A Bayesian Estimation of HANK models with Continuous Time Approach: Comparison between US and Japan

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Abstract

This paper estimates heterogeneous agent New Keynesian (HANK) model for US and Japan through three aggregate observations: real GDP, inflation and interest rate, by adopting combination of easy-to-use computational method for solving the model, developed by Ahn, Kaplan, Moll, Winberry and Wolf (2019), and sequential Monte Carlo (SMC) method with Kalman filter applied for Bayesian estimation with parallel computing. The combination make us enjoy the estimation of HANK just using a Laptop PC, e.g., Mac Book Pro, with MATLAB, neither many-core server computer nor FORTRUN language. We show estimation results of one Asset HANK model, i.e., impulse response, fluctuations of distributions of heterogeneous agent as well as historical decomposition for both countries. Even though using the same model, different data draws different pictures.

Keywords: Hetero-Agent model, Linearization, Model Reduction, Bayesian estimation, Sequential Monte Carlo, Kalman Filter,

JEL Classifications: C32, E32, E62

1 Introduction

Over the last three decades, we have found there is a rapidly increasing macroeconomic literature dealing with the rich heterogeneous households in income, wealth and consumption behavior, thanks to the developing of computational

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algorithm to solve economic models with heterogeneous agents (HA) and aggregate uncertainty started in the middle of 1990's (Algan et a1. 2014). Krusell and Smith (1998) is the one of the pioneering works of the HA models in macroeconomic field.

Krusell and Smith (1998) showed that in benchmark version of the HA model, the aggregate dynamic of output, consumption, and investment in response to a TFP shock, are almost identical to their representative agent (RA) counterpart. However, since the Great Recession, the economic researchers have reported to explore the role of household heterogeneity for business cycles analysis (Mian et al ., 2013). And recently HA models have often showed contrast implications of monetary and fiscal policies rather than do RA model, according to McKay et al. (2015), Aucler (2017), and Kaplan et al. (2018) for monetary policy and Mckay and Reis (2013) and Kaplan and Violante (2014) for fiscal policy.

This paper estimates heterogeneous agent New Keynesian (HANK) model for US and Japan using three aggregate observations: real GDP, inflation and interest rate, using combination of easy-to-use computational method for solving the model, developed by Ahn, Kaplan, Moll, Winberry and Wolf (2019), and sequential Monte Carlo (SMC) method with Kalman filter applied for Bayesian estimation with parallel computing. The combination makes us enjoy the estimation of HANK just using a Laptop PC (mac book pro) with MATLAB, neither server with many cores nor FORTRUN. We show estimation results of one Asset HANK model, impulse response, fluctuations of distributions of heterogeneous agent as well as historical decomposition for both countries. Even though using the same model, different data draws different pictures.

Related Literature

The development of computational algorithm to solve HA model is summarized by Algan et al. (2017). Our study is to combine two methods , i.e., solution of HANK model and estimation of linear large-scale DSGE model. For solution of HANK model in continuous time, we follow algorithm of Achdou et al. (2017), and Ahn, Kaplan, Moll, Winberry and Wolf (2019). One of the key of this approach is linearization of HA models. This linearization builts on the ideas of Reiter (2009) in discrete time. Instead of continuous time, Winberry (2019) also proposes the easy to use solution method of the heterogeneous agent model in descrite time, in which dynamics are calculated by DYNARE.

For Bayesain estimation method by parallel computing, we follow Herbst and Schorfheide (2014, 2015) and Fernandez-Villaverde et al. (2016).

Roadmap

The rest of the paper is organized as follows. Section 2 describes one asset no capital HANK model. Section 3 explains both of solution method and estimation method. In Section 4, emprical results are described. Section 5 concludes.

2 Model

Following Ahn et al. (2019), we set up a one-asset HANK model in continuous time. But we expand this model by adding two aggregate structural shocks besides monetary policy shock. Three aggregate shocks help the model be estimated.

2.1 Environment

Household

In our economy's environment, there is a continuum of households of measure one. Households receive a utility flow from consumption, c_t, and disutility flow from labor suply l_t. We assume that a constant elasticity of intertemporal substitution, γ, and a constant Frish labor supply, ϕ_1. Preferences are time-separable and the future is discounted at rate ρ. To this end, households maximize utility function,

$$\max_{c,\,l} \int_0^\infty e^{-\rho t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} - \phi_0 \frac{l_t^{1+\frac{1}{\phi_1}}}{1+\frac{1}{\phi_1}} \right) dt, \tag{1}$$

where the expectation is taken over realization of idiosyncratic productivity shock, z_t. Households are allowed to hold their wealth as one kind of asset consiting of government bond, a_t. They have heterogenous property with respect to two dimensions: their labor productivity z_t (two types, high and low productivities) and their asset position a_t. The asset positions of heterogenous agents follow a differential equation written as

$$da = (r_t \cdot a_t + (1-\tau) \cdot w_t \cdot z_t \cdot l_t + T_t + \Pi_t - c_t)\, dt, \tag{2}$$

$$a_t > \underline{a}, \tag{3}$$

where τ is tax rate of income, r_t is interest rate, w_t is wage, T_t is lump-sum transfer and Π_t is profit share. $\underline{a}$ denotes lower limit of borrowing constraint.

Firms

There are markets of two kinds of goods, i.e., final goods and intermediate goods. In the market of the final goods, firms produce under perfect competition. In the market of the intermediate goods, firms produce under monopolistically competition.

(a) Final Good Aggregator.

There is a representative final good producer, which produces the final good by combining a continuum of intermedoate input through CES aggregator:

$$Y_t = \left(\int_0^1 y^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{1-\varepsilon}}, \tag{4}$$

where ε is elasticity of demand of intermediate goods.

(b) Intermediate Good Producers

There is a continuum of intermediate good producer firms of measure one. Under monopolistically competitive market, intermediate good producer firms, j, use only labor n but no capital rented from households in competitive input market. Their production functions are given as

$$y_{j,t} = TFP_t \cdot n_{j,t}, \tag{5}$$

where TFP_t is total factor productivity. And this dynamics around steady state, TFP_0, is evolved by an aggregate TFP shock as follows

$$TFP_t = TFP_0 \cdot \varepsilon_{TFP,t}, \tag{6}$$

And the TFP shock follows the Ornstein-Uhlenbeck process,

$$d\varepsilon_{TFP,t} = -\theta_{TFP} \cdot \varepsilon_{TFP,t} + \sigma_{TFP,t} dW_t. \tag{7}$$

Under monopolistically competition, intermediate goods firms choose their price to maximize profits subject to quadratic price adjustment cost, eq.(8), following Rotemberg (1982).

$$\Theta_t \left(\frac{\dot{p}_t}{p_t} \right) = \frac{\theta}{2} \left(\frac{\dot{p}_t}{p_t} \right)^2 Y_t, \tag{8}$$

where Y_t is the aggregate output.

Monetary policy

The monetary authority sets the nominal interest rates i_t on liquid asstes according to the following Taylor-type rule.

$$i_t = \bar{r}_t + \phi_\pi \cdot \pi_t + \phi_y (y_t - \bar{y}) + \varepsilon_{MP,t}, \tag{9}$$

where $\varepsilon_{MP,t}$ is a monetary policy shock which follows the Ornstein-Uhlenbeck process

$$d\varepsilon_{MP,t} = -\theta_{MP} \cdot \varepsilon_{MP,t} + \sigma_{MP,t} dW_t, \tag{10}$$

From Fisher equation, we obtain

$$r_t = i_t - \pi_t. \tag{11}$$

Government

The government raises revenue through a proportional tax income and uses it to finance purchases of final goods G_t, pay lum-sum transfers T_t, and pay interest on it outstanding real debt B_t^g, subject to an intertemporal budget constraint,

$$\dot{B_t^g} + G_t + T_t = \pi_t \int w_t z \, l_t(a,z) \, g_t(a,z) \, da \, dz + r_t B_t^g, \tag{12}$$

where B_t^g, is assumed to follow

$$\dot{B_t^g} = \phi \pi B_t^g. \tag{13}$$

From eq.(12) and eq.(13), lump-sum transfer is derived as

$$T_t = \tau_t \int w_t z_t \, l_t(a,z) \, g_t(a,z) \, da \, dz + r \, B_t^g - G_t - \phi \pi B_t^g, \tag{14}$$

And government expenditure is evolved around steady state G_0 and the Ornstein-Uhlenbeck process,

$$G_t = G_0 \cdot \varepsilon_{FP,t}, \tag{15}$$

$$d\varepsilon_{FP,t} = -\theta_{FP} \cdot \varepsilon_{FP,t} + \sigma_{FP,t} dW_t. \tag{16}$$

The government is the only provider of liquid assets in the economy.

2.2 Equilibrium

Households

From the model setup, we obtain the corresponding Hamilton-Jacobi-Bellman equation (HJB) represented as

$$\begin{aligned}\rho V(a,z) = \max_{c,\,l} \; & u(c,l) + (r \cdot a + (1-\tau) \cdot w \cdot z \cdot l + T + \Pi - c)\partial_a V(a,z) \\ & + \lambda(z) \, (\, V(a,z') \, - \, V(a,z) \,),\end{aligned} \tag{17}$$

where λ is transition probabilities of idiosyncratic labor productivity, z, which is assumed to follow a Poisson process.

Notice that profit of firms are assumed to transferred to households proportional to their income level. This assumption is indicated to minimize the redistribution implied by business cycle fluctuations in profit.

Firms

Intermediate goods firms solve a maximizing problem of profits subject to Rotemberg (1982)-type quadratic price adjustment cost, eq.(8). The solution to dynamic pricing problem yields a New Keynesian Phillips Curve (NKPC). Following Kaplan et al. (2018), we obtain the NKPC as

$$\left(r_t - \frac{\dot{Y}_t}{Y_t}\right)\pi_t = \frac{\varepsilon}{\theta}(m_t - m^*) + \dot{\pi_t}, \tag{18}$$

where marginal cost m_t is assumed to be common across all intermediate good producers. The NKPC indicates that firms pick up prices when their markup $1/m_t$ is below the optimal markup $1/m^*$ in the state of no price regidity where

$$m^* = \frac{\varepsilon - 1}{\varepsilon}. \tag{19}$$

Because $\dot{Y}_t = 0$ in steady state, the NKPC is reduced to

$$r_t \cdot \pi_t = \frac{\varepsilon}{\theta}(m_t - m^*) + \dot{\pi}_t. \tag{20}$$

Market Clearing

The bond market clearing condition is given by

$$B_t^g = \int a\, g_t(a, z)\, da\, dz. \tag{21}$$

Meanwhile, the labor market clearing condition is obtained by

$$\int z\, l_t(a, z)\, g_t(a, z)\, da\, dg = L_t, \tag{22}$$

where L_t is aggegate level of labor demand.

3 Computation Method

3.1 Solution Method

To solve HANK models, we adopt linearization procedure following Ahn et al. (2019). This procedure is organized by the following three steps.

Step 1: Approximation Steady State

For solving the steady state of model with idiosyncratic shocks but without aggregate shocks. In this step, the finite difference methods outlined in Achdou et al (2015) are used, since they are fast, accurate and robust.

$$
\begin{array}{rcl}
\rho v & = & u(v) + A(v;p)v, \\
0 & = & A(v;p)^T g, \\
p & = & F(g)
\end{array}
\tag{23}
$$

where v of the first equation is the value of HJB equation. A is transition matrix. the second equation denotes the steady state of Kolmogorov Forward equation. g is the distibution of heterogenous agents. p in the third equation denotes dynamics of aggregate variables.

Step 2: Linearize Equilibrium Conditions

(a) Linearization In this step, equilibrium conditions around steady states obtained from step 1 are given by

$$
\begin{array}{rcl}
\rho v_t & = & u(v_t) + A(v;p)v_t + \frac{1}{dt} E_t dv_t \,, \\
\frac{dg_t}{dt} & = & A(v;p)^T g_t, \\
dZ_t & = & -\eta Z_t dt + \sigma dW_t, \\
p_t & = & F(g_t;Z_t).
\end{array}
\tag{24}
$$

where Z_t is aggregate shocks. Above non-linear equilibrium conditions, eq. (24), are discretized and a first-order Taylor expansion of them is calculated as eq. (25).

$$
\underbrace{E_t \begin{bmatrix} dv_t \\ dg_t \\ dZ_t \end{bmatrix}}_{N_o \times 1} = \underbrace{\begin{bmatrix} B_{\nu\nu} & B_{\nu\nu}B_{pg} & B_{\nu p}B_{pZ} \\ B_{g\nu} & B_{gg} + B_{gp}B_{pg} & B_{gp}B_{pZ} \\ 0 & 0 & -\eta \end{bmatrix}}_{N_o \times N_o} \underbrace{\begin{bmatrix} v_t \\ g_t \\ Z_t \end{bmatrix}}_{N_o \times 1} dt,
\tag{25}
$$

where size of vector, N_o, is derived from sum of the nodes of value functions: v_t, and densties: g_t, of heterogeneous agents, and the number of aggegate variables and shocks: Z_t.

(b) Model Reduction

However, heterogenous agent models have too large size of variables to estimate it. To calculate solution fast and efficiently, we implement the procedure of *model reduction* in terms of two aspects, (1) value function reduction and (2) distribution reduction, following Ahn et al. (2019).

The procedure of value function reduction is expressed as

$$
\nu_t = X_\nu^T v_t,
\tag{26}
$$

where ν_t is compressed vector of value functions of heterogeneous households. X_ν is a matrix for transforming from the original value functions v_t to ν_t. Similarly, distribution reduction is expressed as

$$
\gamma_t = X_g^T g_t,
\tag{27}
$$

where γ_t is compressed vector of densities of heterogeneous households. X_g is a matrix for transforming from the original densities g_t to γ_t.

To sum up, we obtained model reduction form of linearized equilibrium conditions from eq.(25).

$$E_t \underbrace{\begin{bmatrix} d\nu_t \\ d\gamma_t \\ dZ_t \end{bmatrix}}_{N_R \times 1} = \underbrace{\begin{bmatrix} X_\nu^T B_{\nu\nu} X_\nu & X_\nu^T B_{\nu\nu} B_{pg} X_g & X_\nu^T B_{\nu p} B_{pZ} \\ X_g^T B_{g\nu} X_\nu & X_g^T (B_{gg} + B_{gp} B_{pg}) X_g & X_g^T B_{gp} B_{pZ} \\ 0 & 0 & -\eta \end{bmatrix}}_{N_R \times N_o} \underbrace{\begin{bmatrix} \nu_t \\ \gamma_t \\ Z_t \end{bmatrix}}_{N_o \times 1} dt, \tag{28}$$

where size of vector, N_R, is derived from sum of the nodes of value functions: ν_t, and densties: γ_t, of heterogeneous agents after processing model reduction, and the number of aggegate variables and shocks: Z_t.

Step 3: Solve Linear System

Using the linear system of stochastic differential equations obtained from step 2, we solve the model with standard solving methods for linear representative agent (RA) DSGE models, i.e., Blanchard and Kahn (1980) methods or a Schur decomposition ("gensys" developed by Sims (2002)).

The number of stable roots equals the number of state variables g_t and aggregate shocks Z_t . We can obtain the following solutions with respect to g_t and Z_t.

$$\begin{array}{rcc} v_t & = & D_{\nu g} g_t + D_{\nu Z} Z_t \,, \\ \frac{dg_t}{dt} & = & (B_{gg} B_{gp} + B_{g\nu} D_{\nu g}) g_t + (B_{gp} B_{pZ} + B_{g\nu} D_{\nu Z}) Z_t, \\ dZ_t & = & -\eta Z_t dt + \sigma dW_t \\ p_t & = & B_{pg} g_t + B_{pZ} Z_t, \end{array} \tag{29}$$

where the matrix D_{vg} and D_{vz} are the optimal decision rules.

3.2 Estimation Method

To obtain draws from the posterior distribution of parameters, θ, of a HANK model, we use the *Sequential Monte Carlo* (SMC) sampler combined with *kalman filter*, instead of popular methods such as MCMC sampler. Because MCMC samplers cannot be parallelized for generating the draws, they consume quite a long time. By contrast, the SMC algorithm can be easily done and, in addition, may calculate more accurate approximation of the posterior distribution than the MCMC samplers. We explain the algorithms of the SMC following Herbst and Schorfheide (2014, 2015) and Fernandez-Villaverde et al. (2016).

State Space Model

To estimate above linear dynamic system based on the law of motions, we make a state space form consist of measurement equations and transition equations, after they are discretized.

We have three aggregated observations, i.e., real GDP, inflation and nominal interest rate. The measurement equations of these variables are written as

$$\begin{bmatrix} \triangle \ln(GDP_t) \\ INF_t \\ INT_t \end{bmatrix} = \begin{bmatrix} growth \\ \bar{\pi}_{ss} \\ \bar{i}_{ss} \end{bmatrix} + \begin{bmatrix} y_t - y_{t-1} \\ \pi_t - \pi_{ss} \\ i_t - i_{ss} \end{bmatrix} = C \begin{bmatrix} X_\nu & X_g & 1 \end{bmatrix} \begin{bmatrix} v_t \\ \gamma_t \\ Z_t \end{bmatrix}, \tag{30}$$

where growth stands for average of growth rate of real GDP. π_{ss} and i_{ss} are steady states of inflation and nominal interest rate, respectively. The matrix C denotes linkage relation between observation and state variables which are explained by a transition eqaution discretized from eq. (29).

The Sequential Monte Carlo (SMC)

According to Fernandez-Villaverde et al. (2016), SMC conbines features of classic importance sampling and modern MCMC techniques. The starting point is the creation of a sequence of intermediate or bridge distributions $\{\pi_n(\theta)\}_{n=0}^{N_\phi}$ that converge to the target posterior distribution, $\pi(\theta)$, where we call a sequence: n, as n-th stage.

Suppose ϕ_n, for $n = 0, \cdots, N_\phi$, is a sequence that slowly increases from zero to one. We define a sequence of bridge distributions, $\{\pi_n(\theta)\}_{n=0}^{N_\phi}$, for $n = N_\phi$ and $\phi_n = 1$, as

$$\pi_n(\theta) = \frac{[p(Y|\theta)]^{\phi_n} p(\theta)}{\int [p(Y|\theta)]^{\phi_n} p(\theta) d\theta}, \quad \text{for } n = 0, \cdots, N_\phi,\ \phi_n \uparrow 1,$$

where $p(\theta)$ and $p(Y|\theta)$ are the prior density and likelihood function, respectively. We adopt the *likelihood tempering approach* that generates the bridge distributions, $\{\pi_n(\theta)\}_{n=0}^{N_\phi}$, by taking power transformation of posterior density, $p(Y|\theta)$, with the parameter, ϕ_n, i.e., $[p(Y|\theta)]^{\phi_n}$.

3.3 Data and Calibration and Prior Settings

As data for estimation, we adopt the real per-capita GDP growth ($YGRt$), consumption price index (CPI) inflation ($INFLt$), and nominal interest rates ($INTt$) from 1983:Q2 through 2017:Q2 in Japan. $YGRt$ is quarterly growth computed as the log difference of real GDP divided by the population aged 15 years and over. In obtaining the real GDP we collect from the Cabinet Office's National Accounts and use official 2005 constant price series that cover the period 1994:Q1- 2016:Q2 and merged it with the 2000 constant price series which is available for earlier years. $INFLt$ is the quarter on quarter of CPI from Statistics Bureau and excluded the effects of consumption tax changes. $INTt$ is quarterly averages of monthly uncollateralized call rate obtained from the Bank of Japan. Similary , we adopt real GDP, CPI inflation, FFR between 1983:Q1 and 2018:Q1 for US.

Table 1 represent calibration parameters of the HANK model, while Table 2 shows prior setting of estimated parameters. The calibration parameters are six. And the estimated parameters are twelve. As Table 1, we set the same values as calibration parameters for US and Japan. Notice that a steady state of real interest rate and discount factor ρ are derived in the process of the calculation of steady state in step 1. We select both of asset positions and their densities are approximated as 100 nodes, respectively. And we have two types of labor productivities. Accordingly, we have 400 nodes (= (100*2)+(100*2)). After procedures of the model reduction described in Section 3.1, the 409 nodes (=400+9) of liner system can be shrunk to 64 nodes of them (the size is reduced to about 1/7 from the original that can contribute to improve the speed of estimation). It means that the sizes of vertical vectors, N_R and N_o, are 64 and 409 in eq. (28) of our model, respectively.

[Insert Tables 1 and 2]

4 Emprical Results

4.1 Posterior Estimates

In this section, we describe empirical results of the HANK models for US and Japan. As the setting coefficients of SMC procedure, we choose 20 stages and 4,800 particles. From the particles in the last stage, which are thought to be converged to posterior distributions, we make statistical statement pf posterior estimates as Table 3. And Figure 1 (a) shows histograms and scatter plots of the structural parameters and panel (b) also depicts those of parameters of three aggregate shocks. The marks in US are colored in blue, whereas those in Japan are in red.

As can be seen from figures, intervals of posterior distributions of the parameters in both countries do not seem to be duplicated each other. It indicates that fluctuations of business cycles in both countries are different in terms of heterogeneous agent model even though they are used the same model. For example, in households, the elasticity of intertemporal substitution of consumption is around 1.0 in US, and nearly 1.1 in Japan. On the other hand, Frisch elasticity of labor supply is nearly 0.6 for US, but only half such as 0.3 in Japan. In the case of firms, the posterior mean of parameter of adjustment cost, θ, is about 50% bigger in US than in Japan. It indicates that changing price requires higher cost and promote to be more rigid price setting in US than Japan.

It turns to aggregate shock in Panel (b) of Figure 1. In the case of shocks, we can see more contrast between two countries. The size of θ_{FP} of government spending shock is smaller in US than in Japan. It indecates impact of government spending shock in US is more persistent than Japan. Meanwhile, the size of θ_{TFP} of TFP shock is bigger in US than in Japan. It indecates impact of TFP shock in US is more damp than Japan.

[Insert Table 3]

[Insert Figure 1]

4.2 Value and Policy Functions of Heterogeneous Households

Figure 2 draws estimated value functions and policy functions of consumption, saving and working hours. The red solid and blue dashed lines represent high productivity and low productivity workers, respectively. As Table 3, the posterior means of the deep parameters, i.e., γ andϕ_0, ϕ_1, of heterogeneous households are close between the two countries, although the 90% intervals of them are not duplicated for both countries. As can be seen in the figure, it induces that shapes and sizes of estimated value functions and policy functions of them for two types of workers are very similar in the both countries.

[Insert Figure 2]

4.3 Impules Response of Aggregate Variables to Aggregate Shocks

Figurer 3 shows impulse responses of aggregate variables to three aggregate shocks for 40 quarters horizons in the two countries. In Panel (a), the US is dealt with, whereas Japan is in Panel (b). The blue dashed line represents the monetary policy shock, the red solid line denotes the government spending shocks, and the black dotted line stands for the TFP shock. As shown difference in the sizes of θ_{FP} in both countries, we also confirm that the convergence speed of Government spending shocks are much slower in the US than in Japan in the top middle graphs in (a) and(b) of the figures. On the other hand, the convergence speed of the TFP shock is more rapid in US than Japan. However, the case of monetary policy shock, the speeds are almost the same, say it converge in 20 quarters ahead, in both countries. These features affect for impacts of aggregate variables as shown the second and the third rows' graphs. The impacts to government spending shocks are likely to be larger in the US than Japan, while the impacts to the TFP shock seem to be smaller in the US than Japan.

[Insert Figure 3]

4.4 Impacts of Distribution of Heterogeneous Households to Aggregate Shocks

Now, it turns to the impacts of distributions of heterogeneous households to the aggregate shocks as can be seen from Figure 4. In panel (a), the red dashed and solid lines show the immediate impacts of the distributions of high productivity workers to the three shocks and the steady state of the distributions in US, respectively. The blue dashed and solid lines show those of low productivity workers in US. Meanwhile, in panel (b) those shows in the case of Japan.

In the case of US, the impact of monetary policy shock sifts the distributions to left hand side more strongly than Japan. However, the impact of government spending shock sifts the distributions to left hand side more slight than Japan. In the case of the TFP shock, the effects of the distributions are similar in the two countries.

[Insert Figure 4]

4.5 Variance and Historical Decompositions

Table 4 represents variance decomposition of the three observables for the three aggregate shocks. As seen in the IRFs of Figure 3, the monetary policy shocks have short impact until 20 quarters for all three variables in the both countries. On the other hand, the government spending shock have long impact for all three variables in the US, while the TFP shock have long impact for them in Japan.

This feature is seen in historical decomposition of Figure 5. As Figure 5, in both countries, we found the TFP shock contribute positively to inflation until 2000, negatively after 2000. On the other hand, Government spending shock contribute negatively to inflation until 2000, positively after 2000. In US, the contribution of government spending shock to economic growth play an more important role than in Japan. In Japan, the contribution of TFP shock to economic growth play an more important role than in US.

[Insert Table 4]

[Insert Figure 5]

5 Conclusion

This paper estimates heterogeneous agent New Keynesian (HANK) model for US and Japan using aggregate three observations: real GDP, inflation and interest rate, using combination of easy-to-use computational method for solving the model, developed by Ahn, Kaplan, Moll, Winberry and Wolf (2019), and sequential Monte Carlo (SMC) method with Kalman filter applied for Bayesian estimation with parallel computing. The combination realizes the estimation of HANK just using a Laptop PC (mac book pro) with MATLAB, neither server nor FORTRUN. We show estimation results of one Asset HANK model, impulse response, fluctuations of distributions of heterogeneous agent as well as historical decomposition for both countries. Even though using the same model, different data draws different pictures.

A Appendix

A.1 Algorithm of the Sequential Monte Carlo Method

Suppose ϕ_n, for $n = 0, \cdots, N_\phi$, is a sequence that slowly increases from zero to one. We define a sequence of bridge distributions, $\{\pi_n(\theta)\}_{n=0}^{N_\phi}$, that converge to the target posterior distribution for $n = N_\phi$ and $\phi_n = 1$, as

$$\pi_n(\theta) = \frac{[p(Y|\theta)]^{\phi_n} p(\theta)}{\int [p(Y|\theta)]^{\phi_n} p(\theta) d\theta}, \quad \text{for } n = 0, \cdots, N_\phi, \ \phi_n \uparrow 1,$$

where $p(\theta)$ and $p(Y|\theta)$ are the prior density and likelihood function, respectively. We adopt the likelihood tempering approach that generates the bridge distributions, $\{\pi_n(\theta)\}_{n=0}^{N_\phi}$, by taking power transformation of $p(Y|\theta)$ with the parameter, ϕ_n, i.e., $[p(Y|\theta)]^{\phi_n}$.

The SMC^2 with the likelihood tempering takes the following steps. Let $i \in \{1, \cdots, N_\theta\}$ denote the particles of the parameter sets, θ^i, and $n \in \{0, \cdots, N_\phi\}$ denote the stage of the algorithm. Herbst and Schorfheide (2015) recommend a convex tempering schedule in the form of $\phi_n = (n/N_\phi)^\lambda$ with $\lambda = 2$ for a small-scale DSGE model.

1. Initialize

   (a) Set the initial stage as $n = 0$, and draw the initial particles of parameters, θ_0^i, from a prior distribution $p(\theta)$.

   (b) Set the weight of each particle of the initial stage as $W_0^i = 1$, for $i = 1, \cdots, N_\theta$.

2. For stage $n \in \{1, \cdots, N_\phi\}$ and particle $i \in \{1, \cdots, N_\theta\}$, take the following three steps.

   (a) Correction Step. Calculate the normalized weight, $\tilde{W}_n^i$, for each particle as
   $$\tilde{W}_n^i = \frac{\tilde{w}_n^i W_{n-1}^i}{\frac{1}{N}\sum_{i=1}^N \tilde{w}_n^i W_{n-1}^i} \text{ for } i = 1, \cdots, N_\theta,$$
   where $\tilde{w}_t^i$ is an incremental weight derived from
   $$\tilde{w}_n^i = [p(Y|\theta_{n-1}^i)]^{\phi_n - \phi_{n-1}},$$
   and the likelihood, $\hat{p}(Y|\theta)$, is approximated from the particle filter, which is explained in the next subsection.

   We note that the correction step is a classic importance sampling step, in which particle weights are updated to reflect the stage n distribution, $\pi_n(\theta)$. Because this step does not change the particle value, we can skip this step only by calculating power transformation of $p(Y|\theta)$ with the parameter, ϕ_n.

   (b) Selection (Resampling) Step.

i. Calculate an effective particle sample size, $\widehat{ESS_n}$, which is defined as

$$\widehat{ESS_n} = N_\theta / \left(\frac{1}{N_\theta} \sum_{i=1}^{N_\theta} (\tilde{W}_n^i)^2 \right).$$

ii. If $\widehat{ESS_n} < N_\theta/2$, then resample the particles, $\{\hat{\theta}_n\}_{i=1}^{N_\theta}$, via multinomial resampling and set $W_n^i = 1$.

iii. Otherwise, let $\hat{\theta}_n^i = \theta_{n-1}^i$ and $W_n^i = \tilde{W}_n^i$.

(c) Mutation Step. Propagate the particles $\{\hat{\theta_n^i}, W_n^i\}$ via the random walk MH algorithm with the proposal density,

$$\vartheta | \hat{\theta_n}^i \sim N\left(\hat{\theta_n}^i, c_n^2 \Sigma\left(\hat{\theta_n}\right)\right),$$

where $N(\cdot)$ is the nominal distribution and $\Sigma\left(\hat{\theta_n}\right)$ denotes the covariance matrix of parameter $\hat{\theta_n}$ for all particles $i \in \{1, \cdots, N_\theta\}$ at n-th stage. In order to keep the acceptance rate around 25%, we set a scaling factor c_n for $n > 2$ as

$$c_n = c_{n-1} f(A_{n-1}),$$

where A_n represents the acceptance rate in the mutation step at the n-th stage and the function $f(x)$ is given by

$$f(x) = 0.95 + 0.10 \frac{e^{16(x-0.25)}}{1 + e^{16(x-0.25)}}.$$

3. For the final stage of $n = N_\phi$, calculate the final importance sampling approximation of posterior estimator, $E_\pi[h(\theta)]$, as

$$h_{N_\phi, N_\theta} = \sum_{i=1}^{N_\theta} h(\theta_{N_\phi}^i) W_{N_\phi}^i.$$

We note that, in the final stage, the approximated marginal likelihood of the model is also obtained as a by-product. It can be shown that

$$P_{SMC}(Y) = \prod_{n=1}^{N_\phi} \left(\frac{1}{N_\theta} \sum_{i=1}^{N_\theta} \tilde{w}_n^i W_{n-1}^i \right)$$

converges almost surely to $p(Y)$ as the number of particles $N_\theta \to \infty$.

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Table 1: Calibration Parameters

Parameter	Interpretation	value	Grounds for setting vaule
h	a steady state of working hours	1/3	8 hours per day
ε	elasticity of demand	10	
τ	marginal tax rate	0.2	income tax
Bg/Y	ratio of bond to GDP	6	multiple of quartety GDP
T/Y	ratio of Transfer to GDP	0.06	6% of quarterly GDP in steady state
ϕ	government bond increment rule	0	0% for increment of new Bond issue

Table 2: Prior Settings

Parameter	Interpretation	Distibutions	Mean	St Dev
γ	elastisity of substitution	Normal	1	0.1
$\phi 1$	Frisch elasticity of labor supply	Normal	0.5	0.1
$\phi 0$	fraction of labor disulitity	beta	0.85	0.05
θ	adjustment cost	Normal	100	10
$\phi\pi$	Reaction of inflation inTaylor rule	Normal	1.25	0.2
ϕy	Reaction of output inTaylor rule	Normal	0.1	0.05
σMP	volatility of MP shock	Inverted Gamma	0.025	5
θMP	persistency of MP shock	beta	0.25	0.1
σFP	volatility of FP shock	Inverted Gamma	0.1	5
θFP	persistency of FP shock	beta	0.25	0.1
σTFP	volatility of TFP shock	Inverted Gamma	0.1	5
θTFP	persistency of TFP shock	beta	0.25	0.1

Table 3: Posterior Estimations

(a) US

Parameter	Mean	St Dev	Lower Band	Upper Band
γ	1.008	0.006	1.001	1.016
ϕ1	0.566	0.008	0.555	0.577
ϕ0	0.820	0.002	0.816	0.823
θ	119.433	0.606	118.583	120.380
ϕπ	1.470	0.009	1.457	1.483
ϕy	-0.006	0.001	-0.007	-0.005
σMP	0.021	0.000	0.021	0.022
θMP	0.312	0.004	0.305	0.318
σFP	0.150	0.001	0.149	0.152
θFP	0.054	0.001	0.053	0.055
σTFP	0.109	0.001	0.108	0.111
θTFP	0.139	0.000	0.138	0.139
Marginal Likelihood	-272.59	1.03	-273.88	-271.12

(b) Japan

Parameter	Mean	St Dev	Lower Band	Upper Band
γ	1.089	0.006	1.078	1.098
ϕ1	0.344	0.015	0.322	0.370
ϕ0	0.844	0.003	0.839	0.849
θ	81.088	1.092	79.432	82.928
ϕπ	1.661	0.014	1.638	1.683
ϕy	0.029	0.003	0.024	0.034
σMP	0.027	0.001	0.026	0.027
θMP	0.222	0.005	0.214	0.230
σFP	0.101	0.003	0.096	0.106
θFP	0.226	0.005	0.217	0.233
σTFP	0.105	0.001	0.102	0.107
θTFP	0.066	0.003	0.062	0.070
Marginal Likelihood	-352.79	3.27	-358.34	-347.63

Table 4: Variance Decomposition

(a) US

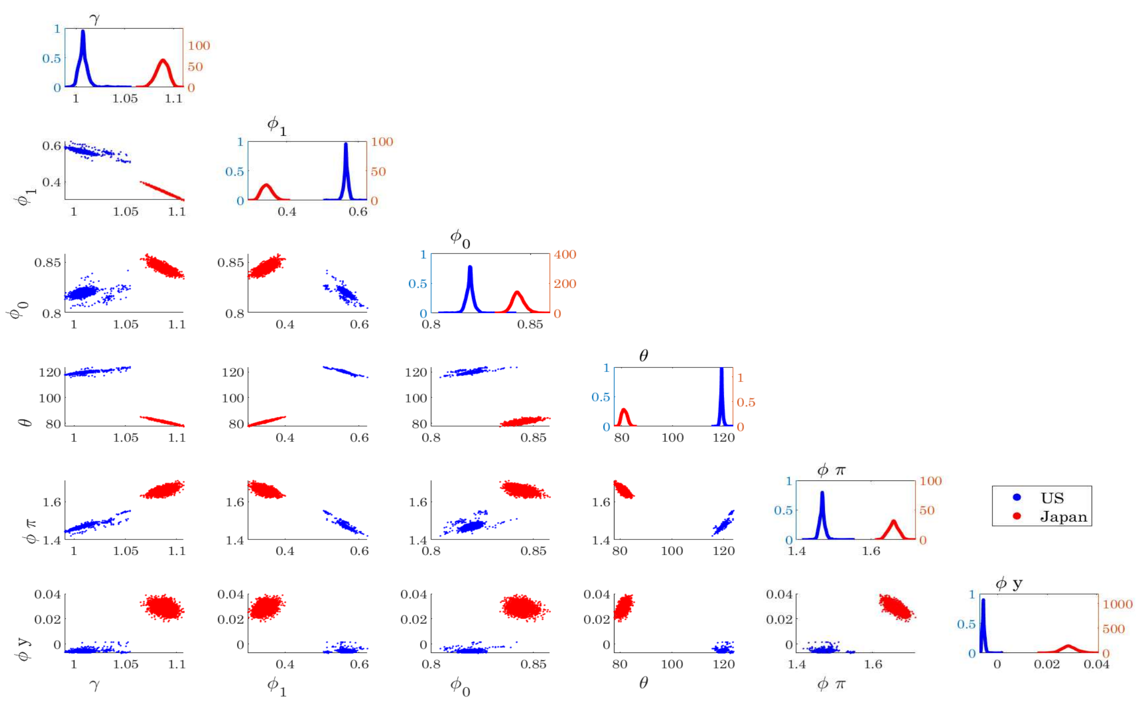
variables	shocks	1Q	5Q	20Q	40Q
Output	MP shock	6.1	1.5	0.0	0.0
	Gov. Sp. Shock	33.3	50.0	91.7	99.6
	TFP shock	60.6	48.5	8.3	0.4
Inflation	MP shock	27.6	9.3	0.0	0.1
	Gov. Sp. Shock	12.8	25.9	80.1	97.6
	TFP shock	59.6	64.7	19.9	2.3
Interest rate	MP shock	4.1	1.1	0.0	0.1
	Gov. Sp. Shock	16.1	27.0	79.1	97.1
	TFP shock	79.9	71.9	20.9	2.4

(b) Japan

variables	shocks	1Q	5Q	20Q	40Q
Output	MP shock	1.9	0.7	0.0	0.0
	Gov. Sp. Shock	10.9	3.8	0.1	0.0
	TFP shock	87.2	95.5	99.9	100.0
Inflation	MP shock	57.0	42.2	1.6	0.0
	Gov. Sp. Shock	25.4	18.1	0.5	0.0
	TFP shock	17.6	39.7	97.8	100.0
Interest rate	MP shock	27.5	18.4	0.6	0.0
	Gov. Sp. Shock	47.7	30.8	0.7	0.0
	TFP shock	24.8	50.7	98.7	100.0

Figure 1: Comparison of Posterior Parameters

(a) Structural Parameters



(b)Parameters of Aggregate Shocks

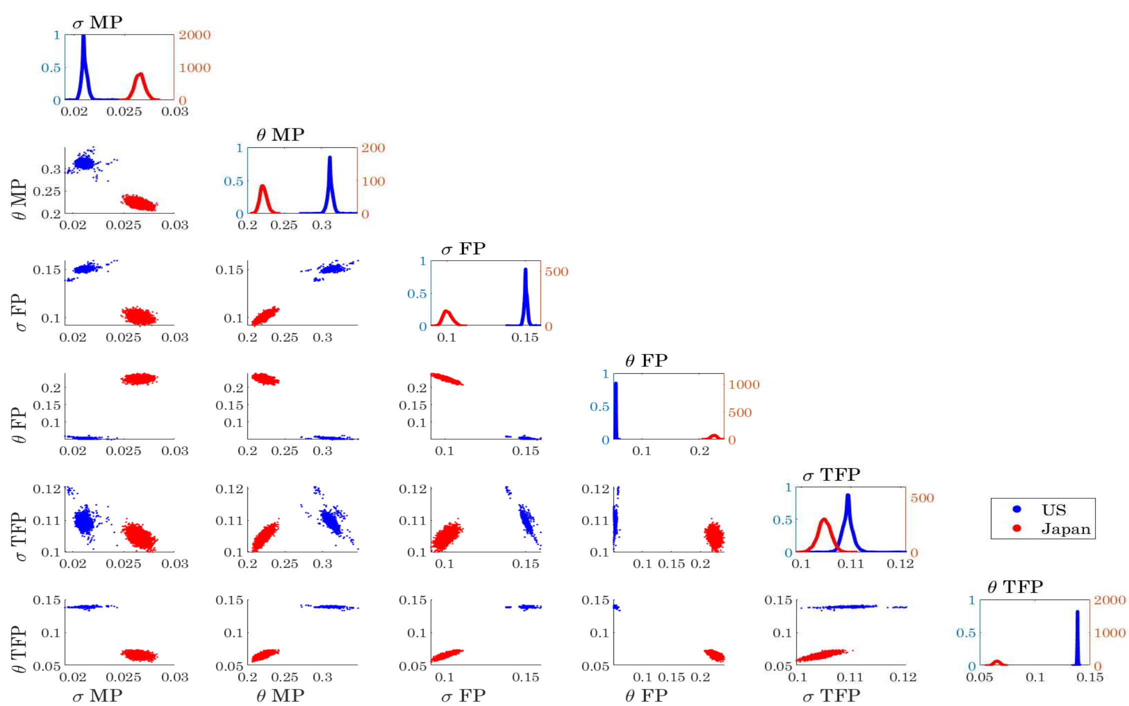
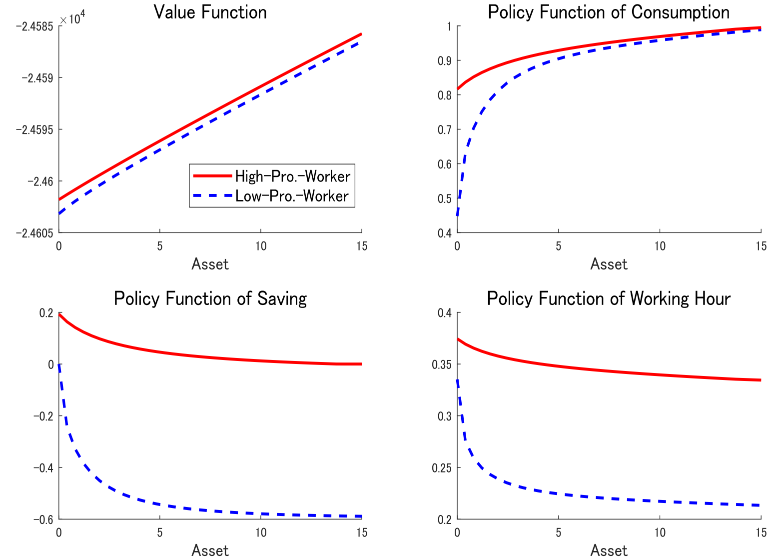


Figure 2: Policy Functions of Hetero Agents

(a) US



(b) Japan

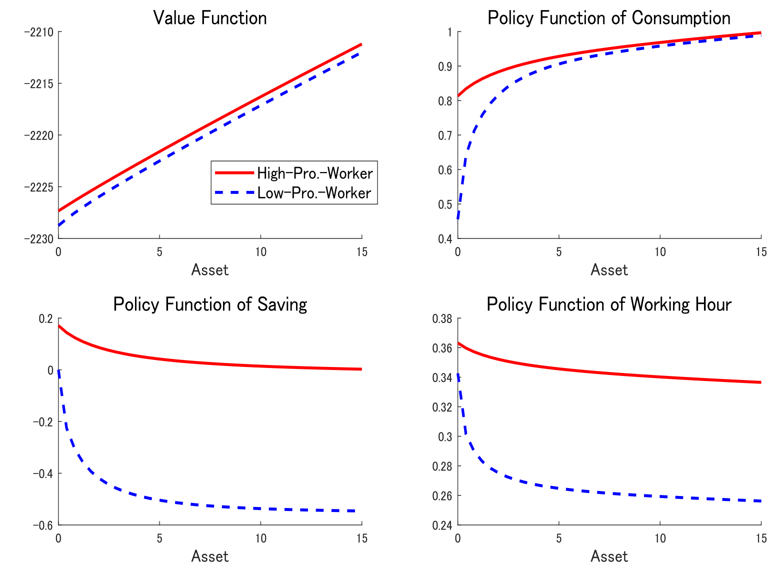
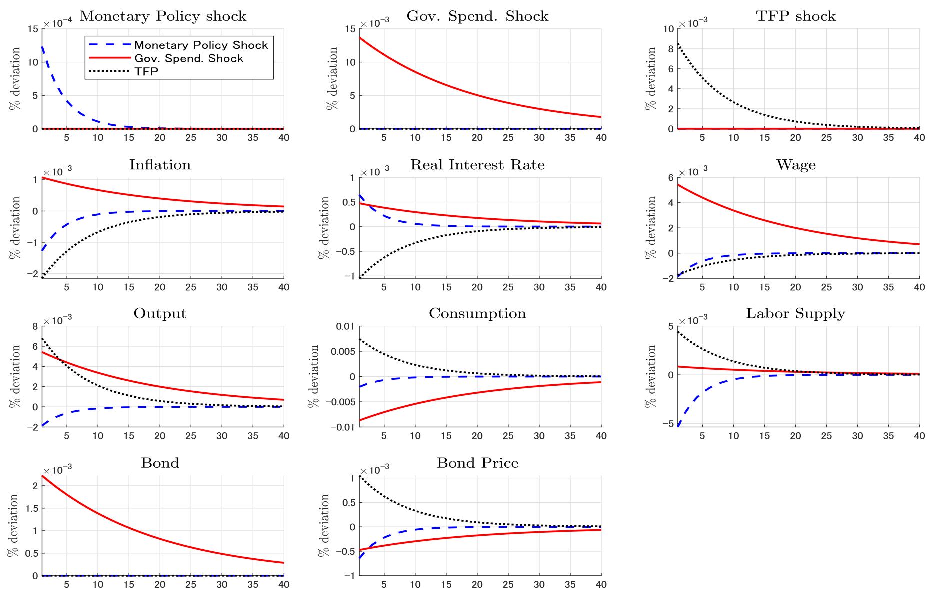


Figure 3: Impulse Response Functions

(a) US



(b) Japan

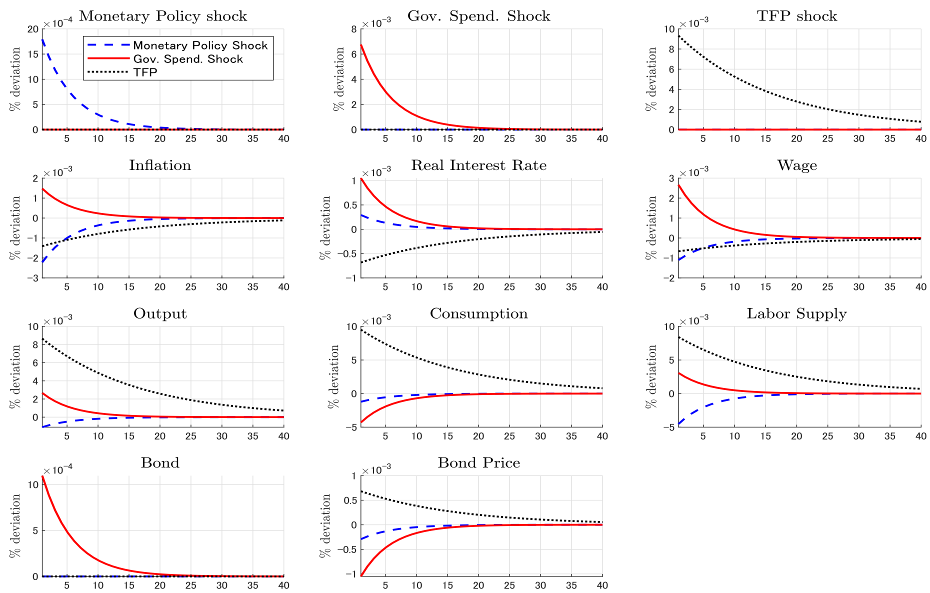
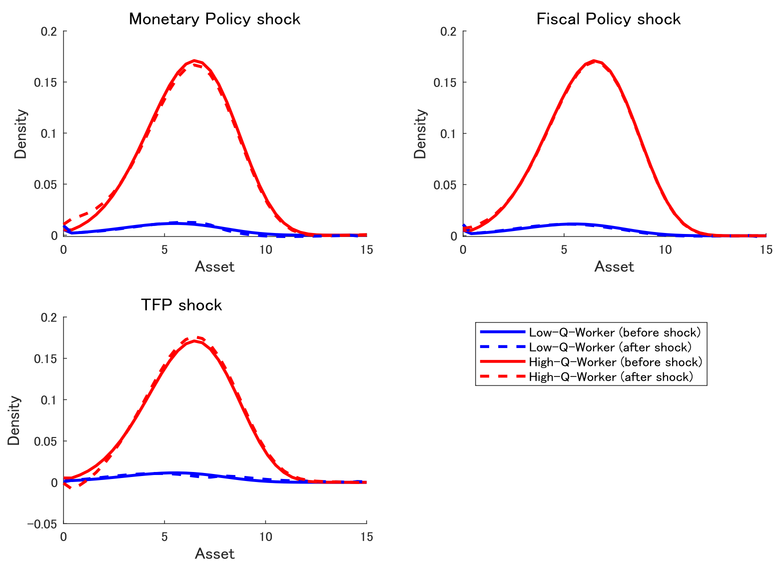


Figure 4: Impacts of Distributions of Hetero Agents to Aggregate Shocks

(a) US



(b) Japan

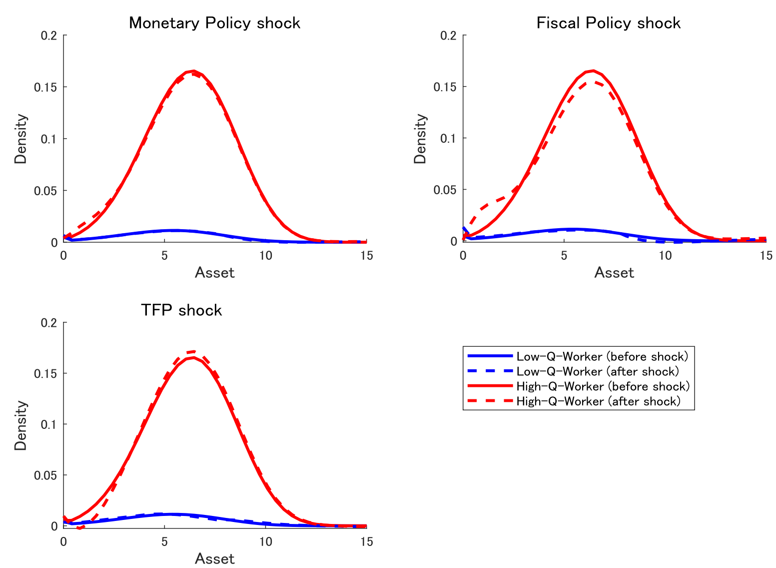
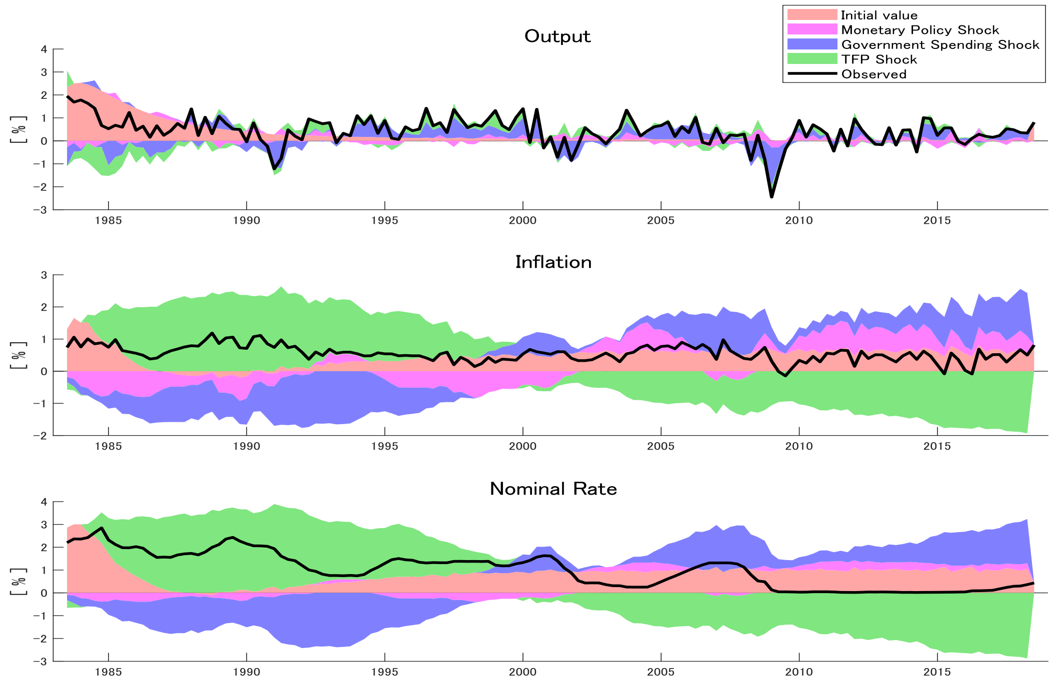


Figure 5: Historical Decompositions

(a) US



(b) Japan