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Abstract

We study a Stackelberg type symmetric dynamic three-players zero-sum game. One player is the leader and two players are followers. All players have symmetric payoff functions. The game is a two-stages game. In the first stage the leader determines the value of its strategic variable. In the second stage the followers determine the values of their strategic variables given the value of the leader's strategic variable. On the other hand, in the static game all players simultaneously determine the values of their strategic variables. We show that if and only if the game is fully symmetric, the Stackelberg equilibrium and the static equilibrium are equivalent.

Keywords: zero-sum game, Stackelberg equilibrium, leader and follower.

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1 Introduction

It is known that in a two-person zero-sum game the equilibrium of the Stackelberg type dynamic game and that of the static game are equivalent. Please see, for example, Korzhyk et. al. (2014), Ponssard and Zamir (1973), Tanaka (2014) and Yin et. al. (2010). We examine this problem in a three-players zero-sum game, and show that the equilibrium of the Stackelberg type dynamic game and that of the static game are equivalent if and only if the game is fully symmetric.

In the next section we show the main result. All players have symmetric payoff functions. One player is the leader and two players are followers. The game is a two-stages game as follows;

1. In the first stage the leader determines the value of its strategic variable.

2. In the second stage the followers determine the values of their strategic variables given the value of the leader's strategic variable.

On the other hand, in the static game all players simultaneously determine the values of their strategic variables. We show that if and only if the game is fully symmetric, the equilibrium of the Stackelberg type dynamic game and that of the static game are equivalent.

As we will show in Section 3 using a model of relative profit maximization in an oligopoly, the Stackelberg equilibrium is not equivalent to the static (Cournot) equilibrium in the following cases which are not fully symmetric.

1. All firms are asymmetric, that is, they have different cost functions.

2. Two followers are symmetric, that is, they have the same cost functions.

3. The leader and one follower are symmetric.

Only if all firms are symmetric, that is, they have the same cost functions, the Stackelberg equilibrium is equivalent to the static (Cournot) equilibrium.

2 Symmetric dynamic zero-sum game

There is a three-players and two-stages game. Players are called Player 1, Player 2 and Player 3. The strategic variable of Player i is $s_i,\ i \in \{1,2,3\}$. The set of strategic variable of Player i is $S_i,\ i \in \{1,2,3\}$, which is a convex and compact set of a linear topological space. One of players is the leader and other players are followers.

The structure of the game is as follows.

1. The first stage

   The leader determines the value of its strategic variable.

2. The second stage

   Followers determine the values of their strategic variables given the value of the leader's strategic variable.

Thus, the game is a Stackelberg type dynamic game. We investigate a sub-game perfect equilibrium of this game.

On the other hand, there is a static game in which three players simultaneously determine the values of their strategic variables.

The payoff of Player i is denoted by $u_i(s_1, s_2, s_3)$. u_i is jointly continuous and differentiable in s_i and s_j, $j \neq i$. We assume

$$u_1(s_1, s_2, s_3) + u_2(s_1, s_2, s_3) + u_3(s_1, s_2, s_3) = 0 \text{ given } (s_1, s_2, s_3).$$

Therefore, the game is a zero-sum game.

We also assume that the game is symmetric in the sense that the payoff functions of all players are symmetric, and assume that the sets of strategic variables for all players are the same. Denote them by S.

We show the following theorem

Theorem 1. *The sub-game perfect equilibrium of the Stackelberg type dynamic zero-sum game with a leader and two followers is equivalent to the equilibrium of the static game.*

Proof. 1. The conditions for the equilibrium of the static game are

$$\frac{\partial u_1(s_1, s_2, s_3)}{\partial s_1} = 0,\ \frac{\partial u_2(s_1, s_2, s_3)}{\partial s_2} = 0,\ \frac{\partial u_3(s_1, s_2, s_3)}{\partial s_3} = 0, \tag{1}$$

given (s_1, s_2, s_3). Since the game is symmetric, we can suppose that there exists a symmetric equilibrium. We write it as (s^*, s^*, s^*) such that $s_1 = s_2 = s_3 = s^*$.

The existence of a symmetric equilibrium is ensured by the fixed point theorem. s^* is obtained as a fixed point of the following function from S to S.

$$f(s) = \arg\max_{s_1 \in S} u_1(s_1, s, s).$$

Assume that $\arg\max_{s_1 \in S} u_1(s_1, s, s)$ is single-valued. Since u_1 is continuous, $f(s)$ is continuous. S is compact. Therefore, $f(s)$ has a fixed point.

Suppose that the leader of the dynamic game is Player 1. Players 2 and 3 are followers. In the second stage of the game Players 2 and 3 determine their strategic variables to maximize their payoffs given the value of the strategic variable of Player 1. The conditions for maximization of the payoffs of Players 2 and 3 are

$$\frac{\partial u_2(s_1, s_2, s_3)}{\partial s_2} = 0, \tag{2a}$$

and

$$\frac{\partial u_3(s_1, s_2, s_3)}{\partial s_3} = 0. \tag{2b}$$

Denote the values of s_2 and s_3 obtained from (2a) and (2b) given s_1 by

$$s_2(s_1) \text{ and } s_3(s_1).$$

By symmetry of the game for Players 2 and 3 we have

$$s_2(s_1) = s_3(s_1),$$

and

$$u_2(s_1, s_2(s_1), s_3(s_1)) = u_3(s_1, s_2(s_1), s_3(s_1)).$$

The responses of $s_2(s_1)$ and $s_3(s_1)$ to a change in s_1 are written as

$$\frac{ds_2}{ds_1} \text{ and } \frac{ds_3}{ds_1}.$$

We assume

$$\frac{ds_2}{ds_1} \neq 0, \frac{ds_2}{ds_1} \neq 1, \frac{ds_3}{ds_1} \neq 0, \frac{ds_3}{ds_1} \neq 1.$$

By symmetry, when $s_2 = s_3$ we have

$$\frac{ds_2}{ds_1} = \frac{ds_3}{ds_1}. \tag{3}$$

Also, we get

$$\frac{\partial u_2(s_1, s_2, s_3)}{\partial s_2} = \frac{\partial u_3(s_1, s_2, s_3)}{\partial s_3}.$$

In the first stage of the game Player 1 determines s_1 to maximize its payoff taking the behaviors of Players 2 and 3 into account. The value of u_1 with $s_2(s_1)$ and $s_3(s_1)$ is

$$u_1(s_1, s_2(s_1), s_3(s_1)).$$

The condition for maximization of u_1 in the dynamic game is

$$\frac{\partial u_1(s_1, s_2, s_3)}{\partial s_1} + \frac{\partial u_1(s_1, s_2, s_3)}{\partial s_2}\frac{ds_2}{ds_1} + \frac{\partial u_1(s_1, s_2, s_3)}{\partial s_3}\frac{ds_3}{ds_1} = 0. \tag{4}$$

By symmetry, when $s_2 = s_3$,

$$\frac{\partial u_1(s_1, s_2, s_3)}{\partial s_2} = \frac{\partial u_1(s_1, s_2, s_3)}{\partial s_3}.$$

From (3)

$$\frac{\partial u_1(s_1,s_2,s_3)}{\partial s_2}\frac{ds_2}{ds_1} = \frac{\partial u_1(s_1,s_2,s_3)}{\partial s_3}\frac{ds_3}{ds_1}.$$

Thus, (4) is rewritten as

$$\frac{\partial u_1(s_1,s_2,s_3)}{\partial s_1} + 2\frac{\partial u_1(s_1,s_2,s_3)}{\partial s_2}\frac{ds_2}{ds_1} = 0. \tag{5}$$

Since the game is zero-sum

$$u_1(s_1,s_2,s_3) + u_2(s_1,s_2,s_3) + u_3(s_1,s_2,s_3) = 0.$$

This means

$$u_1(s_1,s_2,s_3) = -u_2(s_1,s_2,s_3) - u_3(s_1,s_2,s_3).$$

By symmetry for Players 2 and 3, when $s_2 = s_3$,

$$u_1(s_1,s_2,s_3) = -2u_2(s_1,s_2,s_3).$$

Thus,

$$\frac{\partial u_1(s_1,s_2,s_3)}{\partial s_1} = -2\frac{\partial u_2(s_1,s_2,s_3)}{\partial s_1}. \tag{6}$$

2. Now we show that the equilibrium of the static game (s^*,s^*,s^*), where $s_1 = s_2 = s_3 = s^*$, satisfies the conditions for the equilibrium of the Stackelberg type dynamic game.

   Suppose a state such that $s_1 = s_2 = s_3$. By symmetry of the game for Players 1 and 2, $s_1 = s_2$ means

$$\frac{\partial u_2(s_1,s_2,s_3)}{\partial s_1} = \frac{\partial u_1(s_1,s_2,s_3)}{\partial s_2}, \tag{7}$$

   given s_3. From (6) and (7)

$$2\frac{\partial u_1(s_1,s_2,s_3)}{\partial s_2} = -\frac{\partial u_1(s_1,s_2,s_3)}{\partial s_1}.$$

   Substituting this into (5) yields,

$$\frac{\partial u_1(s_1,s_2,s_3)}{\partial s_1} - \frac{\partial u_1(s_1,s_2,s_3)}{\partial s_1}\frac{ds_2}{ds_1} = 0.$$

   Thus,

$$\frac{\partial u_1(s_1,s_2,s_3)}{\partial s_1}\left(1 - \frac{ds_2}{ds_1}\right) = 0$$

   We get

$$\frac{\partial u_1(s_1,s_2,s_3)}{\partial s_1} = 0,$$

because $\frac{ds_2}{ds_1} \neq 1$. Therefore, from (2a) and (2b), this means

$$\frac{\partial u_1(s_1, s_2, s_3)}{\partial s_1} = 0, \quad \frac{\partial u_2(s_1, s_2, s_3)}{\partial s_2} = 0, \quad \frac{\partial u_3(s_1, s_2, s_3)}{\partial s_3} = 0.$$

Since this is equivalent to (1), the equilibrium of the static game (s^*, s^*, s^*) satisfies the conditions for the equilibrium of the Stackelberg type dynamic game.

If the game is not symmetric for Players 1 and 2, (7) does not hold, and then the equilibrium of the dynamic game and that of the static game are not equivalent.

□

3 Example: relative profit maximization in a Stackelberg oligopoly

In the example in this section we consider relative profit maximization in an oligopoly[1].

3.1 Case 1: three firms are different each other

Consider a three firms Stackelberg oligopoly with a homogeneous good. There are Firms A, B and C. The outputs of the firms are x_A, x_B and x_C. The price of the good is p. The inverse demand function is

$$p = a - x_A - x_B - x_C, \ a > 0.$$

The cost functions of the firms are $c_A x_A$, $c_B x_B$ and $c_C x_C$. c_A, c_B and c_C are positive constants. We assume that c_A, c_B and c_C are different each other. The relative profit of Firm A is

$$\varphi_A = px_A - c_A x_A - \frac{1}{2}(px_B - c_B x_B + px_C - c_C x_C).$$

The relative profit of Firm B is

$$\varphi_B = px_B - c_B x_B - \frac{1}{2}(px_A - c_A x_A + px_C - c_C x_C).$$

The relative profit of Firm C is

$$\varphi_C = px_C - c_C x_C - \frac{1}{2}(px_A - c_A x_A + px_B - c_B x_B).$$

[1] About relative profit maximization in an oligopoly please see Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Matsumura, Matsushima and Cato (2013) and Vega-Redondo (1997).

The firms maximize their relative profits. We see

$$\varphi_A + \varphi_B + \varphi_C = 0.$$

Thus, the game is a zero-sum game. Firm A is the leader and Firms B and C are followers. In the first stage of the game Firm A determines x_A, and in the second stage Firms B and C determine x_B and x_C given x_A.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{3a - 5c_A + c_B + c_C}{9},\ x_B = \frac{3a - 5c_B + c_A + c_C}{9},\ x_C = \frac{3a - 5c_C + c_A + c_B}{9}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{24a - 50c_A + 13c_B + 13c_C}{72},$$

$$x_B = \frac{24a - 41c_B + 10c_A + 7c_C}{72},$$

and

$$x_C = \frac{24a - 41c_C + 10c_A + 7c_B}{72}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

3.2 Case 2: the leader and one follower are symmetric

Assume $c_B = c_A$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{3a - 4c_A + c_C}{9},\ x_B = \frac{3a - 4c_A + c_C}{9},\ x_C = \frac{3a - 5c_C + 2c_A}{9}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{24a - 37c_A + 13c_C}{72},$$

$$x_B = \frac{24a - 31c_A + 7c_C}{72},$$

and

$$x_C = \frac{24a - 41c_C + 17c_A}{72}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

3.3 Case 3: two followers are symmetric

Assume $c_C = c_B$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{3a - 5c_A + 2c_B}{9},\ x_B = \frac{3a - 4c_B + c_A}{9},\ x_C = \frac{3a - 4c_B + c_A}{9}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{12a - 25c_A + 13c_B}{36},$$

$$x_B = \frac{12a - 17c_B + 5c_A}{36},$$

and

$$x_C = \frac{12a - 17c_B + 5c_A}{36}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are not equivalent.

3.4 Case 4: all firms are symmetric

Assume $c_B = c_C = c_A$.

Nash equilibrium of the static game

The equilibrium outputs are

$$x_A = \frac{a - c_A}{3},\ x_B = \frac{a - c_A}{3},\ x_C = \frac{a - c_A}{3}.$$

Sub-game perfect equilibrium of the dynamic game

The equilibrium outputs are

$$x_A = \frac{a - c_A}{3},\ x_B = \frac{a - c_A}{3},\ x_C = \frac{a - c_A}{3}.$$

The Nash equilibrium of the static game and the sub-game perfect equilibrium of the dynamic game are equivalent.

4 Concluding Remark

As we said in the introduction, the equivalence of the Stackelberg type dynamic game and the static game in a two-players zero-sum game is a widely known result. But, this problem in a multi-players case has not been analyzed. In this paper we have analyzed a three-players game and a case where payoff functions are differentiable. In the future research we want to extend the analysis in this paper to more general n-players zero-sum game, and want to prove the equivalence of the dynamic game and the static game when payoff functions are not assumed to be differentiable.

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