A Bayesian approach for correcting bias of data envelopment analysis estimators

Zervopoulos, Panagiotis and Emrouznejad, Ali and Sklavos, Sokratis

Department of Management, University of Sharjah, Operations Information Management Department, Aston Business School, NeuroBioAnalysis Lab

January 2019

A Bayesian approach for correcting bias of data envelopment analysis estimators

Panagiotis D. Zervopoulos*

Department of Management

University of Sharjah, Sharjah 27272, United Arab Emirates

Tel.: +971 567159809

pzervopoulos@sharjah.ac.ae

Ali Emrouznejad

Operations & Information Management Department

Aston Business School, Aston University, Birmingham B4 7ET, UK

a.emrouznejad@aston.ac.uk

Sokratis Sklavos

NeuroBioAnalysis Lab, 2 Lefktron St & Michalakopoulou Av., Athens 11527, Greece

sok.sklavos@gmail.com

*Corresponding author

Abstract

The validity of data envelopment analysis (DEA) efficiency estimators depends on the robustness of the production frontier to measurement errors, specification errors and the dimension of the input-output space. It has been proven that DEA estimators, within the interval (0, 1], are overestimated when finite samples are used while asymptotically this bias reduces to zero. The non-parametric literature dealing with bias correction of efficiencies solely refers to estimators that do not exceed one. We prove that efficiency estimators, both lower and higher than one, are biased. A Bayesian DEA method is developed to correct bias of efficiency estimators. This is a two-stage procedure of super-efficiency DEA followed by a Bayesian approach relying on consistent efficiency estimators. This method is applicable to 'small' and 'medium' samples. The new Bayesian DEA method is applied to two data sets of 50 and 100 E.U. banks. The mean square error, root mean square error and mean absolute error of the new method reduce as the sample size increases.

Keywords: Data envelopment analysis; Super-efficiency; Bayesian methods; Statistical inference; Banking

1. Introduction

Data envelopment analysis (DEA) put forth by Charnes et al. (1978) and extended by Banker et al. (1984) is a mathematical programming methodology to evaluate the efficiency of a sample firm relative to a reference set of all sample firms. DEA is a non-parametric approach to construct production frontiers based on observed input and output data of the sample firms. Despite the non-parametric nature of DEA, Banker and Maindiratta (1992), Banker (1993), Sarath and Maindiratta (1997), Banker and Natarajan (2008) provided statistical justification for DEA. In particular, Banker (1993) proved that DEA (with one output and multiple inputs), under the conditions of monotonicity and concavity, yields consistent estimators of the production frontier. The studies of Simar and Wilson (1998, 1999), Kneip et al. (2008, 2011), Kuosmanen and Johnson (2010) and Tsionas and Papadakis (2010) also allowed for inference on DEA efficiency estimators.

The validity of DEA efficiency estimators depends on the robustness of the production frontier to measurement errors, specification errors and the dimension of the input-output space. Banker (1993) was the first to highlight the overestimation of DEA efficiencies when finite samples are used. Banker (1993) and Grosskopf (1996) showed that this bias asymptotically reduces to zero. In line with these studies, Simar (2007) identified an inverse relationship between the rate of convergence of DEA efficiency estimators and the dimensionality of the production set. Simar and Wilson (2015) stated that the true efficiency of a firm is unknown.

Emphasizing DEA, there are six major approaches dealing with the sensitivity of efficiency estimators: (a) Chance Constrained DEA (CCDEA); (b) Two-stage DEA-based methods; (c) Corrected Concave Non-Parametric Least Squares (C^2NLS); (d) Stochastic Non-Smooth Envelopment of Data (StoNED); (e) Bayesian DEA; and (f) bootstrap DEA.

CCDEA (Charnes and Cooper, 1963; Land et al., 1993; Olesen and Petersen, 1995) specifies stochastic production frontiers by replacing the observed input and output data with their randomly distributed counterparts. CCDEA programs are appropriate for dealing with the presence of noise in the data. However, they lack statistical theory. A review of CCDEA is available in Olesen and Petersen (2016). Two-stage DEA-based procedures presented by Banker and Natarajan (2008) for estimating non-parametric stochastic frontiers. In the first stage, a conventional DEA model is applied (e.g. the variable returns to scale (VRS) DEA put forth by Banker et al. (1984)) to estimate the technical efficiency of sample firms. In the second stage, the DEA efficiency estimators obtained from the previous stage are introduced in OLS and maximum likelihood models to yield consistent estimators.

The C^2NLS (Kuosmanen and Johnson, 2010) is a least-square interpretation of the VRS DEA model, which, in contrast to conventional DEA models constructing production frontiers based on dominant firms, uses all available information for estimating a production frontier. Kuosmanen and Johnson (2010) concluded that the C^2NLS estimators outperform DEA estimators when the number of firms are significantly higher than the number of input and output variables while the C^2NLS estimators perform at least as well as the DEA estimators when dimensionality is present. The StoNED method (Kuosmanen and Kortelainen, 2007; Kuosmanen and Kortelainen, 2012) estimates semiparametric frontiers by combining the DEA-style frontier with the Stochastic Frontier Analysis (SFA)-style treatment of inefficiency and noise. StoNED facilitates statistical inference while relying on regularity properties (e.g. free disposability, convexity) and without requiring the assumption of a particular production function.

A Bayesian DEA approach for CCDEA was developed by Tsionas and Papadakis (2010). This method provides statistical inference (e.g. estimation of CCDEA efficiencies based on an estimated prior distribution, construction of confidence intervals) to CCDEA relying on assumptions about the distribution (e.g. multivariate normal) of the (posterior/observed) inputs and outputs. Relying on the distribution of the posterior input and output data, it is possible to estimate the prior distribution of the data and then estimate CCDEA efficiencies. This approach lacks formal statistical justification.

Bootstrap DEA is a widely used method for correcting bias and constructing confidence intervals of efficiency estimators (Kneip et al., 2008). Bootstrap DEA, or smoothed bootstrap, originated from Simar and Wilson (1998), combines both the virtues and limitations of bootstrap and DEA. Smoothed bootstrap relies on pseudo-data obtained from an estimated data generating process (DGP) (Dyson and Shale, 2010). Kneip et al. (2008, 2011) developed improved smoothed bootstrap algorithms providing consistent bias-corrected estimators. Major limitations of the smoothed bootstrap are the considerably large confidence intervals, which make difficult to obtain meaningful comparisons between the sample firms, and unsatisfactory performance when inadequate samples for the dimension of the input-output space are available.

All methods discussed above dealing with the sensitivity of DEA efficiencies refer to estimators lying within the interval (0, 1]. Andersen and Petersen (1993), drawing on the work of Banker and Gifford (1988), presented a super-efficiency DEA model, which makes possible efficiency estimators to exceed unity, unlike the conventional DEA models, as the firm under review is excluded from its own reference set. Super-efficiency DEA procedure is used for ranking efficient units and identifying outliers (Banker and Chang, 2006). However, Banker and Chang (2006) and Banker et al. (2017) found that super-efficiency performs unsatisfactorily in ranking efficiency units. It should be noted that this result has not been tested to cases of multiple inputs and multiple outputs and input values considerably greater than 20. Another issue of the traditional super-efficiency DEA model under VRS is the infeasibility.

The contribution of this work is to provide statistical inference in super-efficiency DEA models. We develop a two-stage Bayesian DEA approach to correct bias of super-efficiency estimators. In the first stage, we use a super-efficiency DEA model while in the second stage we specify consistent super-efficiency estimators. These estimators are introduced in the Bayesian framework to estimate bias-corrected (prior) super-efficiencies. To the best of our knowledge, this is the first work on correction of bias of super-efficiency estimators.

The rest of the paper unfolds as follows. In Section 2, we present the super-efficiency DEA model and analyze the steps of our bias-correction method (i.e. Step 1:

conventional statistical inference; Step 2: Bayesian statistical inference; Step 3: bias correction). In Section 3, we present the two data sets used in this study and analyze the results. Section 4 concludes and discusses future research directions.

2. Methodology

2.1 Super-efficiency DEA

After the work of Andersen and Petersen (1993), many studies appear in the literature (Lovell and Rouse, 2003; Chen, 2005; Li et al., 2007; Ray, 2008; Cook et al., 2009; Chen et al., 2011; Lee et al., 2011; Chen and Liang, 2011) dealing with the measurement of super-efficiency in DEA under the condition of VRS. The latter studies tried to solve the problem of infeasibility of the VRS super-efficiency DEA model.

In this study, we use Chen and Liang (2011)'s model to obtain super-efficiency estimators (θ_j), which is as follows

$$\min \ \tau + M \times \sum_{r=1}^{s} \eta_r$$

$$s.t. \ \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij} \leq (1+\tau) x_{io} \qquad i = 1, 2, ..., m$$

$$\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj} \geq (1-\eta_r) y_{ro} \qquad r = 1, 2, ..., s$$

$$\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1$$

$$\lambda_j \geq 0, \ j \neq o, \ \eta_r \geq 0 \tag{1}$$

where M is a user-defined large positive number (e.g. 10^5).

The super-efficiency estimators are defined as $1 + \tau + \frac{1}{|R|} \sum_{r \in R} \frac{1}{1-\eta_r}$, where R is the set of $\eta_r > 0$ and $|R|$ is the cardinality of the set R.

For the application of this Bayesian DEA method for the correction of bias of the super-efficiency DEA estimators, other super-efficiency models (e.g. Chen, 2005; Li et al., 2007; Ray, 2008; Cook et al., 2009; Chen et al., 2011; Lee et al., 2011) can be used instead of model (1).

2.2 Conventional statistical inference

Let $\Theta = (\theta_j)$, where $\theta_j > 0 \ \ (j = 1, 2, ..., n)$, be a random variable of independently and identically distributed (iid) super-efficiency estimators obtained from model (1). Θ is assumed uniformly distributed from $\theta_L \ (0 < \theta_L < 1)$ to $\theta_U \ (\theta_U > 1)$. The two parameters (i.e. θ_L and θ_U) are unknown.

Acknowledging the probability density function (PDF) of Θ as

$$f_\Theta(\theta_j | \theta_L, \theta_U) = \begin{cases} \dfrac{1}{\theta_U - \theta_L}, & \theta_L \le \theta_j \le \theta_U \\ 0 & , \text{ otherwise} \end{cases} \tag{2}$$

the likelihood function is expressed as follows

$$L(\theta_L, \theta_U | \Theta) \overset{\text{iid}}{=} \prod_{j=1}^{n} f_\Theta(\theta_j | \theta_L, \theta_U) = \frac{1}{(\theta_U - \theta_L)^n}, \ \ \theta_L \le \theta_j \le \theta_U, \ \ j = 1, 2, ..., n \tag{3}$$

By partially differentiating the likelihood function (3)

$$\frac{\partial}{\partial \theta_L} L(\theta_L, \theta_U | \Theta) = \frac{n}{(\theta_U - \theta_L)^{n+1}} > 0$$

$$\frac{\partial}{\partial \theta_U} L(\theta_L, \theta_U | \Theta) = -\frac{n}{(\theta_U - \theta_L)^{n+1}} < 0$$

we find that it is monotone increasing for θ_L and monotone decreasing for θ_U. Hence, the likelihood function (3) is maximized at $\hat{\theta}_L = \min \Theta$ and $\hat{\theta}_U = \max \Theta$.

Taking into account the maximum likelihood estimators (MLE) $\hat{\theta}_L$ and $\hat{\theta}_U$, we define the joint cumulative distribution function (CDF) as follows

$$\begin{aligned} F_{(\hat{\theta}_L, \hat{\theta}_U)}(t, s) &= \text{Prob}(\hat{\theta}_L \le t, \ \hat{\theta}_U \le s) = \text{Prob}(\min \Theta \le t, \ \ \max \Theta \le s) \\ &= \text{Prob}(\max \Theta \le s) - \text{Prob}(\min \Theta \ge t, \ \ \max \Theta \le s) \end{aligned} \tag{4}$$

where

$$\mathrm{Prob}(\max\Theta \le s) = \mathrm{Prob}(\theta_1 \le s, \ldots, \theta_n \le s) \overset{\text{iid}}{=} \left\{\mathrm{Prob}(\theta_1 \le s)\right\}^n$$

$$= \left\{ \int_{\theta_L}^{s} f_\Theta(\theta_1 \,|\, \theta_L, \theta_U)\,\mathrm{d}\theta_1 \right\}^n$$

$$= \left\{ \int_{\theta_L}^{s} \frac{1}{\theta_U - \theta_L}\,\mathrm{d}\theta_1 \right\}^n = \left\{ \frac{s-\theta_L}{\theta_U - \theta_L} \right\}^n, \quad 1 < s \le \theta_U \tag{5}$$

and

$$\mathrm{Prob}(\min\Theta \ge t,\ \max\Theta \le s) = \mathrm{Prob}(t \le \theta_1 \le s, \ldots, t \le \theta_n \le s) \overset{\text{iid}}{=} \left\{\mathrm{Prob}(t \le \theta_1 \le s)\right\}^n$$

$$= \left\{ \int_{t}^{s} f_\Theta(\theta_1 \,|\, \theta_L, \theta_U)\,\mathrm{d}\theta_1 \right\}^n = \left\{ \int_{t}^{s} \frac{1}{\theta_U - \theta_L}\mathrm{d}\theta_1 \right\}^n$$

$$= \left\{ \frac{s-t}{\theta_U - \theta_L} \right\}^n, \quad \theta_L \le t < s \le \theta_U \tag{6}$$

Based on expressions (4)-(6), the joint CDF is written as follows

$$F_{(\hat\theta_L, \hat\theta_U)}(t, s \,|\, \theta_L, \theta_U) = \left\{ \frac{s-\theta_L}{\theta_U - \theta_L} \right\}^n - \left\{ \frac{s-t}{\theta_U - \theta_L} \right\}^n, \quad \theta_L \le t < s \le \theta_U \tag{7}$$

with corresponding joint PDF

$$f_{(\hat\theta_L, \hat\theta_U)}(t, s \,|\, \theta_L, \theta_U) = \frac{\partial^2}{\partial t \partial s} F_{(\hat\theta_L, \hat\theta_U)}(t, s) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s} F_{(\hat\theta_L, \hat\theta_U)}(t, s) \right)$$

$$= \frac{\partial}{\partial t}\left(\frac{n(s-\theta_L)^{n-1}}{(\theta_U - \theta_L)^n} - \frac{n(s-t)^{n-1}}{(\theta_U - \theta_L)^n} \right)$$

$$= \frac{n(n-1)(s-t)^{n-2}}{(\theta_U - \theta_L)^n}, \quad \theta_L \le t < s \le \theta_U \tag{8}$$

The marginal PDF of $\hat\theta_L$ is expressed as follows

$$f_{\hat\theta_L}(t \,|\, \theta_L, \theta_U) = \int_{t}^{\theta_U} f_{(\hat\theta_L, \hat\theta_U)}(t, s \,|\, \theta_L, \theta_U)\,\mathrm{d}s = \int_{t}^{\theta_U} \frac{n(n-1)(s-t)^{n-2}}{(\theta_U - \theta_L)^n}\,\mathrm{d}s$$

$$= \frac{n}{(\theta_U - \theta_L)^n}(\theta_U - t)^{n-1}, \quad \theta_L \le t < \theta_U \tag{9}$$

and the corresponding marginal PDF of $\hat\theta_U$ reads as follows

$$f_{\hat\theta_U}(s \,|\, \theta_L, \theta_U) = \int_{\theta_L}^{s} f_{(\hat\theta_L, \hat\theta_U)}(t, s \,|\, \theta_L, \theta_U)\,\mathrm{d}t = \int_{\theta_L}^{s} \frac{n(n-1)(s-t)^{n-2}}{(\theta_U - \theta_L)^n}\,\mathrm{d}t$$

$$= \frac{n}{(\theta_U - \theta_L)^n}(s-\theta_L)^{n-1}, \quad \theta_L < s \le \theta_U \tag{10}$$

As we noticed in expressions (8)-(10)

$$f_{(\hat{\theta}_L,\hat{\theta}_U)}(t,s|\theta_L,\theta_U) \ne f_{\hat{\theta}_L}(t|\theta_L,\theta_U)\cdot f_{\hat{\theta}_U}(s|\theta_L,\theta_U) \tag{11}$$

Hence, the MLE $\hat{\theta}_L$ and $\hat{\theta}_U$ are not independent.

The expected value of $\hat{\theta}_L$ is (see Appendix 1.1 for formal mathematical analysis)

$$E_n\left\{\hat{\theta}_L\right\} = \int_{\theta_L}^{\theta_U} t\cdot f_{\hat{\theta}_L}(t|\theta_L,\theta_U)\mathrm{d}t = \theta_L\frac{n}{n+1} + \theta_U\frac{1}{n+1} < 1 \tag{12}$$

Based on (12), we conclude that the MLE $\hat{\theta}_L$ is asymptotically unbiased to the parameter θ_L as $\lim_{n\to\infty} E_n\left\{\hat{\theta}_L\right\} = \theta_L$ (13)

The second moment of $\hat{\theta}_L$ is defined as follows (see Appendix 1.2 for formal mathematical analysis)

$$E_n\left\{\hat{\theta}_L^2\right\} = \int_{\theta_L}^{\theta_U} t^2\cdot f_{\hat{\theta}_L}(t|\theta_L,\theta_U)\mathrm{d}t = \theta_L^2 + \frac{2(\theta_U-\theta_L)}{n+1}\cdot\frac{\theta_L(n+1)+\theta_U}{n+2} \tag{14}$$

In addition, the variance of $\hat{\theta}_L$ is

$$\mathrm{Var}_n\left\{\hat{\theta}_L\right\} = E_n\left\{\hat{\theta}_L^2\right\} - E_n^2\left\{\hat{\theta}_L\right\}$$

$$= \theta_L^2 + \frac{2(\theta_U-\theta_L)}{n+1}\cdot\frac{\theta_L(n+1)+\theta_U}{n+2} - \left(\theta_L\frac{n}{n+1} + \theta_U\frac{1}{n+1}\right)^2 \tag{15}$$

It is straightforward that $\lim_{n\to\infty}\mathrm{Var}_n\left\{\hat{\theta}_L\right\} = 0$. (16)

The mean square error (MSE) of $\hat{\theta}_L$ is

$$\mathrm{MSE}_n\left\{\hat{\theta}_L\right\} = \mathrm{bias}_n^2\left\{\hat{\theta}_L\right\} + \mathrm{Var}_n\left\{\hat{\theta}_L\right\} \tag{17}$$

and the bias is defined as follows

$$\text{bias}_n\left\{\hat{\theta}_L\right\} = E_n\left\{\hat{\theta}_L\right\} - \theta_L \tag{18}$$

The MLE $\hat{\theta}_L$ is a consistent estimator of the parameter θ_L.

Proof. Based on expression (13) and (16), we find that $\lim_{n\to\infty} \text{bias}_n\left\{\hat{\theta}_L\right\} = 0$ and $\lim_{n\to\infty} \text{MSE}_n\left\{\hat{\theta}_L\right\} = 0$. □

Likewise, the expected value of $\hat{\theta}_U$ is (the formal mathematical analysis is like that in Appendix 1.1)

$$E_n\left\{\hat{\theta}_U\right\} = \int_{\theta_L}^{\theta_U} s \cdot f_{\hat{\theta}_U}(s|\theta_L, \theta_U)\mathrm{d}s = \theta_U \frac{n}{n+1} + \theta_L \frac{1}{n+1} \tag{19}$$

where $\theta_U > 1 + \frac{1-\theta_L}{n}$ to ensure that the mean value of $\hat{\theta}_U$ is always greater than unity.

Based on expression (19), we conclude that the MLE $\hat{\theta}_U$ is asymptotically unbiased to the parameter θ_U as $\lim_{n\to\infty} E_n\left\{\hat{\theta}_U\right\} = \theta_U$. (20)

The second moment of $\hat{\theta}_U$ is defined as follows (the formal mathematical analysis is like that in Appendix 1.2)

$$E_n\left\{\hat{\theta}_U^2\right\} = \int_{\theta_L}^{\theta_U} s^2 \cdot f_{\hat{\theta}_U}(s|\theta_L, \theta_U)\mathrm{d}s = \theta_U^2 + \frac{2(\theta_U - \theta_L)}{n+1} \cdot \frac{\theta_U(n+1) + \theta_L}{n+2} \tag{21}$$

and its variance is

$$\begin{aligned}\text{Var}_n\left\{\hat{\theta}_U\right\} &= E_n\left\{\hat{\theta}_U^2\right\} - E_n^2\left\{\hat{\theta}_U\right\} \\ &= \theta_U^2 + \frac{2(\theta_U - \theta_L)}{n+1} \cdot \frac{\theta_U(n+1) + \theta_L}{n+2} - \left(\theta_U \frac{n}{n+1} + \theta_L \frac{1}{n+1}\right)^2\end{aligned} \tag{22}$$

It is straightforward that $\lim_{n\to\infty} \text{Var}_n\left\{\hat{\theta}_U\right\} = 0$. (23)

The MSE of $\hat{\theta}_U$ is

$$\text{MSE}_n\left\{\hat{\theta}_U\right\} = \text{bias}_n^2\left\{\hat{\theta}_U\right\} + \text{Var}_n\left\{\hat{\theta}_U\right\} \tag{24}$$

and the bias is defined as follows

$$\text{bias}_n\left\{\hat{\theta}_U\right\} = E_n\left\{\hat{\theta}_U\right\} - \theta_U \tag{25}$$

The MLE $\hat{\theta}_U$ is consistent estimator of the parameter θ_U.

Proof. Based on expressions (20) and (23), we find that $\lim_{n\to\infty} \text{bias}_n\left\{\hat{\theta}_U\right\} = 0$ and $\lim_{n\to\infty} \text{MSE}_n\left\{\hat{\theta}_U\right\} = 0$. □

Using expressions (12) and (19), the unbiased estimators of parameters θ_L and θ_U are $\tilde{\theta}_L = \dfrac{n\hat{\theta}_L - \hat{\theta}_U}{n-1} < \hat{\theta}_L$ (26) and, $\tilde{\theta}_U = \dfrac{n\hat{\theta}_U - \hat{\theta}_L}{n-1} > \hat{\theta}_U$ (27) respectively, which satisfy $E_n\left\{\tilde{\theta}_L\right\} = \theta_L$ (28) and $E_n\left\{\tilde{\theta}_U\right\} = \theta_U$ (29), respectively.

The covariance of $\hat{\theta}_L$ and $\hat{\theta}_U$ is defined as follows (see Appendix 2 for formal mathematical analysis)

$$\begin{aligned} \text{cov}_n(\hat{\theta}_L, \hat{\theta}_U) &= E_n\left\{\hat{\theta}_L\hat{\theta}_U\right\} - E_n\left\{\hat{\theta}_L\right\}E_n\left\{\hat{\theta}_U\right\} \\ &= \theta_L\theta_U + \frac{(\theta_U - \theta_L)^2}{n+2} - \frac{\theta_L\theta_U(n^2+1) + (\theta_L^2 + \theta_U^2)n}{(n+1)^2} \end{aligned} \tag{30}$$

The covariance of $\hat{\theta}_L$ and $\hat{\theta}_U$ facilitates the definition of the variance of the unbiased estimator $\tilde{\theta}_L$. Using property (26), the variance is

$$\begin{aligned} \text{Var}_n\left\{\tilde{\theta}_L\right\} &= \text{Var}_n\left\{\frac{n\hat{\theta}_L - \tilde{\theta}_U}{n-1}\right\} \\ &= \frac{1}{(n-1)^2}\left(n^2\text{Var}_n\left\{\hat{\theta}_L\right\} + \text{Var}_n\left\{\hat{\theta}_U\right\} - 2n\,\text{cov}_n(\hat{\theta}_L, \hat{\theta}_U)\right) \end{aligned} \tag{31}$$

which is asymptotically zero, $\lim_{n\to\infty} \text{Var}_n\left\{\tilde{\theta}_L\right\} = 0$ (32)

In addition, $\text{MSE}_n\left\{\tilde{\theta}_L\right\} = \text{Var}_n\left\{\tilde{\theta}_L\right\}$ (33)

The unbiased estimator $\tilde{\theta}_L$ is consistent to parameter θ_L.

Proof. Based on expressions (28) and (32), we find that $\lim_{n\to\infty} \mathrm{MSE}_n\left\{\tilde{\theta}_L\right\}=0$. □

Likewise, using property (27), the variance of the unbiased estimator $\tilde{\theta}_U$ is

$$\mathrm{Var}_n\left\{\tilde{\theta}_U\right\}=\mathrm{Var}_n\left\{\frac{n\hat{\theta}_U-\tilde{\theta}_L}{n-1}\right\}$$

$$=\frac{1}{(n-1)^2}\left(n^2\mathrm{Var}_n\left\{\hat{\theta}_U\right\}+\mathrm{Var}_n\left\{\hat{\theta}_L\right\}-2n\,\mathrm{cov}_n\left\{\hat{\theta}_L,\hat{\theta}_U\right\}\right) \tag{34}$$

which is asymptotically zero $\lim_{n\to\infty}\mathrm{Var}_n\left\{\tilde{\theta}_U\right\}=0$ (35)

and $\mathrm{MSE}_n\left\{\tilde{\theta}_U\right\}=\mathrm{Var}_n\left\{\tilde{\theta}_U\right\}$ (36)

The unbiased estimator $\tilde{\theta}_U$ is consistent to parameter θ_U.

Proof. Expressions (29) and (35) lead to $\lim_{n\to\infty}\mathrm{MSE}_n\left\{\tilde{\theta}_U\right\}=0$. □

Figure 1 illustrates the performance of the MSE of the unbiased estimators against that of the maximum likelihood estimators. This comparative analysis is obtained from 1,000 Monte Carlo simulations. In detail, we find that $\mathrm{MSE}_n\left\{\tilde{\theta}_L\right\}<\mathrm{MSE}_n\left\{\hat{\theta}_L\right\}$ while $\mathrm{MSE}_n\left\{\hat{\theta}_U\right\}<\mathrm{MSE}_n\left\{\tilde{\theta}_U\right\}$. Therefore, the unbiased estimator $\tilde{\theta}_L$ is better than the corresponding MLE $\hat{\theta}_L$ for the parameter θ_L while the opposite applies for the parameter θ_U.

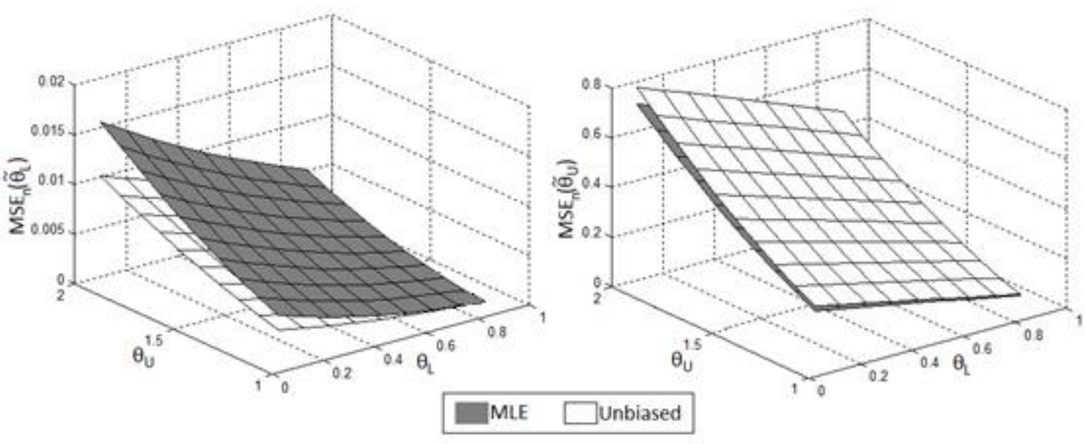

Figure 1. Unbiased estimators vs. maximum likelihood estimators

The covariance of $\tilde{\theta}_L$ and $\tilde{\theta}_U$ is defined as follows (see Appendix 3 for formal mathematical analysis)

$$\mathrm{cov}_n(\tilde{\theta}_L,\tilde{\theta}_U) = E_n\left\{\tilde{\theta}_L\tilde{\theta}_U\right\} - E_n\left\{\tilde{\theta}_L\right\}E_n\left\{\tilde{\theta}_U\right\}$$

$$= \frac{n^2+1}{(n-1)^2}\left(\theta_L\theta_U + \frac{(\theta_U-\theta_L)^2}{n+2}\right) - \frac{n}{(n-1)^2}\left(\theta_L^2+\theta_U^2+\frac{2(\theta_U^2-\theta_L^2)}{(n+1)}\right) - \theta_L\theta_U \quad (37)$$

Based on expression (37) we develop Figure 2 where an inverse relationship between the unbiased estimators $\tilde{\theta}_L$ and $\tilde{\theta}_U$ becomes explicit. These unbiased estimators are asymptotically uncorrelated as expression (37) is asymptotically zero ($\lim_{n\to\infty}\mathrm{cov}_n(\tilde{\theta}_L,\tilde{\theta}_U)=0$)

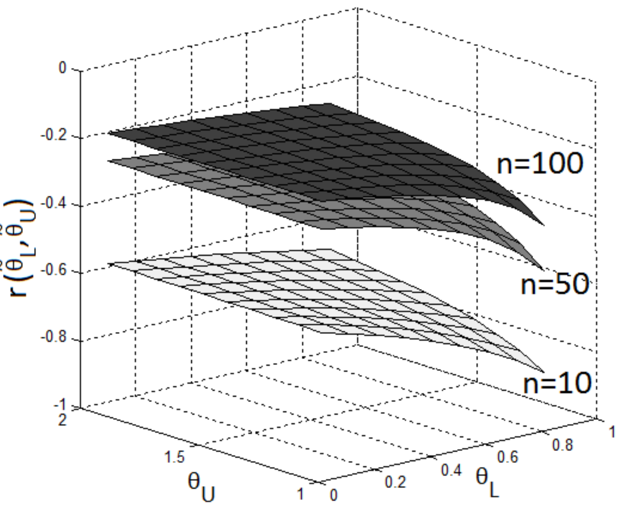

Figure 2. Covariance of the unbiased estimators $\tilde{\theta}_L$ and $\tilde{\theta}_U$

2.3 Bayesian statistical inference

The prior

Let the vector $\Theta=(\Theta_{\mathrm{O}},\Theta_{\mathrm{I}})$ of super-efficiency scores where $\Theta_{\mathrm{O}}=(\theta_1,...,\theta_k)$, $\theta_L \le \Theta_{\mathrm{O}} < 1$, and $\Theta_{\mathrm{I}}=(\theta_{k+1},...,\theta_n)$, $1<\Theta_{\mathrm{I}} \le \theta_U$. In the absence of any information about the distribution of the DEA super-efficiency scores, we assume $\Theta_{\mathrm{O}}\middle|\theta_L$ and $\Theta_{\mathrm{I}}\middle|\theta_U$ to be uniformly distributed with PDF

$$f_{\Theta_O}(\theta_j | \theta_L) = \begin{cases} \dfrac{1}{1-\theta_L}, & \theta_L \le \theta_j < 1, \ j = 1,...,k \\ 0 & , \text{otherwise} \end{cases} \tag{38}$$

$$f_{\Theta_I}(\theta_j | \theta_U) = \begin{cases} \dfrac{1}{\theta_U - 1}, & 1 < \theta_j \le \theta_U, \ j = k+1,...,n \\ 0 & , \text{otherwise} \end{cases} \tag{39}$$

and joint PDF

$$f_{\Theta_O}(\Theta_O | \theta_L) \overset{\text{iid}}{=} \prod_{j=1}^{k} f_{\Theta_O}(\theta_j | \theta_L) = \frac{1}{(1-\theta_L)^k} \tag{40}$$

$$f_{\Theta_I}(\Theta_I | \theta_U) \overset{\text{iid}}{=} \prod_{j=k+1}^{n} f_{\Theta_I}(\theta_j | \theta_U) = \frac{1}{(\theta_U - 1)^{n-k}} \tag{41}$$

The parameter θ_L could be beta-distributed (see Appendix 4) with parameters $\gamma > 0$ and $\delta > 0$.

Assuming the unbiased estimator of θ_L is equivalent to the expected value of the prior beta distribution of θ_L (i.e. $\dfrac{\gamma}{\gamma+\delta}$), we find that δ is an expression of γ

$$\delta = \frac{(1-\tilde{\theta}_L)\gamma}{\tilde{\theta}_L} \tag{42}$$

and the prior beta distribution is reduced to the single parameter γ.

The vector Θ_O has a joint PDF

$$f_{\Theta_O}(\Theta_O) = \int_0^1 f_{\Theta_O|\theta_L}(\Theta_O | \theta_L) f_{\theta_L}(\theta_L) \mathrm{d}\theta_L = \frac{\mathrm{B}(\gamma, \delta - k)}{\mathrm{B}(\gamma, \delta)} \tag{43}$$

where $\delta > k$ (beta function is defined in Appendix 4).

Then, the parameter γ is lower bounded $\gamma > k \dfrac{\tilde{\theta}_L}{1-\tilde{\theta}_L}$ (44)

With respect to Θ_I, the parameter θ_U is assumed to be shifted gamma-distributed (see Appendix 5) with parameters $\alpha > 0$ and $\beta > 0$.

Assuming the unbiased estimator is identical to the expected value $\alpha\beta+1$ of the prior shifted gamma-distribution of θ_U, then the parameter β is an expression of α

$$\beta = \frac{\tilde{\theta}_U - 1}{\alpha} \tag{45}$$

Hence, the prior shifted gamma-distribution is reduced to the single parameter α.

The vector Θ_I has a joint PDF

$$f_{\Theta_\mathrm{I}}(\Theta_\mathrm{I}) = \int_1^\infty f_{\Theta_\mathrm{I}|\theta_U}(\Theta_\mathrm{I}|\theta_U) f_{\theta_U}(\theta_U)\mathrm{d}\theta_U = \frac{\Gamma(\alpha-n+k)}{\Gamma(\alpha)} \tag{46}$$

where $\alpha > n-k$.

Then, the parameter β is upper bounded $\beta < \dfrac{\tilde{\theta}_U - 1}{n-k}$ (47)

The posterior

The Bayesian PDF of $\theta_L|\Theta_\mathrm{O}$ is $f_{\theta_L|\Theta_\mathrm{O}}(\theta_L|\Theta_\mathrm{O}) = \dfrac{f_{\Theta_\mathrm{O}}(\Theta_\mathrm{O}|\theta_L) f_{\theta_L}(\theta_L)}{f_{\Theta_\mathrm{O}}(\Theta_\mathrm{O})}$ (48)

According to the joint PDF (40), the Bayesian PDF (48) becomes

$$\frac{1}{\mathrm{B}(\gamma,\delta-k)}\theta_L^{\gamma-1}(1-\theta_L)^{(\delta-k)-1} \tag{49}$$

which refers to a posterior beta distribution with parameters γ and $\delta-k$.

The posterior beta distribution shifts the corresponding prior beta distribution, with parameters γ and δ, to the right, which is justified by the following expected values

$$E_k\{\theta_L|\Theta_\mathrm{O}\} > E_k\{\theta_L\} \quad \text{as} \quad \frac{\gamma}{\gamma+\delta-k} > \frac{\gamma}{\gamma+\delta} \tag{50}$$

Emphasizing Θ_I, the Bayesian PDF of $\theta_U|\Theta_\mathrm{I}$ is

$$f_{\theta_U|\Theta_I}(\theta_U|\Theta_I) = \frac{f_{\Theta_I}(\Theta_I|\theta_U) f_{\theta_U}(\theta_U)}{f_{\Theta_I}(\Theta_I)} = \frac{(\theta_U - 1)^{\alpha-n+k-1} \exp\left(-\frac{\theta_U - 1}{\beta}\right)}{\Gamma(\alpha - n + k)\beta^{\alpha-n+k}} \quad (51)$$

which refers to a posterior shifted gamma distribution with parameters $\alpha - n + k > 0$ and $\beta > 0$.

The posterior shifted gamma distribution shifts the corresponding prior distribution to the left, which justifies the underestimation of the DEA efficiencies that are greater than one. This underestimation is explained as follows

$$E_{n-k}\{\theta_U|\Theta_I\} < E_{n-k}\{\theta_U\} \text{ as } (\alpha - n + k)\beta + 1 < \alpha\beta + 1 \quad (52)$$

2.4 Bias correction

Let a correction parameter $\xi_L = \tilde{\theta}_L / \hat{\theta}_L$ where $\xi_L < 1$ (53)

Elaborating on expressions (42), (44) and (53), we estimate parameters γ and δ as follows

$$\hat{\gamma} = k\tilde{\theta}_L / (1 - \xi_L) \quad (54)$$

and $\hat{\delta} = \dfrac{(1 - \tilde{\theta}_L)\hat{\gamma}}{\tilde{\theta}_L}$ (55)

Two random data sets of size k are generated, where k expresses the number of DMUs assigned efficiencies (e.g. obtained from DEA program (1)) lower than unity. The first randomly generated data set is drawn from a prior beta distribution of θ_L with parameters $\hat{\gamma}$ and $\hat{\delta}$ (see expressions (54) and (55)). The second randomly generated data set is drawn from a posterior beta distribution $\theta_L|\Theta_O$ with parameters $\hat{\gamma}$ and $\hat{\delta} - k$. The prior to posterior distribution $\Psi_L = \dfrac{\theta_L}{\theta_L|\Theta_O}$ is fitted by a gamma distribution with parameters $z_O > 0$ and $e_O > 0$. The maximum likelihood estimates of these two parameters (i.e. $\hat{z}_O$ and $\hat{e}_O$) are obtained from the MATLAB function *gamfit*. The goodness-of-fit is calculated using the Wilcoxon rank sum test for equal medians.

The corrected estimator is

$$\vartheta_p = \theta_p Gamma(\hat{z}_{\mathrm{O}}, \hat{e}_{\mathrm{O}}),\; p = 1,...,k \;\; (p \subset j) \quad (56)$$

with confidence interval $\bar{\vartheta}_p - \frac{s(\vartheta_p)}{\sqrt{w}}\mathrm{cv} \le \vartheta_p \le \bar{\vartheta}_p + \frac{s(\vartheta_p)}{\sqrt{w}}\mathrm{cv}$ (57)

where w expresses the number of Monte Carlo iterations and cv denotes the critical value of t-distribution with $w-1$ degrees of freedom.

The Monte Carlo simulated mean and standard deviation are defined as follows

$$\bar{\vartheta}_p = w^{-1}\sum_{l=1}^{w}\vartheta_{p,l} \quad (58)$$

and $s(\vartheta_p) = \sqrt{(w-1)^{-1}\sum_{l=1}^{w}(\vartheta_{p,l} - \bar{\vartheta}_p)^2}$ (59)

With respect to the DEA estimators that are greater than unity, we define a correction parameter $\xi_U = \tilde{\theta}_U / \hat{\theta}_U$ that satisfies $\xi_U > 1$ (60)

We already know that $\alpha > n-k$ (46) and $\beta = \frac{\tilde{\theta}_u - 1}{\alpha}$ (45). To estimate the two parameters of the shifted gamma distribution we introduce $\frac{(\tilde{\theta}_U - 1)}{(\tilde{\theta}_U - \hat{\theta}_U)} > 1$ in (46), which leads to $\hat{\alpha} = \frac{(n-k)(\tilde{\theta}_U - 1)}{(\tilde{\theta}_U - \hat{\theta}_U)}$ (61) and $\hat{\beta} = \frac{\tilde{\theta}_U - 1}{\hat{\alpha}}$ (62).

Similar to the correction process followed for the DEA estimators lying within the interval (0,1), for the estimators exceeding one, we generate two random data sets for both the prior shifted gamma distribution with parameters $\hat{\alpha}$ and $\hat{\beta}$, and the posterior shifted gamma distribution with parameters $\hat{\alpha} - n + k$ and $\hat{\beta}$. The prior to posterior distribution $\Psi_U = \frac{\theta_U}{\theta_U | \Theta_{\mathrm{I}}}$ is fitted by a gamma distribution with parameters $z_{\mathrm{I}} > 0$ and $e_{\mathrm{I}} > 0$. The maximum likelihood estimates of these two parameters (i.e. $\hat{z}_{\mathrm{I}}$ and $\hat{e}_{\mathrm{I}}$) are obtained from the MATLAB function *gamfit*. Like above, the goodness-of-fit is calculated using the Wilcoxon rank sum test for equal medians.

The corrected estimator is

$$\vartheta_q = \theta_q Gamma(\hat{z}_1, \hat{e}_1)\,,\ q = 1,\ldots,n-k \quad (q \subset j) \tag{63}$$

with confidence interval $\bar{\vartheta}_q - \frac{s(\vartheta_q)}{\sqrt{w}}\text{cv} \le \vartheta_q \le \bar{\vartheta}_q + \frac{s(\vartheta_q)}{\sqrt{w}}\text{cv}$ (64)

where $\bar{\vartheta}_q = w^{-1}\sum_{l=1}^{w}\vartheta_{q,l}$ and $s(\vartheta_q) = \sqrt{(w-1)^{-1}\sum_{l=1}^{w}(\vartheta_{q,l} - \bar{\vartheta}_q)^2}$.

3. Application to E.U. banks

3.1 Data set and selection of variables

In this study, we used two data sets to test the performance of the Bayesian DEA method in correcting bias of DEA estimators. The first data set consists of 50 banks while the second one is expanded to 100 banks. In practice, the second data set includes 50 new banks in addition to those of the first data set. Both data sets include three inputs (i.e. (a) Deposits & Short-term funding; (b) Equity; (c) Overheads) and two outputs (i.e. (a) Other operating income; (b) Total earning assets). The data come from Orbis Bank Focus (the two data sets are available in the online version of this article; see Table E1).

The size of both samples is considered adequate for the dimension of the input-output space. The sample size (e.g. 50 banks) satisfies the 'rule of thumb' put forth by Cooper et al. (2007): $n \ge \max\{xy,\ 3(x+y)\}$, where n stands for the number of firms and x and y are the number of inputs and output, respectively. However, the DEA efficiency estimators assigned to the banks are expected to be biased as the samples of 50 and 100 firms are regarded as small and medium (Banker et al., 2010).

Descriptive statistics of the two-real-world data sets we used in this study are presented in Table 1.

Table 1. Descriptive statistics

Descriptive Statistics	Input 1 Deposits & Short-term funding (th. USD)	Input2 Equity (th. USD)	Input 3 Overheads (th. USD)	Output 1 Other operating income (th. USD)	Output 2 Total earning assets (th. USD)
			Data set #1		
Min	70,297,253.75	3,206,983.76	441,606.19	39,519.81	78,756,562.06
Max	1,123,647,327.88	107,512,396.64	35,575,451.73	22,855,080.28	1,863,483,249.28
St. Deviation	269,719,167.56	27,256,863.52	8,724,429.52	5,444,032.81	448,968,188.58
N	50				
			Data set #2		
Min	19,990,191.12	489,577.57	28,905.31	24,895.92	19,748,600.00
Max	1,123,647,327.88	107,512,396.64	35,575,451.73	22,855,080.28	1,863,483,249.28
St. Deviation	236,768,787.66	23,063,500.59	7,146,079.09	4,406,831.52	384,593,187.45
N	100				

3.2 Empirical results

The empirical results of the first data set, consisting of 50 banks, are reported in Table 2. The actual efficiencies refer to the results obtained from the traditional super-efficiency model of Andersen and Petersen (1993) and Chen and Liang (2011)'s super-efficiency model (see model (1)). The bias-corrected super-efficiency estimators yielded by our Bayesian DEA approach are presented on the right side column of the actual efficiencies followed by the 95% Monte Carlo confidence intervals of the bias-corrected estimators and the significance of the bias correction process (*p*-value).

According to Table 2, the Bayesian DEA approach presented in Section 2 yields reduced estimators for actual efficiencies lower than one and increased estimators for actual efficiencies exceeding one. The mean bias of efficiency estimators below one is higher (i.e. mean bias: -0.0533; min bias: -0.0325 and max bias: -0.0723) than the bias of the estimators above one (i.e. mean bias: 0.0209; min bias: 0.0137 and max bias: 0.0363). Moreover, in the case of estimators below one, the bias is higher for those getting closer to unity while lower for the estimators with higher deviation from one. The opposite applies to the super-efficiency estimators as the bias becomes lower when the estimator approaches one and higher when it moves far from one. All bias-corrected estimators are statistically significant (p-value <0.001).

In the case of the sample of 50 banks, the mean square error (MSE), root mean square error (RMSE) and mean absolute error (MAE) of the Bayesian DEA estimators are 0.0021, 0.0453 and 0.0416, respectively.

Table 2. Empirical results

#	Banks	Actual efficiencies		Bias-corrected estimators	95% Confidence interval		*p*-value
		Traditional SE	Chen & Liang (2011) SE	Bayesian DEA	Lower	Upper	
1	BNP Paribas	Infeasible	2.2909	2.3210	2.3078	2.3342	<0.001
2	Banco Santander SA	0.7417	0.7417	0.6834	0.6743	0.6925	<0.001
3	Barclays Bank Plc	1.0130	1.0130	1.0268	1.0201	1.0334	<0.001
4	ING Bank NV	1.3874	1.3874	1.4070	1.3980	1.4159	<0.001
5	Lloyds Bank Plc	0.8489	0.8489	0.7888	0.7797	0.7980	<0.001
6	Deutsche Bank AG	1.2194	1.2194	1.2475	1.2403	1.2547	<0.001
7	Société Générale SA	1.2382	1.2382	1.2555	1.2476	1.2633	<0.001
8	Royal Bank of Scotland Plc (The)	0.8295	0.8295	0.7733	0.7624	0.7843	<0.001
9	UniCredit SpA	0.7808	0.7808	0.7282	0.7192	0.7371	<0.001
10	HSBC Bank plc	0.9568	0.9568	0.8943	0.8832	0.9054	<0.001
11	Banco Bilbao Vizcaya Argentaria SA-BBVA	0.7322	0.7322	0.6839	0.6739	0.6939	<0.001
12	Standard Chartered Bank	0.5999	0.5999	0.5557	0.5485	0.5629	<0.001
13	Bank of Scotland Plc	0.7334	0.7334	0.6826	0.6736	0.6917	<0.001
14	Credit Mutuel (Combined - IFRS)	1.1112	1.1112	1.1329	1.1254	1.1405	<0.001
15	Commerzbank AG	0.6867	0.6867	0.6500	0.6420	0.6581	<0.001
16	National Westminster Bank Plc - NatWest	0.5776	0.5776	0.5394	0.5320	0.5468	<0.001
17	Intesa Sanpaolo	1.0047	1.0047	1.0207	1.0136	1.0278	<0.001
18	ABN AMRO Bank NV	0.6262	0.6262	0.5865	0.5793	0.5937	<0.001
19	Credit Agricole Corporate and Investment Bank SA-Credit Agricole CIB	1.4231	1.4231	1.4392	1.4298	1.4486	<0.001
20	Natixis SA	1.1560	1.1560	1.1787	1.1706	1.1867	<0.001
21	Caixabank, S.A.	0.7899	0.7899	0.7291	0.7193	0.7390	<0.001
22	Crédit Industriel et Commercial SA - CIC	0.8282	0.8282	0.7786	0.7696	0.7877	<0.001
23	Banque Fédérative du Crédit Mutuel	1.0571	1.0571	1.0759	1.0691	1.0827	<0.001
24	Danske Bank A/S	1.0505	1.0505	1.0711	1.0637	1.0784	<0.001
25	La Banque Postale	0.9979	0.9979	0.9347	0.9250	0.9445	<0.001
26	UniCredit Bank AG	0.6974	0.6974	0.6513	0.6424	0.6603	<0.001
27	KBC Bank NV	0.6686	0.6686	0.6283	0.6199	0.6367	<0.001
28	Svenska Handelsbanken	0.5994	0.5994	0.5555	0.5490	0.5621	<0.001

29	Banco de Sabadell SA	0.8004	0.8004	0.7395	0.7303	0.7487	<0.001
30	Bankia, SA	0.5714	0.5714	0.5388	0.5319	0.5458	<0.001
31	UniCredit Bank Austria AG-Bank Austria	0.7621	0.7621	0.7053	0.6954	0.7151	<0.001
32	Deutsche Postbank AG	0.7525	0.7525	0.7003	0.6919	0.7088	<0.001
33	ING-DiBa AG	0.5462	0.5462	0.5057	0.4987	0.5128	<0.001
34	Skandinaviska Enskilda Banken AB	1.0449	1.0449	1.0627	1.0556	1.0699	<0.001
35	Banco Popular Espanol SA	0.6505	0.6505	0.6097	0.6017	0.6178	<0.001
36	Le Crédit Lyonnais (LCL) SA	0.8824	0.8824	0.8131	0.8023	0.8239	<0.001
37	ING Belgium SA/NV-ING	0.7142	0.7142	0.6624	0.6545	0.6704	<0.001
38	Banca Monte dei Paschi di Siena SpA-Gruppo Monte dei Paschi di Siena	1.1716	1.1716	1.1977	1.1896	1.2058	<0.001
39	Deutsche Bank Privat-und Geschaftskunden AG	1.5377	1.5377	1.5564	1.5454	1.5675	<0.001
40	Banco BPM SPA	0.9995	0.9995	0.9390	0.9267	0.9512	<0.001
41	Belfius Banque SA/NV-Belfius Bank SA/NV	0.8518	0.8518	0.7938	0.7833	0.8043	<0.001
42	Raiffeisen Bank International AG	0.9183	0.9183	0.8549	0.8444	0.8655	<0.001
43	Bank Austria Creditanstalt AG	0.9260	0.9260	0.8624	0.8511	0.8737	<0.001
44	Dexia Crédit Local SA	2.4948	2.4948	2.5311	2.5143	2.5479	<0.001
45	Caixa Geral de Depositos	1.1182	1.1182	1.1368	1.1305	1.1432	<0.001
46	Allied Irish Banks plc	0.9073	0.9073	0.8350	0.8229	0.8471	<0.001
47	Piraeus Bank SA	0.9216	0.9216	0.8615	0.8509	0.8721	<0.001
48	Abbey National Treasury Services Plc	1.1857	1.1857	1.2032	1.1955	1.2109	<0.001
49	National Bank of Greece SA	0.9647	0.9647	0.8942	0.8826	0.9059	<0.001
50	Deutsche Kreditbank AG (DKB)	1.2066	1.2066	1.2235	1.2147	1.2323	<0.001
	MSE			0.0021			
	RMSE			0.0453			
	MAE			0.0416			

In the case of the extended sample of 100 banks (Table A1 in Appendix 6), the MSE, RMSE and MAE of the Bayesian DEA estimators are 0.0013, 0.0355 and 0.0321, respectively. All three measures are lower than the corresponding measures referring to the sample of 50 banks (Table 2). It should be noted that this Bayesian DEA approach is appropriate for small and medium samples where the estimators, lower and greater than unity, are dependent. Based on expression (37) and Figure 2, it is straightforward that the unbiased estimators are asymptotically uncorrelated.

4. Concluding remarks and future research

The purpose of this study was to develop a Bayesian DEA approach for correcting bias of super-efficiency estimators. The new method uses consistent estimators to the unknown efficiency parameters to correct the bias of the actual efficiencies. The assumptions made for the development of the Bayesian DEA method are regarded as realistic.

The new method draws on Chen and Liang (2011)'s super-efficiency model. However, other super-efficiency models that tackle the infeasibility problem could be used instead. The new Bayesian DEA method is appropriate for small and medium data sets where dependence among the estimators below and above one is present. The empirical results showed a decrease in MSE, RMSE and MAE when the sample size increases. In addition, the range of the 95% Monte Carlo confidence intervals of the estimators decreases while the sample size becomes larger. The empirical analysis presented in this study was based on two real-world data sets of 50 and 100 firms coming from the E.U. banking sector. The two data sets included three inputs and two outputs.

Further research is needed to test the performance of the new Bayesian DEA bias-corrected estimators when real-world data sets of different dimensions and scales than those used in this study are employed. The Bayesian DEA method presented in this work should be modified to become appropriate for large samples as well. Furthermore, the application of outlier-detection methods in conjunction with the new Bayesian DEA method would prevent distortion of the bias-corrected estimators - especially in cases where observed efficiency estimators are significantly higher than one. Addressing these limitations would improve the performance, applicability and generalizability of the new method.

Appendices

Appendix 1.1

The expected value of $\hat{\theta}_L$ is as follows

$$E_n\left\{\hat{\theta}_L\right\} = \int_{\theta_L}^{\theta_U} t \cdot f_{\hat{\theta}_L}(t|\theta_L,\theta_U)\mathrm{d}t = \int_{\theta_L}^{\theta_U} t\frac{n}{(\theta_U-\theta_L)^n}(\theta_U-t)^{n-1}\mathrm{d}t = \frac{1}{(\theta_U-\theta_L)^n}\int_{\theta_L}^{\theta_U} t\frac{\mathrm{d}}{\mathrm{d}t}(-(\theta_U-t)^n)\mathrm{d}t$$

$$= \frac{t(-(\theta_U-t)^n)}{(\theta_U-\theta_L)^n}\Bigg|_{\theta_L}^{\theta_U} + \frac{1}{(\theta_U-\theta_L)^n}\int_{\theta_L}^{\theta_U}(\theta_U-t)^n\mathrm{d}t = \theta_L + \frac{1}{(\theta_U-\theta_L)^n}\int_{\theta_L}^{\theta_U}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{-(\theta_U-t)^{n+1}}{n+1}\right)\mathrm{d}t$$

$$= \theta_L + \frac{1}{(\theta_U-\theta_L)^n}\cdot\frac{-(\theta_U-t)^{n+1}}{n+1}\Bigg|_{\theta_L}^{\theta_U} = \theta_L + \frac{\theta_U-\theta_L}{n+1}$$

$$= \theta_L\frac{n}{n+1} + \theta_U\frac{1}{n+1} < 1$$

Appendix 1.2

The second moment of $\hat{\theta}_L$ is defined as follows

$$E_n\left\{\hat{\theta}_L^2\right\} = \int_{\theta_L}^{\theta_U} t^2 \cdot f_{\hat{\theta}_L}(t|\theta_L,\theta_U)\mathrm{d}t = \theta_L^2 + \frac{2(\theta_U-\theta_L)}{n+1}\cdot\int_{\theta_L}^{\theta_U} t\cdot\frac{(n+1)(\theta_U-t)^n}{(\theta_U-\theta_L)^{n+1}}\mathrm{d}t$$

$$= \theta_L^2 + \frac{2(\theta_U-\theta_L)}{n+1}\cdot E_{n+1}\left\{\hat{\theta}_L\right\} = \theta_L^2 + \frac{2(\theta_U-\theta_L)}{n+1}\cdot\frac{\theta_L(n+1)+\theta_U}{n+2}$$

Appendix 2

The covariance of $\hat{\theta}_L$ and $\hat{\theta}_U$ is defined as follows

$$\mathrm{cov}_n(\hat{\theta}_L,\hat{\theta}_U) = E_n\left\{\hat{\theta}_L\hat{\theta}_U\right\} - E_n\left\{\hat{\theta}_L\right\}E_n\left\{\hat{\theta}_U\right\} \tag{A2.a}$$

where

$$E_n\left\{\hat{\theta}_L\hat{\theta}_U\right\} = \int_{\theta_L}^{\theta_U}\int_{\theta_L}^{s} ts f_{(\hat{\theta}_L,\hat{\theta}_U)}(t,s|\theta_L,\theta_U)\mathrm{d}t\mathrm{d}s = \int_{\theta_L}^{\theta_U}\int_{\theta_L}^{s} ts\frac{n(n-1)(s-t)^{n-2}}{(\theta_U-\theta_L)^n}\mathrm{d}t\mathrm{d}s$$

$$= \frac{n}{(\theta_U-\theta_L)^n}\int_{\theta_L}^{\theta_U} s\int_{\theta_L}^{s} t(n-1)(s-t)^{n-2}\mathrm{d}t\mathrm{d}s \tag{A2.b}$$

$$\int_{\theta_L}^{s} t(n-1)(s-t)^{n-2}\mathrm{d}t = \theta_L(s-\theta_L)^{n-1} + \frac{(s-\theta_L)^n}{n} \tag{A2.c}$$

and

$$\int_{\theta_L}^{\theta_U} s\left(\theta_L(s-\theta_L)^{n-1}+\frac{(s-\theta_L)^n}{n}\right)\mathrm{d}s=\frac{\theta_L\theta_U(\theta_U-\theta_L)^n}{n}+\frac{(\theta_U-\theta_L)^{n+2}}{n(n+1)}-\frac{(\theta_U-\theta_L)^{n+2}}{n(n+1)(n+2)}$$

(A2.d)

By replacing (A2.c) and (A2.d) in (A2.b), we obtain

$$E_n\left\{\hat{\theta}_L\hat{\theta}_U\right\}=\frac{n}{(\theta_U-\theta_L)^n}\left(\theta_L\theta_U\frac{(\theta_U-\theta_L)^n}{n}+\frac{(\theta_U-\theta_L)^{n+2}}{n(n+1)}-\frac{(\theta_U-\theta_L)^{n+2}}{n(n+1)(n+2)}\right)$$

$$=\theta_L\theta_U+\frac{(\theta_U-\theta_L)^2}{n+2} \qquad \text{(A2.e)}$$

Introducing expressions (A2.e), $E_n\left\{\hat{\theta}_L\right\}=\theta_L\frac{n}{n+1}+\theta_U\frac{1}{n+1}$ (12), and $E_n\left\{\hat{\theta}_U\right\}=\theta_U\frac{n}{n+1}+\theta_L\frac{1}{n+1}$ (19) in (A2.a) we find

$$\mathrm{cov}_n\left\{\hat{\theta}_L,\hat{\theta}_U\right\}=\theta_L\theta_U+\frac{(\theta_U-\theta_L)^2}{n+2}-\frac{\theta_L\theta_U(n^2+1)+(\theta_L^2+\theta_U^2)n}{(n+1)^2}$$

Appendix 3

The covariance of $\tilde{\theta}_L$ and $\tilde{\theta}_U$ is defined as follows

$$\mathrm{cov}_n(\tilde{\theta}_L,\tilde{\theta}_U)=E_n\left\{\tilde{\theta}_L\tilde{\theta}_U\right\}-E_n\left\{\tilde{\theta}_L\right\}E_n\left\{\tilde{\theta}_U\right\}$$

Based on expressions (26) and (27), we obtain

$$E_n\left\{\tilde{\theta}_L\tilde{\theta}_U\right\}=E_n\left\{\frac{n\hat{\theta}_L-\hat{\theta}_U}{n-1}\cdot\frac{n\hat{\theta}_U-\hat{\theta}_L}{n-1}\right\}=\frac{1}{(n-1)^2}E_n\left\{n^2\hat{\theta}_L\hat{\theta}_U-n\hat{\theta}_L^2-n\hat{\theta}_U^2+\hat{\theta}_U\hat{\theta}_L\right\}$$

$$=\frac{1}{(n-1)^2}\left\{(n^2+1)E_n(\hat{\theta}_L\hat{\theta}_U)-nE_n(\hat{\theta}_L^2)-nE_n(\hat{\theta}_U^2)\right\} \qquad \text{(A3.a)}$$

given expressions (14), (21) and (A2.e), (A3.a) is rewritten as follows

$$=\frac{n^2+1}{(n-1)^2}\left\{\theta_L\theta_U+\frac{(\theta_U-\theta_L)^2}{n+2}\right\}-\frac{n}{(n-1)^2}\left\{\theta_L^2+\frac{2(\theta_U-\theta_L)}{n+1}\cdot\frac{\theta_L(n+1)+\theta_U}{n+2}\right\}$$

$$-\frac{n}{(n-1)^2}\left\{\theta_U^2+\frac{2(\theta_U-\theta_L)}{n+1}\cdot\frac{\theta_U(n+1)+\theta_L}{n+2}\right\}$$

$$=\frac{n^2+1}{(n-1)^2}\left\{\theta_L\theta_U+\frac{(\theta_U-\theta_L)^2}{n+2}\right\}$$

$$-\frac{n}{(n-1)^2}\left\{\theta_L^2+\theta_U^2+\frac{2(\theta_U-\theta_L)}{(n+1)(n+2)}\left(\theta_L(n+1)+\theta_U+\theta_U(n+1)+\theta_L\right)\right\}$$

where

$$\frac{2(\theta_U-\theta_L)}{(n+1)(n+2)}\left(\theta_L(n+1)+\theta_U+\theta_U(n+1)+\theta_L\right)=2\frac{\theta_U^2-\theta_L^2}{n+1}$$

Since the estimators are unbiased we find

$$\operatorname{cov}_n(\tilde{\theta}_L,\tilde{\theta}_U)=E_n\left\{\tilde{\theta}_L\tilde{\theta}_U\right\}-E_n\left\{\tilde{\theta}_L\right\}E_n\left\{\tilde{\theta}_U\right\}$$

$$=\frac{n^2+1}{(n-1)^2}\left(\theta_L\theta_U+\frac{(\theta_U-\theta_L)^2}{n+2}\right)-\frac{n}{(n-1)^2}\left(\theta_L^2+\theta_U^2+\frac{2(\theta_U^2-\theta_L^2)}{(n+1)}\right)-\theta_L\theta_U$$

Appendix 4

The prior beta distribution of θ_L with parameters $\gamma>0$ and $\delta>0$ has a PDF

$$f_{\theta_L}(\theta_L)=\frac{1}{\mathrm{B}(\gamma,\delta)}\theta_L^{\gamma-1}(1-\theta_L)^{\delta-1},\ \ 0<\theta_L<1$$

with the function

$$\mathrm{B}(\gamma,\delta)=\int_0^1\theta_L^{\gamma-1}(1-\theta_L)^{\delta-1}\mathrm{d}\theta_L$$

Appendix 5

Let a random variable v be gamma-distributed with parameters $a>0$ and $\beta>0$. The PDF is

$$Gamma(v|\alpha,\beta)=\begin{cases}\dfrac{v^{\alpha-1}\exp(-v/\beta)}{\Gamma(\alpha)\beta^\alpha}, & v>0\\ 0, & \text{otherwise}\end{cases}$$

where $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}\exp(-t)\mathrm{d}t$

The expected value of v is $E(v)=\alpha\beta$

The shifted random variable $v_s=v+1$ is shifted gamma-distributed with PDF

$$Gamma_{v_s}(v_s|\alpha,\beta)=Gamma(v_s-1|\alpha,\beta)=\begin{cases}\dfrac{(v_s-1)^{\alpha-1}\exp\left\{-(v_s-1)/\beta\right\}}{\Gamma(\alpha)\beta^\alpha}, & v_s>1\\ 0, & \text{otherwise}\end{cases}$$

Its expected value is $E(v_s)=\alpha\beta+1$

Appendix 6

Table A1. Efficiency estimators

#	Banks	Actual efficiencies		Bias-corrected estimators	95% Confidence interval		*p*-value
		Traditional SE	Chen & Liang (2011) SE	Bayesian DEA	Lower	Upper	
1	BNP Paribas	Infeasible	2.2909	2.3060	2.2971	2.3150	<0.001
2	Banco Santander SA	0.7417	0.7417	0.7089	0.7012	0.7165	<0.001
3	Barclays Bank Plc	1.0130	1.0130	1.0227	1.0187	1.0268	<0.001
4	ING Bank NV	1.3874	1.3874	1.3994	1.3941	1.4047	<0.001
5	Lloyds Bank Plc	0.8489	0.8489	0.7972	0.7893	0.8050	<0.001
6	Deutsche Bank AG	1.2194	1.2194	1.2315	1.2267	1.2363	<0.001
7	Société Générale SA	1.2382	1.2382	1.2442	1.2395	1.2490	0.0067
8	Royal Bank of Scotland Plc (The)	0.8295	0.8295	0.7806	0.7730	0.7882	<0.001
9	UniCredit SpA	0.7808	0.7808	0.7332	0.7261	0.7402	<0.001
10	HSBC Bank plc	0.9568	0.9568	0.9135	0.9047	0.9223	<0.001
11	Banco Bilbao Vizcaya Argentaria SA-BBVA	0.7322	0.7322	0.6974	0.6909	0.7039	<0.001
12	Standard Chartered Bank	0.5999	0.5999	0.5730	0.5675	0.5785	<0.001
13	Bank of Scotland Plc	0.7334	0.7334	0.6963	0.6896	0.7031	<0.001
14	Credit Mutuel (Combined - IFRS)	1.1112	1.1112	1.1186	1.1140	1.1232	<0.001
15	Commerzbank AG	0.6867	0.6867	0.6510	0.6449	0.6572	<0.001
16	National Westminster Bank Plc - NatWest	0.5776	0.5776	0.5403	0.5341	0.5464	<0.001
17	Intesa Sanpaolo	1.0047	1.0047	1.0152	1.0113	1.0191	<0.001
18	ABN AMRO Bank NV	0.6262	0.6262	0.5964	0.5900	0.6029	<0.001
19	Credit Agricole Corporate and Investment Bank SA-Credit Agricole CIB	1.4231	1.4231	1.4368	1.4307	1.4429	<0.001
20	Natixis SA	1.1544	1.1544	1.1637	1.1596	1.1679	<0.001
21	Caixabank, S.A.	0.7850	0.7850	0.7377	0.7297	0.7457	<0.001
22	Crédit Industriel et Commercial SA - CIC	0.7893	0.7893	0.7459	0.7387	0.7531	<0.001
23	Banque Fédérative du Crédit Mutuel	1.0556	1.0556	1.0655	1.0615	1.0694	<0.001
24	Danske Bank A/S	1.0505	1.0505	1.0591	1.0549	1.0633	<0.001
25	La Banque Postale	0.9715	0.9715	0.9132	0.9053	0.9210	<0.001
26	UniCredit Bank AG	0.6644	0.6644	0.6306	0.6238	0.6374	<0.001
27	KBC Bank NV	0.6345	0.6345	0.6031	0.5976	0.6086	<0.001
28	Svenska Handelsbanken	0.5939	0.5939	0.5622	0.5568	0.5675	<0.001
29	Banco de Sabadell SA	0.7385	0.7385	0.6985	0.6908	0.7062	<0.001
30	Bankia, SA	0.5377	0.5377	0.5083	0.5039	0.5128	<0.001

31	UniCredit Bank Austria AG-Bank Austria	0.7204	0.7204	0.6805	0.6733	0.6877	<0.001
32	Deutsche Postbank AG	0.6735	0.6735	0.6362	0.6297	0.6426	<0.001
33	ING-DiBa AG	0.4104	0.4104	0.3888	0.3848	0.3929	<0.001
34	Skandinaviska Enskilda Banken AB	0.9987	0.9987	0.9464	0.9363	0.9565	<0.001
35	Banco Popular Espanol SA	0.5486	0.5486	0.5193	0.5140	0.5245	<0.001
36	Le Crédit Lyonnais (LCL) SA	0.7613	0.7613	0.7181	0.7117	0.7245	<0.001
37	ING Belgium SA/NV-ING	0.5325	0.5325	0.5046	0.4994	0.5099	<0.001
38	Banca Monte dei Paschi di Siena SpA-Gruppo Monte dei Paschi di Siena	1.0716	1.0716	1.0872	1.0826	1.0917	<0.001
39	Deutsche Bank Privat-und Geschaftskunden AG	1.0945	1.0945	1.1038	1.0993	1.1083	<0.001
40	Banco BPM SPA	0.9447	0.9447	0.8974	0.8885	0.9063	<0.001
41	Belfius Banque SA/NV-Belfius Bank SA/NV	0.6500	0.6500	0.6160	0.6099	0.6221	<0.001
42	Raiffeisen Bank International AG	0.6788	0.6788	0.6433	0.6365	0.6502	<0.001
43	Bank Austria Creditanstalt AG	0.6849	0.6849	0.6486	0.6422	0.6549	<0.001
44	Dexia Crédit Local SA	2.4948	2.4948	2.5147	2.5060	2.5234	<0.001
45	Caixa Geral de Depositos	0.8878	0.8878	0.8435	0.8355	0.8516	<0.001
46	Allied Irish Banks plc	0.4556	0.4556	0.4308	0.4266	0.4350	<0.001
47	Piraeus Bank SA	0.3890	0.3890	0.3709	0.3671	0.3747	<0.001
48	Abbey National Treasury Services Plc	0.8700	0.8700	0.8252	0.8179	0.8324	<0.001
49	National Bank of Greece SA	0.4045	0.4045	0.3860	0.3821	0.3898	<0.001
50	Deutsche Kreditbank AG (DKB)	0.5329	0.5329	0.5078	0.5028	0.5129	<0.001
51	Eurobank Ergasias SA	0.4360	0.4360	0.4141	0.4100	0.4181	<0.001
52	Banca Nazionale del Lavoro SpA-BNL	0.5846	0.5846	0.5556	0.5499	0.5612	<0.001
53	HSBC France SA	1.0171	1.0171	1.0253	1.0213	1.0294	<0.001
54	Banco Comercial Português, SA-Millennium bcp	0.9004	0.9004	0.8517	0.8435	0.8598	<0.001
55	Alpha Bank AE	0.4538	0.4538	0.4322	0.4278	0.4367	<0.001
56	SNS Bank N.V.	0.5847	0.5847	0.5487	0.5430	0.5544	<0.001
57	Nykredit Realkredit A/S	1.3530	1.3530	1.3625	1.3570	1.3679	<0.001
58	CACEIS Bank Luxembourg	0.9121	0.9121	0.8682	0.8590	0.8773	<0.001
59	Crédit du Nord SA	0.7184	0.7184	0.6752	0.6685	0.6819	<0.001
60	Ibercaja Banco SAU	1.4488	1.4488	1.4594	1.4534	1.4653	<0.001
61	Kutxabank SA	0.5622	0.5622	0.5332	0.5278	0.5386	<0.001
62	Novo Banco	0.6055	0.6055	0.5708	0.5654	0.5762	<0.001
63	Abanca Corporacion Bancaria SA	0.8723	0.8723	0.8213	0.8133	0.8294	<0.001
64	Clydesdale Bank Plc	0.5678	0.5678	0.5367	0.5313	0.5421	<0.001
65	Bankinter SA	0.6786	0.6786	0.6423	0.6360	0.6486	<0.001
66	Ulster Bank Limited	0.6602	0.6602	0.6238	0.6175	0.6300	<0.001
67	Virgin Money Plc	0.6979	0.6979	0.6578	0.6509	0.6647	<0.001
68	TSB Bank Plc	0.6466	0.6466	0.6122	0.6061	0.6184	<0.001

69	Bank of New York Mellon SA/NV	0.7982	0.7982	0.7551	0.7468	0.7634	<0.001
70	Co-operative Bank Plc (The)	0.6823	0.6823	0.6405	0.6337	0.6474	<0.001
71	BGL BNP Paribas	0.6587	0.6587	0.6214	0.6152	0.6276	<0.001
72	Mediobanca SpA-MEDIOBANCA - Banca di Credito Finanziario Società per Azioni	0.9987	0.9987	0.9464	0.9363	0.9565	<0.001
73	Bank Polska Kasa Opieki SA-Bank Pekao SA	0.6734	0.6734	0.6409	0.6350	0.6469	<0.001
74	Lyonnaise de Banque SA	0.9797	0.9797	0.9293	0.9204	0.9381	<0.001
75	OP Corporate Bank plc	1.3976	1.3976	1.4046	1.4001	1.4092	0.0015
76	Jyske Bank A/S (Group)	0.9055	0.9055	0.8552	0.8472	0.8632	<0.001
77	Ceska Sporitelna a.s.	0.6957	0.6957	0.6592	0.6522	0.6662	<0.001
78	OTP Bank Plc	1.1127	1.1127	1.1203	1.1158	1.1248	<0.001
79	Bank für Arbeit und Wirtschaft und Österreichische Postsparkasse Aktiengesellschaft-BAWAG P.S.K. AG	0.7754	0.7754	0.7301	0.7229	0.7372	<0.001
80	Komercni Banka	0.7662	0.7662	0.7286	0.7217	0.7354	<0.001
81	Banque CIC Est SA	1.0218	1.0218	1.0314	1.0270	1.0358	<0.001
82	SEB AG	0.8577	0.8577	0.8094	0.8014	0.8174	<0.001
83	Banco di Napoli SpA	0.8153	0.8153	0.7719	0.7642	0.7795	<0.001
84	Bank Zachodni WBK S.A.	0.8070	0.8070	0.7577	0.7496	0.7658	<0.001
85	Permanent TSB Plc	0.8130	0.8130	0.7669	0.7598	0.7740	<0.001
86	Sumitomo Mitsui Banking Corporation Europe Limited-SMBCE	0.8248	0.8248	0.7847	0.7768	0.7926	<0.001
87	Ceskoslovenska Obchodni Banka A.S.- CSOB	0.8442	0.8442	0.7966	0.7879	0.8053	<0.001
88	Credito Emiliano SpA-CREDEM	0.8893	0.8893	0.8369	0.8287	0.8450	<0.001
89	KfW Ipex-Bank Gmbh	0.9140	0.9140	0.8631	0.8553	0.8709	<0.001
90	Banca Mediolanum SpA	1.2666	1.2666	1.2777	1.2729	1.2825	<0.001
91	mBank SA	0.8761	0.8761	0.8313	0.8224	0.8401	<0.001
92	Danske Bank Plc	0.9320	0.9320	0.8864	0.8774	0.8953	<0.001
93	Citibank International Limited	0.9092	0.9092	0.8498	0.8419	0.8577	<0.001
94	Banco Cooperativo Espanol	2.4548	2.4548	2.4821	2.4733	2.4910	<0.001
95	CIC Ouest SA	1.1469	1.1469	1.1559	1.1507	1.1611	<0.001
96	Erste Bank der Oesterreichischen Sparkassen AG	0.9982	0.9982	0.9509	0.9425	0.9592	<0.001
97	Montepio Investimento SA	1.0670	1.0670	1.0732	1.0686	1.0777	0.0041
98	Bank of Cyprus Public Company Limited-Bank of Cyprus Group	0.9973	0.9973	0.9410	0.9314	0.9507	<0.001
99	ABH Financial Limited	1.2521	1.2521	1.2654	1.2607	1.2701	<0.001
100	Banca Carige SpA	1.0916	1.0916	1.1000	1.0959	1.1041	<0.001
	MSE			0.0013			
	RMSE			0.0355			
	MAE			0.0321			

References

Andersen, P. & Petersen, N. C. (1993). A procedure for ranking efficient units in data envelopment analysis. *Management Science, 39*, 1261-1264.

Banker, R. D. (1993). Maximum likelihood, consistency and data envelopment analysis: A statistical foundation. *Management Science, 39*, 1265-1273.

Banker, R. D., & Chang, H. (2006). The super-efficiency procedure for outlier identification, not for ranking efficient units. *European Journal of Operational Research, 175*, 1311-1320.

Banker, R. D., & Gifford, J. L. (1988). *A relative efficiency model for the evaluation public health nurse productivity*. Pittsburgh: Mimeo, Carnegie Mellon University.

Banker, R. D., & Maindiratta, A. (1992). Maximum likelihood estimation of monotone and concave production frontiers. *Journal of Productivity Analysis, 3*, 401-415.

Banker, R. D., & Natarajan, R. (2008). Evaluating contextual variables affecting productivity using data envelopment analysis. *Operations Research, 56*, 48-58.

Banker, R. D., Chang, H., & Zhiqiang, Z. (2017). On the use of super-efficiency procedures for ranking efficient units and identifying outliers. *Annals of Operations Research, 250*, 21-35.

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science, 30*, 1078-1092.

Banker, R. D., Zheng, Z., & Natarajan, R. (2010). DEA-based hypothesis tests for comparing two groups of decision making units. *European Journal of Operational Research, 206*, 231-238.

Charnes, A., & Cooper, W. W. (1963). Deterministic equivalents for optimizing and satisficing under chance constraints. *Operations Research, 11*, 18–39.

Charnes, A., Cooper, W.W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research, 2*, 429-444.

Chen, J. X., Deng, M., & Gingras, S. (2011). A modified super-efficiency measure based on simultaneous input–output projection in data envelopment analysis. *Computers & Operations Research, 38*, 496-504.

Chen, Y. (2005). Measuring super-efficiency in DEA in the presence of infeasibility. *European Journal of Operational Research, 161*, 545-551.

Chen, Y., & Liang, L. (2011). Super-efficiency DEA in the presence of infeasibility: One model approach. *European Journal of Operational Research, 213*, 359-360.

Cook, W. D., Liang, L., Zha, Y., & Zhu, J. (2009). A modified super-efficiency DEA model for infeasibility. *Journal of the Operational Research Society, 60*, 276-281.

Cooper, W. W., Seiford, L. M., & Tone, K. (2007). *Data envelopment analysis: A comprehensive text with models*. (2nd ed.). New York: Springer Science + Business Media.

Kneip, A., Simar, L., & Wilson, P. W. (2008). Asymptotics and consistent bootstraps for DEA estimators in nonparametric frontier models. *Econometric Theory, 24*, 1663-1697.

Kneip, A., Simar, L., & Wilson, P. W. (2011). A computationally efficient, consistent bootstrap for inference with non-parametric DEA estimators. *Computational Economics, 38*, 483-515.

Kuosmanen, T., & Johnson, A. L. (2010). Data envelopment analysis as nonparametric least-squares regression. *Operations Research, 58*, 149-160.

Kuosmanen, T., & Kortelainen, M. (2007). *Stochastic nonparametric envelopment of data: Cross-sectional frontier estimation subject to shape constraints*. Economics discussion paper 46, University of Joensuu, Joensuu, Finland.

Kuosmanen, T., & Kortelainen, M. (2012). Stochastic non-smooth envelopment of data: Semi-parametric frontier estimation subject to shape constraints. *Journal of Productivity Analysis, 38*, 11-28.

Land, K. C., Lovell, C. A. K., & Thore, S. (1993). Chance-constraint data envelopment analysis. *Managerial and Decision Economics, 14*, 541–554.

Lee, H. S., Chu, C. W., & Zhu, J. (2011). Super-efficiency DEA in the presence of infeasibility. *European Journal of Operational Research, 212*, 141-147.

Li, S., Jahanshahloo, G. R., & Khodabakhshi, M. (2007). A super-efficiency model for ranking efficient units in data envelopment analysis. *Applied Mathematics and Computation, 184*, 638-648.

Lovell, C. A. K., & Rouse, A. P. B. (2003). Equivalent standard DEA models to provide superefficiency scores. *Journal of the Operational Research Society, 54*, 101-108.

Olesen, O. B., & Petersen, N. C. (1995). Chance-constraint efficiency evaluation. *Management Science, 41*, 442–57.

Olesen, O. B., & Petersen, N. C. (2016). Stochastic data envelopment analysis – A review. *European Journal of Operational Research, 251*, 2-21.

Ray, S. C. (2008). The directional distance function and measurement of super-efficiency: An application to airlines data. *Journal of the Operational Research Society, 59*, 788-797.

Sarath, B., & Maindiratta, A. (1997). On the consistency of maximum likelihood estimation of monotone and concave production frontiers. *Journal of Productivity Analysis, 8*, 239-246.

Simar, L. (2007). How to improve the performances of DEA/FDH estimators in the presence of noise? *Journal of Productivity Analysis, 28*, 183-201.

Simar, L., & Wilson, P. W. (1998). Sensitivity analysis of efficiency scores: How to bootstrap in nonparametric frontier models. *Management Science, 44*, 49-61.

Simar, L., & Wilson, P. W. (1999). Estimating and bootstrapping Malmquist indices. *European Journal of Operational Research, 115*, 459-471.

Simar, L., & Wilson, P. W. (2015). Statistical approaches for non-parametric frontier models: A guided tour. *International Statistical Review, 83*, 77-110.

Tsionas, E. G., & Papadakis, E. N. (2010). A Bayesian approach to statistical inference in stochastic DEA. *Omega, 38*, 309-314.