Value dividends, the Harsanyi set and extensions, and the proportional Harsanyi payoff

Besner, Manfred

HFT Stuttgart, University of Applied Sciences, Schellingstr. 24

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Manfred Besner*

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Abstract

A new concept for TU-values, called value dividends, is introduced. Similar to Harsanyi dividends, value dividends are defined recursively and provide new characterizations of values from the Harsanyi set. In addition, we generalize the Harsanyi set where each of the TU-values from this set is defined by the distribution of the Harsanyi dividends via sharing function systems and give an axiomatic characterization. As a TU value from the generalized Harsanyi set, we present the proportional Harsanyi payoff, a new proportional solution concept. As a side benefit, a new characterization of the Shapley value is proposed. None of our characterizations uses additivity.

Keywords TU-game · Value dividends · (Generalized) Harsanyi set · Weighted Shapley values · (Proportional) Harsanyi payoff · Sharing function systems

1. Introduction

The concept of Harsanyi dividends was introduced by Harsanyi (1959). They can be defined inductively: the dividend of the empty set is zero and the dividend of any other possible coalition of a player set equals the worth of the coalition minus the sum of all dividends of proper subsets of that coalition. Hence, Harsanyi dividends can be interpreted as "the pure contribution of cooperation in a TU-game" (Billot and Thisse, 2005). Harsanyi could show that if the Harsanyi dividends of all possible coalitions are spaced evenly among its members, each player's payoff equals the Shapley value (Shapley, 1953b). The weighted Shapley values (Shapley, 1953a) distribute the Harsanyi dividends proportionally to players' personal given weights. The ratio of the weights of two players is equal for all coalitions containing them. However, sometimes this seems unrealistic. For example, in some coalitions, the influence of one of two players on the other players may be higher than in other coalitions with other players. Hammer et al. (1977) and Vasil'ev (1978)

*M. Besner
Department of Geomatics, Computer Science and Mathematics,
HFT Stuttgart, University of Applied Sciences, Schellingstr. 24
D-70174 Stuttgart, Germany
E-mail: manfred.besner@hft-stuttgart.de

proposed the Harsanyi payoffs[1] which take this into account. There the players' weights, assigned to the Harsanyi dividends via a sharing system, can differ for all coalitions.

Myerson (1980) introduced the balanced contributions axiom which allows, along with efficiency, an elegant axiomatization of the Shapley value. It states for two players i and j that j contributes as much to i's payoff as i contributes to j's payoff. The w-balanced contributions properties, the ratio of two players' payoffs is proportional to their weights, joint with efficiency, characterize the TU-values of an interesting class (Myerson, 1980). These values coincide with the weighted Shapley values (Hart and Mas-Colell, 1989).

So far, no analogous characterization to the above-mentioned characterization of the weighted Shapley values is known for the Harsanyi payoffs. To enable corresponding axiomatizations for the Harsanyi payoffs, we present a new concept, called "value dividends." These are defined inductively: the value dividend of a singleton is the player's payoff in a single-player game and the value dividend of any (non-empty) other coalition to a player represents that player's payoff in the game on the player set of this coalition minus all value dividends to that player in all subgames. We can therefore regard a value dividend as the "pure" payoff to a player that has not yet been realized in a subgame.

Similar to the w-balanced contributions properties, we introduce an axiom called λ-balanced value dividends where λ is a sharing system. If λ has the characteristic that the ratio of two players sharing weights is equal in all coalitions containing them, then, surprisingly, the λ-balanced value dividends property is equivalent to the w-balanced contributions property.

The value dividends allow further axiomatizations of the Harsanyi payoffs and lead to an extension of the Harsanyi set: we provide an axiomatization of the class of generalized Harsanyi payoffs, called generalized Harsanyi set. These values are generally no longer additive and use sharing function systems to distribute the Harsanyi dividends. A central property of all TU-values, discussed in this paper, is the inessential grand coalition property. This property states that in games where the Harsanyi dividend of the grand coalition is zero a players' payoff in the subgames determines the player's payoff.

The generalized Harsanyi set allows proportionality principles in allocation. A common consensus of most proportional sharing rules is the proportional standardness property for two-player games (Ortmann, 2000), which means that the whole must be divided proportionally to the singleton worths of both players. For more than two players, however, there is absolutely no agreement on how proportionality should be applied. Many possibilities are suggested such as the set-valued proper Shapley value (Vorob'ev and Liapunov, 1998; van den Brink et al., 2015) where proportionality is stated by a fixed point argument.

Whereas, as single-valued solutions, the proportional rule (Moriarity, 1975) and, as a value from the generalized Harsanyi set, the proportional Shapley value (Gangolly, 1981; Besner, 2016; Béal et al., 2018) consider only the worths of the singletons as weights, the proportional value (Feldman, 1999; Ortmann, 2000) uses the worths of all coalitions in a recursive formula. As a new representative of the generalized Harsanyi set, we introduce the proportional Harsanyi payoff that also involves for calculating the whole coalition function for a sharing function system in a recursive formula.

The article is organized as follows. Section 2 contains some preliminaries. In Sect. 3, we define the concept of value dividends, present a new characterization of the Harsanyi set via efficiency and λ-balanced value dividends and contrast w-balanced contributions

[1]These TU-values are also known as Harsanyi payoff vectors, sharing values, or Harsanyi solutions.

with λ-balanced value dividends. In Sect. 4, we introduce the inessential grand coalition property that is crucial for the remainder of the article and propose an axiomatization of the Harsanyi payoffs and of the Harsanyi set as a whole. In Sect. 5, we generalize the Harsanyi set and give a class characterization. In Sect. 6, the proportional Harsanyi payoff is defined and axiomatized, and the domain, the value, and reasonable axioms are illustrated using an example. Section 7 concludes and discusses the results. Some extensions of the Harsanyi and the generalized Harsanyi set are briefly presented and we give a quick comparison of the proportional Harsanyi payoff with the Shapley value by a new axiomatization of the Shapley value. Finally, the Appendix (Sect. 8) shows the logical independence of the axioms in our characterizations.

2. Preliminaries

We denote by $\mathbb{N}$ the natural numbers, by $\mathbb{R}$ the real numbers, by $\mathbb{R}_+$ the set of all non-negative real numbers, and by $\mathbb{R}_{++}$ the set of all positive real numbers. Let $\mathfrak{U}$ be a countably infinite set, the universe of all players. We define by $\mathcal{N}$ the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) with player set $N \in \mathcal{N}$ and a **coalition function** $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. We call each subset $S \subseteq N$ a **coalition**, $v(S)$ represents the **worth** of coalition S and we denote by Ω^S the set of all non-empty subsets of S. The set of all TU-games with player set N is denoted by $\mathbb{V}(N)$, if $v(\{i\}) > 0$ for all $i \in N$ or if $v(\{i\}) < 0$ for all $i \in N$, by $\mathbb{V}_0(N)$, and if all $v(S) > 0$ for all $S \in \Omega^N$, by $\mathbb{V}_{0+}(N)$. The **restriction** of (N, v) to a player set $S \in \Omega^N$ is denoted by (S, v). An **unanimity game** (N, u_S), $S \in \Omega^N$, is defined for all $T \subseteq N$ by $u_S(T) = 1$, if $S \subseteq T$, and $u_S(T) = 0$, otherwise; the **null game** $(N, \mathbf{0}^N)$ is given by $\mathbf{0}^N(S) = 0$ for all $S \subseteq N$.

Let $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}(N)$. For all $S \subseteq N$ the **Harsanyi dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\Delta_v(S) := \begin{cases} 0, \text{ if } S = \emptyset, \text{ and} \\ v(S) - \sum_{R \subsetneq S} \Delta_v(R), \text{ otherwise.} \end{cases} \tag{1}$$

A TU-game (N, v) is called **almost positive** if $\Delta_v(S) \geq 0$ for all $S \subseteq N$, $|S| \geq 2$; it is called **totally positive** (Vasil'ev, 1975) if $\Delta_v(S) \geq 0$ for all $S \subseteq N$. We call a totally positive TU-game game (N, v) **strongly positive** if $v(\{i\}) > 0$ for all $i \in N$. The set of all totally positive TU-games is denoted by $\mathbb{V}_+(N)$, and the set of all strongly positive TU-games by $\mathbb{V}_{++}(N)$. A player $i \in N$ is called a **dummy player** in (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$. Two players $i, j \in N$, $i \neq j$, are **symmetric** in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

We define by $W := \{f : \mathfrak{U} \to \mathbb{R}_{++}\}$, $\mathrm{w}_i := \mathrm{w}(i)$ for all $\mathrm{w} \in W$, $i \in \mathfrak{U}$, the collection of all positive **weight systems** on $\mathfrak{U}$. The collection Λ of all **sharing systems** $\lambda \in \Lambda$ on

$\mathcal{N}$ is defined[2] by

$$\Lambda := \left\{ \lambda = (\lambda_{N,i})_{N\in\mathcal{N},\, i\in N} \middle| \sum_{i\in N} \lambda_{N,i} = 1 \text{ and } \lambda_{N,i} \geq 0 \text{ for each } N \in \mathcal{N} \text{ and all } i \in N \right\}.$$

For all $N \in \mathcal{N}$, a **TU-value** φ is an operator that assigns to any $(N, v) \in \mathbb{V}(N)$ a payoff vector $\varphi(N, v) \in \mathbb{R}^N$.

For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and each $\mathrm{w} \in W$, the (positively) **weighted Shapley Value** Sh^{w} (Shapley, 1953a) is defined by

$$Sh_i^{\mathrm{w}}(N, v) := \sum_{S\subseteq N,\, S\ni i} \frac{\mathrm{w}_i}{\sum_{j\in S} \mathrm{w}_j} \Delta_v(S) \text{ for all } i \in N.$$

The set of all weighted Shapley values is also known as **Shapley set**. A special case of a weighted Shapley value, all weights are equal, is the **Shapley value** Sh (Shapley, 1953b), given by

$$Sh_i(N, v) := \sum_{S\subseteq N,\, S\ni i} \frac{\Delta_v(S)}{|S|} \text{ for all } N \in \mathcal{N},\ (N, v) \in \mathbb{V}(N), \text{ and } i \in N.$$

Hammer et al. (1977) and Vasil'ev (1978) introduced independently a set of TU-values, called **Harsanyi set**, also known as **selectope** (Derks et al., 2000). The payoffs are made by distributing the Harsanyi dividends with the help of a sharing system. Each TU-value H^λ, $\lambda \in \Lambda$, in this set, titled **Harsanyi payoff**, is defined by

$$H_i^\lambda(N, v) := \sum_{S\subseteq N,\, S\ni i} \lambda_{S,i}\Delta_v(S), \text{ for all } N \in \mathcal{N},\ (N, v) \in \mathbb{V}(N), \text{ and } i \in N. \tag{2}$$

Obviously, the Shapley set is a proper subset of the Harsanyi set. The following TU-values are not linear and are defined on subsets of $\mathbb{V}(N)$.

For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}_0(N)$, the **proportional rule** π (Moriarity, 1975) is given by

$$\pi_i(N, v) := \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) \text{ for all } i \in N \tag{3}$$

and the **proportional Shapley value** Sh^p (Gangolly, 1981; Besner, 2016; Béal et al., 2018) is defined by

$$Sh_i^p(N, v) := \sum_{S\subseteq N,\, S\ni i} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})} \Delta_v(S) \text{ for all } i \in N. \tag{4}$$

[2]In our opinion, the definition of a weight system on the universe of all players (see, e. g., Casajus, 2018) has some advantages in contrast to the definition on a fixed player set (see, e. g., Kalai and Samet, 1987), especially if other player sets are regarded in such a way that identical players have the same weights in different player sets. In order to have similar advantages for sharing systems, we define these systems on the set of all finite subsets of the universe of all players and not on the set of all subsets of a fixed player set as usually common.

For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}_{0+}(N)$, the **proportional value** P (Feldman, 1999; Ortmann, 2000) is defined inductively for all $i \in S$, $S \subseteq N$, by

$$P_i(S,v) := \begin{cases} v(\{i\}), \text{ if } S = \{i\}, \\ \dfrac{v(S)}{1 + \sum_{j \in S\setminus\{i\}} \dfrac{P_j(S\setminus\{i\},v)}{P_i(S\setminus\{j\},v)}}, \text{ otherwise.} \end{cases} \tag{5}$$

We make use of the following axioms for TU-values φ:

Efficiency, E. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, we have $\sum_{i\in N} \varphi_i(N,v) = v(N)$.

Dummy, D. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $i \in N$ such that i is a dummy player in (N,v), we have $\varphi_i(N,v) = v(\{i\})$.

Homogenity, H. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $i \in N$, and $\alpha \in \mathbb{R}$, we have $\varphi_i(N,\alpha v) = \alpha\varphi_i(N,v)$.

Monotonicity[3]**, M** (Megiddo, 1974)**.** For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $\alpha \in \mathbb{R}_{++}$, we have $\varphi_i(N, v + \alpha \cdot u_N) \geq \varphi_i(N,v)$ for all $i \in N$.

Symmetry, S. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $i,j \in N$ such that i and j are symmetric in (N,v), we have $\varphi_i(N,v) = \varphi_j(N,v)$.

Balanced contributions, BC (Myerson, 1980)**.** For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $i,j \in N$, $i \neq j$, we have $\varphi_i(N,v) - \varphi_i(N\setminus\{j\},v) = \varphi_j(N,v) - \varphi_j(N\setminus\{i\},v)$.

w-balanced contributions, BC$^{\mathrm{w}}$ (Myerson, 1980)**.** For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $i,j \in N$, $i \neq j$, and $\mathrm{w} \in W$, we have

$$\frac{\varphi_i(N,v) - \varphi_i(N\setminus\{j\},v)}{\mathrm{w}_i} = \frac{\varphi_j(N,v) - \varphi_j(N\setminus\{i\},v)}{\mathrm{w}_j}.$$

Proportional standardness, PSt (Ortmann, 2000)**.** For all $N \in \mathcal{N}$, $\{i,j\} \subseteq N$, $i \neq j$, $(\{i,j\},v) \in \mathbb{V}_0(N)$, we have

$$\varphi_i(\{i,j\},v) = \frac{v(\{i\})}{v(\{i\}) + v(\{j\})} v(\{i,j\}).$$

Two further axioms follow which are satisfied by the proportional rule and the proportional Shapley value. The first one states that a player's payoff does not change if another player splits into two new players who together make the same contribution to the game as the original splitting player. We use the following definition.

Definition 2.1. *Let $N, N^j \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $(N^j,v^j) \in \mathbb{V}(N^j)$, $j \in N$, $k,\ell \in \mathfrak{U}$, $k,\ell \notin N$, $N^j = (N\setminus\{j\}) \cup \{k,\ell\}$. The game (N^j,v^j) is called a **corresponding split player game** to (N,v) if for all $S \subseteq N\setminus\{j\}$*

- $v^j(\{k\}) + v^j(\{\ell\}) = v(\{j\})$,
- $v^j(S \cup \{i\}) = v^j(S) + v^j(\{i\})$, *$i \in \{k,\ell\}$,*
- $v^j(S \cup \{k,l\}) = v(S \cup \{j\})$ *and*

[3]In Young (1985) this property is referred to as aggregate monotonicity.

- $v^j(S) = v(S)$.

Player splitting, PSp (Besner, 2019). For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $j \in N$, and corresponding split player games $(N^j, v^j) \in \mathbb{V}(N^j)$ to (N,v), we have $\varphi_i(N,v) = \varphi_i(N^j, v^j)$ for all $i \in N \backslash \{j\}$.

The last property in this section requires the definition of weakly dependent players who are cooperatively productive only in coalitions that include all weakly dependent players.

Definition 2.2. *Let $N \in \mathcal{N}$ and $(N,v) \in \mathbb{V}(N)$. Two players $i,j \in N$, $i \neq j$, are called **weakly dependent** in (N,v) if $v(S \cup \{k\}) = v(S) + v(\{k\})$, $k \in \{i,j\}$, for all $S \subseteq N \backslash \{i,j\}$.*

Weakly dependent players should receive a payoff that is proportional to their singleton worths.

Proportionality, P (Besner, 2019). For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}_0(N)$, $i,j \in N$ such that i and j are weakly dependent in (N,v), we have

$$\frac{\varphi_i(N,v)}{v(\{i\})} = \frac{\varphi_j(N,v)}{v(\{j\})}.$$

3. Value dividends

The Harsanyi dividend of a coalition $S \subseteq N$ can be interpreted as the surplus of the worth of the coalition S versus the sum of all the surpluses of the worths of all proper subsets from S. Similarly, we define for a TU-value φ the value dividend $\Theta_{\varphi_i(S,v)}$ of a coalition $S \subseteq N$ to a player $i \in S$ as the additional payoff to player i in the subgame (S,v) versus the sum of all additional payoffs to player i in all subgames (R,v), $R \subsetneq S$, $R \ni i$. In detail, we have:

Definition 3.1. *For all $N \in \mathcal{N}$ and each $(N,v) \in \mathbb{V}(N), S \subseteq N$, $i \in S$, and TU-Value φ, the **value dividends** $\Theta_{\varphi_i(S,v)}$ are defined inductively by*

$$\Theta_{\varphi_i(S,v)} := \begin{cases} \varphi_i(\{i\},v), & \text{if } S = \{i\}, \\ \varphi_i(S,v) - \sum_{R \subsetneq S,\, R \ni i} \Theta_{\varphi_i(R,v)}, & \text{otherwise.} \end{cases} \tag{6}$$

For efficient TU-values, value dividends have a connection to Harsanyi dividends.

Remark 3.2. *Let $N \in \mathcal{N}$ and $(N,v) \in \mathbb{V}(N)$. By (1), (6), and induction on the size $|S|$, $S \subseteq N$, it is easy to show that we have for an efficient TU-value φ*

$$\sum_{i \in S} \Theta_{\varphi_i(S,v)} = \Delta_v(S) \text{ for all } S \in \Omega^N.$$

3.1. λ-balanced value dividends

We formulate a new axiom for TU-values φ which is related to w-balanced contributions.

λ-balanced value dividends, $\mathbf{BVD}^\lambda$. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $i,j \in N$, and $\lambda \in \Lambda$, we have $\lambda_{N,j}\Theta_{\varphi_i(N,v)} = \lambda_{N,i}\Theta_{\varphi_j(N,v)}$.

The ratio of two players' value dividends of the same coalition equals the ratio of the players' sharing weights if the weights and value dividends are not zero. It turns out that a Harsanyi payoff H^λ is characterized by **E** and $\mathbf{BVD}^\lambda$.

Theorem 3.3. *Let $\lambda \in \Lambda$. H^λ is the unique TU-value that satisfies **E** and $\mathbf{BVD}^\lambda$.*

Proof. I. Let $\lambda \in \Lambda$. It is well-known that H^λ satisfies **E**. Thus, we have only to show that H^λ meets $\mathbf{BVD}^\lambda$. By (2) and (6), we have $\Theta_{H_i^\lambda(N,v)} = \lambda_{N,i}\Delta_v(N)$ for all $i \in N$ and all $N \in \mathcal{N}$. Therefore, it is obvious that $\mathbf{BVD}^\lambda$ is satisfied too.

II. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $S \subseteq N$, and let φ and ϕ be two arbitrary TU-values which satisfy **E** and $\mathbf{BVD}^\lambda$. We show uniqueness by induction on the size $|S|$.

Initialization: If $|S| = 1$, uniqueness is satisfied by **E**.

Induction step: Let $|S| \geq 2$. Assume that equality of the two values holds for all $S' \subsetneq S$, $|S'| \geq 1$, and let $|S| = |S'| + 1$ (IH). Then equality of the value dividends of the two values for all $S' \subsetneq S$ holds too. Let $j \in S$ such that $\lambda_{S,j} \neq 0$. A such j always exists. By $\mathbf{BVD}^\lambda$, we have for all $i \in S$

$$\begin{aligned}
\Theta_{\varphi_i(S,v)} &= \frac{\lambda_{S,i}}{\lambda_{S,j}}\Theta_{\varphi_i(S,v)} \\
\underset{(6)}{\Leftrightarrow} \quad \varphi_i(S,v) - \sum_{\substack{R \subsetneq S,\\ R \ni i}} \Theta_{\varphi_i(R,v)} &= \frac{\lambda_{S,i}}{\lambda_{S,j}}\Bigg[\varphi_j(S,v) - \sum_{\substack{R \subsetneq S,\\ R \ni j}} \Theta_{\varphi_j(R,v)}\Bigg] \qquad (7)
\end{aligned}$$

and analogue

$$\phi_i(S,v) - \sum_{\substack{R \subsetneq S,\\ R \ni i}} \Theta_{\phi_i(R,v)} = \frac{\lambda_{S,i}}{\lambda_{S,j}}\Bigg[\phi_j(S,v) - \sum_{\substack{R \subsetneq S,\\ R \ni j}} \Theta_{\phi_j(R,v)}\Bigg]. \qquad (8)$$

We subtract (8) from (7)

$$\begin{aligned}
&\varphi_i(S,v) - \phi_i(S,v) - \sum_{\substack{R \subsetneq S,\\ R \ni i}} \Theta_{\varphi_i(R,v)} + \sum_{\substack{R \subsetneq S,\\ R \ni i}} \Theta_{\phi_i(R,v)} \\
&\qquad = \frac{\lambda_{S,i}}{\lambda_{S,j}}\Bigg[\varphi_j(S,v) - \phi_j(S,v) - \sum_{\substack{R \subsetneq S,\\ R \ni j}} \Theta_{\varphi_j(R,v)} + \sum_{\substack{R \subsetneq S,\\ R \ni j}} \Theta_{\phi_j(R,v)}\Bigg] \\
\underset{(IH)}{\Leftrightarrow} \quad &\varphi_i(S,v) - \phi_i(S,v) = \frac{\lambda_{S,i}}{\lambda_{S,j}}\big[\varphi_j(S,v) - \phi_j(S,v)\big]. \qquad (9)
\end{aligned}$$

(9) holds for all $i \in S$. We obtain

$$\sum_{i \in S}\big[\varphi_i(S,v) - \phi_i(S,v)\big] = \sum_{i \in S}\frac{\lambda_{S,i}}{\lambda_{S,j}}\big[\varphi_j(S,v) - \phi_j(S,v)\big]. \qquad (10)$$

By **E**, the left side of (10) equals zero. By induction, it follows $\varphi_j(S,v) = \phi_j(S,v)$ for all $j \in S$ and all $S \subseteq N$ with $\lambda_{S,j} \neq 0$. By (9), we have $\varphi_i(S,v) = \phi_i(S,v)$ also for all $i \in S$ with $\lambda_{S,i} = 0$ and uniqueness is shown. □

3.2. w-balanced contributions and λ-balanced value dividends

If players' sharing weights in all coalitions are in the same ratio, a Harsanyi payoff coincides with a weighted Shapley value. For such weights, the w-balanced contributions axiom can be considered as a special case of the λ-balanced value dividends axiom.

Theorem 3.4. *Let* $\mathrm{w} \in W$ *and* $\lambda \in \Lambda$ *such that*

$$\lambda_{N,i} := \frac{\mathrm{w}_i}{\sum_{j\in N} \mathrm{w}_j}, \text{ for all } N \in \mathcal{N},\ |N| \geq 2, \text{ and } i \in N. \tag{11}$$

Then $\boldsymbol{BVD}^{\lambda}$ *is equivalent to* $\boldsymbol{BC}^{\mathrm{w}}$.

Proof. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $i,j \in N$, $i \neq j$, and let w and λ be defined as in Theorem 3.4.

BVD$^{\lambda}$ $\Rightarrow$ **BC**$^{\mathrm{w}}$: By **BVD**$^{\lambda}$ and (6), we have

$$\frac{\varphi_i(N,v) - \sum_{\substack{S\subsetneq N,\\ S\ni i}} \Theta_{\phi_i(S,v)}}{\mathrm{w}_i} = \frac{\varphi_j(N,v) - \sum_{\substack{S\subsetneq N,\\ S\ni j}} \Theta_{\phi_j(S,v)}}{\mathrm{w}_j}$$

$$\Leftrightarrow \frac{\varphi_i(N,v) - \sum_{\substack{S\subseteq N\backslash\{j\},\\ S\ni i}} \Theta_{\phi_i(S,v)} - \sum_{\substack{S\subsetneq N,\\ \{i,j\}\subseteq S}} \Theta_{\phi_i(S,v)}}{\mathrm{w}_i}$$

$$= \frac{\varphi_j(N,v) - \sum_{\substack{S\subseteq N\backslash\{i\},\\ S\ni j}} \Theta_{\phi_j(S,v)} - \sum_{\substack{S\subsetneq N,\\ \{i,j\}\subseteq S}} \Theta_{\phi_j(S,v)}}{\mathrm{w}_j}$$

$$\underset{(\mathbf{BVD}^{\lambda})}{\Leftrightarrow} \frac{\varphi_i(N,v) - \sum_{\substack{S\subseteq N\backslash\{j\},\\ S\ni i}} \Theta_{\phi_i(S,v)}}{\mathrm{w}_i} = \frac{\varphi_j(N,v) - \sum_{\substack{S\subseteq N\backslash\{i\},\\ S\ni j}} \Theta_{\phi_j(S,v)}}{\mathrm{w}_j}$$

$$\underset{(6)}{\Leftrightarrow} \frac{\varphi_i(N,v) - \varphi_i(N\backslash\{j\},v)}{\mathrm{w}_i} = \frac{\varphi_j(N,v) - \varphi_j(N\backslash\{i\},v)}{\mathrm{w}_j}.$$

BC$^{\mathrm{w}}$ $\Rightarrow$ **BVD**$^{\lambda}$: We use induction on the size $|N|$.
Initialization: Let $N = \{i,j\}$. By **BC**$^{\mathrm{w}}$, we have

$$\frac{\varphi_i(N,v) - \varphi_i(N\backslash\{j\},v)}{\mathrm{w}_i} = \frac{\varphi_j(N,v) - \varphi_j(N\backslash\{i\},v)}{\mathrm{w}_j}$$

$$\underset{(6)}{\Leftrightarrow} \frac{\Theta_{\varphi_i(N,v)}}{\mathrm{w}_i} = \frac{\Theta_{\varphi_j(N,v)}}{\mathrm{w}_j}$$

$$\underset{(11)}{\Leftrightarrow} \frac{\Theta_{\varphi_i(N,v)}}{\lambda_{N,i}} = \frac{\Theta_{\varphi_j(N,v)}}{\lambda_{N,j}}.$$

Induction step: Assume that the claim holds true for all player sets N', $N' \subsetneq N$, with $\underset{N'\subsetneq N}{max}|N'| \geq 2$ (IH). By **BC**$^{\mathrm{w}}$, we get

$$\frac{\varphi_i(N,v) - \varphi_i(N\backslash\{j\},v)}{\mathrm{w}_i} = \frac{\varphi_j(N,v) - \varphi_j(N\backslash\{i\},v)}{\mathrm{w}_j}$$

$$\underset{(6)}{\Leftrightarrow} \frac{\Theta_{\varphi_i(N,v)} + \sum_{\substack{S\subsetneq N,\\ S\ni i}} \Theta_{\phi_i(S,v)} - \sum_{\substack{S\subseteq N\backslash\{j\},\\ S\ni i}} \Theta_{\phi_i(S,v)}}{\mathrm{w}_i}$$

$$= \frac{\Theta_{\varphi_j(N,v)} + \sum_{\substack{S\subsetneq N,\\ S\ni j}} \Theta_{\phi_j(S,v)} - \sum_{\substack{S\subseteq N\setminus\{i\},\\ S\ni j}} \Theta_{\phi_j(S,v)}}{\mathrm{w}_j}$$

$$\underset{(11)}{\Leftrightarrow} \quad \frac{\Theta_{\varphi_i(N,v)} + \sum_{\substack{S\subsetneq N,\\ \{i,j\}\subseteq S}} \Theta_{\phi_i(S,v)}}{\lambda_{N,i}} = \frac{\Theta_{\varphi_j(N,v)} + \sum_{\substack{S\subsetneq N,\\ \{i,j\}\subseteq S}} \Theta_{\phi_j(S,v)}}{\lambda_{N,j}}$$

$$\underset{(IH)}{\Leftrightarrow} \quad \frac{\Theta_{\varphi_i(N,v)}}{\lambda_{N,i}} = \frac{\Theta_{\varphi_j(N,v)}}{\lambda_{N,j}}.$$

Since the claim holds for all $N \in \mathcal{N}$, $|N| \geq 2$, equivalence is shown. □

By Theorem 3.4, the following axiom is equivalent to the w-weighted balanced contributions property.

w-weighted balanced value dividends, BVD$^{\mathrm{w}}$. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, $i, j \in N$, and $\mathrm{w} \in W$, we have

$$\frac{\Theta_{\varphi_i(N,v)}}{\mathrm{w}_i} = \frac{\Theta_{\varphi_j(N,v)}}{\mathrm{w}_j}.$$

Especially, the next property is equivalent to the balanced contributions property.

Balanced value dividends, BVD. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i, j \in N$, we have

$$\Theta_{\varphi_i(N,v)} = \Theta_{\varphi_j(N,v)}.$$

Therefore, by Theorems 3.4 and 3.3, we get a corollary that is equivalent to the well-known axiomatization of the weighted Shapley values by efficiency and w-balanced contributions in Myerson (1980) and Hart and Mas-Colell (1989) and, as a special case, of the Shapley value by efficiency and balanced contributions in Myerson (1980).

Corollary 3.5. *Let* $\mathrm{w} \in W$. Sh^{w} *is the unique TU-value that satisfies* ***E*** *and* ***BVD***$^{\mathrm{w}}$. *In particular,* Sh *is the unique TU-value that satisfies* ***E*** *and* ***BVD***.

4. Inessential grand coalition

A TU-game (N, v) is called **inessential** if $v(S) = \sum_{i\in S} v(\{i\})$ for all $S \in \Omega^N$. Note that (N, v) is inessential if and only if $v(S) = \sum_{R\subsetneq S} \Delta_v(R)$ for all $S \subseteq N$, $|S| \geq 2$. We weaken this characteristic for games with at least two players so that the last condition must hold only for the grand coalition: a TU-game (N, v), $|N| \geq 2$, is called an **inessential grand coalition game** if $v(N) = \sum_{S\subsetneq N} \Delta_v(S)$.

The grand coalition is inessential in the sense that $v(N)$ is completely determined by the worths of the proper subsets of N. The following new property for TU-values states that in inessential grand coalition games a player's payoff is completely determined by the player's payoff in all proper subgames.

Inessential grand coalition, IGC. For all $N \in \mathcal{N}$ and all inessential grand coalition games $(N, v) \in \mathbb{V}(N)$, we have $\varphi_i(N, v) = \sum_{S \subsetneq N,\, S \ni i} \Theta_{\varphi_i(S,v)}$ for all $i \in N$.

To axiomatize the proportional Shapley value, Béal et al. (2018) introduced an axiom for two games which only differ in the worth of the grand coalition. Two players' payoff differentials must be proportional to their singleton worths.

Proportional (aggregate) monotonicity, PM (Béal et al., 2018)**.** For all $N \in \mathcal{N}$, $|N| \geq 2$, $(N, v) \in \mathbb{V}_0(N)$, $\alpha \in \mathbb{R}$, and all $i, j \in N$, we have

$$\frac{\varphi_i(N, v) - \varphi_i(N, v + \alpha \cdot u_N)}{v(\{i\})} = \frac{\varphi_j(N, v) - \varphi_j(N, v + \alpha \cdot u_N)}{v(\{j\})}.$$

Similarly, the following property requires that two players' payoff differentials must be proportional to their weights of a sharing system.

λ-balanced monotonicity, $\mathbf{M}^\lambda$. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, $i, j \in N$, $\lambda \in \Lambda$, and $\alpha \in \mathbb{R}$, we have

$$\lambda_{N,j}\big[\varphi_i(N, v) - \varphi_i(N, v + \alpha \cdot u_N)\big] = \lambda_{N,i}\big[\varphi_j(N, v) - \varphi_j(N, v + \alpha \cdot u_N)\big].$$

Theorem 4.1. *Let $\lambda \in \Lambda$. H^λ is the unique TU-value that satisfies **E**, **IGC**, and $\mathbf{M}^\lambda$.*

Proof. I. it is well-known that H^λ satisfies **E** and, by (2) and (6), it is clear that H^λ meets **IGC** and $\mathbf{M}^\lambda$ and existence is shown.

II. For all $N \in \mathcal{N}$, let $(N, v) \in \mathbb{V}(N)$, $\lambda \in \Lambda$, and let φ be a TU-value that satisfies **E**, **IGC**, and $\mathbf{M}^\lambda$. We show uniqueness by induction on the size $|N|$.

Initialization: If $|N| = 1$, uniqueness is satisfied by **E**.

Induction step: Let $|N| \geq 2$. Assume that uniqueness holds for all $N' \subsetneq N$, $|N'| \geq 1$, (IH). Let $j \in N$ such that $\lambda_{N,j} \neq 0$. Note that such a j always exists and that $(N, v - \Delta_v(N) \cdot u_N)$ is an inessential grand coalition game. By $\mathbf{M}^\lambda$, we have for all $i \in N$,

$$\begin{aligned} &\varphi_i(N, v) - \varphi_i(N, v - \Delta_v(N) \cdot u_N) = \frac{\lambda_{N,i}}{\lambda_{N,j}}\big[\varphi_j(N, v) - \varphi_j(N, v - \Delta_v(N) \cdot u_N)\big] \\ \Rightarrow \quad &\sum_{i \in N}\big[\varphi_i(N, v) - \varphi_i(N, v - \Delta_v(N) \cdot u_N)\big] = \sum_{i \in N}\frac{\lambda_{N,i}}{\lambda_{N,j}}\big[\varphi_j(N, v) - \varphi_j(N, v - \Delta_v(N) \cdot u_N)\big]. \end{aligned}$$

By **IGC** and (IH), $\varphi_i(N, v - \Delta_v(N) \cdot u_N)$ is unique for all $i \in N$. Therefore, by **E**, it follows that $\varphi_j(N, v)$ is unique for all $j \in N$ with $\lambda_{N,j} \neq 0$. Thus, by $\mathbf{M}^\lambda$, $\varphi_i(N, v)$ is unique for all $i \in N$ with $\lambda_{N,i} = 0$ too and uniqueness is shown. □

We would like to offer an axiomatic characterization of the Harsanyi set which does not explicitly use the sharing function systems. The next property requires that the ratios of two players' payoff differentials in two different games on the same player set are equal, especially if the payoff differentials are not zero.

Dependent value monotonicity, DVM. For all $N \in \mathcal{N}$, $(N, v), (N, w) \in \mathbb{V}(N)$, $i, j \in N$, and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} &\big[\varphi_i(N, v) - \varphi_i(N, v{+}\alpha \cdot u_N)\big]\big[\varphi_j(N, w) - \varphi_j(N, w + \beta \cdot u_N)\big] \\ &\qquad = \big[\varphi_j(N, v) - \varphi_j(N, v + \alpha \cdot u_N)\big]\big[\varphi_i(N, w) - \varphi_i(N, w + \beta \cdot u_N)\big] \end{aligned}$$

We get an axiomatic characterization of the Harsanyi set.

Theorem 4.2. *A TU-value φ satisfies* **E**, **M**, **DVM**, *and* **IGC** *iff there exists a $\lambda \in \Lambda$, such that $\varphi = H^\lambda$.*

Proof. I. Let $\lambda \in \Lambda$. It is well-known that H^λ satisfies **E** and **M**. By Theorem 4.1, H^λ satisfies **IGC** and $\mathbf{M}^\lambda$. It is obvious that $\mathbf{M}^\lambda$ implies **DVM** and thus existence is shown.

II. For all $N \in \mathcal{N}$, let $(N, v) \in \mathbb{V}(N)$ and let φ be a TU-value that satisfies **E**, **M**, **DVM**, and **IGC**. We show that $\varphi = H^\lambda$ for some $\lambda \in \Lambda$. If $|N| = 1$, we have $\varphi = H^\lambda$ for all $\lambda \in \Lambda$ by **E**. Let now $|N| \geq 2$. By **E** and **M**, we have for all such $N \in \mathcal{N}$ and all $i \in N$, $\varphi_i(N, u_N) - \varphi_i(N, \mathbf{0}^N) = c_{N,i} \in \mathbb{R}_+$ and $\sum_{i \in N} c_{N,i} = 1$. Thus exists a $\lambda \in \Lambda$ with $\lambda_{N,i} := c_{N,i}$ for all $N \in \mathcal{N}$, $|N| \geq 2$, and all $i \in N$. By **DVM**, we get for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, $i, j \in N$, all $\alpha \in \mathbb{R}$, and a $\lambda \in \Lambda$ just defined,

$$\begin{aligned} & \big[\varphi_i(N,v) - \varphi_i(N, v + \alpha \cdot u_N)\big] c_{N,j} = \big[\varphi_j(N,v) - \varphi_j(N, v + \alpha \cdot u_N)\big] c_{N,i} \\ \Leftrightarrow \;\; & \lambda_{N,j}\big[\varphi_i(N,v) - \varphi_i(N, v + \alpha \cdot u_N)\big] = \lambda_{N,i}\big[\varphi_j(N,v) - \varphi_j(N, v + \alpha \cdot u_N)\big]. \end{aligned}$$

Therefore, φ satisfies $\mathbf{M}^\lambda$ and, due to **E**, **IGC**, and Theorem 4.1, φ equals a Harsanyi payoff H^λ with $\lambda \in \Lambda$. □

5. The generalized Harsanyi set

Casajus (2017) introduced a class of TU-values φ^ω, $\omega \in \overline{\Omega}$, $\overline{\Omega} := \{f : \mathbb{R} \times \mathfrak{U} \to \mathbb{R}_{++}\}$, defined by

$$\varphi_i^\omega(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\omega(v(\{i\}), i)}{\sum_{j \in S} \omega(v(\{j\}), j)} \Delta_v(S) \;\; \text{for all } N \in \mathcal{N},\ (N, v) \in \mathbb{V}(N), \text{ and } i \in N.$$

If ω does not depend on v, φ^ω equals a weighted Shapley value. Therefore, we will call the class of all TU-values φ^ω the **generalized Shapley set** and we will denote each value in that class by Sh^ω for each $\omega \in \overline{\Omega}$. This class obviously contains the weighted Shapley values but also non-linear TU-values like the TU-values φ^c which are defined for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and all $c > 0$, by

$$\varphi_i^c(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{|v(\{i\})| + c}{\sum_{j \in S}(|v(\{j\})| + c)} \Delta_v(S) \;\; \text{for all } i \in N. \tag{12}$$

Here the weight function depends on the worth of the singletons. In the following extensions of the Harsanyi set the weights may depend on the entire coalition function.

Let $\widetilde{\mathcal{V}} := \{\widetilde{v} : \mathcal{N} \cup \{\emptyset\} \to \mathbb{R},\ \widetilde{v}(\emptyset) = 0\}$ and let for all $\widetilde{v} \in \widetilde{\mathcal{V}}$, $\mathbb{V}(\mathcal{N}, \widetilde{v}) := \{(N, v^N) \in \mathbb{V}(N) |\, N \in \mathcal{N} \text{ and } v^N(S) = \widetilde{v}(S) \text{ for all } S \subseteq N\}$ be the set of all TU-games $(N, v^N) \in \mathbb{V}(N)$ on all player sets $N \in \mathcal{N}$ such that $v^N(S) = \widetilde{v}(S)$ for all $S \subseteq N$ and all $N \in \mathcal{N}$.

Related to a TU-value, we define: for all $N \in \mathcal{N}$, a **sharing function** ψ^N on N is an operator that assigns to any $(N, v) \in \mathbb{V}(N)$ a sharing vector $\psi^N(v) \in \mathbb{R}_+^N$ such that $\sum_{i \in N} \psi_i^N(v) = 1$. For all $\widetilde{v} \in \widetilde{\mathcal{V}}$, the collection $\Psi(\widetilde{v})$ of all **sharing function systems** $\psi \in \Psi(\widetilde{v})$ is defined by

$$\Psi(\widetilde{v}) := \Big\{\psi = \big(\psi_i^N(v^N)\big)_{N \in \mathcal{N},\, i \in N} \,\big|\, v^N(S) = \widetilde{v}(S) \text{ for all } S \subseteq N \text{ and } N \in \mathcal{N}\Big\}.$$

For each fixed $\widetilde{v}$ or if the sharing functions are constants, a sharing function system coincides with a sharing system. This leads to the naming of the following TU-values. For all $\widetilde{v} \in \widetilde{\mathcal{V}}$, all $(N,v) \in \mathbb{V}(\mathcal{N},\widetilde{v})$, and $\psi \in \Psi(\widetilde{v})$, the **generalized Harsanyi payoff** H^{ψ} is defined by

$$H_i^{\psi}(N,v) := \sum_{S \subseteq N,\, S \ni i} \psi_i^S(v)\Delta_v(S) \ \text{ for all } \ i \in N. \tag{13}$$

We call the class of all generalized Harsanyi payoffs **generalized Harsanyi set**. In the next property the $\lambda \in \Lambda$ in $\mathbf{BVD}^{\lambda}$ will be replaced by a $\psi \in \Psi(\widetilde{v})$.

ψ-balanced value dividends, $\mathbf{BVD}^{\psi}$. For all $\widetilde{v} \in \widetilde{\mathcal{V}}$, all $(N,v) \in \mathbb{V}(\mathcal{N},\widetilde{v})$, and $\psi \in \Psi(\widetilde{v})$, we have $\psi_{N,j}\Theta_{\varphi_i(N,v)} = \psi_{N,i}\Theta_{\varphi_j(N,v)}$.

The following theorem is completely analogous to Theorem 3.3.

Theorem 5.1. *Let $\psi \in \Psi(\widetilde{v})$, $\widetilde{v} \in \widetilde{\mathcal{V}}$. H^{ψ} is the unique TU-value that satisfies **E** and **BVD**$^{\psi}$.*

The proof is omitted since it is straightforward to transfer the proof from Theorem 3.3. We provide a characterization of the generalized Harsanyi set which does not use sharing function systems explicitly. Here, the dependent value monotonicity property in Theorem 4.2 is dropped.

Theorem 5.2. *A TU-value φ satisfies **E**, **IGC**, and **M** iff there exists for all $\widetilde{v} \in \widetilde{\mathcal{V}}$ a $\psi \in \Psi(\widetilde{v})$ such that $\varphi = H^{\psi}$.*

Proof. I. Let $\widetilde{v} \in \widetilde{\mathcal{V}}$, $(N,v) \in \mathbb{V}(\mathcal{N},\widetilde{v})$, and $\psi \in \Psi(\widetilde{v})$. By (13) and (6), it is clear that H^{ψ} satisfies **E**, **IGC**, and **M**.

II. Let $(N,v) \in \mathbb{V}(N)$ and φ a TU-value that satisfies **E**, **IGC**, and **M**. We show uniqueness by induction on the size $|N|$.

Initialization: If $|N| = 1$, uniqueness is satisfied by **E**.

Induction step: Let $|N| \geq 2$. Assume that uniqueness holds for all $N' \subsetneq N$, $|N'| \geq 1$, (IH). Let $(N,w) \in \mathbb{V}(N)$ such that $w(S) := v(S)$ for all $S \subsetneq N$ and $\Delta_w(N) := 0$. Then, by *I.*, (IH), and **IGC**, we get $\varphi_i(N,w) = H_i^{\psi'}(N,w)$ for all $i \in N$ and some $\psi' \in \Psi(\widetilde{v})$ and some $\widetilde{v} \in \widetilde{\mathcal{V}}$ such that $(N,w) \in \mathbb{V}(\mathcal{N},\widetilde{v})$. It follows, as a first case, $\varphi(N,v) = H^{\psi'}(N,v)$ if $v(N) = w(N)$.

If, as a second case, $\Delta_v(N) > 0$, we have, by **E**,

$$\sum_{i \in N} \varphi_i(N,v) = \sum_{i \in N} H^{\psi'}(N,w) + \Delta_v(N).$$

By **M**, we get for all $i \in N$,

$$\begin{aligned}
& \varphi_i(N,v) \geq H_i^{\psi'}(N,w) \\
\Rightarrow \quad & \varphi_i(N,v) = H_i^{\psi'}(N,w) + \chi_i(N,v),\ \chi_i(N,v) \geq 0, \ \text{and} \ \sum_{i \in N} \chi_i(N,v) = \Delta_v(N) \\
\Rightarrow \quad & \varphi_i(N,v) = H_i^{\psi'}(N,w) + \frac{\chi_i(N,v)}{\sum_{j \in N} \chi_j(N,v)}\Delta_v(N) = H_i^{\psi'}(N,w) + \psi_i^N(v)\Delta_v(N).
\end{aligned}$$

$\psi^N(v)$ is a sharing function and is part of a sharing function system $\psi \in \Psi(\widetilde{v})$ with $\psi^S(v) = \psi'^S(v)$ for all $S \subsetneq N$ and sharing functions $\psi'^S(v)$ from a sharing function system $\psi' \in \Psi(\widetilde{v})$. By (13), it follows

$$\varphi_i(N,v) = H_i^{\psi}(N,v) \text{ for all } i \in N.$$

If $\Delta_v(N) < 0$, equality follows analogously. □

6. The proportional Harsanyi payoff

It is clear that the TU-values φ^c from Sect. 5 and the proportional Shapley value Sh^p (on a subset of all TU-games) are also part of the generalized Harsanyi set. One may ask, whether the proportional value P is also a member of the generalized Harsanyi set (on the relevant subset of TU-games). However, it is easy to show that this is not the case[4].

Many scientific studies about the Harsanyi set deal with totally positive games (see, e. g., Vasil'ev and van der Laan, 2002; van den Brink et al., 2014). Also the set-valued proper Shapley value is defined on totally positive games[5] by Vorob'ev and Liapunov (1998). We introduce a new proportional TU-value from the generalized Harsanyi set, defined on the subset of strictly positive games. It satisfies proportional standardness, in harmony with other proportional TU-values as the proportional rule, the proportional value, or the proportional Shapley value.

Definition 6.1. *For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}_{++}(N)$, the* ***proportional Harsanyi payoff*** *H^p is defined inductively for all $i \in S$, $S \subseteq N$, by*

$$H_i^p(S,v) := \begin{cases} v(\{i\}), & \text{if } S = \{i\}, \\ \dfrac{\sum_{R \subsetneq S,\, R \ni i} \Theta_{H_i^p(R,v)}}{\sum_{j \in S} \sum_{R \subsetneq S,\, R \ni j} \Theta_{H_j^p(R,v)}} v(S), & \text{otherwise.} \end{cases} \tag{14}$$

Remarks 6.2. *One easily shows, by Remark 3.2 and induction on the size $|S|$, that H^p is well-defined. Moreover, we have $H_i^p(N,v) > 0$ for all $i \in N$, $(N,v) \in \mathbb{V}_{++}(N)$[6]. By (14), it is obvious that H^p satisfies* ***E*** *and* ***PSt****.*

We present a formula for the proportional Harsanyi payoff that distributes the Harsanyi dividends proportional to players' value dividends and confirms the membership to the Harsanyi set.

Proposition 6.3. *For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}_{++}(N)$, we have*

$$H_i^p(N,v) = \Delta_v(\{i\}) + \sum_{\substack{S \subseteq N \\ S \ni i,\, S \neq \{i\}}} \frac{\sum_{R \subsetneq S,\, R \ni i} \Theta_{H_i^p(R,v)}}{\sum_{j \in S} \sum_{R \subsetneq S,\, R \ni j} \Theta_{H_j^p(R,v)}} \Delta_v(S) \text{ for all } i \in N. \tag{15}$$

[4] Let $(N,v) \in \mathbb{V}(N)$, $N = \{1,2,3\}$, be an inessential grand coalition game, given by $v(\{1\}) = v(\{2\}) = 1$, $v(\{3\}) = 2$, $v(\{1,2\}) = 4$, $v(\{1,3\}) = 3$, $v(\{2,3\}) = 5$ and $v(\{1,2,3\}) = 8$. We obtain $P_1(N,v) = \frac{24}{13} \neq 2 = \sum_{S \subsetneq N} \Theta_{P_1(N,v)}$ and **IGC** is not satisfied.

[5] Van den Brink et al. (2015) generalized the proper Shapley value for monotone TU-games.

[6] This property is called positivity in Derks et al. (2000).

Proof. If $|N| = 1$, the claim is obvious. Let now $|N| \geq 2$. We have for all $i \in N$

$$\begin{aligned}
\Delta_v(\{i\}) \quad &+ \sum_{\substack{S \subseteq N, \\ S \ni i, S \neq \{i\}}} \frac{\sum_{R \subsetneq S,\, R \ni i} \Theta_{H_i^p(R,v)}}{\sum_{j \in S} \sum_{R \subsetneq S,\, R \ni j} \Theta_{H_j^p(R,v)}} \Delta_v(S) \\
\underset{\substack{(1), \mathbf{E} \\ Rem.\, 3.2}}{=} \quad & \Delta_v(\{i\}) + \sum_{\substack{S \subseteq N, \\ S \ni i, S \neq \{i\}}} \frac{\sum_{R \subsetneq S,\, R \ni i} \Theta_{H_i^p(R,v)}}{\sum_{j \in S} \sum_{R \subsetneq S,\, R \ni j} \Theta_{H_j^p(R,v)}} \Big[v(S) - \sum_{R \subsetneq S} \sum_{j \in R} \Theta_{H_j^p(R,v)} \Big] \\
\underset{(14)}{=} \quad & \Delta_v(\{i\}) + \sum_{\substack{S \subseteq N, \\ S \ni i, S \neq \{i\}}} H_i^p(S,v) - \sum_{\substack{S \subseteq N, \\ S \ni i, S \neq \{i\}}} \sum_{\substack{R \subsetneq S, \\ R \ni i}} \Theta_{H_i^p(R,v)} \\
\underset{(6)}{=} \quad & \Delta_v(\{i\}) + \sum_{\substack{S \subseteq N, \\ S \ni i, S \neq \{i\}}} \sum_{\substack{R \subsetneq S, \\ R \ni i}} \Theta_{H_i^p(R,v)} - \sum_{\substack{S \subseteq N, \\ S \ni i, S \neq \{i\}}} \sum_{\substack{R \subsetneq S, \\ R \ni i}} \Theta_{H_i^p(R,v)} \\
\underset{(6)}{=} \quad & H_i^p(N,v).
\end{aligned}$$

□

For games which differ only in the grand coalition, the following axiom requires that players' payoffs remain in the same ratio.

Proportion preservation, PP. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $\alpha \in \mathbb{R}$, we have

$$\varphi_i(N,v)\varphi_j(N,v+\alpha \cdot u_N) = \varphi_j(N,v)\varphi_i(N,v+\alpha \cdot u_N) \text{ for all } i,j \in N.$$

As a consequence of the following axiom, each player's share is independent of the worth of the grand coalition.

Independent share, IS. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $\alpha \in \mathbb{R}$, we have

$$\varphi_i(N,v)[v(N)+\alpha] = \varphi_i(N,v+\alpha \cdot u_N)v(N) \text{ for all } i \in N.$$

This axiom implies, e. g., that if a positive worth of the grand coalition increases, while the worths of all other coalitions remain fixed, then a players' positive payoff increases proportionally to the increase of the worth of the grand coalition.

Remark 6.4. *It is clear that **IS** implies **PP**. One also easily checks that if a TU-value satisfies **E** and **PP**, then it also satisfies **IS**.*

Remark 6.5. *Examination of (3), (5), and (14) shows that the proportional rule π, the proportional value P, and the proportional Harsanyi payoff H^p satisfy **PP** and **IS**, but, by (4), this is not the case for the proportional Shapley value Sh^p.*

The next axiom relates to the λ-balanced value dividends property: two players' value dividends of the grand coalition are in the same proportion as the payoffs if the value dividends and the payoffs are not zero.

Value balanced value dividends, VBVD. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $i,j \in N$, we have

$$\varphi_i(N,v)\Theta_{H_j^p(N,v)} = \varphi_j(N,v)\Theta_{H_i^p(N,v)}. \tag{16}$$

Remark 6.6. *By (6), (16) is equivalent to*

$$\varphi_i(N,v) \sum_{S\subsetneq N,\, S\ni j} \Theta_{H_j^p(S,v)} = \varphi_j(N,v) \sum_{S\subsetneq N,\, S\ni i} \Theta_{H_i^p(S,v)}.$$

Thus, **VBVD** also states that two players' payoffs are in the same proportion as the sums of players' value dividends of all proper subsets of the grand coalition if the value dividends and the payoffs are not zero. Note that **VBVD** holds for all games whereas **PP** and **IS** need games which differ only in the grand coalition. The proportional Harsanyi payoff matches **VBVD** and a lot of other properties.

Proposition 6.7. *Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}_{++}(N)$. H^p satisfies **D**, **M**, **H**, **S**, **VBVD**, **IGC**, **P** and **PSp**.*

Proof. Let $(N,v) \in \mathbb{V}_{++}(N)$. It is well-known that we have for a dummy player $i \in N$ in (N,v), $\Delta_v(S) = 0$ for all $S \subseteq N$, $|S| \geq 2$, $S \ni i$. Thus, **D** follows immediately by (15) and, also by (15), **M** is obviously satisfied. By induction on the size $|N|$ and formula (14) follows **H**. In addition, also as a well-known fact, we have for two symmetric players $i,j \in N$ in (N,v), $\Delta_v(S\cup\{i\}) = \Delta_v(S\cup\{j\})$ for all $S \subseteq N$. Then, it is obvious, by (15), that H^p satisfies **S**. By (14), we can see that H^p matches **VBVD** and, by Definition 3.1, Remark 3.2, and **E**, obviously **IGC** is satisfied.

P: By Lemma 1 in Besner (2019), two players $i,j \in N$ are weakly dependent in (N,v), iff $\Delta_v(S\cup\{k\}) = 0$, $k \in \{i,j\}$, for all $S \subseteq N\backslash\{i,j\}$, $S \neq \emptyset$. By (15), we have $\Theta_{H_i^p(S,v)} = 0$ for all $S \subseteq N$, $S \ni i$, such that $\Delta_v(S) = 0$. By induction on $|N|$, we show that

$$\frac{\Theta_{H_i^p(S,v)}}{v(\{i\})} = \frac{\Theta_{H_j^p(S,v)}}{v(\{j\})} \text{ for all } S \subseteq N,\ i,j \in S. \tag{17}$$

Initialization: If $|N| = 2$, (17) is satisfied by (15) and (6).

Induction step: Let $|N| \geq 2$. Assume that (17) is satisfied for all $N' \subsetneq N$, $|N'| \geq 2$, (IH). Then, by (15) and (IH), we have,

$$\begin{aligned}
H_i^p(N,v) &= \Delta_v(\{i\}) + \sum_{\substack{S\subseteq N,\\ S\ni i,\, S\neq\{i\}}} \frac{\sum_{R\subsetneq S,\, R\ni i} \Theta_{H_i^p(R,v)}}{\sum_{k\in S}\sum_{R\subsetneq S,\, R\ni k} \Theta_{H_k^p(R,v)}} \Delta_v(S) \\
&\underset{\substack{\text{Lemma 1 in}\\ \text{Besner (2019)}}}{=} v(\{i\}) + \sum_{\substack{S\subseteq N,\\ \{i,j\}\subseteq S}} \frac{\sum_{R\subsetneq S,\, \{i,j\}\subset R} \Theta_{H_i^p(R,v)}}{\sum_{k\in S}\sum_{R\subsetneq S,\, \{i,j\}\subset R} \Theta_{H_k^p(R,v)}} \Delta_v(S) \\
&\underset{(IH)}{=} \frac{v(\{i\})}{v(\{j\})} v(\{j\}) + \frac{v(\{i\})}{v(\{j\})} \sum_{\substack{S\subseteq N,\\ \{i,j\}\subseteq S}} \frac{\sum_{R\subsetneq S,\, \{i,j\}\subset R} \Theta_{H_j^p(R,v)}}{\sum_{k\in S}\sum_{R\subsetneq S,\, \{i,j\}\subset R} \Theta_{H_k^p(R,v)}} \Delta_v(S) \\
&= \frac{v(\{i\})}{v(\{j\})} H_j^p(N,v).
\end{aligned}$$

Thus, (17) is satisfied and therefore H^p meets **P** as well.

The proof of **PSp** is omitted since, by (15) and induction on the size $|S|$, $S \subseteq N$, it is straightforward to transfer the proof of Proposition 2 in Besner (2019) for Sh^p to H^p. □

Derks et al. (2000) could show that the Harsanyi set coincides for almost positive games with the core. The **core** of a TU-game $(N, v) \in \mathbb{V}(N)$ is the set $C(N, v) := \{x \in \mathbb{R}^N : x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \in \Omega^N\}$. It is obvious, by **E** and (15), that it is a necessary characteristic of the proportional Harsanyi payoff to be a member of the core. This can be interpreted in such a way that no coalition of players can improve upon or block the payoff.

Remark 6.8. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}_{++}(N)$, we have $H^p(N, v) \in C(N, v)$.*

We have three main results in this section. The first one is related to Theorem 3.3 and shows that the value balanced value dividends axiom is a very strong property.

Theorem 6.9. *Let $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}_{++}(N)$. H^p is the unique TU-value that satisfies **E** and **VBVD**.*

Proof. By Proposition 6.7 and Remarks 6.2, we have only to show uniqueness. Let $(N, v) \in \mathbb{V}_{++}(N)$ and let φ be a TU-value that satisfies **E** and **VBVD**. We use induction on the size $|S|$, $S \subseteq N$.

Initialization: If $|S| = 1$, uniqueness of $\varphi(S, v)$ is satisfied by **E**. Moreover, we have for all $T \subseteq N$, $|T| = |S| + 1$,

$$\sum_{S \subsetneq T,\, S \ni i,} \Theta_{\varphi_i(S,v)} > 0. \tag{18}$$

Induction step: Assume uniqueness and (18) hold for $|S| - 1, |S| \geq 2$ (IH). Then, by (IH), we have $\sum_{R \subsetneq S,\, R \ni i,} \Theta_{\varphi_i(R,v)} > 0$. By (6), it follows for all $i \in S$ and a fixed $j \in S$

$$\varphi_i(S, v) = \frac{\sum_{R \subsetneq S,\, R \ni i,} \Theta_{\varphi_i(R,v)}}{\sum_{R \subsetneq S,\, R \ni j,} \Theta_{\varphi_j(R,v)}} \varphi_j(S, v)$$
$$\Leftrightarrow \quad \sum_{i \in S} \varphi_i(S, v) = \sum_{i \in S} \frac{\sum_{R \subsetneq S,\, R \ni i,} \Theta_{\varphi_i(R,v)}}{\sum_{R \subsetneq S,\, R \ni j,} \Theta_{\varphi_j(R,v)}} \varphi_j(S, v).$$

Thus, by **E** and induction, φ is unique for all $S \subseteq N$ and Theorem 6.9 is shown. □

Our last theorem does not need efficiency.

Theorem 6.10. *Let $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}_{++}(N)$. H^p is the unique TU-value that satisfies **D**, **IGC**, and **IS**.*

Proof. By Proposition 6.7, Remark 6.5, and Remarks 6.2, we have only to show uniqueness. Let $(N, v) \in \mathbb{V}_{++}(N)$ and let φ be a TU-value that satisfies **D**, **IGC**, and **IS**. We use induction on the size $|S|$, $S \subseteq N$.

Initialization: If $|S| = 1$, uniqueness of $\varphi(S, v)$ is satisfied by **D**.

Induction step: Assume uniqueness holds for $|S| - 1, |S| \geq 2$ (IH). Then, by (IH), we have $\sum_{R \subsetneq S,\, R \ni i,} \Theta_{\varphi_i(R,v)}$ is unique for all $i \in S$. Let $(S, w) \in \mathbb{V}_{++}(S)$ such that $\Delta_w(S) = 0$ and $v(R) = w(R)$ for all $R \subsetneq S$. By **IGC**, it follows $\varphi_i(S, w)$ is unique for all $i \in S$. By **IS**, we have for all $i \in S$

$$\varphi_i(S, v) = \frac{v(N)}{w(N)} \varphi_i(S, w).$$

and φ is unique for all $S \subseteq N$ and Theorem 6.10 is shown. □

Remark 6.11. *The proof shows that in Theorem 6.10* ***D*** *can be replaced by any axiom that guarantees that a players' payoff in a singleton game is her worth and that is satisfied by H^P. Therefore, e. g.,* ***E*** *or the inessential game property that states that a player in an inessential game receives her singleton worth can be used instead of* ***D****.*

We have a last characterization of the proportional Harsanyi payoff. It follows immediately from Theorem 6.10 and Remarks 6.4 and 6.11.

Corollary 6.12. *Let $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}_{++}(N)$. H^p is the unique TU-value that satisfies* ***E****,* ***IGC****, and* ***PP****.*

Remark 6.13. *By (13), all values from the generalized Harsanyi set satisfy for all $N \in \mathcal{N}$ on $\mathbb{V}_{++}(N)$,* ***E****,* ***D****, and* ***IGC****. Thus, H^p is the unique value from the Harsanyi set that satisfies* ***PP*** *and* ***IS*** *on $\mathbb{V}_{++}(N)$.*

The following example justifies the value and illustrates the axioms, used in Theorem 6.10 and Corollary 6.12.

Example

A group N of independent carpenters who are not too busy as a one-man business join forces and work together in different groups $S \subseteq N$. The total quantity and size of the orders depends on a number of external factors, such as the macroeconomic cycle, the general interest rate level, the general state of the buildings and so on.

We assume that the share of each carpenter as a one-man company in the total net profit of all one-man companies depends only on her performance. We also assume that orders for larger coalitions depend only on the efficiency of the subgroups and whether the customers receive good value for money from the subgroups. The order volume is not too large, so that all orders, offered for groups of different sizes, can be executed.

We model the situation as a TU-game. Nobody cooperates with a carpenter who has no net profit and therefore, does not work efficiently. Thus, we only regard the player sets N with $v(\{i\}) > 0$ for all players $i \in N$. The worth $v(S)$ of a coalition $S \subseteq N$ equals the sum of all net profits of all subunits of S. The Harsanyi dividend $\Delta_v(S)$ of coalition S is equal to the net profit of the unit S, but only in cases where the players of S work as a unit. Besides, the carpenters won't pay on top. Therefore, we have $\Delta_v(S) \geq 0$ for all $S \subseteq N$, $|S| \geq 2$, and $(N, v) \in \mathbb{V}_{++}(N)$.

The carpenters must agree on how to share the worth of the grand coalition $v(N)$. They want to share the whole worth in such a way that **E** must be satisfied. The carpenters have no way of deciding how to divide the net profit of a unit solely by the performance of the unit itself. They can only take into account the profits of all subunits. Hence, the players' shares on the grand coalition net profits do not depend on the grand coalition's profit as a single unit. The carpenters conclude that each player's share should be independent of the worth of the grand coalition and therefore **IS** must be satisfied. If a grand coalition does not work as a unit, that unit's net profit is zero. Thus, the carpenters agree that in this case the payoff should be equal to the sum of players' payoffs in all proper subunits. Therefore, **IGC** should also be satisfied. Hence, by Theorem 6.10 and Remark 6.11, the proportional Harsanyi value H^p is the method of choice for distributing the profits. Certainly, the carpenters accept **D**, so that a carpenter who does not cooperate with partners in other units should only receive her one-man company result. The carpenters can also conclude, that the payoff in the grand coalition unit should be proportional to the sum of their payoffs in all other subunits and therefore agree, by Remark 6.6, that **VBVD** should be satisfied. After some reflection, all the properties presented in this section seem reasonable to the carpenters.

7. Conclusion and discussion

Most studied TU-values are efficient and a lot of them satisfy monotonicity. Thus, the inessential grand coalition property can be used as a criterion whether such a value has to be in the (generalized) Harsanyi set (see footnote 4).

The article shows a strong connection between values from the (extended generalized) Harsanyi set and value dividends. Also the non-efficient Banzhaf value has a representation with dividends. For all $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}(N)$ the **Banzhaf value** β (Banzhaf, 1965) is given by

$$\beta_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\Delta_v(S)}{2^{|S|-1}} \quad \text{for all } i \in N.$$

Obviously, the Banzhaf value satisfies the inessential grand coalition property and monotonicity. Thus, the superset of the generalized Harsanyi set that does not require efficiency contains the Banzhaf value. This means that in the definition of a sharing function within a sharing function system the condition $\sum_{i \in N} \psi_i^N(v) = 1$ for a sharing vector $\psi^N(v) \in \mathbb{R}_+^N$ must be dropped. In Figure 1/Table 1, the class of these values is marked by φ^χ. However, possible applications of value dividends for axiomatizations of the Banzhaf value are left for further research.

Figure 1: The subset relationships between some classes of TU-values

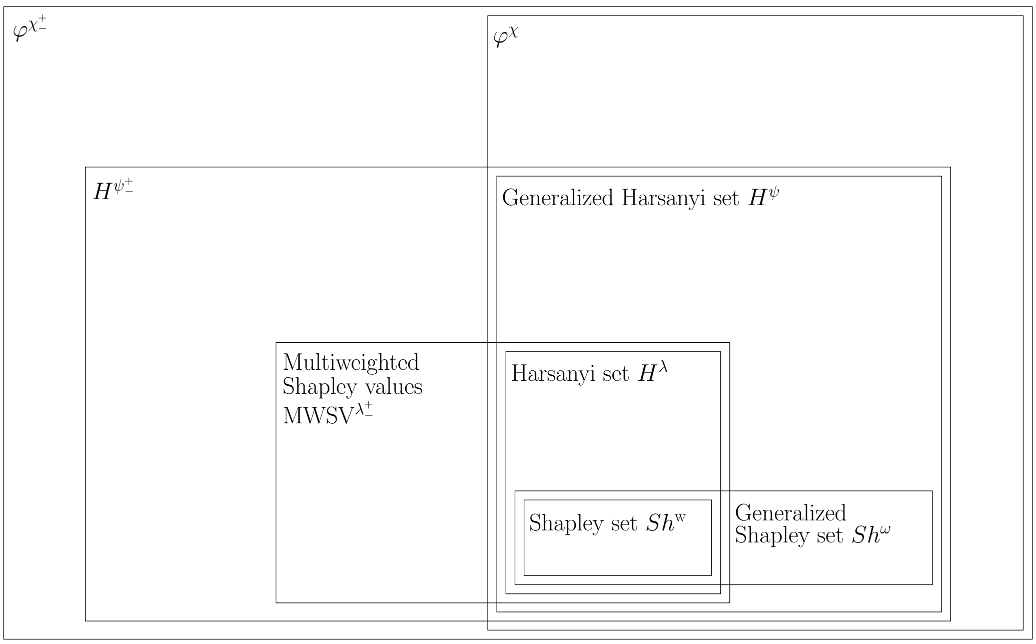

Table 1: Properties of some classes of TU-values[7]

Class	Sh^{w}	Sh^{ω}	H^{λ}	$\mathrm{MWSV}^{\lambda^{+}_{-}}$	H^{ψ}	$H^{\psi^{+}_{-}}$	φ^{χ}	$\varphi^{\chi^{+}_{-}}$
Efficiency	+	+	+	+	+	+	−	−
Null player	+	+	+	+	+	+	+	+
Inessential grand coalition	+	+	+	+	+	+	+	+
Homogenity	+	−	+	+	−	−	−	−
Additivity	+	−	+	+	−	−	−	−
Marginality (Young, 1985)	+	−	+	+	−	−	−	−
Monotonicity	+	+	+	−	+	−	+	−
Strong monotonicity (Young, 1985)	+	−	+	−	−	−	−	−
Weak balanced contributions (Casajus, 2017)	+	+	−	−	−	−	−	−
Dependent value monotonicity	+	−	+	+	−	−	−	−

If we drop in Theorem 4.2 the monotonicity property, we get an axiomatic characterization of the **multiweighted Shapley values**[8] **MWSV**$^{\lambda^{+}_{-}}$(Dragan, 1992). A further removal of the dependent value monotonicity or a removal of the monotonicity property in Theorem 5.1 respectively leads to the characterization of a new class of TU-values (in Figure 1/Table 1 denoted by $H^{\psi^{+}_{-}}$). That means that we allow sharing vectors $\psi^{+N}_{-}(v) \in \mathbb{R}^N$ such that $\sum_{i\in N} \psi_i^{+N}_{-}(v) = 1$. Our last extension, represented in Figure 1/Table 1 by $\varphi^{\chi^{+}_{-}}$, concludes all mentioned classes of TU-values but not the proportional rule and not the proportional Value. This is the class of all TU-values which satisfy the inessential grand coalition property.

Also the TU-values from the Shapley mapping and thus the set-valued proper Shapley value in Vorob'ev and Liapunov (1998) are part (on subsets) of the generalized Harsanyi set (compare Equation (2) in Vorob'ev and Liapunov (1998) with (13)). Here, too, further research must be shifted into the future.

The usage of the proportional Harsanyi payoff is restricted to strongly positive games. This efficient TU-value combines the inessential grand coalition property of the proportional Shapley value with the proportion preservation and independent share property of the proportional rule and the proportional value. The sharing weights are given endogenously and depend on the whole coalition function, not only on the worths of the singletons as by the proportional rule or the proportional Shapley value. Thus, supported by many convincing properties, this value is recommended if the worth of a coalition is dependent from the worths of all subsets of the coalition.

Introducing the proportional value, Ortmann (2000) used similar characterization approaches to that of the Shapley value. In particular, he contrasted efficiency and his preservation of ratios axiom with efficiency and the balanced contributions property and standardness in two player games and consistency with proportional standardness and consistency. An analogous proceeding we can see in Béal et al. (2018) with the proportional Shapley value and the Shapley value. For one such comparison, they introduced the proportional monotonicity property and the following axiom.

[7]All properties of TU-values that are not mentioned in the article are known or easy to verify for the TU-values from the listed classes.

[8]Obviously, (Dragan, 1992) had no knowledge of the Harsanyi set yet, so he showed that many well-known solution concepts are multiweighted Shapley values, but the Harsanyi payoffs were missing. This explains the naming "multiweighted Shapley values" instead of, e. g., "extended Harsanyi payoffs."

Equal (aggregate) monotonicity, EM. (Béal et al., 2018). For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $\alpha \in \mathbb{R}$, we have

$$\varphi_i(N, v) - \varphi_i(N, v + \alpha \cdot u_N) = \varphi_j(N, v) - \varphi_j(N, v + \alpha \cdot u_N) \text{ for all } i, j \in N,$$

Béal et al. (2018) axiomatized the proportional Shapley value by efficiency, dummy player out, weak linearity and proportional monotonicity and contrasted it with an axiomatization of the Shapley value by replacing proportional monotonicity with equal monotonicity.

The question arises whether it is also possible to contrast axiomatizations of the proportional Harsanyi payoff with those of the Shapley value which differ only in one axiom. We can show that the answer is yes. The proportional Harsanyi payoff is uniquely determined by **E**, **IGC** and **PP**. **PP** preserves ratios for games which differ only in the grand coalition. For such games **EM** preserves differences. It follows a "contrasted" axiomatization of the Shapley value.

Theorem 7.1. *Sh is the unique TU-value that satisfies **E**, **IGC**, and **EM**.*

*P*roof. Since it is easy to check that all axioms in theorem 7.1 are satisfied we have only to show uniqueness. Let $(N, v) \in \mathbb{V}(N)$ and let φ be a TU-value that satisfies **E**, **IGC**, and **EM**. We use induction on the size $|S|$, $S \subseteq N$.

Initialization: If $|S| = 1$, uniqueness of $\varphi(S, v)$ is satisfied by **E**.

Induction step: Assume uniqueness holds for $|S| - 1, |S| \geq 2$ (IH). Then, by (IH), we have $\sum_{R \subsetneq S,\, R \ni i,} \Theta_{\varphi_i(R,v)}$ is unique for all $i \in S$. By **IGC**, it follows $\varphi_k(S, v - \Delta_v(S) \cdot u_S)$ is unique for all $k \in S$. By **EM**, we have for all $i, j \in S$

$$\begin{aligned} &\varphi_i(S, v) = \varphi_i(S, v - \Delta_v(S) \cdot u_S) + \varphi_j(S, v) - \varphi_j(S, v - \Delta_v(S) \cdot u_S) \\ \Leftrightarrow \quad &\sum_{k \in S} \varphi_k(S, v) = \sum_{k \in S} \varphi_k(S, v - \Delta_v(S) \cdot u_S) + |S| \cdot \left[\varphi_j(S, v) - \varphi_j(S, v - \Delta_v(S) \cdot u_S)\right] \end{aligned}$$

and φ is unique for all $S \subseteq N$ by **E** and Theorem 7.1 is shown. □

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8. Appendix

We show the logical independence of the axioms used in the characterizations with at least three axioms.

Remark 8.1. *For all $N \in \mathcal{N}$ let $(N, v) \in \mathbb{V}(N)$ and $\lambda \in \Lambda$. The axioms in Theorems 4.1 and 5.2 are logically independent:*

- ***E**: The TU-value $\varphi := 2H^\lambda$ satisfies **IGC** and $\boldsymbol{M^\lambda/M}$ but not **E** in general.*
- ***IGC**: The TU-value φ^λ, defined by $\varphi_i^\lambda(N, v) := \lambda_{N,i} \cdot v(N)$ for all $i \in N$ and all $N \in \mathcal{N}$ satisfies **E** and $\boldsymbol{M^\lambda/M}$ but not **IGC**.*
- *$\boldsymbol{M^\lambda/M}$: The multiweighted Shapley values MWSV satisfy **E** and **IGC** but not $\boldsymbol{M^\lambda/M}$ in general.*

Remark 8.2. *For all $N \in \mathcal{N}$ let $(N, v) \in \mathbb{V}(N)$. The axioms in Theorem 4.2 are logically independent:*

- ***E***: *The TU-value $\varphi := 2Sh$ satisfy **M**, **DVM**, and **IGC** but not **E**.*
- ***M***: *The multiweighted Shapley values satisfy **E**, **DVM**, and **IGC** but not **M** in general.*
- ***DVM***: *The TU-values φ^c, defined by (12), satisfy **E**, **M**, and **IGC** but not **DVM** in general.*
- ***IGC***: *The equal division value ED, defined by $ED_i(N, v) := \frac{v(N)}{|N|}$ for all $i \in N$, satisfies **E**, **M**, and **DVM** but not **IGC**.*

Remark 8.3. *For all $N \in \mathcal{N}$ let $(N, v) \in \mathbb{V}_{++}(N)$. The axioms in Theorem 6.10 and corollary 6.12 are logically independent:*

- ***D/E***: *The TU-value $\varphi := 2H^p$ satisfies **IGC** and **IS/PP** but not **D/E**.*
- ***IGC***: *The proportional value P satisfies **D/E** and **IS/PP** but not **IGC**.*
- ***PP***: *The Shapley value Sh satisfies **E/D** and **IGC** but not **IS/PP**.*

Remark 8.4. *The axioms in Theorem 7.1 are logically independent:*

- ***E***: *The TU-value $\varphi := 2Sh$ satisfies **IGC** and **EM** but not **E**.*
- ***IGC***: *The equal division value ED satisfies **E** and **EM** but not **IGC**.*
- ***EM***: *Each weighted Shapley value $Sh^w \neq Sh$ satisfies **E** and **IGC** but not **EM**.*

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