the results obtained from the two penalty systems ($\Gamma(G, c, r)$ and $\Delta(G, c, r)$). We compare the utility of the planner, the utility of the firms and the delay in the project under the two systems. We first study what happens in Stage 2 and then analyze the whole game.

5.1 Comparing Stage 2

Given a penalty p we compare the equilibria of the subgame obtained in Stage 2 for both $\Gamma(G, c, r)$ and $\Delta(G, c, r)$. Since there can be multiple NE for $\Gamma(G, c, r)$, we assume that firms will play an optimal NE (see Proposition 2). In particular our results also hold when firms play the unique optimistic NE characterized in Proposition 3. Our findings are:

1. The utilities of the planner in the two games are unrelated (in some cases utility is greater in $\Gamma(G, c, r)$ and in other cases in $\Delta(G, c, r)$).

2. The utility of the firms is as great or greater in $\Gamma(G, c, r)$.

   This statement is quite intuitive because in $\Gamma(G, c, r)$ the penalty is applied only when the whole project is delayed, whereas in $\Delta(G, c, r)$ the penalty is always applied.

3. The delay in the project is as long or longer in $\Delta(G, c, r)$.

   This statement seems a little counterintuitive because penalties are applied more often in $\Delta(G, c, r)$ than in $\Gamma(G, c, r)$. Nevertheless, it must be considered that in $\Gamma(G, c, r)$ firms can delay their activities without being punished and can achieve the same utility as when they are punished.

The above findings are formally proved in the following proposition.

Proposition 7 *Let (G, c, r) be a $PERT$ problem with delays and let p be any penalty.*

Let $d^{\Gamma}(p)$ be as follows. When there exists an NE without delays in Stage 2 of $\Gamma(G, c, r)$, let $d^{\Gamma}(p)$ be some of the NE given by Proposition 2. When there is not an NE without delays in Stage 2 of $\Gamma(G, c, r)$, let $d^{\Gamma}(p)$ be some of the NE given by Proposition 1.1.

Let $d^{\Delta}(p)$ be some of the NE in Stage 2 of $\Delta(G, c, r)$ given by Proposition 5.

Let u^{Γ} and u^{Δ} denote the utility functions of $\Gamma(G, c, r)$ and $\Delta(G, c, r)$ respectively.

1. *It is possible that $u_0^{\Gamma}(p, d^{\Gamma}(p)) > u_0^{\Delta}(p, d^{\Delta}(p))$, $u_0^{\Gamma}(p, d^{\Gamma}(p)) < u_0^{\Delta}(p, d^{\Delta}(p))$, and $u_0^{\Gamma}(p, d^{\Gamma}(p)) = u_0^{\Delta}(p, d^{\Delta}(p))$.*

2. *For each $i \in N$, $u_i^{\Gamma}(p, d^{\Gamma}(p)) \geq u_i^{\Delta}(p, d^{\Delta}(p))$.*

3. *$T(d^{\Gamma}(p)) \leq T(d^{\Delta}(p))$.*

Proof of Proposition 7.

We first compute $d^{\Gamma}(p)$. We consider two cases.

- Assume that there exists an NE without delays in Stage 2 of $\Gamma(G, c, r)$. By Proposition 2, $s_i(p) \leq d_i^{\Gamma}(p) \leq D_i$ for each $i \in N$.

- Assume that there is no NE without delays in Stage 2 of $\Gamma(G, c, r)$. By the proof of Proposition 1.1,

$$d_i^{\Gamma}(p) = \begin{cases} 0 & \text{if } r_i \leq p \\ D_i & \text{if } r_i > p. \end{cases}$$

We now compute $d^{\Delta}(p)$. By Proposition 5,

$$d_i^{\Delta}(p) = \begin{cases} 0 & \text{if } r_i \leq p \\ D_i & \text{if } r_i > p. \end{cases}$$

We now prove the statement of the proposition.

1. Example 5. Let $p = 2$ and $c = 5$. Then, $u_0^{\Gamma}(p, d^{\Gamma}(p)) = 0$ and $u_0^{\Delta}(p, d^{\Delta}(p)) = 3$. If we consider $c = 6$, then $u_0^{\Gamma}(p, d^{\Gamma}(p)) = 0$ again, but $u_0^{\Delta}(p, d^{\Delta}(p)) = -2$.

   Example 3. Let $p = 0.8$ and $c = 2$. Then, $u_0^{\Gamma}(p, d^{\Gamma}(p)) = u_0^{\Delta}(p, d^{\Delta}(p)) = 2$.

2. Notice that

$$u_i^{\Delta}(p, d^{\Delta}(p)) = \begin{cases} 0 & \text{if } r_i \leq p \\ (r_i - p)D_i & \text{if } r_i > p. \end{cases}$$

   We consider two cases.

   - Assume that there exists an NE without delays in Stage 2 of $\Gamma(G, c, r)$. By Proposition 2,

$$u_i^{\Gamma}(p, d^{\Gamma}(p)) \geq r_i s_i(p) \geq \begin{cases} 0 & \text{if } r_i \leq p \\ (r_i - p)\, D_i & \text{if } r_i > p. \end{cases}$$

   - Assume that there is no NE without delays in Stage 2 of $\Gamma(G, c, r)$. Since $d^{\Gamma}(p) = d^{\Delta}(p)$ we have that for each $i \in N$, $u_i^{\Gamma}(p, d^{\Gamma}(p)) = u_i^{\Delta}(p, d^{\Delta}(p))$.

3. Again we consider two cases.

   - Assume that there exists an NE without delays in Stage 2 of $\Gamma(G, c, r)$. Thus, $T(d^{\Gamma}(p)) = T \leq T(d^{\Delta}(p))$.
   - Assume that there is no NE without delays in Stage 2 of $\Gamma(G, c, r)$. Since $d^{\Gamma}(p) = d^{\Delta}(p)$ we have that $T(d^{\Gamma}(p)) = T(d^{\Delta}(p))$. ■

5.2 Comparing the whole game

We now compare the NE of $\Gamma(G,c,r)$ and $\Delta(G,c,r)$. Since for $\Gamma(G,c,r)$ there can be multiple NE in Stage 2, we assume firms will play any optimal NE (see Proposition 2). In particular our results also hold when firms play the unique optimistic NE characterized in Proposition 3. Our findings are:

1. The optimal penalty for the planner is as great or greater in $\Delta(G,c,r)$.
2. The utility of the planner is as great or grater in $\Delta(G,c,r)$.
3. The utility of the firms is as great or greater in $\Gamma(G,c,r)$.
4. The delay in the project is unrelated.

For each project, the utility of each firm in $\Gamma(G,c,r)$ is as great or greater than the utility of the firm in $\Delta(G,c,r)$. Thus, firms clearly prefer $\Gamma(G,c,r)$.

For each project, the utility of the planner in $\Delta(G,c,r)$ is as great or greater than the utility of the planner in $\Gamma(G,c,r)$. This theoretical result requires a little more clarification. Although it may suggest that the planner should always apply penalties some clarifications are needed. It could be the case that the utility of the planner is the same in both penalty systems. When the planner is a private institution this may be irrelevant but when it is a public institution it makes sense for such a public institution to apply the penalty only when the whole project is delayed if the utilities of both penalty systems coincide. The reason is that firms are better off under this penalty system, firms are part of society and the public institution represents society [7]. Thus, for public institutions it makes sense to analyze each project separately and then decide which penalty system to apply. If the utility of the planner is strictly greater when penalties are always applied then it should choose that system, but if the planner obtains the same utility then it should choose the other since firms will be better off.

Assume that there is a set of agents (other than the planner and the firms) that need the project to be completed. For instance the project could be the construction of a new hospital, highway, etc. Those agents clearly want the project to be completed as soon as possible, but in such a situation it is not clear which penalty system ($\Gamma(G,c,r)$ or $\Delta(G,c,r)$) is better.

The above findings are proved in the following proposition.

[7] For instance, if firms make more profit they will pay more tax and the funding available to the public institution will be larger.

Proposition 8 *Let (G, c, r) be a $PERT$ problem with delays and p any penalty.*

Let $d^{\Gamma}(p)$ be as follows. When there are NE without delays in Stage 2 of $\Gamma(G, c, r)$, $d^{\Gamma}(p)$ is some of the NE given by Proposition 2. When there is not an NE without delays in Stage 2 of $\Gamma(G, c, r)$, $d^{\Gamma}(p)$ is some of the NE given by Proposition 1.1.

Let $d^{\Delta}(p)$ be some of the NE in Stage 2 of $\Delta(G, c, r)$ given by Proposition 5.

Let u^{Γ} and u^{Δ} denote the utilities functions of $\Gamma(G, c, r)$ and $\Delta(G, c, r)$ respectively. Then

1. *Given*

$$\begin{aligned} p^{\Gamma} &= \min\left\{p : p \in \arg\sup\left\{u_0^{\Gamma}(p, d^{\Gamma}(p)) : p \geq 0\right\}\right\} \text{ and} \\ p^{\Delta} &= \min\left\{p : p \in \arg\sup\left\{u_0^{\Delta}(p, d^{\Delta}(p)) : p \geq 0\right\}\right\}. \end{aligned}$$

   we have that $p^{\Gamma} \leq p^{\Delta}$.

2. $\sup\left\{u_0^{\Gamma}(p, d^{\Gamma}(p)) : p \geq 0\right\} \leq \sup\left\{u_0^{\Delta}(p, d^{\Delta}(p)) : p \geq 0\right\}$.

3. *For each $i \in N$, $u_i^{\Gamma}\left(p^{\Gamma}, d^{\Gamma}(p^{\Gamma})\right) \geq u_i^{\Delta}\left(p^{\Delta}, d^{\Delta}(p^{\Delta})\right)$.*

4. *It is possible that $T\left(d^{\Gamma}(p^{\Gamma})\right) > T\left(d^{\Delta}\left(p^{\Delta}\right)\right)$, $T\left(d^{\Gamma}(p^{\Gamma})\right) = T\left(d^{\Delta}(p^{\Delta})\right)$, or $T\left(d^{\Gamma}(p^{\Gamma})\right) < T\left(d^{\Delta}(p^{\Delta})\right)$.*

Proof of Proposition 8.

1. Two cases are possible.

   (a) $p^{\Gamma} < p_{\Gamma}^{*}$. By the proof of Proposition 4 the project is delayed under $(p^{\Gamma}, d^{\Gamma}(p^{\Gamma}))$. Since $d^{\Gamma}(p) = d^{\Delta}(p)$ when $p < p_{\Gamma}^{*}$, we have that $u_0^{\Gamma}\left(p, d^{\Gamma}(p)\right) = u_0^{\Delta}\left(p, d^{\Delta}(p)\right)$ for any $p < p^{*}$. Thus, $p^{\Gamma} \leq p^{\Delta}$.

   (b) $p^{\Gamma} = p_{\Gamma}^{*}$[8]. By the proof of Proposition 4 the project is not delayed under $\left(p^{\Gamma}, d^{\Gamma}(p^{\Gamma})\right)$. Hence $u_0^{\Gamma}\left(p^{\Gamma}, d^{\Gamma}\left(p^{\Gamma}\right)\right) = 0 \geq u_0^{\Gamma}\left(p, d^{\Gamma}\left(p\right)\right)$ for any $p < p_{\Gamma}^{*}$. We know that $u_0^{\Gamma}\left(p, d^{\Gamma}(p)\right) = u_0^{\Delta}\left(p, d^{\Delta}(p)\right)$ for any $p < p_{\Gamma}^{*}$. Given $p' = \max\{r_j : j \in N\}$ we have that $u_0^{\Delta}\left(p', d^{\Delta}\left(r'\right)\right) = 0$. Then, $p_{\Gamma}^{*} \leq p^{\Delta} \leq p'$.

2. Two cases are possible.

   (a) $p^{\Gamma} < p_{\Gamma}^{*}$. We have seen that $d^{\Gamma}(p) = d^{\Delta}(p)$ when $p < p_{\Gamma}^{*}$. Then, $u_0^{\Gamma}\left(p, d^{\Gamma}(p)\right) = u_0^{\Delta}\left(p, d^{\Delta}(p)\right)$ for any $p < p_{\Gamma}^{*}$ and hence the result holds.

[8] Note that the case $p^{\Gamma} > p_{\Gamma}^{*}$ is not possible because p^{Γ} is the minimum.

(b) $p^{\Gamma} = p^*_{\Gamma}$. We have seen that $\sup\left\{u_0^{\Gamma}\left(p, d^{\Gamma}(p)\right) : p \geq 0\right\} = 0$. Since $u_0^{\Delta}\left(p', d^{\Delta}(p')\right) = 0$ when $p' = \max\{r_j : j \in N\}$, the result holds.

3. Two cases are possible.

By the proof of Proposition 5 we have that

$$u_i^{\Delta}\left(p^{\Delta}, d^{\Delta}(p^{\Delta})\right) = \begin{cases} 0 & \text{if } r_i < p^{\Delta} \\ \left(r_i - p^{\Delta}\right) D_i & \text{if } r_i \geq p^{\Delta}. \end{cases}$$

(a) $p^{\Gamma} < p^*_{\Gamma}$. By the proof of Proposition 1 we have that

$$u_i^{\Gamma}\left(p^{\Gamma}, d^{\Gamma}(p^{\Gamma})\right) = \begin{cases} 0 & \text{if } r_i < p^{\Gamma} \\ \left(r_i - p^{\Gamma}\right) D_i & \text{if } r_i \geq p^{\Gamma}. \end{cases}$$

Since $p^{\Gamma} \leq p^{\Delta}$, the result holds.

(b) $p^{\Gamma} = p^*_{\Gamma}$. By Proposition 2 we have that

$$u_i^{\Gamma}\left(p^{\Gamma}, d^{\Gamma}(p^{\Gamma})\right) \geq r_i s_i(p) \geq \begin{cases} 0 & \text{if } r_i < p^{\Gamma} \\ \left(r_i - p^{\Gamma}\right) D_i & \text{if } r_i \geq p^{\Gamma}. \end{cases}$$

Since $p^{\Gamma} \leq p^{\Delta}$, the result holds.

4. In Example 8. we have that $T\left(d^{\Gamma}(p^{\Gamma})\right) = 6 > 2 = T\left(d^{\Delta}(p^{\Delta})\right)$.

In Example 3 we have that $T\left(d^{\Gamma}(p^{\Gamma})\right) = 10 = T\left(d^{\Delta}(p^{\Delta})\right)$.

Consider the project given by Example 3 where $c = 2$ and

activities	t_i	D_i	r_i
a	10	5	0
b	10	5	0
e	7	5	3

Then $T\left(d^{\Gamma}(p^{\Gamma})\right) = 0 < 2 = T\left(d^{\Delta}(p^{\Delta})\right)$. ■

6 Concluding Remarks

If we consider the results obtained in the paper, we can conclude that the Spanish law does not address the problem of delays in the projects in a proper way. Although a penalty clause explicitly appears in the contracts in order to encourage firms to behave right and not delay their activities, this penalty is a fixed amount that depends on the total cost estimate of the project. Thus, the planner, the public administration in this case, can not change the value of the penalty that may be insufficient to avoid an excessive delay. In comparison the liquidated damages clause in the General Conditions for Construction Contracts - Public Housing programs or the clause suggested by the American Institute of Architects do provide the option of fixing whatever penalty the planner decides, but this penalty can not be excessive. If that is the case the liquidated damages clause, which is lawful, might be considered a penalty clause, which is illegal.

There are situations in which a public organization wants to complete a project and each activity involved in the project is allocated to one firm. That firm outsources to third parties to complete the various activities in the project. This is standard practice in the information technology and construction sectors. Such cases can be included in our model if the planner is considered to be the firm and the public organization does not explicitly appear in the model. In such a case c is the cost per unit of delay that the planner must pay to the public organization if the project is delayed. In these circumstances the public organization cares about the completion time of the project. Since a delay in the project is unrelated in $\Gamma(G, c, r)$ and $\Delta(G, c, r)$, it is not clear which penalty system is better for the public organization.

We assume that the planner does not cancel any contracts with any firm because of excessive delays. Such a situation is quite unusual, at least in the public sector, but it sometimes happens. This situation is not covered by our model [9].

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