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Analysis of dynamic symmetric three-players zero-sum game with a leader and two followers without differentiability of payoff functions

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Abstract

We consider a Stackelberg type symmetric dynamic three-players zero-sum game. One player is the leader and two players are followers. All players have the symmetric payoff functions. The game is a two-stages game. In the first stage the leader determines the value of its strategic variable. In the second stage the followers determine the values of their strategic variables given the value of the leader's strategic variable. On the other hand, in the static game all players simultaneously determine the values of their strategic variable. We do not assume differentiability of players' payoff functions. We show that the sub-game perfect equilibrium of the Stackelberg type symmetric dynamic zero-sum game with a leader and two followers is equivalent to the equilibrium of the static game if and only if the game is fully symmetric.

Keywords: symmetric zero-sum game, Stackelberg equilibrium, leader, follower.

1 Introduction

It is well known that the equilibrium of the Stackelberg type dynamic game and that of the static game are equivalent in a two-person zero-sum game. See, for example, Korzhyk et. al. (2014), Ponssard and Zamir (1973), Tanaka (2014) and Yin et. al. (2010). We examine this problem in a three-players zero-sum game, and show that the equilibrium of the Stackelberg type dynamic zero-sum game and that of the static zero-sum game are equivalent if and only if

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the game is fully symmetric. We do not assume differentiability of players' payoff functions[1]. However, we do not assume that the payoff functions are *not* differentiable. We do not use differentiability of payoff functions.

In the next section we show the main result. All players have symmetric payoff functions. One player is the leader and two players are followers. The game is a two-stages game as follows;

1. In the first stage the leader determines the value of its strategic variable.

2. In the second stage the followers determine the values of their strategic variables given the value of the leader's strategic variable.

On the other hand, in the static game all players simultaneously determine the values of their strategic variables. We show that if the game is fully symmetric, the equilibrium of the Stackelberg type dynamic game and that of the static game are equivalent.

As we will show in Section 3 using a model of relative profit maximization in an oligopoly, the Stackelberg equilibrium is not equivalent to the static (Cournot) equilibrium in the following cases which are not fully symmetric.

1. All firms are asymmetric, that is, they have different cost functions.

2. Two followers are symmetric, that is, they have the same cost functions.

3. The leader and one follower are symmetric.

If and only if all firms are symmetric, that is, they have the same cost functions, the Stackelberg equilibrium is equivalent to the static (Cournot) equilibrium.

2 Symmetric dynamic zero-sum game

There is a three-players and two-stages game. Players are called Player 1, Player 2 and Player 3. The strategic variable of Player i is $s_i,\ i \in \{1,2,3\}$. The set of strategic variable of Player i is $S_i,\ i \in \{1,2,3\}$, which is a convex and compact set of a linear topological space. One of players is the leader and other players are followers.

The structure of the game is as follows.

1. The first stage

   The leader determines the value of its strategic variable.

2. The second stage

   Followers determine the values of their strategic variables given the value of the leader's strategic variable.

[1] In Tanaka (2018) we analyzed a similar problem when payoff functions of players are differentiable.

Thus, the game is a Stackelberg type dynamic game. We investigate a sub-game perfect equilibrium of this game.

On the other hand, there is a static game in which three players simultaneously determine the values of their strategic variables.

The payoff of Player i is denoted by $u_i(s_1, s_2, s_3)$. u_i is jointly continuous in s_i and $s_j,\ j \neq i$. We assume

$$\sum_{i=1}^{n} u_i(s_1, s_2, s_3) = 0 \text{ given } (s_1, s_2, s_3).$$

Therefore, the game is a zero-sum game.

We do not assume differentiability of players' payoff functions[2]. We also assume that the game is symmetric in the sense that the payoff functions of all players are symmetric, and assume that the sets of strategic variables for all players are the same. Denote them by S.

We show the following theorem

Theorem 1. *The sub-game perfect equilibrium of the symmetric Stackelberg type dynamic zero-sum game with a leader and two followers is equivalent to the equilibrium of the static game.*

Proof. (1) Suppose that the leader is Player 1. Let $(s_2(s_1), s_3(s_1))$ be a solution of the following equation;

$$\begin{cases} s_2(s_1) = \arg\max_{s_2 \in S} u_2(s_1, s_2, s_3(s_1)) \\ s_3(s_1) = \arg\max_{s_3 \in S} u_3(s_1, s_2(s_1), s_3), \end{cases}$$

given s_1. Assume that $\arg\max_{s_2 \in S} u_2(s_1, s_2, s_3(s_1))$ and $\arg\max_{s_3 \in S} u_3(s_1, s_2(s_1), s_3)$ are unique. $(s_2(s_1), s_3(s_1))$ is a fixed point of a function of (s, s') from $S \times S$ to $S \times S$;

$$\begin{pmatrix} \arg\max_{s_2 \in S} u_2(s_1, s_2, s') \\ \arg\max_{s_3 \in S} u_3(s_1, s, s_3) \end{pmatrix},$$

given s_1. Since S is compact, $u_2(s_1, s_2, s_3)$ and $u_3(s_1, s_2, s_3)$ are jointly continuous, by the maximum theorem $s_2(s_1)$ is continuous. We have

$$\max_{s_2 \in S} u_2(s_1, s_2, s_3(s_1)) = u_2(s_1, s_2(s_1), s_3(s_1)),$$

and

$$\max_{s_3 \in S} u_3(s_1, s_2(s_1), s_3) = u_3(s_1, s_2(s_1), s_3(s_1)).$$

By symmetry of the game

$$s_2(s_1) = s_3(s_1),$$

and

$$u_2(s_1, s_2(s_1), s_3(s_1)) = u_3(s_1, s_2(s_1), s_3(s_1)),$$

[2]As we said in the introduction, we do not assume that the payoff function is *not* differentiable. We do not use differentiability of payoff functions.

given s_1. $s_1(s_2)$, $s_1(s_3)$, $s_2(s_3)$ and $s_2(s_3)$ are similarly defined, and by symmetry of the game we have

$$s_1(s_2) = s_3(s_2),\ s_1(s_3) = s_2(s_3).$$

$s_2(s_1)$ is also obtained as a fixed point of the following function

$$\max_{s \in S} u_2(s_1, s, s_2(s_1)).$$

(2) The Nash equilibrium of the static game is obtained as a fixed point of a function of (s, s', s'') from $S \times S \times S$ to $S \times S \times S$;

$$\begin{pmatrix} \arg\max_{s_1 \in S} u_1(s_1, s', s'') \\ \arg\max_{s_2 \in S} u_2(s, s_2, s'') \\ \arg\max_{s_3 \in S} u_3(s, s', s_3) \end{pmatrix}.$$

By symmetry of the game for all players we assume that $s_1 = s_2 = s_3$ at the equilibrium. Denote the equilibrium by $(\tilde{s}, \tilde{s}, \tilde{s})$. $\tilde{s}$ is also obtained as a fixed point of the following function.

$$\max_{s \in S} u_1(s, \tilde{s}, \tilde{s}).$$

We assume uniqueness of the Nash equilibrium of the static game. At the equilibrium of the static game $(\tilde{s}, \tilde{s}, \tilde{s})$, we have

$$u_1(\tilde{s}, \tilde{s}, \tilde{s}) > u_1(s, \tilde{s}, \tilde{s}) \text{ for any } s \in S,\ s \neq \tilde{s}, \tag{1}$$

and

$$u_1(\tilde{s}, \tilde{s}, \tilde{s}) = 0.$$

Similarly,

$$u_2(\tilde{s}, \tilde{s}, \tilde{s}) > u_2(\tilde{s}, s, \tilde{s}) \text{ for any } s \in S,\ s \neq \tilde{s},$$

$$u_3(\tilde{s}, \tilde{s}, \tilde{s}) > u_3(\tilde{s}, \tilde{s}, s) \text{ for any } s \in S,\ s \neq \tilde{s},$$

and

$$u_2(\tilde{s}, \tilde{s}, \tilde{s}) = u_3(\tilde{s}, \tilde{s}, \tilde{s}) = 0.$$

Note that

$$s_2(\tilde{s}) = \arg\max_{s_2 \in S} u_2(\tilde{s}, s_2, \tilde{s}) = \tilde{s}.$$

Since the game is zero-sum and symmetric for Players 2 and 3, we have

$$u_1(s, \tilde{s}, \tilde{s}) = -2u_2(s, \tilde{s}, \tilde{s}).$$

Thus, (1) means

$$u_2(s, \tilde{s}, \tilde{s}) > 0.$$

By symmetry for Players 1 and 2, we get

$$u_1(\tilde{s}, s, \tilde{s}) > 0.$$

Therefore,

$$u_1(s, \tilde{s}, \tilde{s}) < 0 < u_1(\tilde{s}, s, \tilde{s}). \tag{2}$$

Similarly,

$$u_1(s, \tilde{s}, \tilde{s}) < 0 < u_1(\tilde{s}, \tilde{s}, s). \tag{3}$$

Also we have

$$|u_1(s, \tilde{s}, \tilde{s})| = 2|u_1(\tilde{s}, s, \tilde{s})| = 2|u_1(\tilde{s}, s, \tilde{s})|. \tag{4}$$

(3) The equilibrium strategy of Player 1 in the dynamic game is written as

$$\arg\max_{s_1 \in S} u_1(s_1, s_2(s_1), s_3(s_1)).$$

Let

$$s_1^* = \arg\max_{s_1 \in S} u_1(s_1, s_2(s_1), s_3(s_1)).$$

$(s_1^*, s_2(s_1^*), s_3(s_1^*))$ is the Stackelberg equilibrium of the dynamic game when Player 1 is the leader. We assume uniqueness of the Stackelberg equilibrium. Similarly, we get s_2^* and s_3^* such that

$$s_2^* = \arg\max_{s_2 \in S} u_2(s_1(s_2), s_2, s_3(s_2)),$$

and

$$s_3^* = \arg\max_{s_3 \in S} u_3(s_1(s_3), s_2(s_3), s_3).$$

s_2^* (s_3^*) is the Stackelberg equilibrium strategy of Player 2 (Player 3) if he is the leader. By symmetry of the game

$$s_1^* = s_2^* = s_3^*.$$

Denote them by s^*.

(4) Since, by symmetry for Players 2 and 3, $s_3(s) = s_2(s)$ for any s, we have

$$s^* = \arg\max_{s \in S} u_1(s, s_2(s), s_2(s)).$$

This is equivalent to

$$u_1(s^*, s_2(s^*), s_2(s^*)) > u_1(s, s_2(s), s_2(s)) \text{ for any } s \in S,\ s \neq s^*.$$

Suppose a state such that $s_1 = s_2 = s_3 = \tilde{s}$. From (2) and (3), for $s \neq \tilde{s}$,

$$u_1(s, \tilde{s}, \tilde{s}) < 0,\ u_1(\tilde{s}, s, \tilde{s}) > 0,\ u_1(\tilde{s}, \tilde{s}, s) > 0.$$

Since $u_1(s_1, s_2, s_3)$ is jointly continuous, there exists a neighborhood $V'(\tilde{s})$ of $\tilde{s}$ such that, for $s' \in V'(\tilde{s}),\ s' \neq \tilde{s}$

$$|u_1(\tilde{s}, s', s')| < |u_1(\tilde{s}, s, \tilde{s})|,$$

and

$$u_1(\tilde{s}, s', s') > 0,$$

for s which satisfies (2) and (3). Since the game is zero-sum,

$$u_1(\tilde{s}, s', s') + u_2(\tilde{s}, s', s') + u_3(\tilde{s}, s', s') = 0.$$

By symmetry

$$u_1(\tilde{s}, s', s') = -2u_2(\tilde{s}, s', s') = -2u_1(s', \tilde{s}, s').$$

Thus,

$$u_1(s', \tilde{s}, s') < 0, \ u_1(s', s', \tilde{s}) < 0.$$

Also we have

$$|u_1(\tilde{s}, s', s')| = 2|u_1(s', \tilde{s}, s')| = 2|u_1(s', s', \tilde{s})|.$$

Since $u_1(s_1, s_2, s_3)$ is jointly continuous, if $V(\tilde{s})$ is sufficiently small, we can assume

$$|u_1(\tilde{s}, s', s') - u_1(\tilde{s}, \tilde{s}, \tilde{s})| \approx 2|u_1(\tilde{s}, s', \tilde{s}) - u_1(\tilde{s}, \tilde{s}, \tilde{s})|.$$

or

$$|u_1(\tilde{s}, s', s')| \approx 2|u_1(\tilde{s}, s', \tilde{s})|.$$

Consequently, from (4)

$$|u_1(\tilde{s}, s', s')| \approx |u_1(s', \tilde{s}, \tilde{s})|.$$

Then, there exists a neighborhood $V(\tilde{s})$ of $\tilde{s}$ such that for $s \in V(\tilde{s})$

$$|u_1(s, s_2(s), s_2(s))| < |u_1(s', \tilde{s}, \tilde{s})|, \ \text{for } s' \in V'(\tilde{s}).$$

It seems to be that

$$|s_2(s) - \tilde{s}| < |s - \tilde{s}|.$$

Since

$$u_1(s, \tilde{s}, \tilde{s}) < 0,$$

and

$$u_1(\tilde{s}, s_2(s), s_2(s)) > 0,$$

we get

$$u_1(s, s_2(s), s_2(s)) < 0.$$

This means

$$u_1(\tilde{s}, \tilde{s}, \tilde{s}) > u_1(s, s_2(s), s_2(s)), \ \text{for } s \in V(s).$$

Thus, $(\tilde{s}, \tilde{s}, \tilde{s})$ is the Stackelberg equilibrium.

We have completed the proof. □ □

3 Example: relative profit maximization in a Stackelberg oligopoly

3.1 Case 1: three firms are different each other

Consider a three firms Stackelberg oligopoly with a homogeneous good[3]. There are Firms A, B and C. The outputs of the firms are x_A, x_B and x_C. The price of the good is p. The inverse demand function is

$$p = a - x_A - x_B - x_C,\ a > 0.$$

The cost functions of the firms are $c_A^2 x_A$, $c_B^2 x_B$ and $c_C^2 x_C$. c_A, c_B and c_C are positive constants. We assume that c_A, c_B and c_C are all different. The relative profit of Firm A is

$$\varphi_A = px_A - c_A x_A^2 - \frac{1}{2}(px_B - c_B x_B^2 + px_C - c_C x_C^2).$$

The relative profit of Firm B is

$$\varphi_B = px_B - c_B x_B^2 - \frac{1}{2}(px_A - c_A x_A^2 + px_C - c_C x_C^2).$$

The relative profit of Firm C is

$$\varphi_C = px_C - c_C x_C^2 - \frac{1}{2}(px_A - c_A x_A^2 + px_B - c_B x_B^2).$$

The firms maximize their relative profits. We see

$$\varphi_A + \varphi_B + \varphi_C = 0.$$

Thus, the game is a zero-sum game. Firm A is the leader and Firms B and C are followers. In the first stage of the game Firm A determines x_A, and in the second stage Firms B and C determine x_B and x_C given x_A.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{a(4c_B+3)(4c_C+3)}{32c_Ac_Bc_C + 32c_Bc_C + 32c_Ac_C + 30c_C + 32c_Ac_B + 30c_B + 30c_A + 27},$$

$$x_B = \frac{a(4c_A+3)(4c_C+3)}{32c_Ac_Bc_C + 32c_Bc_C + 32c_Ac_C + 30c_C + 32c_Ac_B + 30c_B + 30c_A + 27},$$

$$x_C = \frac{a(4c_A+3)(4c_B+3)}{32c_Ac_Bc_C + 32c_Bc_C + 32c_Ac_C + 30c_C + 32c_Ac_B + 30c_B + 30c_A + 27}.$$

[3] In this example payoff functions are differentiable.

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{4a(4c_B + 3)(4c_C + 3)(4c_Bc_C + 4c_C + 4c_B + 3)}{A},$$
$$x_B = \frac{2a(4c_C + 3)(32c_Ac_Bc_C + 24c_Bc_C + 32c_Ac_C + 21c_C + 32c_Ac_B + 21c_B + 30c_A + 18)}{A},$$
$$x_C = \frac{2a(4c_B + 3)(32c_Ac_Bc_C + 24c_Bc_C + 32c_Ac_C + 21c_C + 32c_Ac_B + 21c_B + 30c_A + 18)}{A},$$

where

$$\begin{aligned} A =&512c_Ac_B^2c_C^2 + 512c_B^2c_C^2 + 1024c_Ac_Bc_C^2 + 944c_Bc_C^2 + 512c_Ac_C^2 + 432c_C^2 \\ &+ 1024c_Ac_B^2c_C + 944c_B^2c_C + 1984c_Ac_Bc_C + 1680c_Bc_C + 960c_Ac_C + 747c_C + 512c_Ac_B^2 \\ &+ 432c_B^2 + 960c_Ac_B + 747c_B + 450c_A + 324. \end{aligned}$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

3.2 Case 2: the leader and one follower are symmetric

Assume $c_C = c_A$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{a(4c_B + 3)}{8c_Ac_B + 10c_B + 8c_A + 9},$$
$$x_B = \frac{a(4c_A + 3)}{8c_Ac_B + 10c_B + 8c_A + 9},$$
$$x_C = \frac{a(4c_B + 3)}{8c_Ac_B + 10c_B + 8c_A + 9}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{4a(4c_A + 3)(4c_B + 3)(4c_Ac_B + 4c_B + 4c_A + 3)}{B},$$
$$x_B = \frac{2a(4c_A + 3)(32c_A^2c_B + 56c_Ac_B + 21c_B + 32c_A^2 + 51c_A + 18)}{B},$$
$$x_C = \frac{2a(4c_B + 3)(32c_A^2c_B + 56c_Ac_B + 21c_B + 32c_A^2 + 51c_A + 18)}{B},$$

where

$$B = 512c_A^3c_B^2 + 1536c_A^2c_B^2 + 1456c_Ac_B^2 + 432c_B^2 + 1024c_A^3c_B + 2928c_A^2c_B + 2640c_Ac_B + 747c_B + 512c_A^3 + 1392c_A^2 + 1197c_A + 324.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

3.3 Case 3: two followers are symmetric

Assume $c_C = c_B$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{a(4c_B + 3)}{8c_Ac_B + 8c_B + 10c_A + 9},$$
$$x_B = \frac{a(4c_A + 3)}{8c_Ac_B + 8c_B + 10c_A + 9},$$
$$x_C = \frac{a(4c_A + 3)}{8c_Ac_B + 8c_B + 10c_A + 9}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{2a(2c_B + 1)(2c_B + 3)}{16c_Ac_B^2 + 16c_B^2 + 40c_Ac_B + 35c_B + 25c_A + 18},$$
$$x_B = \frac{2a(4c_Ac_B + 3c_B + 5c_A + 3)}{16c_Ac_B^2 + 16c_B^2 + 40c_Ac_B + 35c_B + 25c_A + 18},$$
$$x_C = \frac{2a(4c_Ac_B + 3c_B + 5c_A + 3)}{16c_Ac_B^2 + 16c_B^2 + 40c_Ac_B + 35c_B + 25c_A + 18}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

3.4 Case 4: all firms are symmetric

Nash equilibrium of the static game

Assume $c_A = c_B = c_C$.

The equilibrium outputs are

$$x_A = \frac{a}{2c_A + 3},\ x_B = \frac{a}{2c_A + 3},\ x_C = \frac{a}{2c_A + 3}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{a}{2c_A + 3},\ x_B = \frac{a}{2c_A + 3},\ x_C = \frac{a}{2c_A + 3}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are equivalent.

4 Concluding Remark

As we said in the introduction, the equivalence of the Stackelberg type dynamic game and the static game in a two-players zero-sum game is a widely known result. But, this problem in a multi-players case has not been analyzed. In this paper we have analyzed a three-players game. In the future research we want to extend the analysis in this paper to more general n-players zero-sum game.

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