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Abstract: Chong (1995) and Bai (1997) proposed a sample splitting method to estimate a multiple-break model. However, their studies focused on stationary time series models, where the identification of the first break depends on the magnitude and the duration of the break, and a testing procedure is needed to assist the estimation of the remaining breaks in subsamples split by the break points found earlier. In this paper, we focus on nonstationary multiple-break autoregressive models. Unlike the stationary case, we show that the duration of a break does not affect if it will be identified first. Rather, it depends on the stochastic order of magnitude of signal strength of the break under the case of constant break magnitude and also the square of the magnitude of the break under the case of shrinking break magnitude. Since the subsamples usually have different stochastic orders in nonstationary autoregressive models with breaks, one can therefore determine which break will be identified first. We apply this finding to the models proposed in Phillips and Yu (2011), Phillips et al. (2011) and Phillips et al. (2015a, 2015b). We provide an estimation procedure as well as the asymptotic theory for the model.

Keywords: Change point, Financial bubble, Least squares estimator, Mildly explosive, Mildly integrated.

JEL Classification: C22

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1 Introduction

The change-point problem has received considerable attention in the literature of econometrics and statistics over the past decades. Many time series data in economics are characterized by single or multiple structural changes (Hansen, 2001), and there is a vast literature on this subject. For example, Bai and Perron (1998) provided the estimation and testing procedures for linear models with multiple structural changes. Harvey et al. (2006), Halunga and Osborn (2012) and Kejriwal et al. (2013) investigated structural changes in persistence. Recent development in this area includes Fryzlewicz and Rao (2014), Cho and Fryzlewicz (2015), Lee et al. (2016), Roy et al. (2017) and Wang and Samworth (2018), who investigated the problem in high-dimensional models.

In this paper, we focus on the statistical inference for nonstationary multiple-break models, since the stationary counterpart has been extensively studied in the literature. For example, Chong (1995) and Bai (1997) proposed a sample splitting method to estimate the breaks one at a time by minimizing the residual sum of squares. In contrast, Bai and Perron (1998) proposed to estimate the breaks simultaneously by minimizing the residual sum of squares. There are pros and cons for the aforementioned estimation procedures. For example, for the simultaneous estimators of breaks, their asymptotic distributions in stationary models are symmetric, but the computational burden is heavy. The least-squares operations are of order $O(T^2)$ even under the most efficient algorithm (Bai and Perron, 2003), where T is the sample size. In contrast, for the sequential estimators of breaks, the computational burden is light (the least-squares operations are of order $O(T)$), but the asymptotic distributions of the estimators are asymmetric. Hence, additional efforts, such as repartitioning the sample, are needed in order to obtain symmetrically asymptotic distributions of the estimators. More importantly, the studies by Chong (1995) and Bai (1997) showed that which break would be identified first depends on the magnitude and the duration of the break, which are unobservable in reality. Hence, a test procedure for breaks is needed to assist in estimating the remaining breaks in the subsamples split by the breaks found earlier. However, the results of Chong (1995) and Bai (1997) are not directly applicable to nonstationary time series models.

The first contribution of this paper is to reveal the key factors determining which break will be identified first in nonstationary autoregressive models with multiple breaks. Unlike the stationary case, we show that the duration of a break does not affect if it will be identified first. Rather, it depends on the stochastic order of magnitude of signal strength of the break under the case of constant break magnitude, and also depends on the square of the magnitude of the break under the case of shrinking break magnitude. Since the subsamples usually have different stochastic orders in nonstationary autoregressive models with breaks, one can therefore determine which break will be identified first. Under this situation, a test procedure for breaks is no longer needed, and the estimation procedure for breaks can therefore be simplified.

The second contribution of this paper is to provide an estimation procedure and the asymptotic theory for a financial bubble process with two breaks by applying our previous finding. This financial bubble process is similar to but more flexible than those proposed in Phillips and Yu (2011), Phillips et al. (2011) and Phillips et al. (2015a, 2015b). It is well known that the global financial crisis of 2008 has a long-lasting negative impact on global economies and asset markets. Central bankers and regulators have made great efforts to understand the formation, evolution and burst of financial bubbles in order to develop early warning systems of financial crises. Researchers have made great contributions to the estimation and detection of bubbles, see Phillips et al. (2011), Phillips and Yu (2011), Homm and Breitung (2012), Shi and Song (2016), Phillips et al. (2015a, 2015b), Harvey et al. (2015, 2017), Harvey et al. (2016) and Phillips and Shi (2018). In the papers of Phillips et al. (2011), Phillips and Yu (2011) and Phillips et al. (2015a, 2015b), the authors proposed an AR(1) model with two changes in the AR parameter at two unknown break dates as follows:

$$y_t = \begin{cases} \beta_1 y_{t-1} + \varepsilon_t, & 1 \leq t \leq k_1^0, \\ \beta_2 y_{t-1} + \varepsilon_t, & k_1^0 + 1 \leq t \leq k_2^0, \\ y^*_{k_1^0} + \sum_{i=k_2^0+1}^{t} \varepsilon_i, & k_2^0 + 1 \leq t \leq T, \end{cases} \tag{1.1}$$

where $\beta_1 = 1, \beta_2 = 1 + c/k_T$ with $c > 0$ and k_T being an increasing sequence of T going to infinity such that $k_T = o(T)$, $y^*_{k_1^0} = y_{k_1^0} + y^*$ with $y^* = O_p(1)$ and $\{\varepsilon_t\}$ being model errors. This model consists of three regimes. The first regime is modeled by a unit root process, which represents the normal market period. The second regime is modeled by a mildly explosive process (Phillips and Magdalinos, 2007a), which represents the bubble expansion period. The third regime is modeled by an abrupt bubble collapse followed by a period of normal market conditions. This model is useful for modeling a financial bubble process from its origination, to expansion, and to its eventual collapse. Phillips et al. (2011), Phillips and Yu (2011) and Phillips et al. (2015a, 2015b) applied this model to NASDAQ data from the 1990s and confirmed Greenspan's declaration of "irrational exuberance" in December 1996.

A similar model was proposed in Harvey et al. (2017). They assumed that $y_t = \mu + u_t$, where μ is a constant, and $\{u_t\}$ contains a bubble process and a collapse process. For this model, the authors applied the least squares method to the differenced data and successfully obtained the consistent estimators for the regime change points. However, in Harvey et al. (2017), the explosive and the stationary AR (1) models, instead of the mildly explosive and the mildly integrated AR(1) models (Phillips and Magdalinos, 2007a), are used to model the bubble expansion process and the bubble collapse process respectively, which makes the model less flexible.

To make Model (1.1) more flexible, Phillips and Shi (2018) suggested the inclusion of an asymptotically negligible drift in the normal market period and the use of a transient mildly integrated process to model the bubble collapse process. Following Phillips and Shi (2018)'s suggestion, we study the following AR(1) model with two unknown break dates, namely,

$$y_t = \begin{cases} cT^{-\eta} + \beta_1 y_{t-1} + \varepsilon_t, & 1 \le t \le k_1^0, \\ \beta_2 y_{t-1} + \varepsilon_t, & k_1^0 + 1 \le t \le k_2^0, \\ \beta_3 y_{t-1} + \varepsilon_t, & k_2^0 + 1 \le t \le T, \end{cases} \tag{1.2}$$

where $c \in R$, $\eta > 1/2$, $\beta_1 = 1$, $\beta_2 = \beta_{2T} = 1 + c_1/k_T$, $\beta_3 = \beta_{3T} = 1 - c_2/h_T$, $c_1 > 0$ and $c_2 > 0$; $\{k_T\}$ and $\{h_T\}$ are two sequences of positive constants increasing to infinity such that $k_T = o(T)$ and $h_T = o(T)$. We denote $k_i^0 = [T\tau_i^0]$, $i = 1, 2$, where $[\cdot]$ denotes the integer part, and the break fractions τ_i^0s are fixed constants between zero and one.

Note that Phillips et al. (2011), Phillips and Yu (2011), Phillips et al. (2015a, 2015b) and Phillips and Shi (2018) focused on real-time bubble detection via recursive right-sided unit root testing procedures. Though consistent estimators of the break fractions can be obtained by these procedures, their convergence rates and the statistical properties of the estimators of the AR parameters are not explored. Hence, one of the aims of this paper is to provide an estimation procedure and an asymptotic theory for Model (1.2).

There are two points worth mentioning. (1) Examining structural changes in autoregressive models is of interest as the time series properties of the model, such as stationarity, may be different before and after the change. As a result, the rates of convergence and the asymptotic distributions of the estimators are difficult to derive (Chong, 2001; Pang et al., 2017). (2) The change-point analysis in this paper differs from that in Bai (1997) in several aspects. First, the model studied in Bai (1997) was a stationary time series model with multiple breaks, while we study a nonstationary time series model with multiple breaks in this paper. Second, a test procedure for breaks is needed to assist in the estimation of breaks in subsamples in Bai (1997), while such a procedure is no longer needed in our paper. Hence, the estimation procedure becomes simpler. Third, Bai (1997) derived the asymptotics for the estimators by analyzing the expectation of the residual sum of squares rather than the residual sum of squares itself. However, it is difficult to calculate the expectation of the residual sum of squares in nonstationary autoregressive models. Thus, we cannot derive the asymptotics for the estimators by following Bai (1997). Instead, we derive the asymptotics by analyzing the residual sum of squares directly, which makes the proofs more complicated and challenging.

The rest of the paper is organized as follows. Section 2 states the assumptions and develops an estimation procedure for the unknown parameters in Model (1.2). Section 3 demonstrates our theoretical findings. Section 4 presents simulation results to examine the finite sample performance of the estimators. Section 5 concludes the paper. The proofs of our theoretical results are relegated to the Appendix.

2 Model Assumptions and Estimation Procedure

2.1 Model Assumptions

For Model (1.2), we make the following assumptions:

- C1: $y_0 = o_p(\sqrt{T})$.
- C2: $\{\varepsilon_t\}$ is a sequence of i.i.d. random variables with mean zero and variance $0 < \sigma^2 < \infty$.
- C3: $\{k_T\}$ and $\{h_T\}$ are two sequences of positive constants increasing to ∞ such that $k_T = o(T)$ and $h_T = o(T)$.
- C4: $0 < \underline{\tau} < \tau_1^0 < \tau_2^0 < \overline{\tau} < 1$.

Remark 2.1 *Assumption C1 implies that y_0 will not affect the asymptotic properties of the estimators of the AR parameters and the break points. The assumption of i.i.d. errors in C2 is only for the convenience of exposition in the proofs. One can extend our results to some cases that allow for dependence of the errors. Interested readers are referred to Phillips and Magdalinos (2007b) and Magdalinos (2012) for details. In addition, the assumption of finite variance in C2 can be relaxed. Our theoretical results will still hold when the assumption of finite variance is replaced by the assumption that the model errors belong to the domain of attraction of the normal law with possible infinite variance. Interested readers are referred* *to Pang et al. (2017) for details. Assumption C3 is the same as that in Phillips and Magdalinos (2007a). Assumption C4 is standard in the change-point literature (Chong, 1995; Bai, 1997; Bai and Perron, 1998; Chong, 2001), which suggests that each regime occupies a non-vanishing proportion of the sample. This is to ensure the identifiability of the AR parameters and the break points.*

2.2 Estimation Procedure

We will conduct a sequential procedure in the estimation of structural changes for Model (1.2). First of all, we rewrite Model (1.2) as follows:

$$y_t = \begin{cases} \beta_1 y_{t-1} + u_t, & 1 \le t \le k_1^0 \\ \beta_2 y_{t-1} + u_t, & k_1^0 + 1 \le t \le k_2^0 \\ \beta_3 y_{t-1} + u_t, & k_2^0 + 1 \le t \le T \end{cases}, \tag{2.1}$$

where $u_t = cT^{-\eta} + \varepsilon_t$ when $t \le k_1^0$ and $u_t = \varepsilon_t$ when $t > k_1^0$. To develop an estimation procedure for Model (2.1), we first compute the difference of the residual sums of squares at k_1^0 and k_2^0. This difference is a result of Lemma A.1 in the Appendix which allows us to develop a sequential estimation procedure. Let $RSS(\tau)$ be the residual sum of squares at the date $[\tau T]$, then it can be shown that

Theorem 2.1 *For Model (2.1), we have*

$$RSS(\tau_1^0) - RSS(\tau_2^0) = \eta_1(\beta_2 - \beta_1) + \eta_2(\beta_3 - \beta_2) + \eta_3(\beta_2 - \beta_1)^2 + \eta_4(\beta_3 - \beta_2)^2 + \Omega_T,$$

where

$$\begin{cases} \eta_1 = 2\left(\dfrac{\sum_{t=1}^{k_1^0} y_{t-1}u_t}{\sum_{t=1}^{k_1^0} y_{t-1}^2} - \dfrac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2} \right) \displaystyle\sum_{t=1}^{k_1^0} y_{t-1}^2 \\ \eta_2 = \dfrac{2\left(\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}u_t - \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t \sum_{t=k_2^0+1}^{T} y_{t-1}^2 \right)}{\sum_{t=k_1^0+1}^{T} y_{t-1}^2} \\ \eta_3 = -\dfrac{\sum_{t=1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2} \\ \eta_4 = \dfrac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=k_1^0+1}^{T} y_{t-1}^2} \end{cases}$$

and

$$\Omega_T = \frac{(\sum_{t=1}^{k_2^0} y_{t-1}u_t)^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2} + \frac{(\sum_{t=k_2^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=1}^{k_1^0} y_{t-1}u_t)^2}{\sum_{t=1}^{k_1^0} y_{t-1}^2} - \frac{(\sum_{t=k_1^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_1^0+1}^{T} y_{t-1}^2}.$$

In general, Ω_T has the smallest stochastic order of magnitude among the five terms in the closed form of $RSS(\tau_1^0) - RSS(\tau_2^0)$, $\eta_1(\beta_2 - \beta_1)$ has a smaller stochastic order of magnitude than $\eta_3(\beta_2 - \beta_1)^2$, and $\eta_2(\beta_3 - \beta_2)$ has a smaller stochastic order of magnitude than $\eta_4(\beta_3 - \beta_2)^2$. For example, suppose Model (1.2) is a stationary model, that is, all β_i's are fixed constants satisfying $|\beta_i| < 1$, then we have

$$\eta_1(\beta_2 - \beta_1) = O_p(\sqrt{T}), \quad \eta_2(\beta_3 - \beta_2) = O_p(\sqrt{T}), \quad \Omega_T = O_p(1)$$

and

$$\eta_3(\beta_2 - \beta_1)^2 = O_p(T), \quad \eta_4(\beta_3 - \beta_2)^2 = O_p(T).$$

Therefore,

$$\begin{aligned} RSS(\tau_1^0) - RSS(\tau_2^0) &= -\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}(\beta_2 - \beta_1)^2(1 + o_p(1)) \\ &\quad + \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=k_1^0+1}^{T} y_{t-1}^2}(\beta_3 - \beta_2)^2(1 + o_p(1)) \\ &= P_2(1 + o_p(1)) - P_1(1 + o_p(1)), \end{aligned} \tag{2.2}$$

where

$$P_1 := \eta_3(\beta_2 - \beta_1)^2 \quad \text{and} \quad P_2 := \eta_4(\beta_3 - \beta_2)^2.$$

The η_3 and η_4 are the signal strength of breaks, and the $(\beta_2 - \beta_1)^2$ and $(\beta_3 - \beta_2)^2$ are the squares of the magnitude of breaks.

Therefore, which break point will be identified first is determined by the stochastic orders of P_1 and P_2. If P_2 has a higher stochastic order of magnitude than P_1, then $RSS(\tau_1^0) - RSS(\tau_2^0)$ will diverge to ∞ in probability, and k_2^0 will be identified first asymptotically. Instead, if P_1 has a higher stochastic order of magnitude than P_2, then $RSS(\tau_1^0) - RSS(\tau_2^0)$ will go to $-\infty$ in probability, and k_1^0 will be identified first asymptotically. However, when P_1 and P_2 have the same stochastic order of magnitude, which break will be identified first depends on the magnitude and the duration of the break, which are unobservable in reality. Therefore, it is difficult to determine which break will be uncovered first, and we need to test and estimate the second break from all subsamples split by the first estimated break point. We provide three illustrative examples below.

Example 1 (a stationary model with two breaks): Suppose all β_i's are fixed constants satisfying $|\beta_i| < 1, i = 1, 2, 3$. Then, we have

$$\frac{P_1}{T} \xrightarrow{p} \sigma^2 \frac{\frac{\tau_1^0}{1-\beta_1^2} \cdot \frac{\tau_2^0-\tau_1^0}{1-\beta_2^2}}{\frac{\tau_1^0}{1-\beta_1^2} + \frac{\tau_2^0-\tau_1^0}{1-\beta_2^2}} (\beta_2 - \beta_1)^2 = \frac{\tau_1^0(\tau_2^0 - \tau_1^0)(\beta_2 - \beta_1)^2}{\tau_1^0(1-\beta_2^2) + (\tau_2^0 - \tau_1^0)(1-\beta_1^2)} \sigma^2$$

and

$$\frac{P_2}{T} \xrightarrow{p} \sigma^2 \frac{\frac{\tau_2^0-\tau_1^0}{1-\beta_2^2} \cdot \frac{1-\tau_2^0}{1-\beta_3^2}}{\frac{\tau_2^0-\tau_1^0}{1-\beta_2^2} + \frac{1-\tau_2^0}{1-\beta_3^2}} (\beta_3 - \beta_2)^2 = \frac{(\tau_2^0 - \tau_1^0)(1-\tau_2^0)(\beta_3 - \beta_2)^2}{(\tau_2^0 - \tau_1^0)(1-\beta_3^2) + (1-\tau_2^0)(1-\beta_2^2)} \sigma^2$$

Therefore, P_1 and P_2 have the same stochastic order of magnitude ($O_p(T)$). A simulation of $RSS(\tau)/T$ with $T = 800$ for this example is plotted in the upper panel of Figure 1. Given that (2.2) is true, if

$$\frac{(1-\tau_2^0)(\beta_3 - \beta_2)^2}{(\tau_2^0 - \tau_1^0)(1-\beta_3^2) + (1-\tau_2^0)(1-\beta_2^2)} < \frac{\tau_1^0(\beta_2 - \beta_1)^2}{\tau_1^0(1-\beta_2^2) + (\tau_2^0 - \tau_1^0)(1-\beta_1^2)}, \tag{2.3}$$

then $RSS(\tau_1^0) - RSS(\tau_2^0) \xrightarrow{p} -\infty$, and k_1^0 will be identified first with probability approaching unity. If the inequality (2.3) is reversed, then $RSS(\tau_1^0) - RSS(\tau_2^0) \xrightarrow{p} \infty$, and k_2^0 will be identified first with probability approaching unity. In the case of equality, k_1^0 and k_2^0 will have the same chance of being identified first asymptotically. Note that condition (2.3) is similar to condition (16) in Chong (1995) and Assumption A.4 in Bai (1997). However, (2.3) is unobservable in reality. Hence, a test procedure is applied to all subsamples split by the first estimated break point in order to find the remaining break point.

Example 2 (a nonstationary model with two breaks): Suppose β_1 is a fixed constant satisfying $|\beta_1| < 1$, $\beta_2 = 1$ and $\beta_3 = 1 - c_2/h_T$, which means the multiple-break model consists of a stationary process, a unit root process and a mildly integrated process. In this case,

$$\begin{cases} \eta_1(\beta_2 - \beta_1) = (O_p(\frac{\sqrt{T}}{T}) - O_p(\frac{T}{T^2})) \cdot O_p(T) = O_p(\sqrt{T}) \\ \eta_2(\beta_3 - \beta_2) = \frac{O_p(T^2\sqrt{Th_T}) - O_p(T \cdot Th_T)}{O_p(T^2) + O_p(Th_T)} \cdot \frac{1}{h_T} = O_p(\sqrt{\frac{T}{h_T}}) \\ \Omega_T = O_p(1) \\ P_1 = \frac{O_p(T) \cdot O_p(T^2)}{O_p(T) + O_p(T^2)} = O_p(T) \\ P_2 = \frac{O_p(T^2) \cdot O_p(Th_T)}{O_p(T^2) + O_p(Th_T)} \cdot \frac{1}{h_T^2} = O_p(\frac{T}{h_T}) \end{cases}$$

by the well-known results of the unit root model and Lemma B.3 in Pang et al. (2017). Hence, (2.2) is true, and P_1 has a higher stochastic order than P_2, which means $RSS(\tau_1^0) - RSS(\tau_2^0) = -P_1(1 + o_p(1)) \xrightarrow{p} -\infty$, and k_1^0 will be uncovered first with probability approaching unity. A simulation of $RSS(\tau)/T$ with $T = 800$ for this example is plotted in the middle panel of Figure 1.

Example 3 (a nonstationary model with two breaks): Suppose $\beta_1 = 1, \beta_2 = 1 + c_1/k_T$ and $\beta_3 = 1 - c_2/h_T$, which suggests that the multiple-break model consists of a unit root process, a mildly explosive process and a mildly integrated process. In this case,

$$\begin{cases} \eta_1(\beta_2 - \beta_1) = (O_p(\frac{T}{T^2}) - O_p(\frac{\beta_2^{k_2^0 - k_1^0}\sqrt{Tk_T}}{\beta_2^{2(k_2^0-k_1^0)}Tk_T})) \cdot O_p(T^2) \cdot O(\frac{1}{k_T}) = O_p(\frac{T}{k_T}) \\ \eta_2(\beta_3 - \beta_2) = \frac{O_p(\beta_2^{2(k_2^0-k_1^0)}Tk_T \cdot \beta_2^{k_2^0-k_1^0}\sqrt{Th_T}) - O_p(\beta_2^{k_2^0-k_1^0}\sqrt{Tk_T}\cdot\beta_2^{2(k_2^0-k_1^0)}Th_T)}{O_p(\beta_2^{2(k_2^0-k_1^0)}Tk_T) + O_p(\beta_2^{2(k_2^0-k_1^0)}Th_T)} \cdot O(\frac{1}{k_T} + \frac{1}{h_T}) \\ \qquad = \begin{cases} O_p(\beta_2^{k_2^0-k_1^0}\sqrt{\frac{T}{h_T}}), & \text{when } h_T = O(k_T) \\ O_p(\beta_2^{k_2^0-k_1^0}\sqrt{\frac{T}{k_T}}), & \text{when } k_T = o(h_T) \end{cases} \\ \Omega_T = O_p(1) \\ P_1 = \frac{O_p(T^2)\cdot O_p(\beta_2^{2(k_2^0-k_1^0)}Tk_T)}{O_p(T^2) + O_p(\beta_2^{2(k_2^0-k_1^0)}Tk_T)} \cdot O(\frac{1}{k_T^2}) = O_p(\frac{T^2}{k_T^2}) \\ P_2 = \frac{O_p(\beta_2^{2(k_2^0-k_1^0)}Tk_T)\cdot O_p(\beta_2^{2(k_2^0-k_1^0)}Th_T)}{O_p(\beta_2^{2(k_2^0-k_1^0)}Tk_T) + O_p(\beta_2^{2(k_2^0-k_1^0)}Th_T)} \cdot O(\frac{1}{k_T} + \frac{1}{h_T})^2 \\ \quad = \begin{cases} O_p(\beta_2^{2(k_2^0-k_1^0)}\frac{T}{h_T}), & \text{when } h_T = O(k_T) \\ O_p(\beta_2^{2(k_2^0-k_1^0)}\frac{T}{k_T}), & \text{when } k_T = o(h_T) \end{cases} \end{cases}$$

by Lemmas A.2-A.4 in the Appendix. Thus, (2.2) is true, and P_2 has a markedly higher stochastic order than P_1, which means $RSS(\tau_1^0) - RSS(\tau_2^0) = P_2(1 + o_p(1)) \xrightarrow{p} \infty$, and k_2^0 will be uncovered first with probability approaching unity. A simulation of $RSS(\tau)/T$ with $T = 800$ for this example is plotted in the bottom panel of Figure 1.

Based on the above analysis, for Model (1.2), we propose the following two-step estimation procedure.

Step 1: For any given $0 < \tau < 1$, denote

$$\hat{\beta}_x(\tau) = \frac{\sum_{t=1}^{[\tau T]} y_t y_{t-1}}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} \quad \text{and} \quad \hat{\beta}_3(\tau) = \frac{\sum_{t=[\tau T]+1}^{T} y_t y_{t-1}}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}.$$

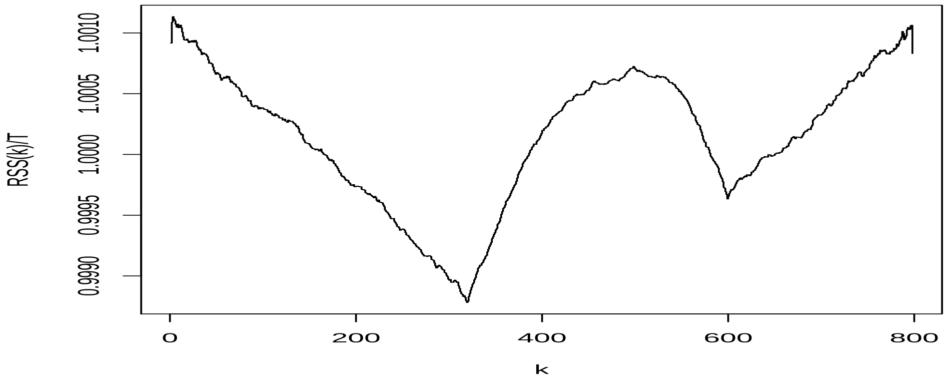
Then the change-point estimator of τ_2^0 is defined as

$$\hat{\tau}_{2,T} = \arg\min_{\tau\in(0,1)} RSS_{2,T}(\tau),$$

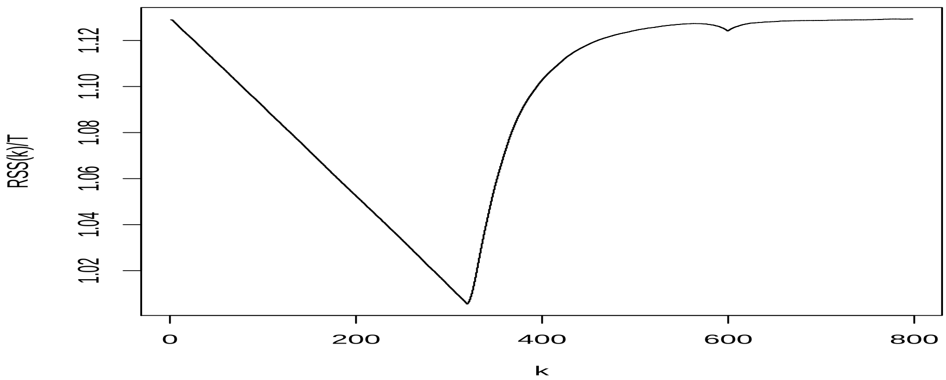
where

$$RSS_{2,T}(\tau) = \sum_{t=1}^{[\tau T]} \Big(y_t - \hat{\beta}_x(\tau) y_{t-1}\Big)^2 + \sum_{t=[\tau T]+1}^{T} \Big(y_t - \hat{\beta}_3(\tau) y_{t-1}\Big)^2.$$

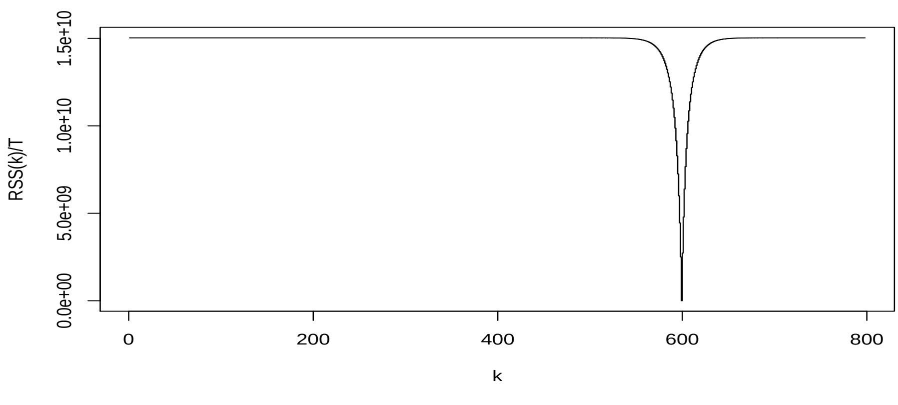
Once we obtain $\hat{\tau}_{2,T}$, the least squares estimator (LSE) of β_3 is represented by $\hat{\beta}_3(\hat{\tau}_{2,T})$, and the LSE of k_2^0 is denoted by $\hat{k}_2 = [\hat{\tau}_{2,T} T]$.



(a) Graph of $RSS(\tau)/T$ for $\beta_1 = 0.7, \beta_2 = 0.8$ and $\beta_3 = 0.7$, $T = 800, k_1^0 = 320$ and $k_2^0 = 600$.



(b) Graph of $RSS(\tau)/T$ for $\beta_1 = 0.5, \beta_2 = 1$ and $\beta_3 = 0.97$, $T = 800, k_1^0 = 320$ and $k_2^0 = 600$.



(c) Graph of $RSS(\tau)/T$ for $\beta_1 = 1, \beta_2 = 1.05$ and $\beta_3 = 0.95$, $T = 800, k_1^0 = 320$ and $k_2^0 = 600$.

Figure 1: Graphs of $RSS(\tau)/T$ for Examples 1-3 (from top to bottom).

Step 2: For any given $0 < \tau < \hat{\tau}_{2,T}$, the LSEs of the AR parameters β_1 and β_2 are given by

$$\hat{\beta}_1(\tau) = \frac{\sum_{t=1}^{[\tau T]} y_t y_{t-1}}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} \quad \text{and} \quad \hat{\beta}_2(\tau) = \frac{\sum_{t=[\tau T]+1}^{\hat{k}_2} y_t y_{t-1}}{\sum_{t=[\tau T]+1}^{\hat{k}_2} y_{t-1}^2},$$

respectively. Then the change-point estimator of τ_1^0 is defined as

$$\hat{\tau}_{1,T} = \arg\min_{\tau \in (0, \hat{\tau}_{2,T})} RSS_{1,T}(\tau),$$

where

$$RSS_{1,T}(\tau) = \sum_{t=1}^{[\tau T]} \Big(y_t - \hat{\beta}_1(\tau) y_{t-1} \Big)^2 + \sum_{t=[\tau T]+1}^{\hat{k}_2} \Big(y_t - \hat{\beta}_2(\tau) y_{t-1} \Big)^2.$$

Once we obtain $\hat{\tau}_{1,T}$, the final LSEs of β_1 and β_2 are represented by $\hat{\beta}_1(\hat{\tau}_{1,T})$ and $\hat{\beta}_2(\hat{\tau}_{1,T})$ respectively, and the LSE of k_1^0 is denoted by $\hat{k}_1 = [\hat{\tau}_{1,T} T]$.

Remark 2.2 *There are two structural changes in Model (2.1), and we estimate them sequentially. In Step 1, we estimate a mis-specified model with one break, then in Step 2 we estimate the other break in the left subsample split by the first break point estimate since we know that k_2^0 is identified first with probability approaching unity. The idea of estimating breaks under a mis-specified model is taken from Chong (1995) and Bai (1997). However, there are three major differences between our work and those of Chong (1995) and Bai (1997). First, the models studied in Chong (1995) and Bai (1997) were both a stationary model with multiple breaks, while the model studied in this paper is a nonstationary model with multiple breaks. Second, a test procedure of breaks is needed in order to locate the remaining breaks in Chong (1995) and Bai (1997), while such a test procedure is no longer needed in our paper. Third, Chong (1995) derived the asymptotics for the estimators by analyzing the probability limit of the criterion function, and Bai (1997) derived the asymptotics for the estimators by analyzing the expectation of the criterion function. However, it is difficult to calculate the probability limit or the expectation of the criterion function in nonstationary autoregressive models. Thus, we cannot derive the asymptotics for the estimators by applying the same arguments in Chong (1995) or in Bai (1997). Instead, we derive the asymptotics by analyzing the criterion function directly in the proofs.*

3 Main Results

We define some notations before proceeding to our main results for Model (1.2). Let $W(\cdot)$ be an independent standard Brownian motion defined on $[0, 1]$, and $W_1(\cdot)$ and $W_2(\cdot)$ be two independent Brownian motions defined on R_+. "$\Rightarrow$" denotes the weak convergence of the associated probability measures. "$\xrightarrow{p}$" denotes convergence in probability, and "$\overset{d}{=}$" means being identical in distribution. The notation $a_T \asymp b_T$ means there exist two positive constants $c_1^{'}$ and $c_2^{'}$ such that $c_1^{'} \leq a_T/b_T \leq c_2^{'}$ for all large T, where a_T and b_T are two positive functions of T. Finally, for Model (1.2), we denote

$$\begin{cases} t_1 = \sqrt{\dfrac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sigma^2}}(\hat{\beta}_1(\hat{\tau}_{1,T}) - \beta_1) \\ t_2 = \sqrt{\dfrac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sigma^2}}(\hat{\beta}_2(\hat{\tau}_{1,T}) - \beta_2) \\ t_3 = \sqrt{\dfrac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sigma^2}}(\hat{\beta}_3(\hat{\tau}_{2,T}) - \beta_3) \end{cases}$$

as the t-ratios of β_1, β_2 and β_3 respectively.

Theorem 3.1 *For Model (1.2), under assumptions C1-C4, the following results hold:*
(a) $\hat{k}_2$ is consistent, but $\hat{k}_1$ is not necessarily consistent, more specifically, when k_T diverges to ∞ such that $k_T = o(T)$, we have

$$\begin{cases} P(\hat{k}_1 \neq k_1^0) \to 0, \quad \textit{when } k_T = o(\sqrt{T}) \\ |\hat{k}_1 - k_1^0| = O_p(1), \quad \textit{when } k_T \asymp \sqrt{T} \\ \dfrac{c_1^2 T^2}{k_T^2}(\hat{\tau}_{1,T} - \tau_1^0) \Rightarrow \arg\max\limits_{\nu \in R} \left\{ \dfrac{W^*(\nu)}{W_1(\tau_1^0)} - \dfrac{|\nu|}{2} \right\}, \quad \textit{when } \sqrt{T} = o(k_T) \\ P(\hat{k}_2 \neq k_2^0) \to 0 \end{cases}, \tag{3.1}$$

where $W^*(\nu)$ is a two-sided Brownian motion on R defined to be $W^*(\nu) = W_1(-\nu)$ for $\nu \leq 0$ and $W^*(\nu) = W_2(\nu)$ for $\nu > 0$.
(b) $\hat{\beta}_1(\hat{\tau}_{1,T})$, $\hat{\beta}_2(\hat{\tau}_{1,T})$ and $\hat{\beta}_3(\hat{\tau}_{2,T})$ are all consistent, and their limiting distributions are respectively given by

$$\begin{cases} k_1^0(\hat{\beta}_1(\hat{\tau}_{1,T}) - \beta_1) \Rightarrow \dfrac{W^2(\tau_1^0) - \tau_1^0}{2\int_0^{\tau_1^0} W^2(s)ds} \overset{d}{=} \dfrac{W^2(1) - 1}{2\int_0^1 W^2(s)ds} \\ \sqrt{\dfrac{\tau_1^0 k_2^0 k_T}{2c_1}} \beta_2^{k_2^0 - k_1^0}(\hat{\beta}_2(\hat{\tau}_{1,T}) - \beta_2) \Rightarrow \xi \\ \sqrt{\dfrac{k_1^0 h_T}{2c_2}} \beta_2^{k_2^0 - k_1^0}(\hat{\beta}_3(\hat{\tau}_{2,T}) - \beta_3) \Rightarrow \zeta \end{cases}, \tag{3.2}$$

where ξ and ζ are two independent standard Cauchy variates.

(c) The limiting distributions of the t-ratios of β_1, β_2 and β_3 are respectively given by

$$\begin{cases} t_1 \Rightarrow \dfrac{W^2(1)-1}{2\sqrt{\int_0^1 W^2(s)ds}} \\ t_2 \Rightarrow N(0,1) \\ t_3 \Rightarrow N(0,1) \end{cases}. \tag{3.3}$$

Without the structural changes, it has been proved by Phillips and Magdalinos (2007a) that the convergence rate of the LSE of the mildly integrated AR parameter β_3 is $\sqrt{Th_T}$ when the initial value of the model is of order $o_p(\sqrt{h_T})$, and the convergence rate of the LSE of the mildly explosive AR parameter β_2 is $k_T\beta_2^T$ when the initial value of the model is of order $o_p(\sqrt{k_T})$. It is surprising to find that, in the presence of structural changes, the convergence rate of the LSE of β_3 can be faster than that of β_2 when $k_T = o(h_T)$. This is due to the difference in the stochastic order of magnitude of initial values across subamples. Note that the stochastic order of magnitude of $y_{k_2^0}$ is higher than that of $y_{k_1^0}$ (see Lemmas A.2 and A.3 in the Appendix for details), which affects the asymptotic properties of LSEs of β_2 and β_3.

As pointed out in Pang et al. (2017), the distribution of $\frac{W^2(1)-1}{2\sqrt{\int_0^1 W^2(s)ds}}$ is markedly less skewed than that of $\frac{W^2(1)-1}{2\int_0^1 W^2(s)ds}$. Moreover, the second and third limiting distributions in (3.2) are both Cauchy, which has an explosive mean and variance, while the second and third limiting distributions in (3.3) are both normal, which has a finite mean and variance. Hence, the t-ratios of β_1, β_2 and β_3 obviously have better estimation accuracy for the AR parameters than the LSEs of β_1, β_2 and β_3. It is recommended to use the t-ratios instead of the LSEs of β_1, β_2 and β_3 to conduct further statistical inference in applications.

The precision of $\hat{k}_1$ and $\hat{k}_2$ mainly depends on the differences of breaks (i.e., $|\beta_2 - \beta_1|$ and $|\beta_3 - \beta_2|$) and their signal strength. Note that for constant break magnitude, since the magnitude of the break is $O(1)$, while the signal strength will have different stochastic orders of magnitude, the magnitude of the break plays no role in the determination of the first identified break. For shrinking breaks, when the signal strength of k_2^0 is strong (see Example 3 in the last section) and the difference between β_2 and β_3 (that is, $c_1/k_T + c_2/h_T$) is large, then k_2^0 can be consistently located for any $k_T = o(T)$ and $h_T = o(T)$. However, since the signal strength of k_1^0 is not strong enough (also see Example 3 in the last section) and the difference between β_1 and β_2 (that is, c_1/k_T) may not be sufficiently large, k_1^0 can only be consistently estimated when $k_T = o(\sqrt{T})$, which means β_1 and β_2 have enough difference.

Remark 3.1 *Note that t_1, t_2 and t_3 in Theorem 3.1 are not pivotal, hence they will be useless in practice. However, by denoting*

$$\begin{cases} t_1^{'} = \sqrt{\dfrac{\sum_{t=1}^{\hat{k}_1} y_{t-1}^2}{\hat{\sigma}^2}}(\hat{\beta}_1(\hat{\tau}_{1,T}) - \beta_1) \\ t_2^{'} = \sqrt{\dfrac{\sum_{t=\hat{k}_1+1}^{\hat{k}_2} y_{t-1}^2}{\hat{\sigma}^2}}(\hat{\beta}_2(\hat{\tau}_{1,T}) - \beta_2) \\ t_3^{'} = \sqrt{\dfrac{\sum_{t=\hat{k}_2+1}^{T} y_{t-1}^2}{\hat{\sigma}^2}}(\hat{\beta}_3(\hat{\tau}_{2,T}) - \beta_3) \end{cases}$$

with

$$\hat{\sigma}^2 = \frac{1}{T}\left\{\sum_{t=1}^{\hat{k}_1}(y_t - \hat{\beta}_1(\hat{\tau}_{1,T})y_{t-1})^2 + \sum_{t=\hat{k}_1+1}^{\hat{k}_2}(y_t - \hat{\beta}_2(\hat{\tau}_{1,T})y_{t-1})^2 + \sum_{t=\hat{k}_2+1}^{T}(y_t - \hat{\beta}_3(\hat{\tau}_{2,T})y_{t-1})^2\right\},$$

it can be proved that

$$\frac{\sum_{t=1}^{\hat{k}_1^0} y_{t-1}^2}{\sum_{t=1}^{k_1^0} y_{t-1}^2} \xrightarrow{p} 1, \quad \frac{\sum_{t=\hat{k}_1^0+1}^{\hat{k}_2^0} y_{t-1}^2}{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2} \xrightarrow{p} 1, \quad \frac{\sum_{t=\hat{k}_2^0+1}^{T} y_{t-1}^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} \xrightarrow{p} 1 \text{ and } \frac{\hat{\sigma}^2}{\sigma^2} \xrightarrow{p} 1.$$

Therefore, part (c) of Theorem 3.1 will still hold when t_1, t_2 and t_3 are replaced by $t_1^{'}, t_2^{'}$ and $t_3^{'}$ respectively. Note that $t_1^{'}, t_2^{'}$ and $t_3^{'}$ can be used directly in applications.

Remark 3.2 *The model studied in this section is closely related to that of Phillips and Shi (2018). In fact, Phillips and Shi (2018) proposed an AR(1) model with three structural changes in the AR parameter to model a bubble process from its origination, expansion, collapse to its reversion to normal behavior. Hence, the bubble process consists of four regimes. The first three regimes are the same as the model studied in this section, and the last regime is a unit root process. It is interesting and important to study the break points of the model proposed in Phillips and Shi (2018). Our results can be applied to the above model. However, the sequential method used in this section heavily relies on the closed forms of the discrepancy of residual sum of squares when the break fraction departs from the true one, and it is extremely difficult and tedious to develop such closed forms for nonstationary processes with three structural changes. We leave this as future work.*

4 Simulations

For empirical applications, we perform the following experiments to see how well the finite sample properties of the estimators in the previous section follow our asymptotic results.

Note that the t-ratios of β_1, β_2 and β_3 have better estimation accuracy for the AR parameters than the LSEs of β_1, β_2 and β_3, and therefore it is recommended to use the t-ratios to conduct statistical inference in empirical applications. As such, we only perform the experiments for parts (a) and (c) of Theorem 3.1. We adopt the parameters similar to those in Pang et al. (2017) in the following experiments. The sample size is set at $T = 800$, the interval $[\underline{\tau}, \overline{\tau}]$ is taken as $[0.05, 0.95]$ (hence the break fraction τ_2^0 is searched within this interval in our experiments), and the two true break fractions are set at $\tau_1^0 = 0.40$ (hence $k_1^0 = 320$) and $\tau_2^0 = 0.75$ (hence $k_2^0 = 600$) respectively. The number of replications is set at $N = 50,000$, $\{y_t\}_{t=1}^T$ are generated from Model (1.2), y_0 is set at zero for simplicity and $\{\varepsilon_t\}_{t=1}^T$ are generated independently from $N(0, 1)$. We also set $c = 1$ and $\eta = 1$ for the drift c/T^η. Moreover, for the parameter β_2, we set $c_1 = 0.85$* and $k_T = T^\alpha$ with $\alpha \in \{0.3, 0.5, 0.7\}$. The case where $\alpha = 0.3$ represents $k_T = o(\sqrt{T})$, the case where $\alpha = 0.5$ represents $k_T \asymp \sqrt{T}$ and the case where $\alpha = 0.7$ represents $\sqrt{T} = o(k_T)$. For the parameter β_3, we set $c_2 = 3$ and $h_T = T^{0.5}$. The graph of the distribution of $\frac{W^2(1)-1}{2\sqrt{\int_0^1 W^2(s)ds}}$ is plotted by dividing the interval $[0, 1]$ into $5,000$ equal-spaced subintervals and use the corresponding Riemann sums to approximate the integral. The number of replications is also set at $N = 50,000$.

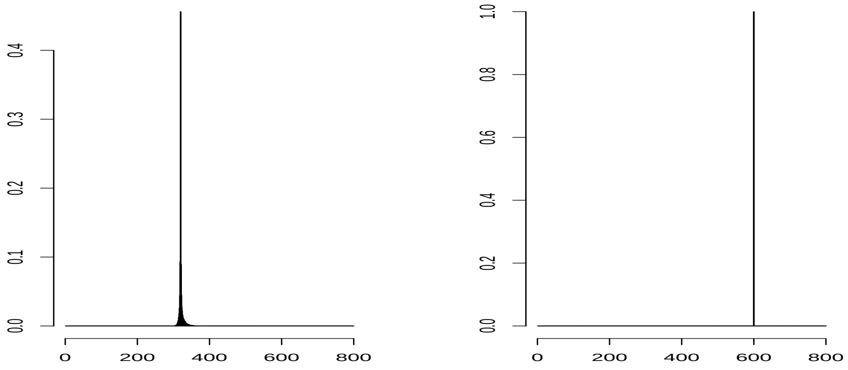
Note that, in our setup, β_1 and β_2 have a large difference ($|\beta_1 - \beta_2| = 0.85/800^{0.3} = 0.114$) when $\alpha = 0.3$, a moderate difference ($|\beta_1 - \beta_2| = 0.85/800^{0.5} = 0.030$) when $\alpha = 0.5$ and a very small difference ($|\beta_1 - \beta_2| = 0.85/800^{0.7} = 0.008$) when $\alpha = 0.7$. Moreover, the difference between β_2 and β_3 is not smaller than $0.85/800^{0.7} + 3/800^{0.5} = 0.114$, meaning that the magnitude of the break $|\beta_3 - \beta_2|$ is sufficiently large.

Figure 2 shows the histograms of $\hat{k}_1$ and $\hat{k}_2$. Part (a) of Theorem 3.1 predicts that $\hat{k}_1$ is a consistent estimator of k_1^0 when $k_T = o(\sqrt{T})$ and has a finite estimation error in probability when $k_T \asymp o(\sqrt{T})$. However, $\hat{k}_1$ has a larger estimation error in probability when $\sqrt{T} = o(k_T)$, whereas $\hat{k}_2$ is always a consistent estimator of k_2^0. These findings are supported by Figure 2.

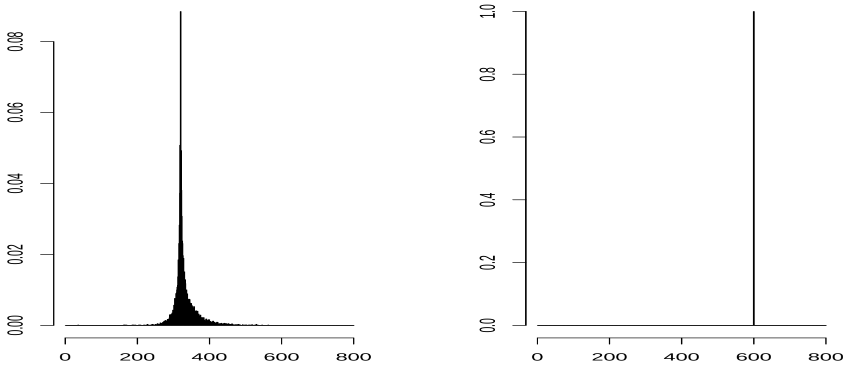
Figure 3 shows the distributions of t_1, t_2 and t_3. Part (c) of Theorem 3.1 predicts that t_1 should follow the Dickey-Fuller t-distribution, and both t_2 and t_3 should follow the normal distribution. These results are supported by Figure 3, except that the distributions of t_1

*As pointed out in Pang et al. (2017), the finite sample distribution of t_2 will suffer from shape distortion for large c_1. This phenomenon can be partially explained by the findings in Anderson (1959), which showed that, in general, the limiting distributions of the LSE and the t-ratio of the AR parameter in an explosive AR(1) model may not exist. Hence, we use $c = 0.85$ in experiments, which guarantees that the mildly explosive AR parameter is not too far away from unity.

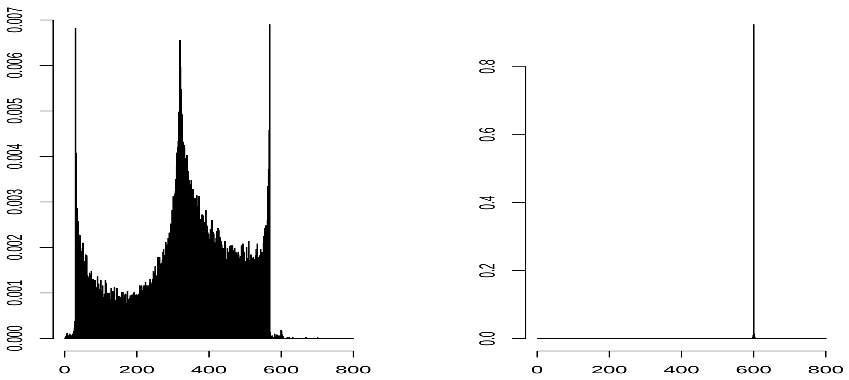
and t_2 when $\alpha = 0.7$ are not very satisfactory due to the close distance between β_1 and β_2.



(a) $\beta_1 = 1, \beta_2 = 1 + c_1/T^\alpha$ with $\alpha = 0.3$ and $\beta_3 = 1 - c_2/\sqrt{T}$

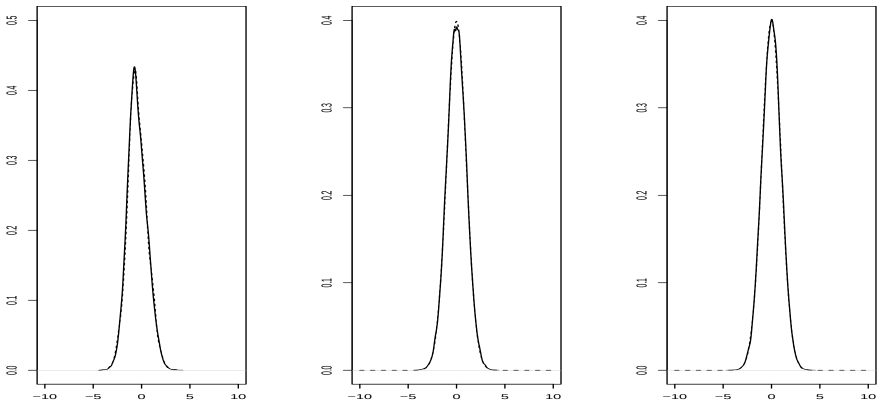


(b) $\beta_1 = 1, \beta_2 = 1 + c_1/T^\alpha$ with $\alpha = 0.5$ and $\beta_3 = 1 - c_2/\sqrt{T}$

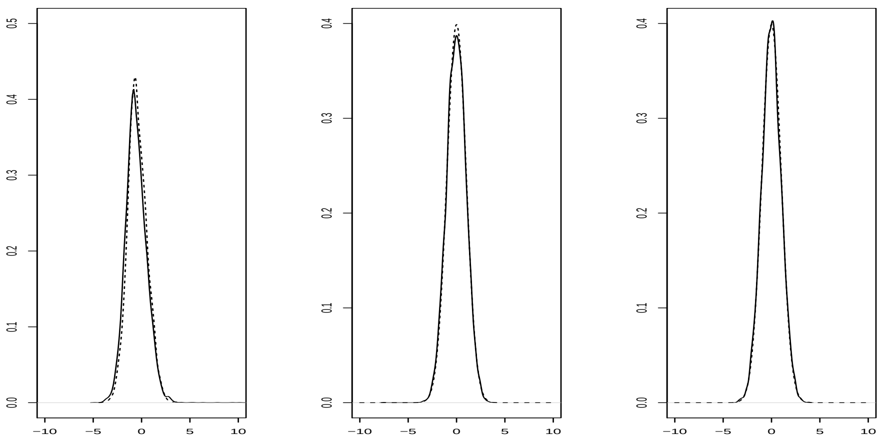


(c) $\beta_1 = 1, \beta_2 = 1 + c_1/T^\alpha$ with $\alpha = 0.7$ and $\beta_3 = 1 - c_2/\sqrt{T}$

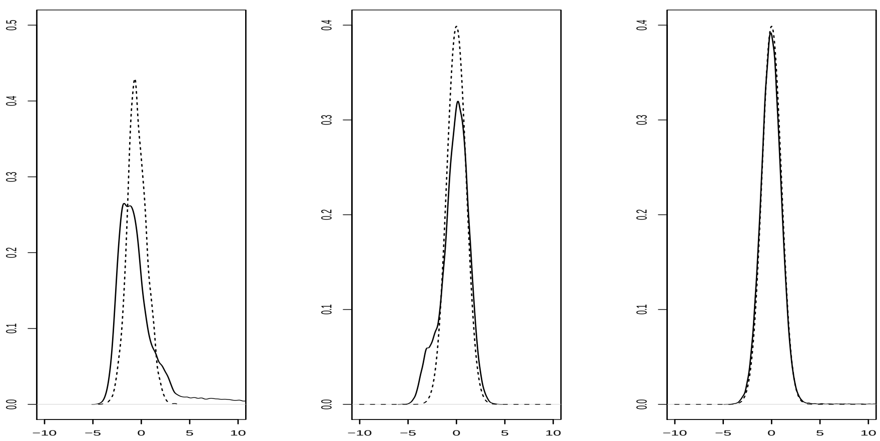
Figure 2: Histograms of $\hat{k}_1$ and $\hat{k}_2$ (from left to right) under the situation where $c_1 = 0.85, c_2 = 3$ and $T = 800$.



(a) $\beta_1 = 1, \beta_2 = 1 + c_1/T^{\alpha}$ with $\alpha = 0.3$ and $\beta_3 = 1 - c_2/\sqrt{T}$



(b) $\beta_1 = 1, \beta_2 = 1 + c_1/T^{\alpha}$ with $\alpha = 0.5$ and $\beta_3 = 1 - c_2/\sqrt{T}$



(c) $\beta_1 = 1, \beta_2 = 1 + c_1/T^{\alpha}$ with $\alpha = 0.7$ and $\beta_3 = 1 - c_2/\sqrt{T}$

Figure 3: The finite sample distributions and the corresponding limiting distributions of t_1, t_2 and t_3 (from left to right) under the situation where $c_1 = 0.85, c_2 = 3$ and $T = 800$. The solid lines represent the graphs when $T = 800$, and the dashed lines represent the graph when $T = \infty$.

5 Conclusions

In this paper, we focus on nonstationary multiple-break autoregressive models and uncover the key factors determining which break point will be identified first. Unlike the stationary cases of Chong (1995) and Bai (1997), our analysis shows that the duration of the break does not affect which break will be uncovered first in the nonstationary case. Rather, it depends on the stochastic order of the break's signal strength under the case of constant break magnitude and also the square of the magnitude of the break under the case of shrinking break. In stationary time series regression models, the signal strength has the same stochastic order of magnitude for each break, so the duration of the break will matter. Since the magnitude and the duration of the break are unobservable in reality, it is difficult to determine which break will be identified first. However, in nonstationary autoregressive models, each subsample has a different stochastic order of magnitude. Hence, we know in advance that the break associated with the subsample that has the highest stochastic order of magnitude of the product of the square of the break magnitude and the signal strength will be uncovered first. This finding allows us to develop an estimation procedure that does not require testing for breaks in the subsamples. As an application of this finding, we revisit the financial bubble model proposed by Phillips and Yu (2011), Phillips et al. (2011) and Phillips et al. (2015a, 2015b). We propose an estimation procedure without the need for estimating the structural changes sequentially by the sample splitting method of Chong (1995) and Bai (1997). The consistency, convergence rates and limiting distributions of the LSEs of the unknown parameters in this model are established. Monte Carlo simulations of the finite sample performance of the estimators provide evidence for our theory. For future work along this line, one may extend our work to nonstationary panel AR models with multiple breaks.

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6 Appendix

In this section, we provide the proof of Theorem 3.1, noting that Theorem 2.1 is just a consequence of Lemma A.1 below. The following observation which will be used frequently in the rest of the paper is a simple generalization of Proposition A.1 in Phillips and Magdalinos (2007a), hence the proof is omitted:

$$\left(\frac{T}{k_T}\right)^a = o(\beta_2^{bT}), \quad \text{for any } a > 0 \text{ and } b > 0. \tag{A.1}$$

The asymptotic analysis for the LSEs of structural changes relies heavily on the closed forms of the discrepancy of residual sum of squares when the break fraction departs from the true one. Hence, we need to develop these closed forms in the presence of two structural changes in AR(1) models.

Lemma A.1 *For Model (2.1), denote*

$$\Omega_T(\tau) = \frac{(\sum_{t=1}^{k_2^0} y_{t-1}u_t)^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2} - \frac{(\sum_{t=1}^{[\tau T]} y_{t-1}u_t)^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} + \frac{(\sum_{t=k_2^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t)^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2},$$

then the following results hold:
(a) for $\tau_1^0 \le \tau \le \tau_2^0$, we have

$$\begin{aligned} & RSS_{2,T}(\tau) - RSS_{2,T}(\tau_2^0) \\ = \;& \eta_1(\tau)(\beta_2 - \beta_1) + \eta_2(\tau)(\beta_3 - \beta_2) + \eta_3(\tau)(\beta_2 - \beta_1)^2 + \eta_4(\tau)(\beta_3 - \beta_2)^2 + \Omega_T(\tau), \end{aligned}$$

where

$$\begin{cases} \eta_1(\tau) = 2\left(\dfrac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \dfrac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\displaystyle\sum_{t=1}^{k_1^0} y_{t-1}^2 \\ \eta_2(\tau) = \dfrac{2\left(\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}u_t - \sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}u_t \sum_{t=k_2^0+1}^{T} y_{t-1}^2\right)}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\ \eta_3(\tau) = -\dfrac{(\sum_{t=1}^{k_1^0} y_{t-1}^2)^2 \sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2 \sum_{t=1}^{[\tau T]} y_{t-1}^2} \\ \eta_4(\tau) = \dfrac{\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \end{cases},$$

(b) for $0 < \tau < \tau_1^0$, we have

$$
\begin{aligned}
& RSS_{2,T}(\tau) - RSS_{2,T}(\tau_2^0) \\
= \ & \theta_1(\tau)(\beta_2-\beta_1) + \theta_2(\tau)(\beta_3-\beta_1) + \theta_3(\tau)(\beta_3-\beta_2) + \theta_4(\tau)(\beta_2-\beta_1)^2 + \theta_5(\tau)(\beta_3-\beta_1)^2 \\
& + \theta_6(\tau)(\beta_3-\beta_2)^2 + \theta_7(\tau)(\beta_2-\beta_1)(\beta_3-\beta_1) + \theta_8(\tau)(\beta_2-\beta_1)(\beta_3-\beta_2) \\
& + \theta_9(\tau)(\beta_3-\beta_1)(\beta_3-\beta_2) + \Omega_T(\tau),
\end{aligned}
$$

where

$$
\begin{cases}
\theta_1(\tau) = 2\dfrac{\sum_{t=1}^{k_1^0} y_{t-1}u_t \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 - \sum_{t=1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2} \\
\qquad + 2\dfrac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t - \sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}u_t \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\
\theta_2(\tau) = 2\dfrac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}u_t - \sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}u_t \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\
\theta_3(\tau) = 2\dfrac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}u_t - \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\
\theta_4(\tau) = \dfrac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2} - \dfrac{\sum_{t=1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2} \\
\theta_5(\tau) = \dfrac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2 \left(\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 + \sum_{t=k_2^0+1}^{T} y_{t-1}^2\right)}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2} \\
\theta_6(\tau) = \dfrac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2 \sum_{t=k_1^0+1}^{T} y_{t-1}^2}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2} \\
\theta_7(\tau) = \dfrac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2 \sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2} \\
\theta_8(\tau) = -\dfrac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2} \\
\theta_9(\tau) = \dfrac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2}
\end{cases},
$$

(c) for $\tau_2^0 < \tau < 1$, we have

$$
\begin{aligned}
& RSS_{2,T}(\tau) - RSS_{2,T}(\tau_2^0) \\
= \ & \gamma_1(\tau)(\beta_2-\beta_1) + \gamma_2(\tau)(\beta_3-\beta_1) + \gamma_3(\tau)(\beta_3-\beta_2) + \gamma_4(\tau)(\beta_2-\beta_1)^2 + \gamma_5(\tau)(\beta_3-\beta_1)^2 \\
& + \gamma_6(\tau)(\beta_3-\beta_2)^2 + \gamma_7(\tau)(\beta_2-\beta_1)(\beta_3-\beta_1) + \gamma_8(\tau)(\beta_2-\beta_1)(\beta_3-\beta_2)
\end{aligned}
$$

$$+\gamma_9(\tau)(\beta_3-\beta_1)(\beta_3-\beta_2)+\Omega_T(\tau),$$

where

$$\begin{cases}
\gamma_1(\tau)=\dfrac{2\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2\left(\sum_{t=1}^{k_1^0}y_{t-1}u_t\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2-\sum_{t=1}^{k_1^0}y_{t-1}^2\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}u_t\right)}{\sum_{t=1}^{[\tau T]}y_{t-1}^2\sum_{t=1}^{k_2^0}y_{t-1}^2}\\
\gamma_2(\tau)=\dfrac{2\left(\sum_{t=1}^{k_1^0}y_{t-1}^2\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}u_t-\sum_{t=1}^{k_1^0}y_{t-1}u_t\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2\right)}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}\\
\gamma_3(\tau)=\dfrac{2\left(\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}u_t-\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}u_t\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2\right)}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}\\
\gamma_4(\tau)=-\dfrac{\sum_{t=1}^{k_1^0}y_{t-1}^2\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2\left(\sum_{t=1}^{k_2^0}y_{t-1}^2+\sum_{t=1}^{[\tau T]}y_{t-1}^2\right)}{(\sum_{t=1}^{[\tau T]}y_{t-1}^2)^2\sum_{t=1}^{k_2^0}y_{t-1}^2}\\
\gamma_5(\tau)=\dfrac{\sum_{t=1}^{k_1^0}y_{t-1}^2\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2\left(\sum_{t=1}^{k_1^0}y_{t-1}^2+\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2\right)}{(\sum_{t=1}^{[\tau T]}y_{t-1}^2)^2}\\
\gamma_6(\tau)=\dfrac{\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2\sum_{t=k_1^0+1}^{[\tau T]}y_{t-1}^2}{(\sum_{t=1}^{[\tau T]}y_{t-1}^2)^2}\\
\gamma_7(\tau)=\dfrac{2\sum_{t=1}^{k_1^0}y_{t-1}^2\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2}{(\sum_{t=1}^{[\tau T]}y_{t-1}^2)^2}\\
\gamma_8(\tau)=-\gamma_7(\tau)\\
\gamma_9(\tau)=\gamma_7(\tau)
\end{cases}.$$

Proof. We first prove part (a). Note that when $\tau_1^0\le\tau\le\tau_2^0$, we have

$$\hat{\beta}_x(\tau)=\frac{\sum_{t=1}^{[\tau T]}y_ty_{t-1}}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}=\beta_1\frac{\sum_{t=1}^{k_1^0}y_{t-1}^2}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}+\beta_2\frac{\sum_{t=k_1^0+1}^{[\tau T]}y_{t-1}^2}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}+\frac{\sum_{t=1}^{[\tau T]}y_{t-1}u_t}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}$$

and

$$\hat{\beta}_3(\tau)=\frac{\sum_{t=[\tau T]+1}^{T}y_ty_{t-1}}{\sum_{t=[\tau T]+1}^{T}y_{t-1}^2}=\beta_2\frac{\sum_{t=[\tau T]+1}^{k_2^0}y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T}y_{t-1}^2}+\beta_3\frac{\sum_{t=k_2^0+1}^{T}y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T}y_{t-1}^2}+\frac{\sum_{t=[\tau T]+1}^{T}y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T}y_{t-1}^2}.$$

The former result implies

$$\begin{cases}
\hat{\beta}_x(\tau)-\beta_1=(\beta_2-\beta_1)\dfrac{\sum_{t=k_1^0+1}^{[\tau T]}y_{t-1}^2}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}+\dfrac{\sum_{t=1}^{[\tau T]}y_{t-1}u_t}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}\\
\hat{\beta}_x(\tau)-\beta_2=(\beta_1-\beta_2)\dfrac{\sum_{t=1}^{k_1^0}y_{t-1}^2}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}+\dfrac{\sum_{t=1}^{[\tau T]}y_{t-1}u_t}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}
\end{cases}\tag{A.2}$$

and

$$\begin{cases}
\hat{\beta}_x(\tau_2^0)-\beta_1=(\beta_2-\beta_1)\dfrac{\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2}{\sum_{t=1}^{k_2^0}y_{t-1}^2}+\dfrac{\sum_{t=1}^{k_2^0}y_{t-1}u_t}{\sum_{t=1}^{k_2^0}y_{t-1}^2}\\
\hat{\beta}_x(\tau_2^0)-\beta_2=(\beta_1-\beta_2)\dfrac{\sum_{t=1}^{k_1^0}y_{t-1}^2}{\sum_{t=1}^{k_2^0}y_{t-1}^2}+\dfrac{\sum_{t=1}^{k_2^0}y_{t-1}u_t}{\sum_{t=1}^{k_2^0}y_{t-1}^2}
\end{cases},\tag{A.3}$$

and the latter result implies

$$\begin{cases} \hat{\beta}_3(\tau)-\beta_2=(\beta_3-\beta_2)\dfrac{\sum_{t=k_2^0+1}^{T}y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T}y_{t-1}^2}+\dfrac{\sum_{t=[\tau T]+1}^{T}y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T}y_{t-1}^2} \\ \hat{\beta}_3(\tau)-\beta_3=(\beta_2-\beta_3)\dfrac{\sum_{t=[\tau T]+1}^{k_2^0}y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T}y_{t-1}^2}+\dfrac{\sum_{t=[\tau T]+1}^{T}y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T}y_{t-1}^2} \end{cases} \tag{A.4}$$

and

$$\begin{cases} \hat{\beta}_3(\tau_2^0)-\beta_2=(\beta_3-\beta_2)+\dfrac{\sum_{t=k_2^0+1}^{T}y_{t-1}u_t}{\sum_{t=k_2^0+1}^{T}y_{t-1}^2} \\ \hat{\beta}_3(\tau_2^0)-\beta_3=\dfrac{\sum_{t=k_2^0+1}^{T}y_{t-1}u_t}{\sum_{t=k_2^0+1}^{T}y_{t-1}^2} \end{cases}. \tag{A.5}$$

In addition, note that

$$\begin{aligned} RSS_{2,T}(\tau) &= \sum_{t=1}^{[\tau T]}\Big(y_t-\hat{\beta}_x(\tau)y_{t-1}\Big)^2+\sum_{t=[\tau T]+1}^{T}\Big(y_t-\hat{\beta}_3(\tau)y_{t-1}\Big)^2 \\ &= \sum_{t=1}^{k_1^0}\Big(u_t-(\hat{\beta}_x(\tau)-\beta_1)y_{t-1}\Big)^2+\sum_{t=k_1^0+1}^{[\tau T]}\Big(u_t-(\hat{\beta}_x(\tau)-\beta_2)y_{t-1}\Big)^2 \\ &\quad +\sum_{t=[\tau T]+1}^{k_2^0}\Big(u_t-(\hat{\beta}_3(\tau)-\beta_2)y_{t-1}\Big)^2+\sum_{t=k_2^0+1}^{T}\Big(u_t-(\hat{\beta}_3(\tau)-\beta_3)y_{t-1}\Big)^2 \\ &= \sum_{t=1}^{T}u_t^2-2(\hat{\beta}_x(\tau)-\beta_1)\sum_{t=1}^{k_1^0}y_{t-1}u_t+(\hat{\beta}_x(\tau)-\beta_1)^2\sum_{t=1}^{k_1^0}y_{t-1}^2 \\ &\quad -2(\hat{\beta}_x(\tau)-\beta_2)\sum_{t=k_1^0+1}^{[\tau T]}y_{t-1}u_t+(\hat{\beta}_x(\tau)-\beta_2)^2\sum_{t=k_1^0+1}^{[\tau T]}y_{t-1}^2 \\ &\quad -2(\hat{\beta}_3(\tau)-\beta_2)\sum_{t=[\tau T]+1}^{k_2^0}y_{t-1}u_t+(\hat{\beta}_3(\tau)-\beta_2)^2\sum_{t=[\tau T]+1}^{k_2^0}y_{t-1}^2 \\ &\quad -2(\hat{\beta}_3(\tau)-\beta_3)\sum_{t=k_2^0+1}^{T}y_{t-1}u_t+(\hat{\beta}_3(\tau)-\beta_3)^2\sum_{t=k_2^0+1}^{T}y_{t-1}^2 \end{aligned}$$

and

$$\begin{aligned} RSS_{2,T}(\tau_2^0) &= \sum_{t=1}^{T}u_t^2-2(\hat{\beta}_x(\tau_2^0)-\beta_1)\sum_{t=1}^{k_1^0}y_{t-1}u_t+(\hat{\beta}_x(\tau_2^0)-\beta_1)^2\sum_{t=1}^{k_1^0}y_{t-1}^2 \\ &\quad -2(\hat{\beta}_x(\tau_2^0)-\beta_2)\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}u_t+(\hat{\beta}_x(\tau_2^0)-\beta_2)^2\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2 \\ &\quad -2(\hat{\beta}_3(\tau_2^0)-\beta_3)\sum_{t=k_2^0+1}^{T}y_{t-1}u_t+(\hat{\beta}_3(\tau_2^0)-\beta_3)^2\sum_{t=k_2^0+1}^{T}y_{t-1}^2. \end{aligned} \tag{A.6}$$

As a result, it follows from (A.2)-(A.6) that

$$
\begin{aligned}
& RSS_{2,T}(\tau) - RSS_{2,T}(\tau_2^0) \\
= \quad & -2\Big((\hat{\beta}_x(\tau)-\beta_1)-(\hat{\beta}_x(\tau_2^0)-\beta_1)\Big)\sum_{t=1}^{k_1^0} y_{t-1}u_t + \Big((\hat{\beta}_x(\tau)-\beta_1)^2-(\hat{\beta}_x(\tau_2^0)-\beta_1)^2\Big)\sum_{t=1}^{k_1^0} y_{t-1}^2 \\
& -2\Big((\hat{\beta}_x(\tau)-\beta_2)-(\hat{\beta}_x(\tau_2^0)-\beta_2)\Big)\sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}u_t + \Big((\hat{\beta}_x(\tau)-\beta_2)^2-(\hat{\beta}_x(\tau_2^0)-\beta_2)^2\Big)\sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}^2 \\
& -2\Big((\hat{\beta}_3(\tau)-\beta_2)-(\hat{\beta}_x(\tau_2^0)-\beta_2)\Big)\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}u_t + \Big((\hat{\beta}_3(\tau)-\beta_2)^2-(\hat{\beta}_x(\tau_2^0)-\beta_2)^2\Big)\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 \\
& -2(\hat{\beta}_3(\tau)-\beta_3)\sum_{t=k_2^0+1}^{T} y_{t-1}u_t + (\hat{\beta}_3(\tau)-\beta_3)^2\sum_{t=k_2^0+1}^{T} y_{t-1}^2 + \frac{(\sum_{t=k_2^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} \\
:= \quad & \eta_1(\tau)(\beta_2-\beta_1)+\eta_2(\tau)(\beta_3-\beta_2)+\eta_3(\tau)(\beta_2-\beta_1)^2+\eta_4(\tau)(\beta_3-\beta_2)^2+\Omega_{1,T}(\tau),
\end{aligned}
$$

where

$$
\begin{aligned}
& \eta_1(\tau) \\
= \quad & -2\left(\frac{\sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=1}^{k_1^0} y_{t-1}u_t + 2\left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}u_t \\
& +2\left(\frac{\sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}^2 \sum_{t=1}^{[\tau T]} y_{t-1}u_t}{(\sum_{t=1}^{[\tau T]} y_{t-1}^2)^2} - \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=1}^{k_2^0} y_{t-1}u_t}{(\sum_{t=1}^{k_2^0} y_{t-1}^2)^2}\right)\sum_{t=1}^{k_1^0} y_{t-1}^2 \\
& -2\left(\frac{\sum_{t=1}^{k_0} y_{t-1}^2 \sum_{t=1}^{[\tau T]} y_{t-1}u_t}{(\sum_{t=1}^{[\tau T]} y_{t-1}^2)^2} - \frac{\sum_{t=1}^{k_0} y_{t-1}^2 \sum_{t=1}^{k_2^0} y_{t-1}u_t}{(\sum_{t=1}^{k_2^0} y_{t-1}^2)^2}\right)\sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}^2 \\
& -2\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}u_t + 2\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2 \sum_{t=1}^{k_2^0} y_{t-1}u_t}{(\sum_{t=1}^{k_2^0} y_{t-1}^2)^2}\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 \\
= \quad & 2\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=1}^{k_1^0} y_{t-1}^2,
\end{aligned}
$$

$$
\begin{aligned}
\eta_2(\tau) \quad = \quad & -2\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}u_t + 2\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2 \sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2)^2}\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 \\
& +2\frac{\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\sum_{t=k_2^0+1}^{T} y_{t-1}u_t - 2\frac{\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 \sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2)^2}\sum_{t=k_2^0+1}^{T} y_{t-1}^2 \\
= \quad & \frac{2\left(\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}u_t - \sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}u_t \sum_{t=k_2^0+1}^{T} y_{t-1}^2\right)}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2},
\end{aligned}
$$

$$
\begin{aligned}
\eta_3(\tau) &= \left(\left(\frac{\sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right) \sum_{t=1}^{k_1^0} y_{t-1}^2 \\
&\quad + \left(\left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right) \sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}^2 - \left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2 \sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 \\
&= -\frac{(\sum_{t=1}^{k_1^0} y_{t-1}^2)^2 \sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2 \sum_{t=1}^{[\tau T]} y_{t-1}^2},
\end{aligned}
$$

$$
\begin{aligned}
\eta_4(\tau) &= \left(\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2 \sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 + \left(\frac{\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2 \\
&= \frac{\sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}
\end{aligned}
$$

and

$$
\begin{aligned}
& \Omega_{1,T}(\tau) \\
= & -2\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right) \sum_{t=1}^{k_1^0} y_{t-1}u_t \\
& + \left(\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right) \sum_{t=1}^{k_1^0} y_{t-1}^2 \\
& -2\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right) \sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}u_t \\
& + \left(\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right) \sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}^2 \\
& -2\left(\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} - \frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right) \sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}u_t \\
& + \left(\left(\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right) \sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2 \\
& -2\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \sum_{t=k_2^0+1}^{T} y_{t-1}u_t + \left(\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2 + \frac{(\sum_{t=k_2^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} \\
= & \frac{(\sum_{t=1}^{k_2^0} y_{t-1}u_t)^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2} - \frac{(\sum_{t=1}^{[\tau T]} y_{t-1}u_t)^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} + \frac{(\sum_{t=k_2^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t)^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\
= & \Omega_T(\tau).
\end{aligned}
$$

These prove part (a).

To prove part (b), note that when $0 < \tau < \tau_1^0$,

$$\hat{\beta}_x(\tau) = \frac{\sum_{t=1}^{[\tau T]} y_t y_{t-1}}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} = \beta_1 + \frac{\sum_{t=1}^{[\tau T]} y_{t-1} u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2},$$

we have

$$\hat{\beta}_x(\tau) - \beta_1 = \frac{\sum_{t=1}^{[\tau T]} y_{t-1} u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}, \tag{A.7}$$

and

$$\begin{aligned} \hat{\beta}_3(\tau) &= \frac{\sum_{t=[\tau T]+1}^{T} y_t y_{t-1}}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\ &= \beta_1 \frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} + \beta_2 \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} + \beta_3 \frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} + \frac{\sum_{t=[\tau T]+1}^{T} y_{t-1} u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}, \end{aligned}$$

which gives

$$\begin{cases} \hat{\beta}_3(\tau) - \beta_1 = (\beta_2 - \beta_1) \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} + (\beta_3 - \beta_1) \frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} + \frac{\sum_{t=[\tau T]+1}^{T} y_{t-1} u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\ \hat{\beta}_3(\tau) - \beta_2 = (\beta_1 - \beta_2) \frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} + (\beta_3 - \beta_2) \frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} + \frac{\sum_{t=[\tau T]+1}^{T} y_{t-1} u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\ \hat{\beta}_3(\tau) - \beta_3 = (\beta_1 - \beta_3) \frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} + (\beta_2 - \beta_3) \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} + \frac{\sum_{t=[\tau T]+1}^{T} y_{t-1} u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \end{cases} \tag{A.8}$$

In addition, since

$$\begin{aligned} RSS_{2,T}(\tau) &= \sum_{t=1}^{[\tau T]} \left(y_t - \hat{\beta}_x(\tau) y_{t-1} \right)^2 + \sum_{t=[\tau T]+1}^{T} \left(y_t - \hat{\beta}_3(\tau) y_{t-1} \right)^2 \\ &= \sum_{t=1}^{[\tau T]} \left(u_t - (\hat{\beta}_x(\tau) - \beta_1) y_{t-1} \right)^2 + \sum_{t=[\tau T]+1}^{k_1^0} \left(u_t - (\hat{\beta}_3(\tau) - \beta_1) y_{t-1} \right)^2 \\ &\quad + \sum_{t=k_1^0+1}^{k_2^0} \left(u_t - (\hat{\beta}_3(\tau) - \beta_2) y_{t-1} \right)^2 + \sum_{t=k_2^0+1}^{T} \left(u_t - (\hat{\beta}_3(\tau) - \beta_3) y_{t-1} \right)^2 \\ &= \sum_{t=1}^{T} u_t^2 - 2(\hat{\beta}_x(\tau) - \beta_1) \sum_{t=1}^{[\tau T]} y_{t-1} u_t + (\hat{\beta}_x(\tau) - \beta_1)^2 \sum_{t=1}^{[\tau T]} y_{t-1}^2 \\ &\quad -2(\hat{\beta}_3(\tau) - \beta_1) \sum_{t=[\tau T]+1}^{k_1^0} y_{t-1} u_t + (\hat{\beta}_3(\tau) - \beta_1)^2 \sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \\ &\quad -2(\hat{\beta}_3(\tau) - \beta_2) \sum_{t=k_1^0+1}^{k_2^0} y_{t-1} u_t + (\hat{\beta}_3(\tau) - \beta_2)^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \\ &\quad -2(\hat{\beta}_3(\tau) - \beta_3) \sum_{t=k_2^0+1}^{T} y_{t-1} u_t + (\hat{\beta}_3(\tau) - \beta_3)^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2, \end{aligned}$$

using (A.3) and (A.5)-(A.8), we have

$$
\begin{aligned}
& RSS_{2,T}(\tau) - RSS_{2,T}(\tau_2^0) \\
= \;& -2\left((\hat{\beta}_x(\tau)-\beta_1)-(\hat{\beta}_x(\tau_2^0)-\beta_1)\right)\sum_{t=1}^{[\tau T]} y_{t-1}u_t + \left((\hat{\beta}_x(\tau)-\beta_1)^2-(\hat{\beta}_x(\tau_2^0)-\beta_1)^2\right)\sum_{t=1}^{[\tau T]} y_{t-1}^2 \\
& -2\left((\hat{\beta}_3(\tau)-\beta_1)-(\hat{\beta}_x(\tau_2^0)-\beta_1)\right)\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}u_t + \left((\hat{\beta}_3(\tau)-\beta_1)^2-(\hat{\beta}_x(\tau_2^0)-\beta_1)^2\right)\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \\
& -2\left((\hat{\beta}_3(\tau)-\beta_2)-(\hat{\beta}_x(\tau_2^0)-\beta_2)\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t + \left((\hat{\beta}_3(\tau)-\beta_2)^2-(\hat{\beta}_x(\tau_2^0)-\beta_2)^2\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \\
& -2\left((\hat{\beta}_3(\tau)-\beta_3)-(\hat{\beta}_3(\tau_2^0)-\beta_3)\right)\sum_{t=k_2^0+1}^{T} y_{t-1}u_t + \left((\hat{\beta}_3(\tau)-\beta_3)^2-(\hat{\beta}_3(\tau_2^0)-\beta_3)^2\right)\sum_{t=k_2^0+1}^{T} y_{t-1}^2 \\
:= \;& \theta_1(\tau)(\beta_2-\beta_1)+\theta_2(\tau)(\beta_3-\beta_1)+\theta_3(\tau)(\beta_3-\beta_2)+\theta_4(\tau)(\beta_2-\beta_1)^2+\theta_5(\tau)(\beta_3-\beta_1)^2 \\
& +\theta_6(\tau)(\beta_3-\beta_2)^2+\theta_7(\tau)(\beta_2-\beta_1)(\beta_3-\beta_1)+\theta_8(\tau)(\beta_2-\beta_1)(\beta_3-\beta_2) \\
& +\theta_9(\tau)(\beta_3-\beta_1)(\beta_3-\beta_2)+\Omega_{2,T}(\tau),
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_1(\tau) \;=\;& 2\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\sum_{t=1}^{[\tau T]} y_{t-1}u_t - 2\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\sum_{t=1}^{[\tau T]} y_{t-1}^2 \\
& -2\left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}-\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}u_t \\
& +2\left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}-\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \\
& +2\left(\frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}-\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t \\
& -2\left(\frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}-\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \\
=\;& 2\left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}u_t\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2-\sum_{t=1}^{k_1^0} y_{t-1}^2\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right. \\
& \left.+\frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t-\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}u_t\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right),
\end{aligned}
$$

$$
\theta_2(\tau) = -2\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}u_t + 2\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2
$$

$$
\begin{aligned}
&\quad +2\frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\sum_{t=k_2^0+1}^{T} y_{t-1}u_t - 2\frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\sum_{t=k_2^0+1}^{T} y_{t-1}^2 \\
&= 2\frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}u_t - \sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}u_t \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2},
\end{aligned}
$$

$$
\begin{aligned}
\theta_3(\tau) &= -2\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t + 2\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \\
&\quad +2\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\sum_{t=k_2^0+1}^{T} y_{t-1}u_t - 2\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\sum_{t=k_2^0+1}^{T} y_{t-1}^2 \\
&= 2\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}u_t - \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2},
\end{aligned}
$$

$$
\begin{aligned}
\theta_4(\tau) &= -\left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\sum_{t=1}^{[\tau T]} y_{t-1}^2 + \left(\left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right)\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \\
&\quad + \left(\left(\frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \\
&= \frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=[\tau T]+1}^{k_2^0} y_{t-1}^2}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2} - \frac{\sum_{t=1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2},
\end{aligned}
$$

$$
\begin{aligned}
\theta_5(\tau) &= \left(\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 + \left(\frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2\sum_{t=k_2^0+1}^{T} y_{t-1}^2 \\
&= \frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2 \left(\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 + \sum_{t=k_2^0+1}^{T} y_{t-1}^2\right)}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
\theta_6(\tau) &= \left(\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 + \left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2\sum_{t=k_2^0+1}^{T} y_{t-1}^2 \\
&= \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2 \sum_{t=k_1^0+1}^{T} y_{t-1}^2}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2},
\end{aligned}
$$

$$
\theta_7(\tau) = \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2 \sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2},
$$

$$\theta_8(\tau) = -\frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2},$$

$$\theta_9(\tau) = \frac{\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\left(\sum_{t=[\tau T]+1}^{T} y_{t-1}^2\right)^2}$$

and

$$\begin{aligned}
\Omega_{2,T}(\tau) &= -2\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=1}^{[\tau T]} y_{t-1}u_t \\
&\quad -2\left(\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} - \frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}u_t \\
&\quad -2\left(\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} - \frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t \\
&\quad -2\left(\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} - \frac{\sum_{t=k_2^0+1}^{T} y_{t-1}u_t}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}\right)\sum_{t=k_2^0+1}^{T} y_{t-1}u_t \\
&\quad +\left(\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right)\sum_{t=1}^{[\tau T]} y_{t-1}^2 \\
&\quad +\left(\left(\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right)\sum_{t=[\tau T]+1}^{k_1^0} y_{t-1}^2 \\
&\quad +\left(\left(\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \\
&\quad +\left(\left(\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}u_t}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}\right)^2\right)\sum_{t=k_2^0+1}^{T} y_{t-1}^2 \\
&= \frac{(\sum_{t=1}^{k_2^0} y_{t-1}u_t)^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2} - \frac{(\sum_{t=1}^{[\tau T]} y_{t-1}u_t)^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} + \frac{(\sum_{t=k_2^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t)^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\
&= \Omega_T(\tau).
\end{aligned}$$

These prove part (b).

To prove part (c), note that when $\tau_2^0 < \tau < 1$, we have

$$\hat{\beta}_x(\tau) = \frac{\sum_{t=1}^{[\tau T]} y_t y_{t-1}}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} = \beta_1\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} + \beta_2\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} + \beta_3\frac{\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} + \frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}$$

and

$$\hat{\beta}_3(\tau) = \frac{\sum_{t=[\tau T]+1}^{T} y_t y_{t-1}}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} = \beta_3 + \frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}.$$

The former result implies

$$\left\{\begin{array}{l}\hat{\beta}_x(\tau)-\beta_1=(\beta_2-\beta_1)\dfrac{\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}+(\beta_3-\beta_1)\dfrac{\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}+\dfrac{\sum_{t=1}^{[\tau T]}y_{t-1}u_t}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}\\ \hat{\beta}_x(\tau)-\beta_2=(\beta_1-\beta_2)\dfrac{\sum_{t=1}^{k_1^0}y_{t-1}^2}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}+(\beta_3-\beta_2)\dfrac{\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}+\dfrac{\sum_{t=1}^{[\tau T]}y_{t-1}u_t}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}\\ \hat{\beta}_x(\tau)-\beta_3=(\beta_1-\beta_3)\dfrac{\sum_{t=1}^{k_1^0}y_{t-1}^2}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}+(\beta_2-\beta_3)\dfrac{\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}+\dfrac{\sum_{t=1}^{[\tau T]}y_{t-1}u_t}{\sum_{t=1}^{[\tau T]}y_{t-1}^2}\end{array}\right., \tag{A.9}$$

and the latter result implies

$$\hat{\beta}_3(\tau)-\beta_3=\frac{\sum_{t=[\tau T]+1}^{T}y_{t-1}u_t}{\sum_{t=[\tau T]+1}^{T}y_{t-1}^2}. \tag{A.10}$$

In addition, we have

$$\begin{aligned}RSS_{2,T}(\tau) &= \sum_{t=1}^{[\tau T]}\Big(y_t-\hat{\beta}_x(\tau)y_{t-1}\Big)^2+\sum_{t=[\tau T]+1}^{T}\Big(y_t-\hat{\beta}_3(\tau)y_{t-1}\Big)^2\\ &= \sum_{t=1}^{k_1^0}\Big(u_t-(\hat{\beta}_x(\tau)-\beta_1)y_{t-1}\Big)^2+\sum_{t=k_1^0+1}^{k_2^0}\Big(u_t-(\hat{\beta}_x(\tau)-\beta_2)y_{t-1}\Big)^2\\ &\quad+\sum_{t=k_2^0+1}^{[\tau T]}\Big(u_t-(\hat{\beta}_x(\tau)-\beta_3)y_{t-1}\Big)^2+\sum_{t=[\tau T]+1}^{T}\Big(u_t-(\hat{\beta}_3(\tau)-\beta_3)y_{t-1}\Big)^2\\ &= \sum_{t=1}^{T}u_t^2-2(\hat{\beta}_x(\tau)-\beta_1)\sum_{t=1}^{k_1^0}y_{t-1}u_t+(\hat{\beta}_x(\tau)-\beta_1)^2\sum_{t=1}^{k_1^0}y_{t-1}^2\\ &\quad-2(\hat{\beta}_x(\tau)-\beta_2)\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}u_t+(\hat{\beta}_x(\tau)-\beta_2)^2\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2\\ &\quad-2(\hat{\beta}_x(\tau)-\beta_3)\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}u_t+(\hat{\beta}_x(\tau)-\beta_3)^2\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2-\frac{(\sum_{t=[\tau T]+1}^{T}y_{t-1}u_t)^2}{\sum_{t=[\tau T]+1}^{T}y_{t-1}^2}.\end{aligned}$$

Thus, it follows from (A.3), (A.5), (A.6), (A.9) and (A.10) that

$$\begin{aligned}&RSS_{2,T}(\tau)-RSS_{2,T}(\tau_2^0)\\ =\;&-2\Big((\hat{\beta}_x(\tau)-\beta_1)-(\hat{\beta}_x(\tau_2^0)-\beta_1)\Big)\sum_{t=1}^{k_1^0}y_{t-1}u_t+\Big((\hat{\beta}_x(\tau)-\beta_1)^2-(\hat{\beta}_x(\tau_2^0)-\beta_1)^2\Big)\sum_{t=1}^{k_1^0}y_{t-1}^2\\ &-2\Big((\hat{\beta}_x(\tau)-\beta_2)-(\hat{\beta}_x(\tau_2^0)-\beta_2)\Big)\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}u_t+\Big((\hat{\beta}_x(\tau)-\beta_2)^2-(\hat{\beta}_x(\tau_2^0)-\beta_2)^2\Big)\sum_{t=k_1^0+1}^{k_2^0}y_{t-1}^2\\ &-2(\hat{\beta}_x(\tau)-\beta_3)\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}u_t+(\hat{\beta}_x(\tau)-\beta_3)^2\sum_{t=k_2^0+1}^{[\tau T]}y_{t-1}^2\end{aligned}$$

$$
\begin{aligned}
& +\frac{(\sum_{t=k_2^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t)^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\
:= \; & \gamma_1(\tau)(\beta_2-\beta_1) + \gamma_2(\tau)(\beta_3-\beta_1) + \gamma_3(\tau)(\beta_3-\beta_2) + \gamma_4(\tau)(\beta_2-\beta_1)^2 + \gamma_5(\tau)(\beta_3-\beta_1)^2 \\
& +\gamma_6(\tau)(\beta_3-\beta_2)^2 + \gamma_7(\tau)(\beta_2-\beta_1)(\beta_3-\beta_1) + \gamma_8(\tau)(\beta_2-\beta_1)(\beta_3-\beta_2) \\
& +\gamma_9(\tau)(\beta_3-\beta_1)(\beta_3-\beta_2) + \Omega_{3,T}(\tau),
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_1(\tau) &= -2\left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=1}^{k_1^0} y_{t-1}u_t \\
&\quad +2\left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t \\
&\quad +2\left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=1}^{k_1^0} y_{t-1}^2 \\
&\quad -2\left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} - \frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \\
&= \frac{2\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2\left(\sum_{t=1}^{k_1^0} y_{t-1}u_t\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 - \sum_{t=1}^{k_1^0} y_{t-1}^2\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t\right)}{\sum_{t=1}^{[\tau T]} y_{t-1}^2\sum_{t=1}^{k_2^0} y_{t-1}^2},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_2(\tau) &= -2\frac{\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=1}^{k_1^0} y_{t-1}u_t + 2\frac{\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=1}^{k_1^0} y_{t-1}^2 \\
&\quad +2\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}u_t - 2\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2 \\
&= \frac{2\left(\sum_{t=1}^{k_1^0} y_{t-1}^2\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}u_t - \sum_{t=1}^{k_1^0} y_{t-1}u_t\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2\right)}{\sum_{t=1}^{[\tau T]} y_{t-1}^2},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_3(\tau) &= -2\frac{\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t + 2\frac{\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \\
&\quad +2\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}u_t - 2\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2 \\
&= \frac{2\left(\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}u_t - \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2\right)}{\sum_{t=1}^{[\tau T]} y_{t-1}^2},
\end{aligned}
$$

$$
\gamma_4(\tau) = \left(\left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2 - \left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right)\sum_{t=1}^{k_1^0} y_{t-1}^2
$$

$$
\begin{aligned}
&+\left(\left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2-\left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \\
= &-\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2 \left(\sum_{t=1}^{k_2^0} y_{t-1}^2+\sum_{t=1}^{[\tau T]} y_{t-1}^2\right)}{(\sum_{t=1}^{[\tau T]} y_{t-1}^2)^2 \sum_{t=1}^{k_2^0} y_{t-1}^2},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_5(\tau) &= \left(\frac{\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2 \sum_{t=1}^{k_1^0} y_{t-1}^2+\left(\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2 \sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2 \\
&= \frac{\sum_{t=1}^{k_1^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2 \left(\sum_{t=1}^{k_1^0} y_{t-1}^2+\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2\right)}{(\sum_{t=1}^{[\tau T]} y_{t-1}^2)^2},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_6(\tau) &= \left(\frac{\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2+\left(\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2 \sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2 \\
&= \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2 \sum_{t=k_1^0+1}^{[\tau T]} y_{t-1}^2}{(\sum_{t=1}^{[\tau T]} y_{t-1}^2)^2},
\end{aligned}
$$

$$
\gamma_7(\tau)=2\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\frac{\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=1}^{k_1^0} y_{t-1}^2=\frac{2\sum_{t=1}^{k_1^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2}{(\sum_{t=1}^{[\tau T]} y_{t-1}^2)^2},
$$

$$
\gamma_8(\tau)=-2\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\frac{\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2=-\gamma_7(\tau),
$$

$$
\gamma_9(\tau)=2\frac{\sum_{t=1}^{k_1^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2=\gamma_7(\tau)
$$

and

$$
\begin{aligned}
&\Omega_{3,T}(\tau) \\
= &-2\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}-\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=1}^{k_1^0} y_{t-1}u_t \\
&-2\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}-\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t-2\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}u_t \\
&+\left(\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2-\left(\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right)\sum_{t=1}^{k_1^0} y_{t-1}^2 \\
&+\left(\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2-\left(\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right)^2\right)\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2+\left(\frac{\sum_{t=1}^{[\tau T]} y_{t-1}u_t}{\sum_{t=1}^{[\tau T]} y_{t-1}^2}\right)^2\sum_{t=k_2^0+1}^{[\tau T]} y_{t-1}^2
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{(\sum_{t=k_2^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t)^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\
= \; & \frac{(\sum_{t=1}^{k_2^0} y_{t-1}u_t)^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2} - \frac{(\sum_{t=1}^{[\tau T]} y_{t-1}u_t)^2}{\sum_{t=1}^{[\tau T]} y_{t-1}^2} + \frac{(\sum_{t=k_2^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=[\tau T]+1}^{T} y_{t-1}u_t)^2}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \\
= \; & \Omega_T(\tau).
\end{aligned}
$$

These prove part (c). □

The five lemmas below are needed in the proof of Theorem 3.1.

Lemma A.2 *For Model (2.1), under assumptions C1-C4, the following results hold jointly:*

(a) $\frac{1}{T}\sum_{t=1}^{k_1^0} y_{t-1}u_t \Rightarrow \frac{\sigma^2}{2}(W^2(\tau_1^0) - \tau_1^0)$,

(b) $\frac{1}{T^2}\sum_{t=1}^{k_1^0} y_{t-1}^2 \Rightarrow \sigma^2 \int_0^{\tau_1^0} W^2(s)ds$,

(c) $\frac{y_{k_1^0}}{\sqrt{T}} \Rightarrow \sigma W(\tau_1^0)$.

Proof. To prove part (a), note that

$$
y_t = y_0 + \sum_{i=1}^{t} u_i = \frac{tc}{T^\eta} + y_0 + \sum_{i=1}^{t} \varepsilon_i, \quad 0 \le t \le k_1^0; \tag{A.11}
$$

it is obvious that

$$
\begin{aligned}
\frac{1}{T}\sum_{t=1}^{k_1^0} y_{t-1}u_t &= \frac{1}{T}\sum_{t=1}^{k_1^0} \left(\frac{(t-1)c}{T^\eta} + y_0 + \sum_{i=1}^{t-1} \varepsilon_i \right) \left(\frac{c}{T^\eta} + \varepsilon_t\right) \\
&= \frac{1}{T}\sum_{t=1}^{k_1^0} \left(\sum_{i=1}^{t-1} \varepsilon_i \right) \varepsilon_t + o_p(1)
\end{aligned}
$$

by assumption C1 and the fact that $\eta > 1/2$. Then, applying the standard results in the unit root literature, we have

$$
\frac{1}{T}\sum_{t=1}^{k_1^0} y_{t-1}u_t \Rightarrow \frac{\sigma^2}{2}\int_0^{\tau_1^0} W(s)dW(s) \overset{d}{=} \frac{\sigma^2}{2}(W^2(\tau_1^0) - \tau_1^0).
$$

Part (b) can be proved in a similar manner, thus the details are omitted. Part (c) is implied by (A.11) and the functional central limit theorem.

It is not difficult to see that parts (a), (b) and (c) hold jointly. □

Lemma A.3 *For Model (2.1), under assumptions C1-C4, the following results hold jointly:*

(a) $\frac{1}{\beta_2^{k_2^0-k_1^0}\sqrt{Tk_T}}\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t \Rightarrow \sigma^2 W(\tau_1^0)X,$

(b) $\frac{1}{\beta_2^{2(k_2^0-k_1^0)}Tk_T}\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \Rightarrow \frac{\sigma^2}{2c_1}W^2(\tau_1^0),$

(c) $\frac{y_{k_2^0}}{\beta_2^{k_2^0-k_1^0}\sqrt{T}} \Rightarrow \sigma W(\tau_1^0),$

where X *is a random variable following* $N(0, \frac{1}{2c_1})$ *and independent of* $W(\tau_1^0)$.

Proof. We first prove part (a). In view of part (c) of Lemma A.2, Lemma C.1 in Pang et al. (2017) and the observation (A.1), one can show that

$$\begin{aligned}
\frac{1}{\beta_2^{k_2^0-k_1^0}\sqrt{Tk_T}}\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}u_t &= \frac{1}{\beta_2^{k_2^0-k_1^0}\sqrt{Tk_T}}\sum_{t=k_1^0+1}^{k_2^0}\left(\beta_2^{t-1-k_1^0}y_{k_1^0}+\sum_{i=k_1^0+1}^{t-1}\beta_2^{t-1-i}u_i\right)u_t \\
&= \frac{y_{k_1^0}}{\beta_2^{k_2^0-k_1^0}\sqrt{Tk_T}}\sum_{t=k_1^0+1}^{k_2^0}\beta_2^{t-1-k_1^0}u_t + o_p(1) \\
&= \frac{y_{k_1^0}}{\sqrt{T}}\cdot\frac{1}{\sqrt{k_T}}\sum_{i=1}^{k_2^0-k_1^0}\beta_2^{i-1-(k_2^0-k_1^0)}u_{k_1^0+i} + o_p(1) \\
&\Rightarrow \sigma^2 W(\tau_1^0)X. \qquad (A.12)
\end{aligned}$$

It is clear that $W(\tau_1^0)$ and X are independent of each other.

To prove part (b), applying part (c) of Lemma A.2 again, one can show that

$$\begin{aligned}
\frac{1}{\beta_2^{2(k_2^0-k_1^0)}Tk_T}\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 &= \frac{y_{k_1^0}^2}{\beta_2^{2(k_2^0-k_1^0)}Tk_T}\sum_{t=k_1^0+1}^{k_2^0}\beta_2^{2(t-1-k_1^0)} + o_p(1) \\
&= \frac{y_{k_1^0}^2}{T}\cdot\frac{1}{k_T\beta_2^{2(k_2^0-k_1^0)}}\frac{\beta_2^{2(k_2^0-k_1^0)}-1}{\beta_2^2-1} + o_p(1) \\
&\Rightarrow \frac{\sigma^2}{2c_1}W^2(\tau_1^0). \qquad (A.13)
\end{aligned}$$

We now prove part (c). Note that

$$\begin{aligned}
\frac{y_{k_2^0}}{\beta_2^{k_2^0-k_1^0}\sqrt{T}} &= \frac{1}{\beta_2^{k_2^0-k_1^0}\sqrt{T}}\left(\beta_2^{k_2^0-k_1^0}y_{k_1^0}+\sum_{t=k_1^0+1}^{k_2^0}\beta_2^{k_2^0-t}u_t\right) \\
&= \frac{y_{k_1^0}}{\sqrt{T}} + o_p(1) \\
&\Rightarrow \sigma W(\tau_1^0).
\end{aligned}$$

It is easy to see that parts (a), (b) and (c) hold jointly. □

Lemma A.4 *For Model (2.1), under assumptions C1-C4, the following results hold jointly:*

(a) $\dfrac{1}{\beta_2^{k_2^0-k_1^0}\sqrt{Th_T}}\displaystyle\sum_{t=k_2^0+1}^{T} y_{t-1}u_t \Rightarrow \sigma^2 W(\tau_1^0)Z,$

(b) $\dfrac{1}{\beta_2^{2(k_2^0-k_1^0)}Th_T}\displaystyle\sum_{t=k_2^0+1}^{T} y_{t-1}^2 \Rightarrow \dfrac{\sigma^2}{2c_2}W^2(\tau_1^0),$

where Z is a random variable following $N(0,\frac{1}{2c_2})$ and independent of $W(\tau_1^0)$.

Proof. To prove part (a), using part (c) of Lemma A.3 and Lemma A.1 in Pang et al. (2017), we have

$$\begin{aligned}
\frac{1}{\beta_2^{k_2^0-k_1^0}\sqrt{Th_T}}\sum_{t=k_2^0+1}^{k_3^0} y_{t-1}u_t &= \frac{1}{\beta_2^{k_2^0-k_1^0}\sqrt{Th_T}}\sum_{t=k_2^0+1}^{k_3^0}\left(\beta_3^{t-1-k_2^0}y_{k_2^0}+\sum_{i=k_2^0+1}^{t-1}\beta_3^{t-1-i}u_i\right)u_t \\
&= \frac{y_{k_2^0}}{\beta_2^{k_2^0-k_1^0}\sqrt{Th_T}}\sum_{t=k_2^0+1}^{k_3^0}\beta_3^{t-1-k_2^0}u_t + o_p(1) \\
&= \frac{y_{k_2^0}}{\beta_2^{k_2^0-k_1^0}\sqrt{T}}\cdot\frac{1}{\sqrt{h_T}}\sum_{j=1}^{(\tau_3^0-\tau_2^0)T}\beta_3^{(\tau_3^0-\tau_2^0)T-j}u_{k_3^0-j+1} + o_p(1) \\
&\Rightarrow \sigma^2 W(\tau_1^0)Z.
\end{aligned}$$

It is clear that Z and $W(\tau_1^0)$ are independent of each other.

To prove part (b), note that

$$\begin{aligned}
\frac{1}{\beta_2^{2(k_2^0-k_1^0)}Th_T}\sum_{t=k_2^0+1}^{k_3^0} y_{t-1}^2 &= \frac{1}{\beta_2^{2(k_2^0-k_1^0)}Th_T}\sum_{t=k_2^0+1}^{k_3^0}\left(\beta_3^{t-1-k_2^0}y_{k_2^0}+\sum_{i=k_2^0+1}^{t-1}\beta_3^{t-1-i}u_i\right)^2 \\
&= \frac{y_{k_2^0}^2}{\beta_2^{2(k_2^0-k_1^0)}T}\cdot\frac{1}{h_T}\sum_{t=k_2^0+1}^{k_3^0}\beta_3^{2(t-1-k_2^0)} + o_p(1) \\
&= \frac{y_{k_2^0}^2}{\beta_2^{2(k_2^0-k_1^0)}T}\cdot\frac{1-\beta_3^{2(k_3^0-k_2^0)}}{h_T(1-\beta_3^2)} + o_p(1) \\
&\Rightarrow \frac{\sigma^2}{2c_2}W^2(\tau_1^0)
\end{aligned}$$

by part (c) of Lemma A.3 again.

It is easy to see that parts (a) and (b) hold jointly. $\square$

To find out the leading terms in $RSS_{2,T}(\tau)-RSS_{2,T}(\tau_2^0)$ when τ departs from τ_2^0, we have the following lemma:

Lemma A.5 *Denote* $\kappa_T = |\beta_3 - \beta_2| = |\beta_{3T} - \beta_{2T}|$ *and*

$$\begin{cases} B_{1T} = \{m : m \in Z_T, k_1^0 \le m < k_2^0 - M_T\} \\ B_{2T} = \{m : m \in Z_T, 1 \le m < k_1^0\} \\ B_{3T} = \{m : m \in Z_T, k_2^0 + M_T < m \le T\} \end{cases}$$

with $M_T > 0$ *such that* $M_T \to \infty$ *at an arbitrary slow pace;* Z_T *denotes the set* $\{0, 1, 2, \cdots, T\}$. *For Model (2.1), under assumptions C1-C4, we have*

(a) for $\tau_1^0 \le \tau \le \tau_2^0$,

$$\begin{cases} \sup_{m \in B_{1T}} \left|\eta_i(\frac{m}{T})/\eta_4(\frac{m}{T})\right| = o_p(\kappa_T^2), \quad i = 1, 2, 3 \\ \sup_{m \in B_{1T}} \left|\Omega_T(\frac{m}{T})/\eta_4(\frac{m}{T})\right| = o_p(\kappa_T^2) \end{cases},$$

(b) for $0 < \tau < \tau_1^0$,

$$\begin{cases} \sup_{m \in B_{2T}} \left|\theta_i(\frac{m}{T})/\theta_6(\frac{m}{T})\right| = o_p(\kappa_T^2), \quad i = 1, 2, 3, 4, 5, 7, 8, 9 \\ \sup_{m \in B_{2T}} \left|\Omega_T(\frac{m}{T})/\theta_6(\frac{m}{T})\right| = o_p(\kappa_T^2) \end{cases},$$

(c) for $\tau_2^0 < \tau < 1$,

$$\begin{cases} \sup_{m \in B_{3T}} \left|\gamma_i(\frac{m}{T})/\gamma_6(\frac{m}{T})\right| = o_p(\kappa_T^2), \quad i = 1, 2, 3, 4, 5, 7, 8, 9 \\ \sup_{m \in B_{3T}} \left|\Omega_T(\frac{m}{T})/\gamma_6(\frac{m}{T})\right| = o_p(\kappa_T^2) \end{cases}.$$

Proof. To prove part (a), note that

$$0 \ne \kappa_T = |\beta_{3T} - \beta_{2T}| = \begin{cases} O(\frac{1}{k_T}), & \text{when } k_T = O(h_T) \\ O(\frac{1}{h_T}), & \text{when } h_T = o(k_T) \end{cases}$$

and

$$\begin{aligned} \sup_{m \in B_{1T}} \left|\frac{1}{\eta_4(\frac{m}{T})}\right| &= \sup_{m \in B_{1T}} \frac{\sum_{t=m+1}^T y_{t-1}^2}{\sum_{t=m+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^T y_{t-1}^2} \\ &= \left(1 + \frac{\sum_{t=k_2^0+1}^T y_{t-1}^2}{\sum_{t=k_2^0-M_T}^{k_2^0} y_{t-1}^2}\right) \frac{1}{\sum_{t=k_2^0+1}^T y_{t-1}^2} \\ &= \left(1 + O_p(\frac{\beta_2^{2(k_2^0-k_1^0)} T h_T}{\beta_2^{2(k_2^0-k_1^0)} T M_T})\right) \cdot O_p(\frac{1}{\beta_2^{2(k_2^0-k_1^0)} T h_T}) \\ &= O_p(\frac{1}{\beta_2^{2(k_2^0-k_1^0)} T M_T}) \end{aligned} \tag{A.14}$$

by Lemmas A.3 and A.4. For the term $\sup_{m \in B_{1T}} |\eta_1(\frac{m}{T})/\eta_4(\frac{m}{T})|$, noting that

$$\sup_{m \in B_{1T}} \left|\eta_1(\frac{m}{T})\right| = \sup_{m \in B_{1T}} \left|\left(\frac{\sum_{t=1}^m y_{t-1}u_t}{\sum_{t=1}^m y_{t-1}^2} - \frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right) \sum_{t=1}^{k_1^0} y_{t-1}^2\right|$$

$$
\begin{aligned}
&\leq \sup_{m\in B_{1T}} \left|\frac{\sum_{t=1}^{m} y_{t-1}u_t}{\sum_{t=1}^{m} y_{t-1}^2}\right| \sum_{t=1}^{k_1^0} y_{t-1}^2 + \left|\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right| \sum_{t=1}^{k_1^0} y_{t-1}^2 \\
&\leq \sup_{m\in B_{1T}} \sqrt{\frac{\sum_{t=1}^{m} u_t^2}{\sum_{t=1}^{m} y_{t-1}^2}} \sum_{t=1}^{k_1^0} y_{t-1}^2 + \left|\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right| \sum_{t=1}^{k_1^0} y_{t-1}^2 \\
&\leq \sqrt{\frac{\sum_{t=1}^{k_2^0} u_t^2}{\sum_{t=1}^{k_1^0} y_{t-1}^2}} \sum_{t=1}^{k_1^0} y_{t-1}^2 + \left|\frac{\sum_{t=1}^{k_2^0} y_{t-1}u_t}{\sum_{t=1}^{k_2^0} y_{t-1}^2}\right| \sum_{t=1}^{k_1^0} y_{t-1}^2 \\
&= O_p(\sqrt{\frac{T}{T^2}}T^2) + O_p(\frac{\beta_2^{k_2^0-k_1^0}\sqrt{Tk_T}}{\beta_2^{2(k_2^0-k_1^0)}Tk_T}T^2) \\
&= O_p(T^{3/2})
\end{aligned}
$$

by Lemmas A.2 and A.3, we have

$$
\sup_{m\in B_{1T}} \left|\frac{\eta_1(\frac{m}{T})}{\eta_4(\frac{m}{T})}\right| \leq \sup_{m\in B_{1T}} \left|\eta_1(\frac{m}{T})\right| \cdot \sup_{m\in B_{1T}} \left|\frac{1}{\eta_4(\frac{m}{T})}\right| \leq O_p(T^{3/2}) \cdot O_p(\frac{1}{\beta_2^{2(k_2^0-k_1^0)}TM_T}) = o_p(\kappa_T^2).
$$

For the term $\sup_{m\in B_{1T}} |\eta_2(\frac{m}{T})/\eta_4(\frac{m}{T})|$, it is clear that

$$
\begin{aligned}
\sup_{m\in B_{1T}} \left|\eta_2(\frac{m}{T})\right| &= \sup_{m\in B_{1T}} \left|\frac{\left(\sum_{t=m+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}u_t - \sum_{t=m+1}^{k_2^0} y_{t-1}u_t \sum_{t=k_2^0+1}^{T} y_{t-1}^2\right)}{\sum_{t=m+1}^{T} y_{t-1}^2}\right| \\
&\leq \sup_{m\in B_{1T}} \left|\frac{\sum_{t=m+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}u_t}{\sum_{t=m+1}^{T} y_{t-1}^2}\right| + \sup_{m\in B_{1T}} \left|\frac{\sum_{t=m+1}^{k_2^0} y_{t-1}u_t \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=m+1}^{T} y_{t-1}^2}\right| \\
&\leq \left|\sum_{t=k_2^0+1}^{T} y_{t-1}u_t\right| + \sup_{m\in B_{1T}} \left|\sum_{t=m+1}^{k_2^0} y_{t-1}u_t\right| \\
&\leq O_p(\beta_2^{k_2^0-k_1^0}\sqrt{Th_T}) + \sup_{m\in B_{1T}} \sqrt{\sum_{t=m+1}^{k_2^0} y_{t-1}^2 \sum_{t=m+1}^{k_2^0} u_t^2} \\
&\leq O_p(\beta_2^{k_2^0-k_1^0}\sqrt{Th_T}) + \sqrt{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0} u_t^2} \\
&= O_p(\beta_2^{k_2^0-k_1^0}\sqrt{Th_T}) + O_p(\sqrt{\beta_2^{2(k_2^0-k_1^0)}Tk_TT}) \\
&= O_p(\beta_2^{k_2^0-k_1^0}T\sqrt{k_T}).
\end{aligned}
$$

Then, it follows from (A.14) that

$$
\sup_{m\in B_{1T}} \left|\frac{\eta_1(\frac{m}{T})}{\eta_4(\frac{m}{T})}\right| \leq O_p(\beta_2^{k_2^0-k_1^0}T\sqrt{k_T}) \cdot O_p(\frac{1}{\beta_{2T}^{2(k_2^0-k_1^0)}TM_T}) = o_p(\kappa_T^2).
$$

For the term $\sup_{m\in B_{1T}} |\eta_3(\frac{m}{T})/\eta_4(\frac{m}{T})|$, we have

$$
\sup_{m\in B_{1T}} \left|\frac{\eta_3(\frac{m}{T})}{\eta_4(\frac{m}{T})}\right| = \sup_{m\in B_{1T}} \frac{(\sum_{t=1}^{k_1^0} y_{t-1}^2)^2 \sum_{t=m+1}^{k_2^0} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2 \sum_{t=1}^{m} y_{t-1}^2} \cdot \frac{\sum_{t=m+1}^{T} y_{t-1}^2}{\sum_{t=m+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2}
$$

$$
\begin{aligned}
&= \frac{(\sum_{t=1}^{k_1^0} y_{t-1}^2)^2 \sum_{t=k_1^0+1}^{T} y_{t-1}^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2 \sum_{t=1}^{k_1^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2} \\
&= O_p(\frac{T^4(\beta_2^{2(k_2^0-k_1^0)} Tk_T + \beta_2^{2(k_2^0-k_1^0)} Th_T)}{\beta_2^{2(k_2^0-k_1^0)} Tk_T T^2 \beta_2^{2(k_2^0-k_1^0)} Th_T}) \\
&= o_p(\kappa_T^2).
\end{aligned}
$$

To examine the stochastic order of the term $\sup_{m\in B_{1T}} |\Omega_T(\frac{m}{T})/\eta_4(\frac{m}{T})|$, we first apply Lemmas A.2-A.4 to have

$$
\frac{(\sum_{t=1}^{k_2^0} y_{t-1}u_t)^2}{\sum_{t=1}^{k_2^0} y_{t-1}^2} = O_p(1) \quad \text{and} \quad \frac{(\sum_{t=k_2^0+1}^{T} y_{t-1}u_t)^2}{\sum_{t=k_2^0+1}^{T} y_{t-1}^2} = O_p(1).
$$

Second, applying Cauchy-Schwarz inequality, we have

$$
\sup_{m\in B_{1T}} \left| \frac{(\sum_{t=1}^{m} y_{t-1}u_t)^2}{\sum_{t=1}^{m} y_{t-1}^2} \right| \leq \sup_{m\in B_{1T}} \sum_{t=1}^{m} u_t^2 \leq \sum_{t=1}^{T} u_t^2 = O_p(T)
$$

and

$$
\sup_{m\in B_{1T}} \left| \frac{(\sum_{t=m+1}^{T} y_{t-1}u_t)^2}{\sum_{t=m+1}^{T} y_{t-1}^2} \right| \leq \sup_{m\in B_{1T}} \sum_{t=m+1}^{T} u_t^2 \leq \sum_{t=1}^{T} u_t^2 = O_p(T).
$$

Therefore,

$$
\sup_{m\in B_{1T}} \left| \Omega_T(\frac{m}{T}) \right| \leq O_p(T),
$$

which together with (A.14) yield

$$
\sup_{m\in B_{1T}} \left| \frac{\Omega_T(\frac{m}{T})}{\eta_4(\frac{m}{T})} \right| = o_p(\kappa_T^2).
$$

These prove part (a).

Part (b) and part (c) can be proved in a similar manner, hence the details are omitted for the sake of brevity. □

Lemma A.6 *For Model (2.1), under assumptions C1-C4, we have, for any fixed integer* $m \geq 0$,

(a) $\frac{1}{\beta_2^{2(k_2^0-k_1^0)} T(\beta_3-\beta_2)^2} \Big(RSS_{2,T}(\tau_2^0 - \frac{m}{T}) - RSS_{2,T}(\tau_2^0) \Big) \Rightarrow m\sigma^2 W^2(\tau_1^0)$,

(b) $\frac{1}{\beta_2^{2(k_2^0-k_1^0)} T(\beta_3-\beta_2)^2} \Big(RSS_{2,T}(\tau_2^0 + \frac{m}{T}) - RSS_{2,T}(\tau_2^0) \Big) \Rightarrow m\sigma^2 W^2(\tau_1^0)$.

Proof. To prove part (a), by the similar arguments in the proof of Lemma A.5, one can show that

$$
RSS_{2,T}(\tau_2^0 - \frac{m}{T}) - RSS_{2,T}(\tau_2^0)
$$

$$
\begin{aligned}
&= \eta_4(\tau_2^0 - \frac{m}{T})(\beta_3-\beta_2)^2(1+o_p(1)) \\
&= \frac{\sum_{t=k_2^0-m+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=k_2^0-m+1}^{T} y_{t-1}^2}(\beta_3-\beta_2)^2(1+o_p(1)) \\
&= \sum_{t=k_2^0-m+1}^{k_2^0} y_{t-1}^2(\beta_3-\beta_2)^2(1+o_p(1))
\end{aligned}
$$

by Lemmas A.3 and A.4. Then, we immediately have

$$
\begin{aligned}
&\frac{1}{\beta_2^{2(k_2^0-k_1^0)}T(\beta_3-\beta_2)^2}\Big(RSS_{2,T}(\tau_2^0-\frac{m}{T}) - RSS_{2,T}(\tau_2^0)\Big) \\
= \; & \frac{1}{\beta_2^{2(k_2^0-k_1^0)}T}\sum_{t=k_2^0-m+1}^{k_2^0} y_{t-1}^2(1+o_p(1)) \\
= \; & \frac{m y_{k_2^0}^2}{\beta_2^{2(k_2^0-k_1^0)}T}(1+o_p(1)) \\
\Rightarrow \; & m\sigma^2 W^2(\tau_1^0)
\end{aligned}
$$

by Lemma A.3 again. This proves part (a).

To prove part (b), similarly, it can be shown that

$$
\begin{aligned}
&RSS_{2,T}(\tau_2^0+\frac{m}{T}) - RSS_{2,T}(\tau_2^0) \\
= \; & \gamma_6(\tau_2^0+\frac{m}{T})(\beta_3-\beta_2)^2(1+o_p(1)) \\
= \; & \frac{\sum_{t=k_1^0+1}^{k_2^0} y_{t-1}^2 \sum_{t=k_2^0+1}^{k_2^0+m} y_{t-1}^2 \sum_{t=k_1^0+1}^{k_2^0+m} y_{t-1}^2}{(\sum_{t=1}^{k_2^0+m} y_{t-1}^2)^2}(\beta_3-\beta_2)^2(1+o_p(1)) \\
= \; & \sum_{t=k_2^0+1}^{k_2^0+m} y_{t-1}^2(\beta_3-\beta_2)^2(1+o_p(1))
\end{aligned}
$$

by Lemmas A.2 and A.3. Then, we immediately have

$$
\begin{aligned}
&\frac{1}{\beta_2^{2(k_2^0-k_1^0)}T(\beta_3-\beta_2)^2}\Big(RSS_{2,T}(\tau_2^0+\frac{m}{T}) - RSS_{2,T}(\tau_2^0)\Big) \\
= \; & \frac{1}{\beta_2^{2(k_2^0-k_1^0)}T}\sum_{t=k_2^0+1}^{k_2^0+m} y_{t-1}^2(1+o_p(1)) \\
= \; & \frac{m y_{k_2^0}^2}{\beta_2^{2(k_2^0-k_1^0)}T}(1+o_p(1)) \\
\Rightarrow \; & m\sigma^2 W^2(\tau_1^0)
\end{aligned}
$$

by Lemma A.3 again. This proves part (b). □

Proof of Theorem 3.1. We prove the last results in (3.1)-(3.3) first. To prove the last result in (3.1), we first prove the following result:

$$|\hat{\tau}_{2,T} - \tau_2^0| = O_p(1/T). \tag{A.15}$$

To this end, we shall use the contradiction argument. Suppose (A.15) is not true, then there exists an integer sequence $M_T > 0$ such that $M_T \to \infty, M_T = o(k_T)$, $M_T = o(h_T)$, and

$$\liminf_{T\to\infty} P(|\hat{k}_2 - k_2^0| > M_T) \geq \alpha, \tag{A.16}$$

where α is a positive constant in $(0, 1]$. Recall the definitions of κ_T, Z_T, B_{1T}, B_{2T} and B_{3T} in Lemma A.5 and define

$$B_{0T} = \{m : m \in Z_T, k_2^0 - M_T \leq m \leq k_2^0 + M_T\}.$$

Note that

$$\begin{aligned} P(|\hat{k}_2 - k_2^0| > M_T) &= P(\inf_{m\in B_{1T}\cup B_{2T}\cup B_{3T}} RSS_{2,T}(\frac{m}{T}) < \inf_{m\in B_{0T}} RSS_{2,T}(\frac{m}{T})) \\ &\leq P(\inf_{m\in B_{1T}\cup B_{2T}\cup B_{3T}} RSS_{2,T}(\frac{m}{T}) < RSS_{2,T}(\tau_2^0)) \\ &\leq \sum_{i=1}^{3} P(\inf_{m\in B_{iT}} (RSS_{2,T}(\frac{m}{T}) - RSS_{2,T}(\tau_2^0)) < 0). \end{aligned} \tag{A.17}$$

To examine the term $P(\inf_{m\in B_{1T}}(RSS_{2,T}(\frac{m}{T}) - RSS_{2,T}(\tau_2^0)) < 0)$, using part (a) of Lemma A.1, part (a) of Lemma A.5 and the fact that both $|\beta_2 - \beta_1|$ and $|\beta_3 - \beta_2|$ approach zero, we have for large T that

$$\begin{aligned} & P(\inf_{m\in B_{1T}} (RSS_{2,T}(\frac{m}{T}) - RSS_{2,T}(\tau_2^0)) < 0) \\ = \;& P(\inf_{m\in B_{1T}} (\eta_1(\frac{m}{T})(\beta_2 - \beta_1) + \eta_2(\frac{m}{T})(\beta_3 - \beta_2) + \eta_3(\frac{m}{T})(\beta_2 - \beta_1)^2 + \eta_4(\frac{m}{T})(\beta_3 - \beta_2)^2 \\ & \quad + \Omega_T(\frac{m}{T})) < 0) \\ = \;& P(\inf_{m\in B_{1T}} (\eta_4(\frac{m}{T})(\frac{\eta_1(\frac{m}{T})}{\eta_4(\frac{m}{T})}(\beta_2 - \beta_1) + \frac{\eta_2(\frac{m}{T})}{\eta_4(\frac{m}{T})}(\beta_3 - \beta_2) + \frac{\eta_3(\frac{m}{T})}{\eta_4(\frac{m}{T})}(\beta_2 - \beta_1)^2 + (\beta_3 - \beta_2)^2 \\ & \quad + \frac{\Omega_T(\frac{m}{T})}{\eta_4(\frac{m}{T})})) < 0) \\ \leq \;& P(\inf_{m\in B_{1T}} \eta_4(\frac{m}{T})((\beta_3 - \beta_2)^2 - \sum_{i=1}^{3} \sup_{m\in B_{1T}} |\frac{\eta_i(\frac{m}{T})}{\eta_4(\frac{m}{T})}| - \sup_{m\in B_{1T}} |\frac{\Omega_T(\frac{m}{T})}{\eta_4(\frac{m}{T})}|) < 0) \\ = \;& P(\inf_{m\in B_{1T}} \eta_4(\frac{m}{T})(\beta_3 - \beta_2)^2(1 + o_p(1)) < 0) \\ = \;& o(1) \end{aligned} \tag{A.18}$$

since

$$\inf_{m\in B_{1T}} \eta_4(\frac{m}{T}) \geq \frac{y_{k_2^0-1}^2 \sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sum_{t=k_1^0+1}^{T} y_{t-1}^2} = O_p(\frac{\beta_2^{2(k_2^0-k_1^0)} T h_T}{k_T + h_T}),$$

which suggests $\inf_{m\in B_{1T}} \eta_4(\frac{m}{T})(\beta_3-\beta_2)^2$ will diverge to infinity in probability by (A.1).

Similarly, one can use part (b) of Lemma A.1 and part (b) of Lemma A.5 to obtain

$$P(\inf_{m\in B_{2T}}(RSS_{2,T}(\frac{m}{T})-RSS_{2,T}(\tau_2^0))<0)=o(1) \tag{A.19}$$

and use part (c) of Lemma A.1 and part (c) of Lemma A.5 to obtain

$$P(\inf_{m\in B_{3T}}(RSS_{2,T}(\frac{m}{T})-RSS_{2,T}(\tau_2^0))<0)=o(1). \tag{A.20}$$

The details are omitted. Now, combining (A.17)-(A.20) together leads to

$$P(|\hat{k}_2-k_2^0|>M_T)=o(1),$$

which contradicts (A.16). Thus, (A.15) is proved.

Next, we will improve the result (A.15). Given (A.15), for any $\eta>0$, there exists an $M>0$ such that $P(|\hat{k}_2-k_2^0|>M)<\eta$. Therefore,

$$\begin{aligned}
& P(\hat{k}_2\neq k_2^0) \\
= \;& P(|\hat{k}_2-k_2^0|>M)+P(|\hat{k}_2-k_2^0|\leq M, \hat{k}_2\neq k_2^0) \\
\leq \;& \eta+\sum_{m=1}^{M}P(RSS_{2,T}(\tau_2^0-\frac{m}{T})-RSS_{2,T}(\tau_2^0)<0)+\sum_{m=1}^{M}P(RSS_{2,T}(\tau_2^0+\frac{m}{T})-RSS_{2,T}(\tau_2^0)<0) \\
= \;& \eta+\sum_{m=1}^{M}P(\frac{1}{\beta_2^{2(k_2^0-k_1^0)}T(\beta_3-\beta_2)^2}(RSS_{2,T}(\tau_2^0-\frac{m}{T})-RSS_{2,T}(\tau_2^0))<0) \\
& +\sum_{m=1}^{M}P(\frac{1}{\beta_2^{2(k_2^0-k_1^0)}T(\beta_3-\beta_2)^2}(RSS_{2,T}(\tau_2^0+\frac{m}{T})-RSS_{2,T}(\tau_2^0))<0) \\
= \;& o(1)
\end{aligned}$$

by Lemma A.6, the finiteness of M and the arbitrariness of η. Hence, the last result in (3.1) is proved.

Based on the above result, one can easily prove that the limiting distributions of $\hat{\beta}_3(\hat{\tau}_{2,T})$ and $\hat{\beta}_3(\tau_2^0)$ are the same by using the arguments in the proof of Theorem 4 in Chong (2001). Then, applying Lemma A.4, we have

$$\sqrt{\frac{k_1^0h_T}{2c_2}}\beta_2^{k_2^0-k_1^0}(\hat{\beta}_3(\tau_2^0)-\beta_3)=\sqrt{\frac{k_1^0}{2Tc_2}}\cdot\frac{\frac{1}{\beta_2^{k_2^0-k_1^0}\sqrt{Th_T}}\sum_{t=k_2^0+1}^{T}y_{t-1}u_t}{\frac{1}{\beta_2^{2(k_2^0-k_1^0)}Th_T}\sum_{t=k_2^0+1}^{T}y_{t-1}^2}\Rightarrow\zeta,$$

implying

$$\sqrt{\frac{k_1^0h_T}{2c_2}}\beta_2^{k_2^0-k_1^0}(\hat{\beta}_3(\hat{\tau}_{2,T})-\beta_3)\Rightarrow\zeta.$$

The last result in (3.2) is proved.

Similarly, using Lemma A.4 again, we have

$$\sqrt{\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sigma^2}}(\hat{\beta}_3(\tau_2^0) - \beta_3) = \frac{\frac{1}{\sigma^2 \beta_2^{k_2^0 - k_1^0}\sqrt{Th_T}} \sum_{t=k_2^0+1}^{T} y_{t-1}u_t}{\sqrt{\frac{1}{\sigma^2 \beta_2^{2(k_2^0 - k_1^0)} Th_T} \sum_{t=k_2^0+1}^{T} y_{t-1}^2}} \Rightarrow N(0,1),$$

which implies

$$t_3 = \sqrt{\frac{\sum_{t=k_2^0+1}^{T} y_{t-1}^2}{\sigma^2}}(\hat{\beta}_3(\hat{\tau}_{2,T}) - \beta_3) \Rightarrow N(0,1).$$

The last result in (3.3) is proved.

To prove the remaining results in Theorem 3.1, note that τ_1^0, β_1 and β_2 are estimated using the subsample $\{y_1, \cdots, y_{\hat{k}_2}\}$ and we have proven that $P(\hat{k}_2 \neq k_2^0) \to 0$. Hence, the asymptotic properties of the estimators of τ_1^0, β_1 and β_2 obtained through the subsample $\{y_1, \cdots, y_{\hat{k}_2}\}$ are the same as those of the estimators of τ_1^0, β_1 and β_2 obtained through the subsample $\{y_1, \cdots, y_{k_2^0}\}$ by the similar arguments in the proof of Theorem 4 in Chong (2001). The asymptotic properties of the LSEs of the break point k_1^0 and the two AR parameters β_1 and β_2 under the subsample $\{y_1, \cdots, y_{k_2^0}\}$, which contains a unit root model and a mildly explosive AR(1) model, follow immediately from Theorem 1.3 and Lemmas B.1 and C.2 in Pang et al. (2017). □