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Abstract

We study a Stackelberg type symmetric dynamic multi-players zero-sum game. One player is the leader and other players are followers. All players have symmetric payoff functions. The game is a two-stages game. In the first stage the leader determines the value of its strategic variable. In the second stage the followers determine the values of their strategic variables given the value of the leader's strategic variable. On the other hand, in the static game all players simultaneously determine the values of their strategic variables. We show that if and only if the game is fully symmetric, the Stackelberg equilibrium and the static equilibrium are equivalent.

Keywords: Stackelberg equilibrium, static equilibrium, multi-players zero-sum game.

1 Introduction

This paper investigates the relation between the equilibrium of the Stackelberg type dynamic game and the equilibrium of the static game in a multi-players zero-sum game, and show that if and only if the game is fully symmetric, the equilibrium of the Stackelberg type dynamic game and the equilibrium of the static game are equivalent. In a two-person zero-sum game the equilibrium of the Stackelberg type dynamic game and the equilibrium of the static game are equivalent[1]. WE extend this analysis to a general multi-players zero-sum game.

In the next section, using a model of relative profit maximization in an oligopoly with four firms, we show that the Stackelberg equilibrium is not equivalent to the static (Cournot) equilibrium in the following cases which are not fully symmetric.

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[1]Please see Korzhyk et. al. (2014), Ponssard and Zamir (1973), Tanaka (2014) and Yin et. al. (2010).

1. All firms are asymmetric, that is, they have different cost functions.
2. Two followers are symmetric, that is, they have the same cost functions.
3. Three followers are symmetric.
4. The leader and one follower are symmetric.
5. The leader and two followers are symmetric.

The Stackelberg equilibrium is equivalent to the static (Cournot) equilibrium if and only if all firms are symmetric, that is, they have the same cost functions.

In Section 3 we show the main result. All players have symmetric payoff functions. One player is the leader and other players are followers. The game is a two-stages game as follows;

1. In the first stage the leader determines the value of its strategic variable.
2. In the second stage the followers determine the values of their strategic variables given the value of the leader's strategic variable.

On the other hand, in the static game all players simultaneously determine the values of their strategic variables. We show that the equilibrium of the Stackelberg type dynamic game and the equilibrium of the static game are equivalent if and only if the game is fully symmetric.

2 Example: relative profit maximization in a Stackelberg oligopoly

In the example in this section we consider relative profit maximization in an oligopoly[2].

2.1 Case 1: four firms are different each other

Suppose a four firms Stackelberg oligopoly with a homogeneous good. There are Firms A, B, C and D. The outputs of the firms are x_A, x_B, x_C and x_D. The price of the good is p. The inverse demand function is

$$p = a - x_A - x_B - x_C - x_D,\ a > 0.$$

The cost functions of the firms are $c_A x_A$, $c_B x_B$, $c_C x_C$ and $c_D x_D$. c_A, c_B, c_C and c_D are positive constants. We assume that c_A, c_B, c_C and c_D are different each other. The relative profit of Firm A is

$$\varphi_A = px_A - c_A x_A - \frac{1}{3}(px_B - c_B x_B + px_C - c_C x_C + px_D - c_D x_D).$$

[2] About relative profit maximization in an oligopoly see Matsumura, Matsushima and Cato (2013), Vega-Redondo (1997), Satoh and Tanaka (2014a) and Satoh and Tanaka (2014b).

The relative profit of Firm B is

$$\varphi_B = px_B - c_B x_B - \frac{1}{3}(px_A - c_A x_A + px_C - c_C x_C + px_D - c_D x_D).$$

The relative profit of Firm C is

$$\varphi_C = px_C - c_C x_C - \frac{1}{3}(px_A - c_A x_A + px_B - c_B x_B + px_D - c_D x_D),$$

The relative profit of Firm D is

$$\varphi_D = px_D - c_D x_D - \frac{1}{3}(px_A - c_A x_A + px_B - c_B x_B + px_C - c_C x_C).$$

The firms maximize their relative profits. We see

$$\varphi_A + \varphi_B + \varphi_C + \varphi_D = 0.$$

Thus, the game is a zero-sum game. Firm A is the leader and Firms B, C and D are followers. In the first stage of the game Firm A determines x_A, and in the second stage Firms B, C and D determine x_B, x_C and x_D given x_A.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{2a + c_D + c_C + c_B - 5c_A}{8}, x_B = \frac{2a + c_D + c_C - 5c_B + c_A}{8},$$
$$x_C = \frac{2a + c_D - 5c_C + c_B + c_A + 2a}{8}, x_D = \frac{2a - 5c_D + c_C + c_B + c_A}{8}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{18a + 19c_D + 19c_C + 19c_B - 75c_A}{72},$$
$$x_B = \frac{18a + 7c_D + 7c_C - 47c_B + 15c_A}{72},$$
$$x_C = \frac{18a + 7c_D - 47c_C + 7c_B + 15c_A}{72},$$
$$x_D = \frac{18a - 47c_D + 7c_C + 7c_B + 15c_A}{72}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

2.2 Case 2: the leader and one follower are symmetric

Assume $c_D = c_A$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{2a + c_C + c_B - 4c_A}{8}, x_B = \frac{2a + c_C - 5c_B + 2c_A}{8},$$
$$x_C = \frac{2a - 5c_C + c_B + 2c_A}{8}, x_D = \frac{2a + c_C + c_B - 4c_A}{8}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{18a + 19c_C + 19c_B - 56c_A}{72},$$
$$x_B = \frac{18a + 7c_C - 47c_B + 22c_A}{72},$$
$$x_C = \frac{18a - 47c_C + 7c_B + 22c_A}{72},$$
$$x_D = \frac{18a + 7c_C + 7c_B - 32c_A}{72}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

2.3 Case 3: two followers are symmetric

Assume $c_D = c_C$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{2a + 2c_C + c_B - 5c_A}{8}, x_B = \frac{2a + 2c_C - 5c_B + c_A}{8},$$
$$x_C = \frac{2a - 4c_C + c_B + c_A}{8}, x_D = \frac{2a - 4c_C + c_B + c_A}{8}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{18a + 38c_C + 19c_B - 75c_A}{72},$$
$$x_B = \frac{18a + 14c_C - 47c_B + 15c_A}{72},$$
$$x_C = \frac{18a - 40c_C + 7c_B + 15c_A}{72},$$
$$x_D = \frac{18a - 40c_C + 7c_B + 15c_A}{72}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

2.4 Case 4: the leader and two followers are symmetric

Assume $c_D = c_C = c_A$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{2a + c_B - 3c_A}{8}, x_B = \frac{2a - 5c_B + 3c_A}{8},$$
$$x_C = \frac{2a + c_B - 3c_A}{8}, x_D = \frac{2a + c_B - 3c_A}{8}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{18a + 19c_B - 37c_A}{72},$$
$$x_B = \frac{18a - 47c_B + 29c_A}{72},$$
$$x_C = \frac{18a + 7c_B - 25c_A}{72},$$
$$x_D = \frac{18a + 7c_B - 25c_A}{72}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

2.5 Case 5: three followers are symmetric

Assume $c_D = c_C = c_B$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{2a + 3c_B - 5c_A}{8}, x_B = \frac{2a - 3c_B + c_A}{8},$$
$$x_C = \frac{2a - 3c_B + c_A}{8}, x_D = \frac{2a - 3c_B + c_A}{8}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{6a + 19c_B - 25c_A}{24},$$
$$x_B = \frac{6a - 11c_B + 5c_A}{24},$$
$$x_C = \frac{6a - 11c_B + 5c_A}{24},$$
$$x_D = \frac{6a - 11c_B + 5c_A}{24}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

2.6 Case 6: all firms are symmetric

Assume $c_B = c_C = c_D = c_A$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{a - c_A}{4}, x_B = \frac{a - c_A}{4}, x_C = \frac{a - c_A}{4}, x_D = \frac{a - c_A}{4}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{a - c_A}{4}, x_B = \frac{a - c_A}{4}, x_C = \frac{a - c_A}{4}, x_D = \frac{a - c_A}{4}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are equivalent.

3 Symmetric dynamic zero-sum game

There is an n-players and two-stages game. Players are called Player i, $i \in \{1, 2, \ldots, n\}$. The strategic variable of Player i is s_i, $i \in \{1, 2, \ldots, n\}$. The set of strategic variable of Player i is S_i, $i \in \{1, 2, \ldots, n\}$, which is a convex and compact set of a linear topological space. One of players is the leader and other players are followers.

The structure of the game is as follows.

1. The first stage

   The leader determines the value of its strategic variable.

2. The second stage

   Followers determine the values of their strategic variables given the value of the leader's strategic variable.

Thus, the game is a Stackelberg type dynamic game. We investigate a sub-game perfect equilibrium of this game.

On the other hand, there is a static game in which all players simultaneously determine the values of their strategic variables.

The payoff of Player i is denoted by $u_i(s_1, s_2, \ldots, s_n)$. u_i is jointly continuous and differentiable in s_i and s_j, $j \neq i$. We assume

$$\sum_{i=1}^{n} u_i(s_1, s_2, \ldots, s_n) = 0 \text{ given } (s_1, s_2, \ldots, s_n).$$

Therefore, the game is a zero-sum game.

We also assume that the game is symmetric in the sense that the payoff functions of all players are symmetric, and assume that the sets of strategic variables for all players are the same. Denote them by S.

We show the following theorem

Theorem 1. *The sub-game perfect equilibrium of the symmetric Stackelberg type dynamic zero-sum game is equivalent to the equilibrium of the static game.*

Proof. 1. The conditions for the equilibrium of the static game are

$$\frac{\partial u_i(s_1, s_2, \dots, s_n)}{\partial s_i} = 0,\ i \in \{1, 2, \dots, n\}, \tag{1}$$

given $(s_1, s_2, \dots, s_n)$. Since the game is symmetric, we can suppose that there exists a symmetric equilibrium. We write it as $(s^*, s^*, \dots, s^*)$ such that $s_i = s^*$ for all $i \in \{1, 2, \dots, n\}$.

The existence of a symmetric equilibrium is ensured by the fixed point theorem. s^* is obtained as a fixed point of the following function from S to S.

$$f(s) = \arg\max_{s_i \in S} u_i(s, \dots, s, s_i, s, \dots, s).$$

Assume that $\arg\max_{s_i \in S} u_i(s, \dots, s, s_i, s, \dots, s)$ is single-valued. Since u_i is continuous, $f(s)$ is continuous. S is compact. Therefore, $f(s)$ has a fixed point.

2. Suppose that the leader of the dynamic game is Player 1. Other players are followers. In the second stage of the game the followers determine their strategic variables to maximize their payoffs given the value of the strategic variable of Player 1. The conditions for maximization of the payoffs of the followers are

$$\frac{\partial u_j(s_1, s_2, \dots, s_n)}{\partial s_j} = 0,\ j \in \{2, 3, \dots, n\}. \tag{2}$$

Denote the values of $s_j,\ j \in \{2, 3, \dots, n\}$, obtained from (2) given s_1 by

$$s_j(s_1),\ j \in \{2, 3, \dots, n\}.$$

By symmetry of the game for the players other than Player 1 we have

$$s_j(s_1) = s_k(s_1),\ k \neq j,\ j, k \in \{2, 3, \dots, n\}.$$

The responses of $s_j(s_1)$ to a change in s_1 for each $j \in \{2, 3, \dots, n\}$ is written as

$$\frac{ds_j}{ds_1},\ j \in \{2, 3, \dots, n\}.$$

We assume

$$\frac{ds_j}{ds_1} \neq 0, \left|\frac{ds_j}{ds_1}\right| < 1,\ j \in \{2, 3, \dots, n\}.$$

By symmetry, when all $s_j,\ j \in \{2, 3, \dots, n\}$ are equal, we have

$$\frac{ds_j}{ds_1} = \frac{ds_k}{ds_1},\ k \neq j,\ j, k \in \{2, 3, \dots, n\}. \tag{3}$$

Also, we get

$$\frac{\partial u_j(s_1, s_2, \dots, s_n)}{\partial s_j} = \frac{\partial u_k(s_1, s_2, \dots, s_n)}{\partial s_k},\ k \neq j,\ j, k \in \{2, 3, \dots, n\}.$$

In the first stage of the game Player 1 determines s_1 to maximize its payoff taking the behaviors of other players into account. The value of u_1 with $s_j(s_1),\ j \in \{2, 3, \dots, n\}$ is

$$u_1(s_1, s_2(s_1), s_3(s_1), \dots, s_n(s_i)).$$

The condition for maximization of u_1 in the dynamic game is

$$\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_1} + \sum_{j=2}^{n} \frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_j} \frac{ds_j}{ds_1} = 0. \tag{4}$$

By symmetry when $s_j = s_k$,

$$\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_j} = \frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_k},\ k \neq j,\ k, j \in \{2, 3, \dots, n\}.$$

From (3)

$$\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_j} \frac{ds_j}{ds_1} = \frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_k} \frac{ds_k}{ds_1},\ j \neq k,\ j, k \in \{2, 3, \dots, n\}.$$

Thus, when all $s_j,\ j \in \{2, 3, \dots, n\}$ are equal, (4) is rewritten as

$$\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_1} + (n-1)\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_j} \frac{ds_j}{ds_1} = 0. \tag{5}$$

Since the game is zero-sum

$$u_1(s_1, s_2, \dots, s_n) + \sum_{j=2}^{n} u_j(s_1, s_2, \dots, s_n) = 0.$$

This means

$$u_1(s_1, s_2, \dots, s_n) = -\sum_{j=2}^{n} u_j(s_1, s_2, \dots, s_n).$$

By symmetry for the players other than Player 1, when $s_j = s_k,\ j, k \in \{2, 3, \dots, n\}$,

$$u_1(s_1, s_2, \dots, s_n) = -(n-1)u_j(s_1, s_2, \dots, s_n).$$

Thus,

$$\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_1} = -(n-1)\frac{\partial u_j(s_1, s_2, \dots, s_n)}{\partial s_1}. \tag{6}$$

3. We show that the equilibrium of the static game $(s^*, s^*, \dots, s^*)$, where $s_i = s^*$ for all $i \in \{1, 2, \dots, n\}$, satisfies the conditions for the equilibrium of the Stackelberg type dynamic game.

   Suppose a state such that $s_1 = s_2 = \cdots = s_n$. By symmetry of the game, $s_i = s_j,\ j \neq i,\ i, j \in \{1, 2, \dots, n\}$ means

$$\frac{\partial u_j(s_1, s_2, \dots, s_n)}{\partial s_1} = \frac{\partial u_1(s_1, s_2, s_3)}{\partial s_j},\ j \in \{2, 3, \dots, n\}. \tag{7}$$

   From (6) and (7)

$$-(n-1)\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_j} = \frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_1}.$$

   Substituting this into (5) yields,

$$\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_1} - \frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_1}\frac{ds_j}{ds_1} = 0.$$

   Thus,

$$\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_1}\left(1 - \frac{ds_j}{ds_1}\right) = 0.$$

   We get

$$\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_1} = 0,$$

   because $\left|\frac{ds_j}{ds_1}\right| < 1,\ j \neq 1$. From (2), this means

$$\frac{\partial u_1(s_1, s_2, \dots, s_n)}{\partial s_1} = 0,\ \frac{\partial u_j(s_1, s_2, \dots, s_n)}{\partial s_j} = 0,\ j \in \{2, 3, \dots, n\}.$$

   Since this is equivalent to (1), the equilibrium of the static game $(s^*, s^*, \dots, s^*)$ satisfies the conditions for the equilibrium of the Stackelberg type dynamic game.

   If the game is not symmetric for Player 1 and another player, (7) does not hold, and then the equilibrium of the dynamic game and that of the static game are not equivalent.

   □

4 Concluding Remark

As we said in the introduction, the equivalence of the Stackelberg type dynamic game and the static game in a two-players zero-sum game is a widely known result. But, this problem in a multi-players case has not been analyzed. In this paper we have analyzed a multi-players game and a case where payoff functions are differentiable. In the future research we want to prove the equivalence of the dynamic game and the static game when payoff functions are not assumed to be differentiable.

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