The folk rule through a painting procedure for minimum cost spanning tree problems with multiple sources

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Abstract

We consider minimum cost spanning tree problems with multiple sources. We propose a cost allocation rule based on a painting procedure. Agents paint the edges on the paths connecting them to the sources. We prove that the painting rule coincides with the folk rule. Finally, we provide an axiomatic characterization.

Keywords: minimum cost spanning tree problems with multiple sources, painting rule, axiomatic characterization.

1. Introduction

We study situations where a group of agents need services provided by several sources. Agents need to be connected, directly or indirectly, to all sources. Every connection is costly. Situations of this kind are called minimum cost spanning tree problems with multiple sources and are extensions of the classical minimum cost spanning tree problem (where there is a single source).

The first issue addressed is to find the least costly networks connecting all agents with all sources. Obviously, such a network is a tree. It can also be found in polynomial time using the same algorithms as in the classical problem (e.g., Kruskal (1956) and Prim (1957)).

The second issue addressed is how to allocate the cost of the tree obtained among the agents. Several papers have studied this issue in minimum cost spanning tree problems, but as far as we know only three have considered it in the

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case of multiple sources. Rosenthal (1987) and Kuipers (1997) study a situation slightly different from this paper, whereas Bergantiños et al. (2017) study the same situation as we present here. Rosenthal (1987) considers situations where all sources provide the same service and agents want to be connected to at least one of them. He considers a cooperative game and studies the core of that game. Kuipers (1997) considers situations where each source offers a different service and each agent needs to be connected to a subset of the sources. He also considers a cooperative game and seeks to determine under what conditions the core is non-empty. Bergantiños et al. (2017) study the same situation as in this paper. They extend different definitions of the folk rule, defined for classical minimum cost spanning tree problems, to the case of multiple sources. They also present some axiomatic characterizations of the folk rule.

In classical minimum cost spanning tree problems the folk rule is one of the most important rules. It has been studied in several papers, including Bergantiños and Kar (2010), Bergantiños et al. (2010, 2011, 2014), Bergantiños and Vidal-Puga (2007, 2009), Branzei et al. (2004), and Tijs et al. (2006).

Our paper is closely related to that of Bergantiños et al. (2014). They study a general framework of connection problems involving a single source, which contains classical minimum cost spanning tree problems. They propose a cost allocation rule, called the *painting rule* because it can be interpreted through a painting story. The idea is the following: start with a tree t; for each agent, identify the unique path in t from that agent to the source. Agents start painting the first edge on that path. Following a protocol, an agent continues painting until all edges on her path have been painted. They also give some axiomatic characterizations of the painting rule. They prove that the painting rule coincides with the folk rule in classical minimal cost spanning tree problem. Thus, they obtain a new way of computing the folk rule and a new axiomatic characterization.

The first objective of this paper is to extend the definition of the painting rule to the case of minimum cost spanning tree problems with multiple sources. The main problem that arises when doing this is that given a tree and an agent, several paths in the tree could connect the agent to a source. In order to avoid this problem, we define a two-phase procedure: In Phase 1, given a tree t, we compute a tree t^* with the same cost as t such that t^* is also a tree when it is restricted to the set of sources. Notice that for each agent there is a unique path in t^* connecting the agent with the set of all sources. In Phase 2 we apply the ideas of the painting rule to the tree t^*. This extension of the painting rule is not straightforward because it could depend on the tree t considered initially and the tree t^* computed in Phase 1, which is not determined solely by t. In Proposition 2 we prove that for each tree t and t^* considered, the painting rule always coincide with the folk rule. Thus, the painting rule is independent of the trees t and t^* considered.

The second objective of the paper is to provide an axiomatic characterization of the folk rule in minimum cost spanning trees with multiple sources. We do this with the properties of cost monotonicity, symmetry, cone-wise additivity, isolated agents, and equal treatment of source costs. The first three properties are quite standard in the literature and are defined as in the classical minimum cost spanning tree problem. Cost monotonicity says that if the connection cost between two nodes increases, no agent can be better off. Symmetry says that agents with the same connection costs to other nodes must pay the same. Cone-wise additivity says that the rule should be additive on the cost function when it is restricted to cones. The isolated agents property is inspired by the property (also called isolated agents) introduced in Bergantiños et al. (2014). Nevertheless, the extension is not so straightforward as with the previous ones. An agent is isolated when all her connection costs are the same and she does not benefit from connecting to the sources through the rest of the agents. If all sources can be connected to one another for free, then an isolated agent should only pay her connection cost to any node. Equal treatment of source costs is a property defined only in the case of multiple sources. It is introduced in Bergantiños et al. (2017) and states that if the connection cost between two sources increases then all agents must be affected in the same way.

The paper is organized as follows. Section 2 introduces minimum cost spanning tree problems with multiple sources. Section 3 introduces the painting rule. Section 4 gives the axiomatic characterization.

2. The minimum cost spanning tree problem with multiple sources

We consider situations where a group of nodes N (called agents) wants to be connected to a set of suppliers M (called sources).

Let $N = \{1, ..., n\}$ be the finite set of agents and $M = \{a_1, ..., a_m\}$ the finite set of sources. There is a *cost matrix* $C = (c_{ij})_{i,j \in N \cup M}$ over $N \cup M$ representing the cost of the direct link between any pair of nodes, with $c_{ji} = c_{ij} \geq 0$ and $c_{ii} = 0$, for all $i, j \in N \cup M$. We denote by $\mathcal{C}^{N \cup M}$ the set of all cost matrices over $N \cup M$.

A *minimum cost spanning tree problem with multiple sources* (briefly, a problem) is a triple (N, M, C) where N is the set of agents, M is the set of sources and $C \in \mathcal{C}^{N \cup M}$ is the cost matrix. If $c_{ij} \in \{0, 1\}$, for all $i, j \in N \cup M$, then (N, M, C) is called a *simple problem.*

An *edge* is a non-ordered pair (i, j) such that $i, j \in N \cup M$. Sometimes we write ij instead of (i, j). A *network* g is a subset of edges. The cost associated with a network g is defined as

$$c(N, M, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there are no ambiguities, we write $c(g)$ or $c(C, g)$ instead of $c(N, M, C, g)$.

Given a network g and any pair of nodes i and j, a *path* from i to j in g is a sequence of distinct edges $g_{ij} = \{(i_{h-1}, i_h)\}_{h=1}^{q}$ satisfying that $(i_{h-1}, i_h) \in g$ for all $h = 1, ..., q$, $i = i_0$ and $j = i_q$. A *cycle* is a path from i to i with at least two edges. A *tree* is a graph without cycles that connects all the elements of $N \cup M$.

Two nodes i, j are *connected* in g if there exists a path from i to j. We say that $S \subseteq N \cup M$ is a *connected component* on g if every $i, j \in S$ are connected in g and S is maximal, *i.e.*, for each $T \in N \cup M$ with $S \subsetneq T$ there exist $k, l \in T$, $k \neq l$, such that k and l are not connected in g.

Let (N, M, C) be a simple problem. We denote by $g^{0,C}$ the network induced by the edges with zero cost. Namely, $g^{0,C} = \{(i,j) : i, j \in N \cup M \text{ and } c_{ij} = 0\}$. We say that $i, j \in S \subseteq N \cup M$ are (C,S)*-connected* if i and j are connected in $g^{0,C}$. We say that S is a C*-component* if S is a connected component on $g^{0,C}$.

The first issue addressed in the literature is how to find a tree with the lowest associated cost (which is not necessarily unique). This problem is polynomial and the algorithms of Kruskal (1956) and Prim (1957) enable such a tree, which is called *minimal tree* (*mt*), to be computed. We denote by $m(N, M, C)$ the cost of any *mt* in (N, M, C).

Let (N, M, C) be a problem and t a minimal tree in (N, M, C). For each $i, j \in N \cup M$ we denote by t_{ij} the unique path in t joining i and j. Bird (1976) defines the *minimal network* associated with the minimal tree t as the problem (N, M, C^t), where $c^t_{ij} = \max_{(k,l) \in t_{ij}} c_{kl}$. It is well known that C^t is independent of the chosen t. Then, the *irreducible problem* (N, M, C^*) of (N, M, C) is defined as the minimal network associated with any minimal tree of (N, M, C).

After obtaining a minimal tree, the second issue addressed is how to divide its cost among the agents. A cost allocation rule (briefly, a rule) is a mapping f that associates a vector $f(N, M, C) \in \mathbb{R}^N$ with each problem (N, M, C) such that $\sum_{i \in N} f_i(N, M, C) = m(N, M, C)$. The i-th element of $f(N, M, C)$ denotes the payment of agent $i \in N$.

One of the most popular rules in the classical minimum cost spanning tree problem (*mcstp*) is the folk rule. Bergantiños et al. (2017) extend the definition of the folk rule to the problem with multiple sources and provide several ways to obtain it. One of them is through cone-wise decomposition.

Norde et al. (2004) prove that every classical *mcstp* can be written as a non-negative combination of classical simple problems. What follows is an adaptation of this result to our context.

Lemma 1. *For each problem (N, M, C), there exists a positive number $m(C) \in \mathbb{N}$, a sequence $\{C^q\}_{q=1}^{m(C)}$ of simple cost matrices and a sequence $\{x^q\}_{q=1}^{m(C)}$ of non-negative real numbers satisfying three conditions:*

(1) $C = \sum\limits_{q=1}^{m(C)} x^q C^q$.

(2) *For each $q \in \{1, \ldots, m(C)\}$, there exists a network g^q such that $c^q_{ij} = 1$ if $(i,j) \in g^q$ and $c^q_{ij} = 0$ otherwise.*

(3) *Take $q \in \{1, \ldots, m(C)\}$ and $\{i, j, k, l\} \subset N \cup M$. If $c_{ij} \leq c_{kl}$, then $c^q_{ij} \leq c^q_{kl}$.*

Let (N, M, C) be a simple problem and $P = \{S_1, ..., S_p\}$ the partition of $N \cup M$ in C-components. Bergantiños et al. (2017) define the folk rule F for simple problems as follows.

$$F_i(N,M,C) = \begin{cases} \dfrac{|S_k \in P : S_k \cap M \neq \varnothing| - 1}{|N|}, & \text{if } S(i,P) \cap M \neq \varnothing \\ \dfrac{1}{|S(i,P)|} + \dfrac{|S_k \in P : S_k \cap M \neq \varnothing| - 1}{|N|}, & \text{otherwise,} \end{cases}$$

where $S(i,P)$ is the element of P to which i belongs to. Then, the folk rule for a general problem (N,M,C) is defined as

$$F(N,M,C) = \sum_{q=1}^{m(C)} x^q F(N,M,C^q).$$

3. The painting procedure

Given a fixed tree t, Bergantiños et al. (2014) provide an algorithm to define a rule through a painting procedure in the classical *mcstp*. They motivate it as follows.

" In order to illustrate the procedure used to obtain the rule, assume that the nodes represent the houses of the different agents and the edges are the canals which connect them to an irrigation point. These canals need painting and there is only one machine to do this for each one. The machines cannot be moved to another canal and all of them work at the same speed. At every stage, each agent is assigned to an edge while the path from his house to the source has not been completely painted. The canals in t have painters assigned to them if the painting has not been completed. In each step, the agents assigned to an edge which is not completely painted share equally the time the painting machine is in operation. This can be read as their paying the same cost in that segment. At stage 1, each agent is assigned to the first edge in the unique path in t from his house to the source. At stage s, each agent is assigned to the first unpaid edge in this unique path. If all edges in such a path have already been paid for in the previous stages, then this agent has finished his job. The procedure ends when all edges have been paid for completely."

We seek to apply the procedure described above to the case of multiple sources. The main problem that arises is that with multiple sources, given a tree t and an agent i, several paths in t could connect agent i to a source in M. Assume that in the tree t all sources are directly connected to one another (namely t_M, the restriction of t to M, is also a tree). In this case, there is only one path in t to connect each agent to the nearest source.

Our idea for extending the definition of Bergantiños et al. (2014) to the case of multiple sources is the following. First, given a problem (N,M,C) and an *mt* t in (N,M,C), we compute a tree t^* in (N,M,C^*) with the same cost as t such that t^*_M is also a tree. Second, we divide the cost of $t^* \backslash t^*_M$ using the same procedure as in Bergantiños et al. (2014) and the cost of t^*_M is divided equally among all agents.

We now give an example where we explain the above procedure intuitively. It is presented formally below.

Example 1. *Let* $N = \{1, 2, 3, 4\}$, $M = \{a_1, a_2, a_3, a_4\}$, $c_{3a_3} = 1$, $c_{14} = 2$, $c_{23} = 3$, $c_{4a_4} = 4$, $c_{34} = 5$, $c_{1a_1} = 6$, $c_{a_2a_3} = 7$ *and* $c_{ij} = 10$ *otherwise. The minimal tree* t *for this problem is represented in Figure 1.*

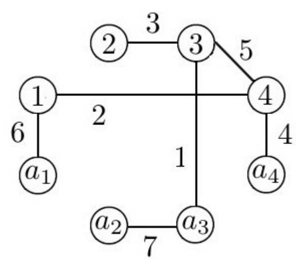

Figure 1: Minimal tree for (N, M, C).

Notice that the sources are not directly connected to one another. Every agent (except for agent 2*) has several paths in* t *connecting her to a source. For instance, agent* 1 *could connect to source* a_1 *through path* $\{(1, a_1)\}$ *or could connect to source* a_4 *through path* $\{(1, 4), (4, a_4)\}$.

We now construct the tree t^*. *We first connect sources* a_1 *and* a_3[1]. *We remove from* t *the most expensive edge on the unique path in* t *joining* a_1 *and* a_3, *which is edge* $(1, a_1)$. *We add to* t *the edge* (a_1, a_3) *and we change its cost from 10 to 6 (the cost of edge* $(1, a_1)$*).*

We now connect sources a_3 *and* a_4. *We remove from* t *the most expensive edge on the unique path in* t *joining* a_3 *and* a_4, *which is edge* $(3, 4)$. *We add to* t *the edge* (a_3, a_4) *and we change its cost from 10 to 5 (the cost of edge* $(3, 4)$*). Figure 2 shows the modified tree.*

In this tree, each agent has a unique path to the set of sources. The path for agent 1 is $\{(1, 4), (4, a_4)\}$, *for agent 2 it is* $\{(2, 3), (3, a_3)\}$, *for agent 3 it is* $\{(3, a_3)\}$ *and for agent 4 it is* $\{(4, a_4)\}$. *Then, the original idea of the painting procedure can be applied.*

Stage 1. Agent 1 *selects edge* $(1, 4)$, *agent* 2 *selects* $(2, 3)$, *agent* 3 *selects* $(3, a_3)$, *and agent* 4 *selects* $(4, a_4)$. *Thus, agent* 3 *paints edge* $(3, a_3)$ *completely and agents* 1, 2 *and* 4 *paint one unit of their edges. Thus, agent* 3 *is already connected to source* a_3 *and she is removed from the procedure.*

Stage 2. Agents 1, 2 *and 4 select the same edges as in Stage 1. Edge* $(1, 4)$ *is completely painted by agent* 1. *One more unit of edges* $(2, 3)$ *and* $(4, a_4)$ *is painted by agent* 2 *and* 4, *respectively.*

[1] Note that this procedure depends on the sources chosen for connecting. For instance, instead of joining sources a_1 and a_3 it is possible to join sources a_1 and a_4. Later we prove that the cost allocation is independent of the choices made.

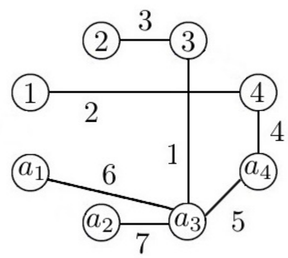

Figure 2: Alternative tree.

Stage 3. Agent 2 *keeps selecting edge* $(2,3)$ *and agents* 1 *and* 4 *select edge* $(4,a_4)$*. Agent 2 paints one unit of edge* $(2,3)$*. Agents* 1 *and* 4 *paint* $\frac{1}{2}$ *of edge* $(4,a_4)$*. Thus, edge* $(2,3)$ *is completely painted and agent* 2 *is therefore connected to source* a_3 *(through agent* 3*) and she is removed from the procedure.*

Stage 4. Agents 1 *and* 4 *keep selecting edge* $(4,a_4)$*. Each agent paints* $\frac{1}{2}$ *of edge* $(4,a_4)$*, which is now completely painted. Then, both agents are connected to source* a_4 *and removed from the procedure.*

Stage 5. The edges connecting the sources ((a_1,a_3)*,* (a_2,a_3) *and* (a_3,a_4)*) are painted by all agents.*

Table 1 summarizes this procedure.

Agent →	Agent 1		Agent 2		Agent 3		Agent 4	
Stage ↓	Edge	Amount	Edge	Amount	Edge	Amount	Edge	Amount
Stage 1	$(1,4)$	1	$(2,3)$	1	$(3,a_3)$	1	$(4,a_4)$	1
Stage 2	$(1,4)$	1	$(2,3)$	1			$(4,a_4)$	1
Stage 3	$(4,a_4)$	$\frac{1}{2}$	$(2,3)$	1			$(4,a_4)$	$\frac{1}{2}$
Stage 4	$(4,a_4)$	$\frac{1}{2}$					$(4,a_4)$	$\frac{1}{2}$
Stage 5	t^*_M	$\frac{6+7+5}{4}$	t^*_M	$\frac{6+7+5}{4}$	t^*_M	$\frac{6+7+5}{4}$	t^*_M	$\frac{6+7+5}{4}$
Total		$\frac{15}{2}$		$\frac{15}{2}$		$\frac{11}{2}$		$\frac{15}{2}$

Table 1: Summary of the painting procedure.

We now formally introduce the procedure explained in Example 1. We consider a two-phase procedure. In the first phase, given any *mt* t, we construct a tree t^* with the same cost as t and where all the sources are connected to one another. In the second phase we apply the painting procedure as in Bergantiños et al. (2014).

Phase 1: Constructing the tree

Given a *mcstp* with multiple sources (N,M,C) and a minimal tree t in (N,M,C), let $P(t_M)=\{S_1,...,S_{m(t)}\}$ denote the partition of M in connected components induced by t_M.

We consider an algorithm to construct a minimal tree t^* of the irreducible problem (N, M, C^*).

We start with $t^0 = t$. Assume that stage β is defined, for all $\beta \leq \delta - 1$.

Stage δ: We have two cases,

- $P(t_M^{\delta-1}) = \{M\}$. The algorithm ends and $t^* = t^{\delta-1}$.
- $P(t_M^{\delta-1}) \neq \{M\}$. We define

$$E(t^{\delta-1}) = \{(i_{h-1}, i_h)\}_{h=1}^q$$

  as the unique path from $\bigcup_{r=1}^{\delta} S_r$ to $S_{\delta+1}$ in $t^{\delta-1}$, with $i_0 \in \bigcup_{r=1}^{\delta} S_r$, $i_q \in S_{\delta+1}$, $i_1 \notin \bigcup_{r=1}^{\delta} S_r$ and $i_{q-1} \notin S_{\delta+1}$.

  Let (i, j) be the most expensive edge in $E(t^{\delta-1})$ (if there are several edges, then select just one). Namely,

$$c_{ij} = \max_{(k,l)\in E(t^{\delta-1})} \{c_{kl}\}.$$

  We now define,

$$t^{\delta} = t^{\delta-1} \backslash (i, j) \cup (i_0, i_q).$$

This process is completed in a finite number of stages (exactly at $m(t) - 1$ stages and $1 \leq m(t) \leq m$). The tree t^* is a *mt* for (N, M, C^*). Besides $c(C^*, t^*) = c(C, t)$ and t_M^* is also a tree.

Notice that given a tree t, several trees t^* could be obtained through this procedure.

We now formally apply Phase 1 to Example 1. We start with

$$t^0 = t = \{(1, a_1), (1, 4), (4, a_4), (2, 3), (3, a_3), (3, 4), (a_2, a_3)\}.$$

Stage 1:

- $P(t_M^0) = \{\{a_1\}, \{a_2, a_3\}, \{a_4\}\}$. Then
  $E(t^0) = \{(a_1, 1), (1, 4), (4, 3), (3, a_3)\}$.
  The most expensive edge in $E(t^0)$ is $(1, a_1)$. Thus

$$t^1 = \{(a_1, a_3), (1, 4), (4, a_4), (2, 3), (3, a_3), (3, 4), (a_2, a_3)\}.$$

Stage 2:

- $P(t_M^1) = \{\{a_1, a_2, a_3\}, \{a_4\}\}$. Then
  $E(t^1) = \{(a_3, 3), (3, 4), (4, a_4)\}$.
  The most expensive edge in $E(t^1)$ is $(3, 4)$. Thus

$$t^2 = \{(a_1, a_3), (1, 4), (4, a_4), (2, 3), (3, a_3), (a_3, a_4), (a_2, a_3)\}.$$

Stage 3:

- $P(t_M^2) = \{\{a_1, a_2, a_3, a_4\}\}$. Then the algorithm ends and $t^* = t^2$.

We know formally define the second phase of our procedure. This phase is obtained by applying the same ideas as in the painting procedure of Bergantiños et al. (2014).

Phase 2: Painting the tree.

Let t^* be an mt in (N, N, C^*) satisfying that t_M^* is a tree over M and $c(N, M, C^*, t^*) = m(N, M, C)$. By Phase 1 we know that such tree exists. We take

- $e_i^0\ (C, t^*) = \varnothing$ for all $i \in N$. In general, $e_i^\delta(C, t^*)$ denotes the edge of t^* assigned to agent i at stage δ. Agent i will pay part of the cost of this edge.

- $c^0(C, t^*) = 0$ and $c^\delta(C, t^*)$ represents the part of the cost of each edge that it is paid at stage δ.

- $p_i^0(C, t^*) = 0$ for all $i \in N$. In general, $p_i^\delta(C, t^*)$ is the cost that agent i pays at stage δ.

- $E^0(C, t^*) = t^* \backslash t_M^*$ and $E^\delta(C, t^*)$ is the set of unpaid edges of $t^* \backslash t_M^*$ at stage δ.

When no confusion arises we will write e_i^δ, $e_i^\delta(C)$ or $e_i^\delta(t^*)$ instead of $e_i^\delta(C, t^*)$. We will do the same with $c^\delta(C, t^*)$, $p_i^\delta(C, t^*)$ and $E^\delta(C, t^*)$. Assume that stage β is defined, for all $\beta \leq \delta - 1$.

Stage δ:

- For each $i \in N$, let e_i^δ be the first edge in the unique path in t^* from i to M belonging to $E^{\delta-1}$. If all edges in such path are not in $E^{\delta-1}$, take $e_i^\delta = \varnothing$.

- For each $(i, j) \in E^{\delta-1}$ we define

$$N_{ij}^\delta = \{k \in N : e_k^\delta = (i, j)\}$$

  and

$$c^\delta = \min \left\{ c_{ij} - \sum_{r=0}^{\delta-1} c^r : (i, j) \in E^{\delta-1} \right\}.$$

- For each $i \in N$, we define

$$p_i^\delta = \begin{cases} \dfrac{c^\delta}{\left| N_{e_i^\delta}^\delta \right|}, & \text{if } e_i^\delta \neq \varnothing \\ 0, & \text{otherwise.} \end{cases}$$

- We define
$$E^{\delta} = \left\{ (i,j) \in E^{\delta-1} : \sum_{r=0}^{\delta} c^r < c_{ij} \right\}.$$

This procedure ends when we find a stage $\gamma(C,t^*)$ ($\gamma(C)$, $\gamma(t^*)$ or γ when no confusion arises) such that $E^{\gamma} = \varnothing$. Since $E^0 = t^* \backslash t^*_M$, $E^{\delta+1} \subset E^{\delta}$ and $E^{\delta+1} \neq E^{\delta}$, γ is finite.

Stage $\gamma + 1$. The cost of all edges on t^*_M, $c(t^*_M) = \sum_{(i,j)\in t^*_M} c^*_{ij}$, is divided equally among all agents. Then,

$$p_i^{\gamma+1} = \frac{c(t^*_M)}{|N|}.$$

For each problem (N, M, C), each *mt* t, and each $i \in N$, we define the panting rule $f_i^{P,t}$ as
$$f_i^{P,t}(N,M,C) = \sum_{\delta=1}^{\gamma+1} p_i^{\delta}(C,t^*).$$

Note that this definition depends on trees t and t^* considered.

We now formally apply Phase 2 to Example 1. We start with:

- e_1^0, e_2^0, e_3^0, $e_4^0 = \varnothing$.
- $c^0 = 0$.
- p_1^0, p_2^0, p_3^0, $p_4^0 = 0$.
- $E^0 = \{(1,4),(4,a_4),(2,3),(3,a_3)\}$.

Stage 1:

- $e_1^1 = (1,4)$, $e_2^1 = (2,3)$, $e_3^1 = (3,a_3)$ and $e_4^1 = (4,a_4)$.
- $N_{14}^1 = \{1\}$, $N_{23}^1 = \{2\}$, $N_{3a_3}^1 = \{3\}$ and $N_{4a_4}^1 = \{4\}$.
- $c^1 = \min\{c_{14}, c_{23}, c_{3a_3}, c_{4a_4}\} = \min\{2,3,1,4\} = 1$.
- p_1^1, p_2^1, p_3^1, $p_4^1 = 1$.
- $E^1 = \{(1,4),(4,a_4),(2,3)\}$.

Stage 2:

- $e_1^2 = (1,4)$, $e_2^2 = (2,3)$, $e_3^2 = \varnothing$ and $e_4^2 = (4,a_4)$.
- $N_{14}^2 = \{1\}$, $N_{23}^2 = \{2\}$ and $N_{4a_4}^2 = \{4\}$.
- $c^2 = \min\{c_{14}-1, c_{23}-1, c_{4a_4}-1\} = \min\{2-1, 3-1, 4-1\} = 1$.
- $p_1^2 = 1$, $p_2^2 = 1$, $p_3^2 = 0$ and $p_4^2 = 1$.

- $E^2 = \{(2,3),(4,a_4)\}$.

Stage 3:

- $e_1^3 = (4,a_4)$, $e_2^3 = (2,3)$, $e_3^3 = \varnothing$ and $e_4^3 = (4,a_4)$.
- $N_{23}^3 = \{2\}$ and $N_{4a_4}^3 = \{1,4\}$.
- $c^3 = \min\{c_{23}-2, c_{4a_4}-2\} = \min\{3-2, 4-2\} = 1$.
- $p_1^3 = \frac{1}{2}$, $p_2^3 = 1$, $p_3^3 = 0$ and $p_4^3 = \frac{1}{2}$.
- $E^3 = \{(4,a_4)\}$.

Stage 4:

- $e_1^4 = (4,a_4)$, $e_2^4 = \varnothing$, $e_3^4 = \varnothing$ and $e_4^4 = (4,a_4)$.
- $N_{4a_4}^4 = \{1,4\}$.
- $c^4 = \min\{c_{4a_4}-3\} = \min\{4-3\} = 1$.
- $p_1^4 = \frac{1}{2}$, $p_2^4 = 0$, $p_3^4 = 0$ and $p_4^4 = \frac{1}{2}$.
- $E^4 = \varnothing$. Thus, $\gamma = 4$.

Stage 5: For each $i \in N$,

$$p_i^5 = \frac{c(t_M^*)}{4} = \frac{18}{4} = \frac{9}{2}.$$

Then,

$$\begin{aligned}
f_1^P(N,M,C) &= 1+1+\frac{1}{2}+\frac{1}{2}+\frac{9}{2} = \frac{15}{2}, \\
f_2^P(N,M,C) &= 1+1+1+0+\frac{9}{2} = \frac{15}{2}, \\
f_3^P(N,M,C) &= 1+0+0+0+\frac{9}{2} = \frac{11}{2}, \\
f_4^P(N,M,C) &= 1+1+\frac{1}{2}+\frac{1}{2}+\frac{9}{2} = \frac{15}{2}.
\end{aligned}$$

We now show that the solution does not actually depend on the minimal tree t considered initially and the tree t^* defined in Phase 1. To that end, we introduce two propositions.

Proposition 1. *Let (N,M,C) and (N,M,C') be two mcstp with multiple sources satisfying that there is an order σ over the set of edges of $N \cup M$ such that for all $i,j,k,l \in N \cup M$ satisfying that $\sigma(i,j) < \sigma(k,l)$, then $c_{ij} \le c_{kl}$ and $c'_{ij} \le c'_{kl}$. Let t be a minimal tree in C, C', and $C+C'$. Then,*

$$f^{P,t}(N,M,C+C') = f^{P,t}(N,M,C) + f^{P,t}(N,M,C').$$

Proof. Applying Phase 1 to t, we can obtain a common *mt* t^* for (N, M, C^*), (N, M, C'^*) and $(N, M, C^* + C'^*)$.

We now compute Phase 2. First, consider the case when for all $i, j, k, l \in N \cup M$ satisfying that $\sigma(i,j) < \sigma(k,l)$, then $c_{ij} < c_{kl}$ and $c'_{ij} < c'_{kl}$. Thus $c_{ij} + c'_{ij} < c_{kl} + c'_{kl}$. For all $i \in N$, let $i^M \in N \cup M$ denote the immediate successor of i in the unique path from i to M in t^*. Without loss of generality, we assume that $c_{ii^M} < c_{jj^M}$ when $i < j$, for all $i, j \in N$. Then,

Stage 1:

- $\forall i \in N$, $e_i^1(C) = e_i^1(C') = e_i^1(C + C') = (i, i^M)$.
- $\forall i \in N$, $N_{ii^M}^1(C) = N_{ii^M}^1(C') = N_{ii^M}^1(C + C') = \{i\}$.
- $c^1(C) = \min\limits_{i \in N}\{c_{ii^M}\} = c_{11^M}$,
  $c^1(C') = \min\limits_{i \in N}\{c'_{ii^M}\} = c'_{11^M}$ and
  $c^1(C + C') = \min\limits_{i \in N}\{c_{ii^M} + c'_{ii^M}\} = c_{11^M} + c'_{11^M}$.
- $\forall i \in N$, $p_i^1(C) = c_{11^M}$, $p_i^1(C') = c'_{11^M}$ and $p_i^1(C + C') = c_{11^M} + c'_{11^M}$.
- $E^1(C) = E^1(C') = E^1(C + C') = \{(i, i^M)\}_{i=2}^{|M|}$.

  Then, for all $i \in N$, $p_i^1(C + C') = p_i^1(C) + p_i^1(C')$.

Stage 2:

- $\forall i \in N \backslash 1$, $e_i^2(C) = e_i^1(C)$, $e_i^2(C') = e_i^1(C')$ and $e_i^2(C + C') = e_i^1(C + C')$. If $1^M \in M$ then $e_i^2(C) = e_i^2(C') = e_i^2(C + C') = \emptyset$. If $1^M \notin M$ then $e_i^2(C) = e_i^2(C') = e_i^2(C + C') = e_{1^M}^1(C)$.
  Then, $\forall i \in N$, $e_i^2(C) = e_i^2(C') = e_i^2(C + C')$.
- $N_{ii^M}^2(C) = N_{ii^M}^2(C') = N_{ii^M}^2(C + C')$, for all $i \in N \backslash \{1\}$.
- $c^2(C) = \min\limits_{i \in N \backslash \{1\}}\{c_{ii^M} - c^1(C)\} = c_{22^M} - c_{11^M}$,
  $c^2(C') = \min\limits_{i \in N \backslash \{1\}}\{c'_{ii^M} - c^1(C')\} = c'_{22^M} - c'_{11^M}$ and
  $c^2(C + C') = \min\limits_{i \in N \backslash \{1\}}\{c_{ii^M} + c'_{ii^M} - c^1(C + C')\} = c_{22^M} + c'_{22^M} - (c_{11^M} + c'_{11^M})$.
- $\forall i \in N$, $p_i^2(C) = \dfrac{c_{22^M} - c_{11^M}}{|N_{e_i^2}^2(C)|}$, $p_i^2(C') = \dfrac{c'_{22^M} - c'_{11^M}}{|N_{e_i^2}^2(C')|}$ and
  $p_i^2(C + C') = \dfrac{c_{22^M} + c'_{22^M} - (c_{11^M} + c'_{11^M})}{|N_{e_i^2}^2(C + C')|}$.
- $E^2(C) = E^2(C') = E^2(C + C') = \{(i, i^M)\}_{i=2}^{|M|}$.

  Then $\forall i \in N$, $p_i^2(C + C') = p_i^2(C) + p_i^2(C')$.

Repeating this argument, we can prove that $\gamma(C) = \gamma(C') = \gamma(C + C')$ and that for each stage $\delta = 1, ..., \gamma$ and for every $i \in N$, we have that $p_i^{\gamma}(C + C') = p_i^{\gamma}(C) + p_i^{\gamma}(C')$. Besides, for every $i \in N$, $p_i^{\gamma+1}(C) = \dfrac{c(t_M^*)}{|N|}$, $p_i^{\gamma+1}(C') = \dfrac{c'(t_M^*)}{|N|}$ and $p_i^{\gamma+1}(C + C') = \dfrac{c(t_M^*) + c'(t_M^*)}{|N|}$. Thus,

$$f^{P,t}(N, M, C + C') = f^{P,t}(N, M, C) + f^{P,t}(N, M, C').$$

Now, consider the general case when, if $\sigma(i, j) < \sigma(k, l)$, then $c_{ij} \leq c_{kl}$ and $c'_{ij} \leq c'_{kl}$. Let C^{ε} and C'^{ε} be two cost functions such that:

- For each $i, j \in N \cup M$, $c_{ij} - \varepsilon \leq c^{\varepsilon}_{ij} \leq c_{ij} + \varepsilon$ and $c'_{ij} - \varepsilon \leq c'^{\varepsilon}_{ij} \leq c'_{ij} + \varepsilon$
- If $\sigma(i, j) < \sigma(k, l)$ then $c^{\varepsilon}_{ij} < c^{\varepsilon}_{kl}$ and $c'^{\varepsilon}_{ij} < c'^{\varepsilon}_{kl}$.
- t is a minimal tree in C^{ε}, C'^{ε}, and $C^{\varepsilon} + C'^{\varepsilon}$.

Notice that C^{ε} and C'^{ε} satisfy the condition in the first case studied. So, $f^{P,t}(N, M, C^{\varepsilon} + C'^{\varepsilon}) = f^{P,t}(N, M, C^{\varepsilon}) + f^{P,t}(N, M, C'^{\varepsilon})$.

Finally, taking into account the definition of the rule $f^{P,t}$, we have that $\lim_{\varepsilon \to 0} f^{P,t}(N, M, C^{\varepsilon}) = f^{P,t}(N, M, C)$, $\lim_{\varepsilon \to 0} f^{P,t}(N, M, C'^{\varepsilon}) = f^{P,t}(N, M, C')$ and $\lim_{\varepsilon \to 0} f^{P,t}(N, M, C^{\varepsilon} + C'^{\varepsilon}) = f^{P,t}(N, M, C + C')$. Thus,

$$f^{P,t}(N, M, C + C') = f^{P,t}(N, M, C) + f^{P,t}(N, M, C'). \qquad \square$$

We now prove that for each problem (N, M, C) and every minimal tree t the painting rule associated with t coincides with the folk rule. Thus, the painting rule is well defined and is independent of the minimal tree t and the tree t^* computed in Phase 1.

Proposition 2. *For every problem (N, M, C) and every minimal tree t for (N, M, C),*

$$f^{P,t}(N, M, C) = F(N, M, C).$$

Proof. By Lemma 1, we know that $C = \sum_{q=1}^{m(C)} x^q C^q$ where for each q, (N, M, C^q) is a simple problem. Besides t is a minimal tree for each (N, M, C^q). By Proposition 1 and the definition of the folk rule F, it is enough to prove that $f^{P,t}(N, M, C^q) = F(N, M, C^q)$ when (N, M, C^q) is a simple problem and t is a minimal tree in (N, M, C^q).

Let t^* be a tree obtained on Phase 1. For all $i \in N$, let $i^M \in N \cup M$ denote the immediate successor of i in the unique path from i to M in t^*. Now, we apply the procedure of Phase 2:

Stage 1: Take $i \in N$.

- $\forall i \in N$, $e_i^1(C^q, t^*) = (i, i^M)$.
- $\forall i \in N$, $N^1_{ii^M}(C^q, t^*) = \{i\}$.

Let $P = \{S_1, ..., S_p\}$ be the partition of $N \cup M$ in C^q-components. We consider several cases:

Case 1: $S(i,P) \cap M \neq \varnothing$, for all $i \in N$. Then,

- $c^1(C^q, t^*) = 0$.
- $\forall i \in N$, $p_i^1(C^q, t^*) = 0$.
- $E^1(C^q, t^*) = \varnothing$.

Then, $\gamma = 1$ and $\forall i \in N$,

$$p_i^2(C^q, t^*) = \frac{|S_k : S_k \cap M \neq \varnothing| - 1}{|N|}.$$

Thus, $\forall i \in N$,

$$f_i^{P,t}(N, M, C^q) = \frac{|S_k : S_k \cap M \neq \varnothing| - 1}{|N|} = F(N, M, C^q).$$

Case 2: $|S(i,P)| = 1$, for all $i \in N$. Then $S(i,P) \cap M = \varnothing$, $\forall i \in N$. Now

- $c^1(C^q, t^*) = 1$.
- $\forall i \in N$, $p_i^1(C^q, t^*) = 1 = \dfrac{1}{|S(i,P)|}$.
- $E^1(C^q, t^*) = \varnothing$.

As in the first case, $\gamma = 1$ and $\forall i \in N$

$$p_i^2(C^q, t^*) = \frac{|S_k : S_k \cap M \neq \varnothing| - 1}{|N|}.$$

Therefore

$$f_i^{P,t}(N, M, C^q) = \frac{1}{|S(i,P)|} + \frac{|S_k : S_k \cap M \neq \varnothing| - 1}{|N|} = F(N, M, C^q).$$

Case 3: Otherwise.

- $c^1(C^q, t^*) = 0$.
- $\forall i \in N$, $p_i^1(C^q, t^*) = 0$.
- $E^1(C^q, t^*) = \{(i, i^M) \in E^0 : c^q_{ii^M} = 1\} \neq \varnothing$.

Stage 2:

- Let $i \in N$. If $S(i,P) \cap M \neq \varnothing$, then $e_i^2(C^q, t^*) = \varnothing$. If $S(i,P) \cap M = \varnothing$, there exists a unique $j \in S(i,P)$ such that $(j, j^M) \in E^1$. Thus $e_i^2(C^q, t^*) = (j, j^M)$.

- $N^2_{e^2_i}(C^q, t^*) = S(i, P)$.
- $c^2(C^q, t^*) = 1$.
- For each $i \in N$,

$$p^2_i(C^q, t^*) = \begin{cases} 0, & \text{if } S(i,P) \cap M \neq \varnothing \\ \dfrac{1}{|S(i,P)|}, & \text{otherwise.} \end{cases}$$

- $E^2(C^q, t^*) = \varnothing$.

In this case, $\gamma = 2$ and $\forall i \in N$

$$p^3_i(C^q, t^*) = \frac{|S_k : S_k \cap M \neq \varnothing| - 1}{|N|}.$$

Then,

$$f^{P,t}_i(N, M, C^q) = \begin{cases} \dfrac{|S_k : S_k \cap M \neq \varnothing| - 1}{|N|}, & \text{if } S(i,P) \cap M \neq \varnothing \\ \dfrac{1}{|S(i,P)|} + \dfrac{|S_k : S_k \cap M \neq \varnothing| - 1}{|N|}, & \text{otherwise.} \end{cases}$$

Therefore, $f^{P,t}_i(N, M, C^q) = F_i(N, M, C^q)$, for all $i \in N$. □

Since the rule coincides with the folk rule, which does not depend on the tree t chosen, the rule can be denoted by f^P instead of $f^{P,t}$.

Bergantiños et al. (2017) extend the folk rule for *mcstp* with multiple sources using four approaches: As the Shapley value of the irreducible game (Bergantiños and Vidal-Puga (2007)), as an obligation rule (Tijs et al. (2006) and Bergantiños and Kar (2010)), as a partition rule (Bergantiños et al. (2010, 2011)), and through a cone-wise decomposition (Branzei et al. (2004) and Bergantiños and Vidal-Puga (2009)). Thus, the painting rule is a new way of calculating the extension of the folk rule to this context. The main advantage of this approach is that it makes it very clear that the allocation of an agent given by the folk rule depends only on her path to the sources and the connection cost between them in the irreducible problem.

4. An axiomatic characterization

This section presents an axiomatic characterization of the painting rule. In their Corollary 1, Bergantiños et al. (2014) characterize the folk rule in classical minimum cost spanning tree problems with the properties of cost monotonicity, symmetry, cone-wise additivity, and isolated agents. We extend this characterization to the case of multiple sources by considering these four axioms and adding a new one called equal treatment of source costs. The definition of the properties of cost monotonicity, symmetry, and cone-wise additivity in the case of multiple sources is the same as in the classical case. The definition of isolated agents is not straightforward. Equal treatment of source costs is a property defined only in the case of multiple sources.

A rule f for *mcstp* with multiple sources satisfies:

Cone-wise additivity (CA). Let (N, M, C) and (N, M, C') be two *mcstp* with multiple sources satisfying that there is an order σ over the set of edges of $N \cup M$ such that for all $i, j, k, l \in N \cup M$ satisfying that $\sigma(i, j) < \sigma(k, l)$, then $c_{ij} \leq c_{kl}$ and $c'_{ij} \leq c'_{kl}$. Thus,

$$f(N, M, C + C') = f(N, M, C) + f(N, M, C').$$

CA says that the rule should be additive on the cost function C when restricted to cones.

Cost monotonicity (CM). For all (N, M, C) and (N, M, C') such that $C \leq C'$, then

$$f(N, M, C) \leq f(N, M, C').$$

CM says that if a certain number of connection costs increase and the rest (if any) remain the same, no agent should end up better off.

Symmetry (SYM). For all (N, M, C) and all $i, j \in N$ such that $c_{ik} = c_{jk}$, $\forall k \in (N \cup M)\backslash\{i, j\}$, then

$$f_i(N, M, C) = f_j(N, M, C).$$

If two agents are symmetrical with respect to their connection costs, SYM says that they should pay the same.

The next property is inspired by the isolated agents property introduced in Bergantiños et al. (2014) for source connection problems.

An agent $i \in N$ is called *isolated* in a problem (N, M, C) if $c_{ij} = x$, for all $j \in (N \cup M)\backslash\{i\}$ and $c_{jk} \leq x$, for all $j, k \in (N \cup M)\backslash\{i\}$. Notice that if agent i is isolated, then agent i does not benefit from connecting to the sources through agents in $N\backslash\{i\}$.

Isolated agents (IA). For all (N, M, C) such that for all $k, l \in M$, there is a path from k to l, g_{kl}, such that $c(g_{kl}) = 0$,

$$f_i(N, M, C) = x,$$

for every isolated agent $i \in N$.

If there is a way of connecting all sources to one another for free (not necessarily directly), an isolated agent should only pay her connection cost to any node.

Equal treatment of sources costs (ETSC). For each pair of problems (N, M, C) and (N, M, C') such that there exist $k, l \in M$, $k \neq l$, such that $c_{kl} < c'_{kl}$ and $c_{ij} = c'_{ij}$ otherwise, then for each $i, j \in N$

$$f_i(N, M, C') - f_i(N, M, C) = f_j(N, M, C') - f_j(N, M, C).$$

This property was introduced in Bergantiños et al. (2017). It says that if the cost between two sources increases, then all agents should be affected in the same way.

In the next theorem, we present the characterization of the painting rule.

Theorem 1. *The painting rule f^P is the unique rule satisfying CA, CM, SYM, IA and ETSC.*

Proof. First we prove that the painting rule satisfies the five properties. Bergantiños et al. (2017) proved that the folk rule satisfies *CA CM*, *SYM* and *ETSC*. By Proposition 2, f^P satisfies *CA CM*, *SYM* and *ETSC*.

We now prove that f^P satisfies *IA*. Let $i \in N$ be an isolated agent for a problem (N, M, C). Let t be a minimal tree for (N, M, C). We can take t in such a way that no agent in $N \backslash \{i\}$ is connected to any source through agent i. Namely, for each $j \in N \backslash \{i\}$ and each $k \in M$, $i \notin t_{jk}$.

Since there is a path at cost zero to join together every two sources, the tree obtained in Phase 1, t^*, is such that $c(t^*_M) = 0$.

We now apply Phase 2. Since no agent is connected to the source through agent i and $c_{ik} = x \geq c_{jk}$, $\forall j, k \in (N \cup M) \backslash \{i\}$, we have that, for each $\delta = 1, ..., \gamma$, $e_i^\delta = (i, i^M)$ and $e_j^\delta \neq (i, i^M)$, for all $j \in N \backslash \{i\}$.

Then,

$$f_i^P(N, M, C) = c_{ii^M} + \frac{c(t^*_M)}{|N|} = x + 0 = x.$$

Thus, f^P satisfies *IA*.

We now prove the uniqueness. Let f be a rule satisfying the properties of Theorem 1. By *CA*, it is enough to prove that $f = f^P$ in simple problems.

Let (N, M, C) be a simple problem and $P = \{S_1, ..., S_p\}$ the set of C-components. Consider the next cost function:

$$c'_{ij} = \begin{cases} c_{ij}, & \text{if } \{i, j\} \cap N \neq \varnothing \\ 0, & \text{otherwise.} \end{cases}$$

We have a simple problem (N, M, C') such that all sources are connected to one another at cost zero and $C \geq C'$.

For each $S_k \in P$ such that $S_k \cap M = \varnothing$, we define a pair of cost function as follows:

$$c^k_{ij} = \begin{cases} 1, & \text{if } \{i, j\} \cap S_k \neq \varnothing \\ 0, & \text{otherwise} \end{cases}$$

and

$$c'^k_{ij} = \begin{cases} c'_{ij}, & \text{if } \{i, j\} \cap S_k \neq \varnothing \\ 0, & \text{otherwise.} \end{cases}$$

We first analyze how f works on (N, M, C^k) and (N, M, C'^k). Let $S_k \in P$ with $S_k \cap M = \varnothing$ and $i \in N$.

- On (N, M, C^k). If $i \in S_k$, i is an isolated agent. By *IA*, $f_i(N, M, C^k) = 1$, for all $i \in S_k$. Besides, $m(N, M, C^k) = |S_k|$. Since all agents in $N \backslash S_k$ are symmetric, $f_i(N, M, C^k) = 0$, for all $i \notin S_k$. This is,

$$f_i(N, M, C^k) = \begin{cases} 1, & \text{if } i \in S_k \\ 0, & \text{otherwise.} \end{cases}$$

- On (N, M, C'^k). We have that $C'^k \leq C^k$. If $i \notin S_k$, by *CM*, $f_i(N, M, C'^k) \leq f_i(N, M, C^k) = 0$. It is straightforward to see that if a rule satisfies *CM* and *SYM*, then it should be non-negative. Then, $f_i(N, M, C^k) = 0$ if $i \notin S_k$. All agents on S_k are symmetric and $m(N, M, C'^k) = 1$. Thus,

$$f_i(N, M, C'^k) = \begin{cases} \dfrac{1}{|S_k|}, & \text{if } i \in S_k \\ 0, & \text{otherwise.} \end{cases}$$

Take $i \in N$ and let $S(i, P)$ denote the C-component to which i belongs. We consider two cases.

- $S(i, P) \cap M = \varnothing$. Since $C' \geq C'^k$ and *CM*,

$$f_i(N, M, C') \geq f_i(N, M, C'^k) = \frac{1}{|S(i, P)|}.$$

- $S(i, P) \cap M \neq \varnothing$. Since a rule satisfying *CM* and *SYM* should be non-negative

$$f_i(N, M, C') \geq 0.$$

Taking into account that $m(N, M, C') = |S_k \in P : S_k \cap M = \varnothing|$ and

$$\sum_{i \in N} f_i(N, M, C') \geq \sum_{i \in N | S(i,P) \cap M = \varnothing} \frac{1}{|S(i, P)|} = |S_k \in P : S_k \cap M = \varnothing|,$$

we conclude that

$$f_i(N, M, C') = \begin{cases} \dfrac{1}{|S(i, P)|}, & \text{if } S(i, P) \cap M = \varnothing \\ 0, & \text{otherwise.} \end{cases}$$

Finally, notice that the cost functions C and C' are as the definition of *ETSC*. Then, for all $i, j \in N$,

$$f_i(N, M, C) - f_i(N, M, C') = f_j(N, M, C) - f_j(N, M, C').$$

Fix $i \in N$,

$$
\begin{aligned}
|N|[f_i(N,M,C) - f_i(N,M,C')] &= \sum_{j\in N}[f_j(N,M,C) - f_j(N,M,C')] \\
&= \sum_{j\in N} f_j(N,M,C) - \sum_{j\in N} f_j(N,M,C') \\
&= |P| - 1 - (|P| - |S_k \in P : S_k \cap M \neq \varnothing|) \\
&= |S_k \in P : S_k \cap M \neq \varnothing| - 1.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f_i(N,M,C) &= \frac{|S_k \in P : S_k \cap M \neq \varnothing| - 1}{|N|} + f_i(N,M,C') \\
&= \begin{cases} \dfrac{|S_k \in P : S_k \cap M \neq \varnothing| - 1}{|N|} + \dfrac{1}{|S(i,P)|}, & \text{if } S(i,P) \cap M = \varnothing \\ \dfrac{|S_k \in P : S_k \cap M \neq \varnothing| - 1}{|N|}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Therefore, $f(N,M,C) = F(N,M,C)$. By Proposition 2, $f(N,M,C) = f^P(N,M,N)$. □

Next we prove that all properties are needed in the previous characterization.

CA is independent of the other properties. Consider the rule f^e defined in Bergantiños et al. (2017) when they prove that *CA* is independent of the properties they use in Theorem 2. f^e satisfies all properties but *CA*.

CM is independent of the other properties. Given a problem (N,M,C), let t be a *mt* of (N,M,C) and t^* a *mt* of (N,M,C^*) obtained through Phase 1. We now consider the following classical *mcstp* $(N_0,\overline{C})$, where $\bar{c}_{0i} = \max\{c^*_{kl} : (k,l) \in t^*_{ij}$ for some $j \in M$ and $k,l \in N\}$ and $\bar{c}_{ij} = c^*_{ij}$, for all $i,j \in N$.

For a classical problem with a single *mt*, Bird (1976) proposed a rule called the *Bird rule*. This rule is obtained by requiring each agent to pay the total cost of the first edge in her unique path to the source. Dutta and Kar (2004) extended the Bird rule when there is more than one *mt* (an extension we denote as B). This rule is the average of the allocations given by the Bird rule on all the minimal trees associated with Prim's algorithm.

We now extend it to our setting in the following way:

$$f^B(N,M,C) = B(N_0,\overline{C}) + \frac{c(t^*_M)}{|N|}.$$

f^B satisfies all properties but *CM*.

SYM is independent of the other properties. For each problem (N,M,C) and each $\delta = 1,...,n+m-1$, let (i^δ, j^δ) denote the edge selected by Kruskal's algorithm at stage δ and g^δ be the set of all edges selected according to Kruskal's algorithm until stage δ (included). Besides $P(g^{\delta})$ denotes the partition of $N \cup M$ in connected components induced by g^{δ}.

Given a partition P we define the function α as

$$\alpha_i(P) = \begin{cases} \dfrac{|S_k \in P : S_k \cap M \neq \varnothing| - 1}{|N|} + 1, & \text{if } S(i,P) \cap M = \varnothing \\ & \text{and } i \leq j, \forall j \in S(i,P) \\ \dfrac{|S_k \in P : S_k \cap M \neq \varnothing| - 1}{|N|}, & \text{otherwise,} \end{cases}$$

Thus we define the rule f^{α} such that for each problem (N,M,C) and each $i \in N$,

$$f_i^{\alpha}(N,M,C) = \sum_{\delta=1}^{n+m-1} c_{i^{\delta} j^{\delta}} [\alpha_i(P(g^{\delta-1})) - \alpha_i(P(g^{\delta}))].$$

f^{α} satisfies all properties but *SYM*.

IA is independent of the other properties. Let E be the rule in which the cost of the minimal tree is divided equally among all agents. Namely, for each problem (N,M,C) and each $i \in N$,

$$E_i(N,M,C) = \frac{m(N,M,C)}{|N|}.$$

This rule satisfies all properties but *IA*.

ETSC is independent of the other properties. Let (N,M,C) be a problem. If $N = \{1,2\}$ and $M = \{a_1,a_2\}$, let us define the sets $N' = \{1,2,a_2\}$ and $M' = \{a_1\}$. Then, for every $i \in N$, we define the rule

$$f_i(N,M,C) = \begin{cases} f_i^P(N',M',C) + \dfrac{f_{a_2}^P(N',M',C)}{2}, & \text{if } N = \{1,2\} \text{ and } M = \{a_1,a_2\} \\ f_i^P(N,M,C), & \text{otherwise.} \end{cases}$$

This rule satisfies all properties but *ETSC*.

We end this paper by mentioning other properties mey by the rule. These properties are introduced in Bergantiños et al. (2017).

Independence of irrelevant trees (IIT). For each (N,M,C) and (N,M,C'), if they have a common minimal tree t such that $c_{ij} = c'_{ij}$ for each $(i,j) \in t$, then $f(N,M,C) = f(N,M,C')$.

This property requires the cost allocation chosen by a rule to depend only on the edges that belong to a minimal tree.

Core selection (CS). For each (N,M,C) and each $S \subseteq N$, $\sum_{i \in S} f_i(N,M,C) \leq m(S,M,C)$.

CS implies that no coalition of agents would be better off by constructing their own minimal tree.

Separability (SEP). For each (N,M,C) and each $S \subseteq N$, if $m(N,M,C) = m(S,M,C) + m(N \backslash S,M,C)$, then

$$f_i(N,M,C) = \begin{cases} f_i(S,M,C), & \text{if } i \in S, \\ f_i(N \backslash S,M,C), & \text{if } i \in N \backslash S. \end{cases}$$

Two subsets of agents, S and $N \backslash S$ can be connected to all the sources either separately or jointly. This property implies that if the minimal costs in two situations are the same, agents will pay the same in both circumstances.

Population monotonicity (PM). For each (N,M,C), each $S \subset T \subseteq N$ and each $i \in S$, $f_i(S,M,C) \geq f_i(T,M,C)$.

If new agents join the problem, then no agent in the original problem should be worse off.

These properties are not completely independent. The following proposition summarizes the relations between the properties seen in this paper.

Proposition 3. *(i) CM implies IIT.*

(ii) CS implies AI.

(iii) SEP implies AI.

(iv) PM implies SEP, CS and IA.

Proof. (i) It appears in Bergantiños et al. (2017).

(ii) It is easy to see that *CS* implies *IA* noticing that, if $i \in N$ is an isolated agent in (N,M,C), then $m(N,M,C) = m(N \backslash \{i\},M,C) + x$. Since $\sum_{j \in N \backslash \{i\}} f_j(N,M,C) \leq m(N \backslash \{i\},M,C)$ and $f_i(N,M,C) \leq x$, we have that $f_i(N,M,C) = x$.

(iii) It is similar to Case (ii).

(iv) Bergantiños et al. (2017) prove that *PM* implies *SEP* and *CS*. By (ii), *PM* also implies *IA*. □

Bergantiños et al. (2017) also provide two characterizations of the folk rule in minimum cost spanning tree problems with multiple sources. As in Theorem 1, they use *CA*, *SYM*, and *ETSC*. In both cases they also use *IIT* and complete one characterization with *CS* and the other with *SEP*. Thus, the three characterizations are unrelated. Namely, no characterization is a consequence of another.

Apart from this, the proof of uniqueness in the characterization of this paper and the proof of uniqueness in the characterizations of Bergantiños et al. (2017) are also unrelated. In all three cases the first step is the same. By CA we can consider only simple games. But now the arguments are completely different. In this paper we consider the problems C', C'^k, and C^k and depending on how a rule works in such problems uniqueness is obtained. Bergantiños et al. (2017) obtain uniqueness by considering the expression of the folk rule as an obligation rule.

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