Stackelberg equilibrium of dynamic symmetric multi-players zero-sum game with a leader and followers without differentiability of payoff functions

Tanaka, Yasuhito

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Stackelberg equilibrium of dynamic symmetric multi-players zero-sum game with a leader and followers without differentiability of payoff functions*

Yasuhito Tanaka†
Faculty of Economics, Doshisha University,
Kamigyo-ku, Kyoto, 602-8580, Japan.

Abstract

This paper studies a Stackelberg type symmetric dynamic n-players zero-sum game. There is one leader and $n-1$ followers. Players have the symmetric payoff functions. The game is a two-stages game. In the first stage the leader determines the value of its strategic variable. In the second stage the followers determine the values of their strategic variables given the value of the leader's strategic variable. In the static game, on the other hand, all players simultaneously determine the values of their strategic variable. We do not assume differentiability of payoff functions. This paper shows that the sub-game perfect equilibrium of the Stackelberg type symmetric dynamic zero-sum game is equivalent to the equilibrium of the static game if and only if the game is fully symmetric.

Keywords: Stackelberg equilibrium, leader, follower, dynamic symmetric zero-sum game.

1 Introduction

We examine the relation between the Stackelberg equilibrium of dynamic game and the equilibrium of the static game in a multi-players zero-sum game, and show that the Stackelberg equilibrium of a dynamic zero-sum game and the equilibrium of the static zero-sum game are equivalent if and only if the game is fully symmetric. The Stackelberg equilibrium of dynamic

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†yatanaka@mail.doshisha.ac.jp

game and the equilibrium of the static game are equivalent in a two-person zero-sum game[1]. We extend this analysis to more general multi-players zero-sum game. We do not assume differentiability of payoff functions. However, we do not assume that the payoff functions are *not* differentiable. We do not use differentiability of payoff functions.

In the next section, using a model of relative profit maximization in an oligopoly with four firms, we show that the Stackelberg equilibrium is not equivalent to the static (Cournot) equilibrium in the following cases which are not fully symmetric.

1. All firms are asymmetric, that is, they have different cost functions.
2. Two followers are symmetric, that is, they have the same cost functions.
3. Three followers are symmetric.
4. The leader and one follower are symmetric.
5. The leader and two followers are symmetric.

The Stackelberg equilibrium is equivalent to the static (Cournot) equilibrium if and only if all firms are symmetric, that is, they have the same cost functions.

In Section 3 we show the main result. All players have symmetric payoff functions. One player is the leader and other players are followers. The game is a two-stages game as follows;

1. In the first stage the leader determines the value of its strategic variable.
2. In the second stage the followers determine the values of their strategic variables given the value of the leader's strategic variable.

On the other hand, in the static game all players simultaneously determine the values of their strategic variables. We show that the equilibrium of the Stackelberg type dynamic game and the equilibrium of the static game are equivalent if the game is fully symmetric.

2 Example: relative profit maximization in a Stackelberg oligopoly

In the example in this section we consider relative profit maximization in an oligopoly[2].

[1]Please see, for example, Korzhyk et. al. (2014), Ponssard and Zamir (1973), Tanaka (2014) and Yin et. al. (2010)

[2]About relative profit maximization in an oligopoly see Matsumura, Matsushima and Cato (2013), Vega-Redondo (1997), Satoh and Tanaka (2014a) and Satoh and Tanaka (2014b). In this example payoff functions are differentiable.

2.1 Case 1: four firms are different each other

Suppose a four firms Stackelberg oligopoly with a homogeneous good. There are Firms A, B, C and D. The outputs of the firms are x_A, x_B, x_C and x_D. The price of the good is p. The inverse demand function is

$$p = a - x_A - x_B - x_C - x_D,\ a > 0.$$

The cost functions of the firms are $c_A x_A^2$, $c_B x_B^2$, $c_C x_C^2$ and $c_D x_D^2$. c_A, c_B, c_C and c_D are positive constants. We assume that c_A, c_B, c_C and c_D are different each other. The relative profit of Firm A is

$$\varphi_A = px_A - c_A x_A^2 - \frac{1}{3}(px_B - c_B x_B^2 + px_C - c_C x_C^2 + px_D - c_D x_D^2).$$

The relative profit of Firm B is

$$\varphi_B = px_B - c_B x_B^2 - \frac{1}{3}(px_A - c_A x_A^2 + px_C - c_C x_C^2 + px_D^2 - c_D x_D^2).$$

The relative profit of Firm C is

$$\varphi_C = px_C - c_C x_C^2 - \frac{1}{3}(px_A - c_A x_A^2 + px_B - c_B x_B^2 + px_D - c_D x_D^2),$$

The relative profit of Firm D is

$$\varphi_D = px_D - c_D x_D^2 - \frac{1}{3}(px_A - c_A x_A^2 + px_B - c_B x_B^2 + px_C - c_C x_C^2).$$

The firms maximize their relative profits. We see

$$\varphi_A + \varphi_B + \varphi_C + \varphi_D = 0.$$

Thus, the game is a zero-sum game. Firm A is the leader and Firms B, C and D are followers. In the first stage of the game Firm A determines x_A, and in the second stage Firms B, C and D determine x_B, x_C and x_D given x_A.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{ac_B(c_C(27c_D + 18) + 18c_D + 12) + a(c_C(18c_D + 12) + 12c_D + 8)}{\Delta_1},$$
$$x_B = \frac{ac_C(c_A(27c_D + 18) + 18c_D + 12) + a(c_A(18c_D + 12) + 12c_D + 8)}{\Delta_1},$$
$$x_C = \frac{ac_B(c_A(27c_D + 18) + 18c_D + 12) + a(c_A(18c_D + 12) + 12c_D + 8)}{\Delta_1},$$
$$x_D = \frac{ac_B((27c_A + 18)c_C + 18c_A + 12) + a((18c_A + 12)c_C + 12c_A + 8)}{\Delta_1},$$

where

$$\begin{aligned}\Delta_1 =& c_B(c_C(c_A(54c_D + 54) + 54c_D + 48) + c_A(54c_D + 48) + 48c_D + 40) \\ &+ c_C(c_A(54c_D + 48) + 48c_D + 40) + c_A(48c_D + 40) + 40c_D + 32.\end{aligned}$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$\begin{aligned} x_A =& \frac{1}{\Delta_{1s}}[3a(3c_B+2)(3c_C+2)(3c_D+2)(27c_Bc_Cc_D+27c_Cc_D+27c_Bc_D+20c_D \\ &+27c_Bc_C+20c_C+20c_B+12)], \end{aligned}$$

$$\begin{aligned} x_B =& \frac{1}{\Delta_{1s}}[3a(3c_C+2)(3c_D+2)(81c_Ac_Bc_Cc_D+54c_Bc_Cc_D+81c_Ac_Cc_D \\ &+42c_Cc_D+81c_Ac_Bc_D+42c_Bc_D+72c_Ac_D+32c_D+81c_Ac_Bc_C+42c_Bc_C+72c_Ac_C \\ &+32c_C+72c_Ac_B+32c_B+60c_A+24)], \end{aligned}$$

$$\begin{aligned} x_C =& \frac{1}{\Delta_{1s}}[3a(3c_B+2)(3c_D+2)(81c_Ac_Bc_Cc_D+54c_Bc_Cc_D+81c_Ac_Cc_D \\ &+42c_Cc_D+81c_Ac_Bc_D+42c_Bc_D+72c_Ac_D+32c_D+81c_Ac_Bc_C+42c_Bc_C+72c_Ac_C \\ &+32c_C+72c_Ac_B+32c_B+60c_A+24)], \end{aligned}$$

$$\begin{aligned} x_D =& \frac{1}{\Delta_{1s}}[3a(3c_B+2)(3c_C+2)(81c_Ac_Bc_Cc_D+54c_Bc_Cc_D+81c_Ac_Cc_D+42c_Cc_D+81c_Ac_Bc_D \\ &+42c_Bc_D+72c_Ac_D+32c_D+81c_Ac_Bc_C+42c_Bc_C+72c_Ac_C+32c_C+72c_Ac_B+32c_B \\ &+60c_A+24)], \end{aligned}$$

where

$$\begin{aligned} \Delta_{1s} =& 2(2187c_Ac_B^2c_C^2c_D^2+2187c_B^2c_C^2c_D^2+4374c_Ac_Bc_C^2c_D^2+3807c_Bc_C^2c_D^2 \\ &+2187c_Ac_C^2c_D^2+1620c_C^2c_D^2+4374c_Ac_B^2c_Cc_D^2+3807c_B^2c_Cc_D^2+8262c_Ac_Bc_Cc_D^2 \\ &+6264c_Bc_Cc_D^2+3888c_Ac_Cc_D^2+2556c_Cc_D^2+2187c_Ac_B^2c_D^2+1620c_B^2c_D^2+3888c_Ac_Bc_D^2 \\ &+2556c_Bc_D^2+1728c_Ac_D^2+1008c_D^2+4374c_Ac_B^2c_C^2c_D+3807c_B^2c_C^2c_D+8262c_Ac_Bc_C^2c_D \\ &+6264c_Bc_C^2c_D+3888c_Ac_C^2c_D+2556c_C^2c_D+8262c_Ac_B^2c_Cc_D+6264c_B^2c_Cc_D \\ &+14904c_Ac_Bc_Cc_D+9936c_Bc_Cc_D+6696c_Ac_Cc_D+3936c_Cc_D+3888c_Ac_B^2c_D+2556c_B^2c_D \\ &+6696c_Ac_Bc_D+3936c_Bc_D+2880c_Ac_D+1520c_D+2187c_Ac_B^2c_C^2+1620c_B^2c_C^2 \\ &+3888c_Ac_Bc_C^2+2556c_Bc_C^2+1728c_Ac_C^2+1008c_C^2+3888c_Ac_B^2c_C+2556c_B^2c_C \\ &+6696c_Ac_Bc_C+3936c_Bc_C+2880c_Ac_C+1520c_C+1728c_Ac_B^2 \\ &+1008c_B^2+2880c_Ac_B+1520c_B+1200c_A+576). \end{aligned}$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

2.2 Case 2: the leader and one follower are symmetric

Assume $c_D = c_A$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{a(3c_B + 2)(3c_C + 2)}{2(9c_Ac_Bc_C + 9c_Bc_C + 9c_Ac_C + 8c_C + 12c_Ac_B + 10c_B + 10c_A + 8)},$$

$$x_B = \frac{a(3c_A + 2)(3c_C + 2)}{2(9c_Ac_Bc_C + 9c_Bc_C + 9c_Ac_C + 8c_C + 12c_Ac_B + 10c_B + 10c_A + 8)},$$

$$x_C = \frac{a(3c_A + 2)(3c_B + 2)}{2(9c_Ac_Bc_C + 9c_Bc_C + 9c_Ac_C + 8c_C + 12c_Ac_B + 10c_B + 10c_A + 8)},$$

$$x_D = \frac{a(3c_A + 2)(3c_B + 2)}{2(9c_Ac_Bc_C + 9c_Bc_C + 9c_Ac_C + 8c_C + 12c_Ac_B + 10c_B + 10c_A + 8)}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{3a(3c_B + 2)(27c_Bc_C^2 + 27c_C^2 + 54c_Bc_C + 40c_C + 20c_B + 12)}{\Delta_2},$$

$$x_B = \frac{3a(3c_C + 2)(27c_Ac_Bc_C + 18c_Bc_C + 27c_Ac_C + 14c_C + 36c_Ac_B + 16c_B + 30c_A + 12)}{\Delta_2},$$

$$x_C = \frac{3a(3c_B + 2)(27c_Ac_Bc_C + 18c_Bc_C + 27c_Ac_C + 14c_C + 36c_Ac_B + 16c_B + 30c_A + 12)}{\Delta_2},$$

$$x_D = \frac{3a(3c_B + 2)(27c_Ac_Bc_C + 18c_Bc_C + 27c_Ac_C + 14c_C + 36c_Ac_B + 16c_B + 30c_A + 12)}{\Delta_2},$$

where

$$\begin{aligned}\Delta_2 =& 2(243c_Ac_B^2c_C^2 + 243c_B^2c_C^2 + 486c_Ac_Bc_C^2 + 423c_Bc_C^2 + 243c_Ac_C^2 \\ &+ 180c_C^2 + 648c_Ac_B^2c_C + 522c_B^2c_C + 1188c_Ac_Bc_C + 828c_Bc_C + 540c_Ac_C + 328c_C \\ &+ 432c_Ac_B^2 + 252c_B^2 + 720c_Ac_B + 380c_B + 300c_A + 144).\end{aligned}$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

2.3 Case 3: two followers are symmetric

Assume $c_D = c_C$.

Nash equilibrium of the static game

The equilibrium outputs are

$$
\begin{aligned}
x_A &= \frac{a(3c_B+2)(3c_C+2)}{2(9c_Ac_Bc_C+12c_Bc_C+9c_Ac_C+10c_C+9c_Ac_B+10c_B+8c_A+8)},\\
x_B &= \frac{a(3c_A+2)(3c_C+2)}{2(9c_Ac_Bc_C+12c_Bc_C+9c_Ac_C+10c_C+9c_Ac_B+10c_B+8c_A+8)},\\
x_C &= \frac{a(3c_A+2)(3c_B+2)}{2(9c_Ac_Bc_C+12c_Bc_C+9c_Ac_C+10c_C+9c_Ac_B+10c_B+8c_A+8)},\\
x_D &= \frac{a(3c_B+2)(3c_C+2)}{2(9c_Ac_Bc_C+12c_Bc_C+9c_Ac_C+10c_C+9c_Ac_B+10c_B+8c_A+8)}.
\end{aligned}
$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$
\begin{aligned}
x_A =& \frac{1}{\Delta_3}[3a(3c_A+2)(3c_B+2)(3c_C+2)(27c_Ac_Bc_C+27c_Bc_C+27c_Ac_C\\
&+20c_C+27c_Ac_B+20c_B+20c_A+12)],
\end{aligned}
$$

$$
\begin{aligned}
x_B =& \frac{1}{\Delta_3}[3a(3c_A+2)(3c_C+2)(81c_A^2c_Bc_C+135c_Ac_Bc_C+42c_Bc_C+81c_A^2c_C+114c_Ac_C+32c_C\\
&+81c_A^2c_B+114c_Ac_B+32c_B+72c_A^2+92c_A+24)],
\end{aligned}
$$

$$
\begin{aligned}
x_C =& \frac{1}{\Delta_3}[3a(3c_A+2)(3c_B+2)(81c_A^2c_Bc_C+135c_Ac_Bc_C+42c_Bc_C+81c_A^2c_C+114c_Ac_C+32c_C\\
&+81c_A^2c_B+114c_Ac_B+32c_B+72c_A^2+92c_A+24)],
\end{aligned}
$$

$$
\begin{aligned}
x_D =& \frac{1}{\Delta_3}[3a(3c_B+2)(3c_C+2)(81c_A^2c_Bc_C+135c_Ac_Bc_C+42c_Bc_C+81c_A^2c_C+114c_Ac_C+32c_C\\
&+81c_A^2c_B+114c_Ac_B+32c_B+72c_A^2+92c_A+24)],
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_3 =& 2(2187c_A^3c_B^2c_C^2+6561c_A^2c_B^2c_C^2+5994c_Ac_B^2c_C^2+1620c_B^2c_C^2\\
&+4374c_A^3c_Bc_C^2+12069c_A^2c_Bc_C^2+10152c_Ac_Bc_C^2+2556c_Bc_C^2+2187c_A^3c_C^2\\
&+5508c_A^2c_C^2+4284c_Ac_C^2+1008c_C^2+4374c_A^3c_B^2c_C+12069c_A^2c_B^2c_C+10152c_Ac_B^2c_C\\
&+2556c_B^2c_C+8262c_A^3c_Bc_C+21168c_A^2c_Bc_C+16632c_Ac_Bc_C+3936c_Bc_C+3888c_A^3c_C\\
&+9252c_A^2c_C+6816c_Ac_C+1520c_C+2187c_A^3c_B^2+5508c_A^2c_B^2+4284c_Ac_B^2\\
&+1008c_B^2+3888c_A^3c_B+9252c_A^2c_B+6816c_Ac_B+1520c_B+1728c_A^3\\
&+3888c_A^2+2720c_A+576).
\end{aligned}
$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

2.4 Case 4: the leader and two followers are symmetric

Assume $c_D = c_C = c_A$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{a(3c_B + 2)}{2(3c_A c_B + 5c_B + 3c_A + 4)}, x_B = \frac{a(3c_A + 2)}{2(3c_A c_B + 5c_B + 3c_A + 4)},$$
$$x_C = \frac{a(3c_B + 2)}{2(3c_A c_B + 5c_B + 3c_A + 4)}, x_D = \frac{a(3c_B + 2)}{2(3c_A c_B + 5c_B + 3c_A + 4)}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{3a(3c_B + 2)(27c_A^2 c_B + 54c_A c_B + 20c_B + 27c_A^2 + 40c_A + 12)}{\Delta_4},$$
$$x_B = \frac{3a(3c_A + 2)(27c_A^2 c_B + 54c_A c_B + 16c_B + 27c_A^2 + 44c_A + 12)}{\Delta_4},$$
$$x_C = \frac{3a(3c_B + 2)(27c_A^2 c_B + 54c_A c_B + 16c_B + 27c_A^2 + 44c_A + 12)}{\Delta_4},$$
$$x_D = \frac{3a(3c_B + 2)(27c_A^2 c_B + 54c_A c_B + 16c_B + 27c_A^2 + 44c_A + 12)}{\Delta_4},$$

where

$$\begin{aligned}\Delta_4 =& 2(243c_A^3 c_B^2 + 891c_A^2 c_B^2 + 954c_A c_B^2 + 252c_B^2 + 486c_A^3 c_B + 1611c_A^2 c_B + 1548c_A c_B \\ &+ 380c_B + 243c_A^3 + 720c_A^2 + 628c_A + 144).\end{aligned}$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

2.5 Case 5: three followers are symmetric

Assume $c_D = c_C = c_B$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{a(3c_B + 2)}{2(3c_Ac_B + 3c_B + 5c_A + 4)}, x_B = \frac{a(3c_A + 2)}{2(3c_Ac_B + 3c_B + 5c_A + 4)},$$
$$x_C = \frac{a(3c_A + 2)}{2(3c_Ac_B + 3c_B + 5c_A + 4)}, x_D = \frac{a(3c_A + 2)}{2(3c_Ac_B + 3c_B + 5c_A + 4)}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{3a(c_B + 2)(3c_B + 1)}{2(9c_Ac_B^2 + 9c_B^2 + 30c_Ac_B + 23c_B + 25c_A + 12)},$$
$$x_B = \frac{3a(3c_Ac_B + 2c_B + 5c_A + 2)}{2(9c_Ac_B^2 + 9c_B^2 + 30c_Ac_B + 23c_B + 25c_A + 12)},$$
$$x_C = \frac{3a(3c_Ac_B + 2c_B + 5c_A + 2)}{2(9c_Ac_B^2 + 9c_B^2 + 30c_Ac_B + 23c_B + 25c_A + 12)},$$
$$x_D = \frac{3a(3c_Ac_B + 2c_B + 5c_A + 2)}{2(9c_Ac_B^2 + 9c_B^2 + 30c_Ac_B + 23c_B + 25c_A + 12)}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

2.6 Case 6: all firms are symmetric

Assume $c_B = c_C = c_D = c_A$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{a}{2(c_A + 2)}, x_B = \frac{a}{2(c_A + 2)}, x_C = \frac{a}{2(c_A + 2)}, x_D = \frac{a}{2(c_A + 2)}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{a}{2(c_A + 2)}, x_B = \frac{a}{2(c_A + 2)}, x_C = \frac{a}{2(c_A + 2)}, x_D = \frac{a}{2(c_A + 2)}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are equivalent.

3 Symmetric dynamic zero-sum game

There is an n-players and two-stages game. Players are called Player 1, 2, ..., n. The strategic variable of Player i is $s_i,\ i \in \{1, 2, \dots, n\}$. The set of strategic variable of Player i is $S_i,\ i \in \{1, 2, \dots, n\}$, which is a convex and compact set of a linear topological space. One of players is the leader and other players are followers.

The structure of the game is as follows.

1. The first stage

   The leader determines the value of its strategic variable.

2. The second stage

   Followers determine the values of their strategic variables given the value of the leader's strategic variable.

Thus, the game is a Stackelberg type dynamic game. We investigate a sub-game perfect equilibrium of this game.

On the other hand, there is a static game in which all players simultaneously determine the values of their strategic variables.

The payoff of Player i is denoted by $u_i(s_1, s_2, \dots, s_n)$. u_i is jointly continuous in s_i and all $s_j,\ j \neq i$. We assume

$$\sum_{i=1}^{n} u_i(s_1, s_2, \dots, s_n) = 0 \text{ given } (s_1, s_2, \dots, s_n).$$

Therefore, the game is a zero-sum game.

We assume that the game is symmetric in the sense that the payoff functions of all players are symmetric, and assume that the sets of strategic variables for all players are the same. Denote them by S. We do not assume differentiability of players' payoff functions[3].

We show the following theorem

Theorem 1. *The sub-game perfect equilibrium of the symmetric Stackelberg type dynamic zero-sum game is equivalent to the equilibrium of the static game.*

Proof. (1) Suppose that the leader is Player 1. Let $(s_2(s_1), s_3(s_1), \dots, s_n(s_1))$ be a solution of the following equation;

$$\begin{cases} s_2(s_1) = \arg\max_{s_2 \in S} u_2(s_1, s_2, s_3(s_1), \dots s_n(s_1)) \\ s_3(s_1) = \arg\max_{s_3 \in S} u_3(s_1, s_2(s_1), s_3, \dots s_n(s_1)) \\ \dots \\ s_n(s_1) = \arg\max_{s_n \in S} u_n(s_1, s_2(s_1), s_3(s_1), \dots s_n) \end{cases}$$

[3]As we said in the introduction, we do not assume that the payoff function is *not* differentiable. We do not use differentiability of payoff functions.

given s_1. Assume that $\arg\max_{s_i \in S} u_i(s_1, s_2(s_1), \dots, s_i, \dots, s_n(s_1))$ for $i \in \{2, 3, \dots, n\}$ are unique. Since S is compact, $u_i(s_1, s_2, \dots, s_n)$ for all $i \in \{1, 2, \dots, n\}$ are jointly continuous, by the maximum theorem $s_2(s_1)$, $s_3(s_1)$, $\dots$, $s_n(s_1)$ are continuous. We have

$$\max_{s_i \in S} u_i(s_1, s_2(s_1), \dots, s_i, \dots, s_n(s_1)) = u_i(s_1, s_2(s_1), \dots, s_i(s_1), \dots, s_n(s_1)),\ i \in \{2, \dots, n\}.$$

By symmetry of the game

$$s_2(s_1) = s_3(s_1) = \dots = s_n(s_1),$$

and

$$u_2(s_1, s_2(s_1), \dots, s_n(s_1)) = u_3(s_1, s_2(s_1), \dots, s_n(s_1)) = \dots = u_n(s_1, s_2(s_1), \dots, s_n(s_1)),$$

given s_1. $s_1(s_2)$, $s_3(s_2)$, $\dots$, $s_n(s_2)$, $s_2(s_3)$, $\dots$, $s_n(s_3)$, $\dots$, $s_1(s_n)$, $\dots$, $s_{n-1}(s_n)$ are similarly defined. By symmetry of the game we have

$$s_1(s_2) = s_3(s_2) = \dots = s_n(s_2),\ s_2(s_3) = \dots = s_n(s_3),\ \dots,\ s_1(s_n) = \dots = s_{n-1}(s_n).$$

$s_2(s_1)$ is also obtained as a fixed point of the following function

$$\max_{s \in S} u_2(s_1, s, s_2(s_1), \dots, s_2(s_1)).$$

(2) The Nash equilibrium of the static game is obtained as a fixed point of a function from S^n to S^n;

$$\begin{pmatrix} \arg\max_{s_1 \in S} u_1(s_1, s^{(1)}, s^{(2)}, \dots, s^{(n)}) \\ \arg\max_{s_2 \in S} u_2(s^{(1)}, s_2, \dots, s^{(n)}) \\ \dots \\ \arg\max_{s_n \in S} u_n(s^{(1)}, s^{(2)}, \dots, s_n) \end{pmatrix}.$$

By symmetry of the game for all players we assume that $s_1 = s_2 = \dots = s_n$ at the equilibrium. Denote the equilibrium by $(\tilde{s}, \tilde{s}, \dots, \tilde{s})$. $\tilde{s}$ is also obtained as a fixed point of the following function.

$$\max_{s \in S} u_1(s, \tilde{s}, \tilde{s}, \dots, \tilde{s}).$$

We assume uniqueness of the Nash equilibrium of the static game. At the equilibrium of the static game $(\tilde{s}, \tilde{s}, \dots, \tilde{s})$, we have

$$u_1(\tilde{s}, \tilde{s}, \dots, \tilde{s}) > u_1(s, \tilde{s}, \dots, \tilde{s}) \text{ for any } s \in S,\ s \neq \tilde{s}, \tag{1}$$

and

$$u_1(\tilde{s}, \tilde{s}, \dots, \tilde{s}) = 0.$$

Similarly,

$$u_i(\tilde{s}, \dots, \tilde{s}, \dots, \tilde{s}) > u_i(\tilde{s}, \dots, \tilde{s}, \dots, s, \dots, \tilde{s}, \dots, \tilde{s}) \text{ for any } s \in S,\ s \neq \tilde{s},\ s_i = s,$$

$$u_i(\tilde{s}, \dots, \tilde{s}, \dots \tilde{s}) = 0$$

for $i \in \{2, \dots, n\}$. Note that

$$s_2(\tilde{s}) = \arg\max_{s_2 \in S} u_2(\tilde{s}, s_2, \tilde{s}, \dots, \tilde{s}) = \tilde{s},$$

and so on. Since the game is zero-sum and symmetric, we have

$$u_1(s, \tilde{s}, \dots, \tilde{s}) = -(n-1)u_2(s, \tilde{s}, \dots, \tilde{s}).$$

Thus, (1) means

$$u_2(s, \tilde{s}, \dots, \tilde{s}) > 0.$$

By symmetry, we get

$$u_1(\tilde{s}, s, \tilde{s}, \dots, \tilde{s}) > 0.$$

Therefore,

$$u_1(s, \tilde{s}, \dots, \tilde{s}) < 0 < u_1(\tilde{s}, s, \tilde{s}, \dots, \tilde{s}). \tag{2}$$

Similarly,

$$u_1(s, \tilde{s}, \dots, \tilde{s}) < 0 < u_1(\tilde{s}, \dots, \tilde{s}, s, \tilde{s}, \dots, \tilde{s}),\ s_i = s,\ i \in \{3, 4, \dots, n\}. \tag{3}$$

Also we have

$$|u_1(s, \tilde{s}, \dots, \tilde{s})| = (n-1)|u_1(\tilde{s}, s, \tilde{s}, \dots, \tilde{s})|. \tag{4}$$

(3) The equilibrium strategy of Player 1 in the dynamic game is written as

$$\arg\max_{s_1 \in S} u_1(s_1, s_2(s_1), \dots, s_n(s_1)).$$

Let

$$s_1^* = \arg\max_{s_1 \in S} u_1(s_1, s_2(s_1), \dots, s_n(s_1)).$$

$(s_1^*, s_2(s_1^*), \dots, s_n(s_1^*))$ is a Stackelberg equilibrium of the dynamic game when Player 1 is the leader. We assume uniqueness of the Stackelberg equilibrium. Similarly, we get s_i^* such that

$$s_i^* = \arg\max_{s_i \in S} u_i(s_1(s_i), \dots, s_{i-1}(s_i), s_i, s_{i+1}(s_i), \dots, s_n(s_i)).$$

By symmetry of the game

$$s_1^* = s_2^* = \cdots = s_n^*.$$

Denote them by s^*.

(4) Since, by symmetry for Players 2 to n, $s_n(s) = s_{n-1}(s) = \cdots = s_2(s)$ for any s, we have

$$s^* = \arg\max_{s \in S} u_1(s, s_2(s), \dots, s_2(s)).$$

This is equivalent to

$$u_1(s^*, s_2(s^*), \dots, s_2(s^*)) > u_1(s, s_2(s), \dots, s_2(s)) \text{ for any } s \in S,\ s \neq s^*.$$

Suppose a state such that $s_1 = s_2 = \cdots = \tilde{s}$. From (2) and (3), for $s \neq \tilde{s}$,

$$u_1(s, \tilde{s}, \dots, \tilde{s}) < 0, \dots, u_1(\tilde{s}, \dots, \tilde{s}, s, \tilde{s}, \dots, \tilde{s}) > 0\,(s_i = s),\ \dots,\ u_1(\tilde{s}, \dots, \tilde{s}, s) > 0\,(s_n = s).$$

Since $u_1(s_1, s_2, \dots, s_n)$ is jointly continuous, there exists a neighborhood $V'(\tilde{s})$ of $\tilde{s}$ such that, for $s' \in V'(\tilde{s}),\ s' \neq \tilde{s}$

$$|u_1(\tilde{s}, s', \dots, s')| < |u_1(\tilde{s}, s, \tilde{s}, \dots, \tilde{s})|,$$

and

$$u_1(\tilde{s}, s', \dots, s') > 0,$$

for s which satisfies (2) and (3). Since the game is zero-sum,

$$u_1(\tilde{s}, s', \dots, s') + u_2(\tilde{s}, s', \dots, s') + u_3(\tilde{s}, s', \dots, s') + \cdots + u_n(\tilde{s}, s', \dots, s') = 0.$$

By symmetry

$$u_1(\tilde{s}, s', \dots, s') = -(n-1)u_2(\tilde{s}, s', \dots, s') = -(n-1)u_1(s', \tilde{s}, s', \dots, s').$$

Thus,

$$u_1(s', \tilde{s}, s', \dots, s') < 0\ (s_2 = \tilde{s}), \dots, u_1(s', s', \dots, s', \tilde{s}) < 0\ (s_n = \tilde{s}).$$

Also we have

$$|u_1(\tilde{s}, s', \dots, s')| = (n-1)|u_1(s', \tilde{s}, s', \dots, s')|.$$

Since $u_1(s_1, s_2, \dots, s_n)$ is jointly continuous, if $V(\tilde{s})$ is sufficiently small, we can assume

$$|u_1(\tilde{s}, s', \dots, s') - u_1(\tilde{s}, \tilde{s}, \dots, \tilde{s})| \approx (n-1)|u_1(\tilde{s}, s', \tilde{s}, \dots, \tilde{s}) - u_1(\tilde{s}, \tilde{s}, \dots, \tilde{s})|.$$

or

$$|u_1(\tilde{s}, s', \dots, s')| \approx (n-1)|u_1(\tilde{s}, s', \tilde{s}, \dots, \tilde{s})|.$$

Consequently, from (4)

$$|u_1(\tilde{s}, s', \dots, s')| \approx |u_1(s', \tilde{s}, \dots, \tilde{s})|.$$

There exists a neighborhood $V(\tilde{s})$ of $\tilde{s}$ such that for $s \in V(\tilde{s})$

$$|u_1(s, s_2(s), \dots, s_2(s))| < |u_1(s', \tilde{s}, \tilde{s}, \dots, \tilde{s})|, \text{ for } s' \in V'(\tilde{s}).$$

It seems to be that

$$|s_2(s) - \tilde{s}| < |s - \tilde{s}|.$$

Since

$$u_1(s, \tilde{s}, \dots, \tilde{s}) < 0,$$

and

$$u_1(\tilde{s}, s_2(s), \dots, s_2(s)) > 0,$$

we get

$$u_1(s, s_2(s), \dots, s_2(s)) < 0.$$

This means

$$u_1(\tilde{s}, \tilde{s}, \dots, \tilde{s}) > u_1(s, s_2(s), \dots, s_2(s)), \text{ for } s \in V(s).$$

Thus, $(\tilde{s}, \tilde{s}, \dots, \tilde{s})$ is the Stackelberg equilibrium.
We have completed the proof. □

4 Concluding Remark

As we said in the introduction, the equivalence of the Stackelberg type dynamic game and the static game in a two-players zero-sum game is a widely known result. But, this problem in a multi-players case has not been analyzed. In this paper we have analyzed a general n-players game.

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