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# Emergence of Urban Landscapes: Equilibrium Selection in a Model of Internal Structure of the Cities<sup>\*†</sup>

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**Abstract:** This paper addresses a longstanding stability issue of equilibria in a seminal model in spatial economic theory, making use of the potential game approach. The model explains the formation of multiple business centers in cities as an equilibrium outcome under the presence of commuting costs of households and positive production externalities between firms. We first show that the model can be viewed as a large population (nonatomic) potential game. To elucidate properties of stable spatial equilibria in the model, we select global maximizers of the potential function, which are known to be globally stable under various learning dynamics. We find that the formation of business centers (agglomeration of firms) is possible only when the commuting costs of households are sufficiently low and that the size (number) of business centers increases (decreases) monotonically as communication between firms becomes easier. Our results indicate a new range of applications, i.e., spatial economic models, for the theory of potential games.

**Keywords:** Agglomeration; multiple equilibria; equilibrium selection; potential game; global stability.

**JEL Classification:** C62, C72, C73, R14

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# 1 Introduction

Understanding the structures and functions of the rich internal landscape of cities has been a formidable task that has attracted many economists (see surveys by, e.g., [Anas, Arnott and Small, 1998](#); [Duranton and Puga, 2015](#)). The most simplified model for describing the internal structure of an urban region is to suppose, in the first place, that there exists a central business district, toward which all workers in the city are assumed to commute. Then, the internal structure of the city, in terms of the residential density of households, is determined by the trade-off between commuting cost and land rent. This model—the monocentric city model developed by [Alonso \(1964\)](#); [Muth \(1969\)](#); [Mills \(1967\)](#)—has been the workhorse of economic analysis of urban land use. In the monocentric city model, urban spatial structure is supposed to be formed by disincentives for the concentration of agents, or *dispersion forces*; households in the framework are trying to reduce both the *land rent* paid and the *commuting costs* that they must bear.

Another strand of research was initiated by the seminal work of [Beckmann \(1976\)](#), emphasizing the role of nonmarket interactions, or *agglomeration economies*, in the formation of urban spatial structures. In this literature, the center of a city is no longer given *a priori*. The formation of a single major concentration in an urban area is described as the equilibrium outcome under the presence of distance-decaying positive externalities: agents are supposed to prefer proximity to other agents. In these models, the role of commuting costs is abstracted away. The spatial structure of a city is delineated by the balance of the two opposing forces: a dispersion force induced by land rent and an *agglomeration force* arising from positive externalities that stem from the spatial concentration of agents.

The innovation of [Fujita and Ogawa \(1982\)](#) was to integrate the two strands of urban models by considering the three mechanisms discussed above: the agglomerative force from positive externalities between agents, the dispersion force induced by the commuting costs of households, and land rent. As the first model that is capable of explaining the formation of *polycentric* urban spatial structure as an equilibrium outcome, the contribution of [Fujita and Ogawa \(1982\)](#) (henceforth FO) has been substantial.<sup>1</sup>

Despite a number of simplifying assumptions, the FO model is known for its intractability. In particular, prevented by the lack of effective methods for stability analysis, the stability of spatial equilibria has not been addressed even after nearly forty years since it was originally proposed. Because of the existence of agglomeration economies, the multiplicity of equilibria is inherent in the FO model; the numerical analyses conducted by FO illustrated that numerous possible equilibrium spatial configurations may coexist. Because some of the equilibria may never be attained under any behaviorally natural assumptions pertaining to dynamics, equilibrium refinement is crucial for drawing reliable implications. Given the huge influence of the FO model, the absence of investigations into stability is becoming increasingly uncomfortable.

To address this issue, we introduce *global stability analysis* into the FO model. It is made

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<sup>1</sup>Most notably, the seminal ideas suggested in the FO model have been introduced in the state-of-the-art quantitative spatial economic literature, including the works by [Ahlfeldt, Redding, Sturm and Wolf \(2015\)](#) and [Owens, Rossi-Hansberg and Sarte \(2017\)](#) (see the survey by [Redding and Rossi-Hansberg, 2017](#)).

possible by the fact, shown in this paper, that the FO model is a *potential game*, or a game associated with scalar-valued functions that characterize its incentive structure (Monderer and Shapley, 1996; Sandholm, 2001).

In potential games, sensible approaches are available for the characterization and selection of equilibria. First, the set of Nash equilibria of a potential game coincides with that of Karush–Kuhn–Tucker (KKT) points of the maximization problem of the associated potential function. Second, the local maximizers of the potential function are *locally stable* under various myopic learning dynamics including the best response dynamic (Gilboa and Matsui, 1991) and other behaviorally plausible dynamics. Third, the global maximizer(s) of the potential function is (are) *globally stable* in the sense of *stochastic stability* under the Logit choice rule (Blume, 1993) or, alternatively, in the sense of selection under perfect foresight dynamics (Oyama, 2009a,b).

In this paper, we employ the last selection result. By selecting spatial configurations that globally maximize the potential function, one can obtain globally stable equilibria in the FO model. To concretely demonstrate the utility of our approach, we study the properties of spatial equilibria in a specific, stylized economy, namely, one-dimensional circular geography. We find that (i) the formation of a (single or multiple) business center can be a globally stable equilibrium outcome; that (ii) the formation of business center(s) can be globally stable only when the commuting costs of households are sufficiently low; and that (iii) the business centers in globally stable equilibria monotonically decrease in number, while their size increases, whenever either the commuting costs of households goes lower monotonically or the level of agglomeration externalities between firms diminishes monotonically. The results are consistent with intuition as well as conjectural predictions in the prior literature.

The rest of the paper is organized as follows. Section 2 relates our contribution to the extant literature. Section 3 formulates the model. Section 4 introduces a decomposition of the FO model into the short- and long-run equilibrium problem. Sections 5 to 7 are devoted to the study of a specific geography, i.e., circular geography as found in Mossay and Picard (2011), to illustrate the effectiveness of the potential game approach in the analysis of the FO model. Section 8 concludes the paper.

## 2 Related literature

Through a series of papers Ogawa and Fujita (1980) and Fujita and Ogawa (1982), Masahisa Fujita and Hideaki Ogawa initiated the study of urban spatial equilibrium models with multiple types of agents and economics of agglomeration.<sup>2</sup> The framework has been extended in many directions, including two-dimensional space (Ogawa and Fujita, 1989); a monopolistic competition framework (Fujita, 1988); multi-unit firms (Ota and Fujita, 1993); and full-fledged general equilibrium frameworks (Lucas, 2001; Lucas and Rossi-Hansberg, 2002; Berliant, Peng and Wang, 2002; Mossay, Picard and Tabuchi, 2017). The latest generation of quantitative spatial economic models (e.g., Ahlfeldt et al., 2015; Owens et al., 2017), implicitly or explicitly, inherit many elements from these theories. These studies have significantly contributed

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<sup>2</sup>The study of socially optimal urban spatial structure under similar assumptions had been explored by other researchers, e.g., Imai (1982).

toward enhancing the reduced-form insights suggested by FO. However, their exploration of the properties of spatial equilibria for their models has been inadequate because of the lack of systematic analytical methods. In particular, investigations into stability are virtually absent. As we discussed, equilibrium selection is vital to the drawing of robust insights in the face of the possibility of multiple equilibria. This paper is the first step in tackling the stability issue of equilibria in FO-type frameworks. Section 8 briefly discusses that the potential game approach employed in the present paper is efficacious in the analysis of models with more general assumptions than the original FO model.

For concrete analyses conducted in Sections 5 to 7, we employ circular geography, which is one of the canonical set-ups for theoretical investigation (e.g., Papageorgiou and Smith, 1983; Krugman, 1993; Mossay and Picard, 2011; Akamatsu, Takayama and Ikeda, 2012; Ikeda, Akamatsu and Kono, 2012; Blanchet, Mossay and Santambrogio, 2016; Osawa, Akamatsu and Takayama, 2017) as it abstracts away various effects of geographical asymmetries. In what related to our results, Mossay and Picard (2011) and Blanchet et al. (2016) analyzed a model with a *single type* of mobile agents (Beckmann-type model), in contrast to the FO model where there are *two types* of mobile agents. They showed that the formation of symmetric disjointed cities that are equidistantly placed on the circumference is the only possible equilibrium outcome. In our context, the multiple-city equilibrium may be interpreted as the formation of multiple business centers in a city. Highlighted in the study is the intrinsic multiplicity of equilibria—not only rotationally symmetric spatial configurations over a circle but also equilibria that differ in the number of cities can coexist. With regard to this, equilibrium selection is essential. The stability of multiple-city equilibria is, however, unknown.<sup>3</sup>

We formulate a discrete-space version of the FO model where the city is divided into a finite number of cells to avoid technical complications. In Section 7, however, we turn our attention to the continuous limit where the number of cells approaches infinity to approximate the original continuous-space formulation. The analysis in the section relates to nonatomic games with continuous strategy sets and a continuum of agents. This class of games, along with associated learning dynamics, was recently proposed by Cheung and Lahkar (2018). It would be possible to directly analyze the continuous-space model as an instance of this class of games.<sup>4</sup> However, the characterization of the global stability of equilibria for this class of games is absent in the literature. In addition, the FO model incorporates the solving of an optimal transport problem (Santambrogio, 2015) as a lower-level problem (see Section 7) which may introduce another source of technical complications. We, therefore, resort to the “continuous approximation of a finite-strategy game” interpretation to focus on more general insights. Further developments in the theory of learning dynamics in games that have continuous strategy sets and a continuum of agents are crucial to the study spatial economic models that typically suppose continuous

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<sup>3</sup>Blanchet et al. (2016) characterizes spatial equilibria of Beckmann (1976)’s and other models as a variational problem regarding a specific functional. The variational characterization would also be employed for the stability analysis of the equilibria. In fact, Akamatsu, Fujishima and Takayama (2017) studies stability of equilibria for a discrete-space version of Beckmann (1976)’s model employing a variational formulation.

<sup>4</sup>A rigorous analysis of continuous-space model has been presented by Carlier and Ekeland (2007). Although their analysis is by far general in its assumptions compared to other studies, most of the analysis focuses on qualitative analysis (existence and uniqueness).

space.

Our approach toward equilibrium selection is, as discussed in Section 1, based on a global stability analysis under existence of a potential function. In the context of network games (Jackson, 2010), where the multiplicity of equilibria is pervasive, equilibrium selection that is based on global stability concepts (most notably, the stochastic stability concept) is one of the standard recipes for theoretical and numerical investigations (see, e.g., Wallace and Young, 2015, for a survey). The present paper provides an example for another range of applications, i.e., spatial equilibrium models with agglomeration economies, where the occurrence of multiple equilibria is ubiquitous, so that equilibrium refinement that is based on global stability can effectively remediate the elucidation of the properties of the models. In particular, Section 6 illustrates that the FO model can in fact have *multiple locally stable equilibria*. Under such circumstances, equilibrium selection based on global stability can make the discussion cleaner without affecting the basic implications of the model.

### 3 The model

Building on Fujita and Ogawa (1982), we introduce a discrete-space model of the internal structure of cities. We then show that the set of equilibria of the model is characterized by an optimization problem.

#### 3.1 Setup

Consider a city that consists of a set of discrete cells indexed by  $i, j, \dots \in \mathcal{I} \equiv \{1, 2, \dots, I\}$ . Each cell  $i \in \mathcal{I}$  is endowed with  $a_i > 0$  units of a fixed supply of land area where we suppose  $\sum_{i \in \mathcal{I}} a_i = 1$ . We denote  $\mathbf{a} \equiv (a_i)_{i \in \mathcal{I}}$ . Land is owned by absentee landlords who spend their rental revenues outside the city. The opportunity cost of land is normalized to 0. We suppose that there are two types of mobile agents: households and firms.

Households freely migrate from the outside world where their reservation utility is normalized to be 0. The total mass of households in the city is denoted by  $N$ . Each household is endowed with a single unit of labor that is supplied to firms and compensated by a wage. Each household inelastically consumes one unit of land for household purposes. Supposing quasilinear utility, the indirect utility of a household located in cell  $i$  and commuting to  $j$  is given by the following:

$$v_{ij} = w_j - t_{ij} - r_i, \quad (1)$$

where  $w_j \geq 0$  and  $r_i \geq 0$  are endogenously determined market wages prevailing in cell  $j$  and the prevailing land rent in cell  $i$ , respectively, and  $t_{ij} \geq 0$  is the commuting cost from  $i$  to  $j$  in the monetary unit. We denote  $\mathbf{w} \equiv (w_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^I$ ,  $\mathbf{r} \equiv (r_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^I$ , and  $\mathbf{t} \equiv (t_{ij})_{ij \in \mathcal{I} \times \mathcal{I}} \in \mathbb{R}_+^{I^2}$ . Also, the mass of households that commute from cell  $i$  to  $j$  is denoted by  $n_{ij} \geq 0$ . The commuting pattern as a whole is denoted by  $\mathbf{n} \equiv (n_{ij})_{ij \in \mathcal{I} \times \mathcal{I}} \in \mathbb{R}_+^{I^2}$ .

We assume that there is a fixed mass of  $M$  business firms. Each business firm produces a

single unit of goods exported to the outside world under a fixed price prevailing there. Each firm requires one unit of land and  $\phi > 0$  units of labor to operate. The profit of a firm located in cell  $i$  in reduced form is given by

$$\pi_i = A_i - \phi w_i - r_i, \quad (2)$$

where  $A_i$  expresses the level of production in cell  $i$ .

Let  $m_i \geq 0$  be the mass of firms in  $i \in \mathcal{I}$ , and  $\mathbf{m} \equiv (m_i)_{i \in \mathcal{I}}$  denotes the spatial distribution of firms. Then, as each firm uses a single unit of land in production, the set of all possible spatial configurations of firms is given by the following:

$$\Delta \equiv \left\{ \mathbf{m} \in \mathbb{R}_+^I \mid \sum_{i \in \mathcal{I}} m_i = M, 0 \leq m_i \leq a_i \ \forall i \in \mathcal{I} \right\}. \quad (3)$$

We suppose that there is an external economy in production that arises from nonmarket interactions: firms are supposed to produce more goods when they are close to other firms. Let  $d_{ij}$  be the strength of the spatial spillover of positive externalities from  $j$  to  $i$ . We assume that  $A_i$  is expressed by the following:

$$A_i(\mathbf{m}) = \sum_{j \in \mathcal{I}} d_{ij} m_j. \quad (4)$$

We impose the following assumptions on  $\mathbf{D} = [d_{ij}]$  so that the above expression actually expresses the positive externalities of agglomeration.

**Assumption 1.**  $\mathbf{D} = [d_{ij}]$  satisfies the following conditions.

- (i)  $\mathbf{D}$  is symmetric:  $d_{ij} = d_{ji} \ \forall i, j \in \mathcal{I}$ .
- (ii)  $\mathbf{D}$  is positive definite with respect to  $T\Delta \equiv \{\boldsymbol{\epsilon} = (\epsilon_i) \in \mathbb{R}^I \mid \sum_{i \in \mathcal{I}} \epsilon_i = 0\}$ , i.e.,  $\boldsymbol{\epsilon}^\top \mathbf{D} \boldsymbol{\epsilon} > 0$  for all nonzero  $\boldsymbol{\epsilon} \in T\Delta$ .

For example, Assumption 1 is satisfied by a common specification in the spatial interaction modeling literature  $d_{ij} = e^{-\tau \ell_{ij}}$  where  $\tau > 0$  is a distance-decay parameter and  $\ell_{ij}$  is the geographical distance from  $i$  to  $j$ .<sup>5</sup> The first condition (i) requires that the amount of spatial spillover that a firm induces is symmetric for all pairs of  $i$  and  $j$ . It is satisfied by various specifications of spatial spillover that define  $d_{ij}$  by monotonic transformations of  $\ell_{ij}$ . The second condition (ii) implies *self-reinforcing externalities*. Consider that a hypothetical firm leaves a cell and joins another. Then, under Assumption 1 (ii), the improvements in the positive externalities of the cell to which the firm switches dominate those found in one left.

## 3.2 Equilibrium

In equilibrium, every household should maximize its own utility so that there is no incentive to change its residential location or job location. Moreover, every firm should maximize its own

<sup>5</sup>For instance, [Ahlfeldt et al. \(2015\)](#) employs the exponential specification to express agglomeration economies.

profit by choosing its location. These conditions are expressed as follows:

$$\text{(utility maximization)} \quad \begin{cases} v^* = v_{ij} = w_j - t_{ij} - r_i & \text{if } n_{ij} > 0 \\ v^* \geq v_{ij} = w_j - t_{ij} - r_i & \text{if } n_{ij} = 0 \end{cases} \quad \forall ij \in \mathcal{I} \times \mathcal{I}, \quad (5)$$

$$\text{(profit maximization)} \quad \begin{cases} \pi^* = \pi_i = A_i(\mathbf{m}) - r_i - \phi w_i & \text{if } m_i > 0 \\ \pi^* \geq \pi_i = A_i(\mathbf{m}) - r_i - \phi w_i & \text{if } m_i = 0 \end{cases} \quad \forall i \in \mathcal{I}, \quad (6)$$

where  $v^* = 0$  and  $\pi^* \geq 0$  are the reservation utility and equilibrium profit levels, respectively. The land and labor markets are assumed to be perfectly competitive. The markets clear when the following conditions are met:

$$\text{(land market clearing)} \quad \begin{cases} m_i + \sum_{j \in \mathcal{I}} n_{ij} = a_i & \text{if } r_i > 0 \\ m_i + \sum_{j \in \mathcal{I}} n_{ij} \leq a_i & \text{if } r_i = 0 \end{cases} \quad \forall i \in \mathcal{I}, \quad (7)$$

$$\text{(labor market clearing)} \quad \begin{cases} \sum_{i \in \mathcal{I}} n_{ij} = \phi m_j & \text{if } w_j > 0 \\ \sum_{i \in \mathcal{I}} n_{ij} \geq \phi m_j & \text{if } w_j = 0 \end{cases} \quad \forall j \in \mathcal{I}. \quad (8)$$

Last, we must require

$$\text{(conservation)} \quad \sum_{i \in \mathcal{I}} m_i = M, \quad (9)$$

as we assume that the total mass of firms in the city is fixed.

To summarize, equilibrium in the model is defined as follows:

**Definition 1.** For a given set of the exogenous parameters  $(\mathbf{D}, \mathbf{t}, \mathbf{a}, \phi)$ , a *spatial equilibrium* is a collection of variables  $(\mathbf{n}, \mathbf{m}, \mathbf{r}, \mathbf{w}, \pi^*)$  satisfying the conditions (5), (6), (7), (8), and (9).

From (7) and (8) it must be noted that  $M + N \leq 1$  and  $N \geq \phi M$ , implying that  $M \leq \frac{1}{1+\phi} \in (0, 1)$ . This condition arises because of the normalization of land endowment  $\{a_i\}$ . Otherwise, it should read  $\sum_{i \in \mathcal{I}} a_i \geq N + M \geq (1 + \phi)M$ , which requires that the total land endowment meet the total demand from all agents (firms and households) in a city in equilibrium; the area of the city should increase with the mass of its residents. In addition, in equilibrium, every household in the city should supply its labor to some firm; letting  $N^*$  be the total equilibrium mass of households, we must have  $N^* = \phi M$ .

### 3.3 Associated optimization problem

It turns out that the set of spatial equilibria in the model is associated with an optimization problem. Let  $f : \mathbb{R}^I \rightarrow \mathbb{R}$  and  $h : \mathbb{R}^{I^2} \rightarrow \mathbb{R}$  be defined as follows:

$$f(\mathbf{m}) \equiv \oint A(\omega) d\omega = \frac{1}{2} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} d_{ij} m_i m_j, \quad (10)$$

$$h(\mathbf{n}) \equiv \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} t_{ij} n_{ij}. \quad (11)$$

Given  $f$  and  $h$ , we have the following characterization of the spatial equilibria.<sup>6</sup>

**Proposition 1.** *Suppose Assumption 1 (i). Then, any spatial equilibrium of the model is a KKT point of the following optimization problem:*

$$[\mathcal{P}_0] \quad \max_{(\mathbf{m}, \mathbf{n}) \geq \mathbf{0}} f(\mathbf{m}) - h(\mathbf{n}) \quad (12a)$$

$$\text{s.t.} \quad m_i + \sum_{j \in \mathcal{I}} n_{ij} \leq a_i \quad [r_i] \quad \forall i \in \mathcal{I}, \quad (12b)$$

$$\sum_{i \in \mathcal{I}} n_{ij} \geq \phi m_j \quad [w_j] \quad \forall j \in \mathcal{I}, \quad (12c)$$

$$\sum_{i \in \mathcal{I}} m_i = M, \quad [\pi^*] \quad (12d)$$

where  $r_i$ ,  $w_j$ , and  $\pi^*$  are determined as the Lagrange multiplier for the land supply constraint (12b), the labor supply constraint (12c), and the conservation constraint (12d), respectively.

Observe that the objective function of the optimization problem reflects the structure of the model. On the one hand, the first term  $f(\mathbf{m})$  is a measure of firms' merit from agglomeration. We note that  $f(\mathbf{m})$  is strictly convex on  $\Delta$  under Assumption 1 (ii).

**Lemma 1.** *Suppose Assumption 1 (ii). Then  $f(\mathbf{m})$  is strictly convex.*

Therefore, the first term of the objective function (12a) always prefers agglomeration toward a smaller number of cells, which is consistent with intuition.

On the other hand,  $h(\mathbf{n})$  is the total commuting cost that is spent in the city for a given  $\mathbf{n}$ . Thus,  $[\mathcal{P}_0]$  is a problem that pertains to maximizing the merit of agglomeration on the side of firms, while at the same time reducing the commuting costs of households.

**Remark 1.** Although related,  $[\mathcal{P}_0]$  is not a welfare maximization problem. The welfare maximization problem for the model would be as follows:

$$[\mathcal{W}] \quad \max_{(\mathbf{m}, \mathbf{n}) \geq \mathbf{0}} 2f(\mathbf{m}) - h(\mathbf{n}) \quad \text{s.t.} \quad (12b), (12c), \text{ and } (12d) \quad (13)$$

because the surplus of agglomeration is given by  $\sum_{i \in \mathcal{I}} A_i(\mathbf{m}) m_i = 2f(\mathbf{m})$ . Since  $[\mathcal{W}]$  has a larger weight on  $f(\mathbf{m})$  than  $[\mathcal{P}_0]$  does and  $f(\mathbf{m})$  prefers agglomeration, the agglomeration of firms is inadequate in equilibrium when compared with the social optimum.

## 4 Short- and long-run equilibria

To elucidate the structure of the problem  $[\mathcal{P}_0]$ , we consider decomposing the original equilibrium problem into an equivalent two-step problem by supposing that the adjustment of

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<sup>6</sup>All proofs are relegated to Appendix A.

households' decision is faster than that of firms. In the short run, the spatial distribution of firms  $m \in \Delta$  is fixed, with only households choosing their residential location and workplace. In the long run, firms choose locations that maximize their profit, while respecting the short-run equilibrium.

#### 4.1 Short-run equilibrium

Given  $m \in \Delta$ , the short-run equilibrium considers households' decisions pertaining to optimal job and residential location choice.

**Definition 2** (Short-run equilibrium). Fix a given set of parameters  $(t, a, \phi)$  and a spatial distribution of firms  $m \in \Delta$ . Then, a *short-run equilibrium* is a collection of variables  $(n, \check{r}, \check{w})$  satisfying conditions (5), (7), and (8), where we replace  $(r, w)$  with  $(\check{r}, \check{w})$ .

In Definition 2, we replace  $r$  and  $w$  with  $\check{r} = (\check{r}_i)$  and  $\check{w} = (\check{w}_i)$ , respectively, to indicate that they are short-run equilibrium variables. We note that  $\check{r}$  and  $\check{w}$  do not necessarily coincide with long-run equilibrium values of land rent and wages because firms' strategies are fixed in the short run.

The short-run equilibrium reduces to an optimization problem.

**Proposition 2.** *The short-run equilibrium for a given  $m \in \Delta$  has the following properties:*

(a) *Any short-run equilibrium commuting pattern  $n$  is a KKT point for the following problem:*

$$[\mathcal{S}] \quad \min_{n \geq 0} h(n) \quad \text{s.t.} \quad (12b) \text{ and } (12c). \quad (14)$$

(b) *Any short-run equilibrium value of  $(\check{r}, \check{w})$  is the KKT point for the following problem:*

$$[\mathcal{D}] \quad \max_{(\check{r}, \check{w}) \geq 0} h^*(\check{r}, \check{w}) \equiv \phi \sum_{i \in \mathcal{I}} \check{w}_i m_i - \sum_{i \in \mathcal{I}} \check{r}_i (a_i - m_i), \quad (15a)$$

$$\text{s.t.} \quad 0 \geq \check{w}_j - t_{ij} - \check{r}_i \quad \forall i, j \in \mathcal{I}, \quad (15b)$$

where we normalize  $\check{r}$  and  $\check{w}$  by letting  $\min_{i \in \mathcal{I}} \{\check{r}_i\} = 0$ .

Observe that  $[\mathcal{S}]$  is the minimization of total commuting costs across the city while satisfying the land and labor market constraints.  $[\mathcal{D}]$  is the dual problem for  $[\mathcal{S}]$ . The first term of its objective is the total land rent paid, and the second term is the total wages received by households. Thus,  $[\mathcal{D}]$  is the maximization of the surplus of households, while keeping utility of households below the reservation utility level 0. It determines the highest bid  $\check{r}$  for land and lowest possible wage  $\check{w}$  to compensate for commuting costs and land rent.

With  $[\mathcal{S}]$  being a linear programming problem,  $n$  might not be uniquely determined. However, the optimal value uniquely exists because (12b), (12c), and the nonnegativity constraint of  $n$  together define a closed and convex set. We can therefore regard it as a function of  $m$ . Let  $\check{h}(m)$  be the optimal value of  $[\mathcal{S}]$  as a function of  $m$ . Then, it coincides with the optimal value

of  $[\mathcal{D}]$  because of the strong duality of linear programming:

$$\check{h}(\mathbf{m}) \equiv \min_{\mathbf{n}} h(\mathbf{n}) = \max_{\check{\mathbf{r}}, \check{\mathbf{w}}} h^*(\check{\mathbf{r}}, \check{\mathbf{w}}).$$

From the dual representation  $[\mathcal{D}]$ ,  $\check{h}(\mathbf{m})$  is a point-wise maximum of the affine functions of  $\mathbf{m}$ ; thus, it is convex in  $\mathbf{m}$ .

**Corollary 1.**  $\check{h}(\mathbf{m})$  is convex.

Intuitively, the convexity of  $\check{h}(\mathbf{m})$  means that if firms are concentrated in a smaller number of cells, it induces a larger total commuting costs of households in short-run equilibrium.

It is noted that the envelope theorem implies the following relation:

$$\frac{\partial \check{h}(\mathbf{m})}{\partial m_i} = \phi \check{w}_i(\mathbf{m}) + \check{r}_i(\mathbf{m}). \quad (16)$$

The right-hand side coincides with the firms' (minimized) short-run costs. The profit of firms in cell  $i$  in the short-run equilibrium is given by the following:

$$\check{\pi}_i(\mathbf{m}) \equiv A_i(\mathbf{m}) - \phi \check{w}_i(\mathbf{m}) - \check{r}_i(\mathbf{m}). \quad (17)$$

## 4.2 Long-run equilibrium

Given the short-run equilibrium land rent  $\check{r}(\mathbf{m})$  and wage  $\check{w}(\mathbf{m})$ , the profit-maximization condition for firms (6) may be rewritten as follows:

$$\begin{cases} \pi^* = \pi_i = A_i(\mathbf{m}) - \phi \check{w}_i(\mathbf{m}) - \check{r}_i(\mathbf{m}) - \hat{r}_i & \text{if } m_i > 0 \\ \pi^* \geq \pi_i = A_i(\mathbf{m}) - \phi \check{w}_i(\mathbf{m}) - \check{r}_i(\mathbf{m}) - \hat{r}_i & \text{if } m_i = 0 \end{cases} \quad \forall i \in \mathcal{I}, \quad (18)$$

where  $\hat{r}_i \geq 0$  is the additional land rent paid by firms in the long run. The market land rent  $r_i$  of the whole problem (7) is, thus, decomposed as  $r_i = \hat{r}_i + \check{r}_i$ . With regard to  $\hat{\mathbf{r}} = (\hat{r}_i)$ , the following condition should be met:

$$\text{(land market clearing in the long-run)} \quad \begin{cases} m_i = a_i & \text{if } \hat{r}_i > 0 \\ m_i \leq a_i & \text{if } \hat{r}_i = 0 \end{cases} \quad \forall i \in \mathcal{I}. \quad (19)$$

It requires that if households cannot afford to reside in cell  $i$ , (i.e., if  $\hat{r}_i > 0$  so that  $r_i = \check{r}_i + \hat{r}_i > \check{r}_i$ ), the cell is completely occupied by the firms in the long run. Similarly, if  $m_i \leq a_i$  and  $\sum_{j \in \mathcal{I}} n_{ij} \geq 0$ , the land rent paid by firms and households must be the same:  $\hat{r}_i = 0$  so that  $r_i = \check{r}_i$ .

Summing up, the long-run equilibrium is defined as follows:

**Definition 3** (Long-run equilibrium). Fix a given set of parameters  $(\mathbf{D}, \mathbf{t}, \mathbf{a}, \phi)$ . A long-run equilibrium is a collection of variables  $(\mathbf{m}, \hat{\mathbf{r}}, \pi^*)$  satisfying (18), (19), and (9), where the associated short-run equilibrium is defined for the parameters  $(\mathbf{t}, \mathbf{a}, \phi)$ .

Long-run equilibria are characterized by an optimization problem.

**Proposition 3.** Let  $g(\mathbf{m}) \equiv f(\mathbf{m}) - \check{h}(\mathbf{m})$ . Any long-run equilibrium spatial distribution  $\mathbf{m}$  is a KKT point for the following maximization problem:

$$[\mathcal{P}] \quad \max_{\mathbf{m} \in \Delta} g(\mathbf{m}). \quad (20)$$

It is easy to show that any long-run equilibrium spatial distribution of firms, given its associated short-run equilibrium, is an equilibrium solution for the original problem (Definition 1). In fact, the decomposition of the short- and long-run problems is basically entailed in the following rearrangement:

$$\begin{aligned} [\mathcal{P}_0] &= \max_{(\mathbf{m}, \mathbf{n}) \geq \mathbf{0}} \{f(\mathbf{m}) - h(\mathbf{n}) \mid \text{s.t. (12b), (12c), (12d)}\} \\ &= \max_{\mathbf{m} \in \Delta} \left\{ f(\mathbf{m}) - \min_{\mathbf{n} \geq \mathbf{0}} \{h(\mathbf{n}) \mid \text{s.t. (12b), (12c)}\} \right\} \\ &= \max_{\mathbf{m} \in \Delta} \{f(\mathbf{m}) - \check{h}(\mathbf{m})\} \\ &= [\mathcal{P}], \end{aligned}$$

where the second equality is justified by the fact that the lower-level problem is a simple linear programming problem.

Although equivalent, the property of the problem  $[\mathcal{P}]$  is simpler to interpret and analyze than that of  $[\mathcal{P}_0]$ . For instance, in  $[\mathcal{P}]$ , the objective function  $g(\mathbf{m})$  is nonconvex because it is the difference of convex functions; it clearly indicates the possible multiplicity of long-run equilibria without the need of deriving any concrete examples.

### 4.3 Stability of long-run equilibria

Proposition 3 shows that the long-run equilibrium problem (Definition 3) is a potential game in the following definition.

**Definition 4** (Nonatomic potential game). A *nonatomic game* is a triplet  $\mathcal{G} = (\mathcal{S}, \mathcal{X}, \mathbf{F})$  of a finite strategy set  $\mathcal{S} = \{1, 2, \dots, S\}$  where  $S$  denotes the number of strategies; closed and convex state space  $\mathcal{X} \subset \mathbb{R}_+^S$ ; and payoff function  $\mathbf{F} : \mathcal{X} \rightarrow \mathbb{R}^S$ . A nonatomic game  $\mathcal{G}$  is a *potential game* if there exists a scalar-valued function  $p : \mathcal{X} \rightarrow \mathbb{R}$  such that the following holds true:

$$\frac{\partial p(\mathbf{x})}{\partial x_i} - \frac{\partial p(\mathbf{x})}{\partial x_j} = F_i(\mathbf{x}) - F_j(\mathbf{x}) \quad \forall i, j \in \mathcal{S}, \forall \mathbf{x} \in \mathcal{X}. \quad (21)$$

**Corollary 2.** The long-run equilibrium problem is a potential game with strategy set  $\mathcal{I}$ , state space  $\Delta$ , payoff function  $\check{\pi} : \Delta \rightarrow \mathbb{R}^I$ , and the associated potential function  $g : \Delta \rightarrow \mathbb{R}$ .

Note that the payoff function for firms is the short-run profit defined by (17). The short-run profit function satisfies (21) because upon recalling the relation (16), we have the following:

$$\frac{\partial g(\mathbf{m})}{\partial m_i} = A_i(\mathbf{m}) - \phi \check{w}_i(\mathbf{m}) - \check{r}_i(\mathbf{m}) = \check{\pi}_i(\mathbf{m}). \quad (22)$$

For the interior of  $\Delta$ , we have  $\hat{r} = \mathbf{0}$ , and thus,  $\pi(\mathbf{m}) = \check{\pi}(\mathbf{m})$ ; therefore (21) holds true for the long-run profit function  $\pi(\mathbf{m})$  as well. At a long-run equilibrium where  $m_i = a_i$  for some cells,  $\hat{r}_i$  is determined so as to cancel out any firms' extra profit and to ensure that the long-run payoff is given by some equal level  $\pi^*$  for all  $i$  with  $m_i > 0$ ,

$$\hat{r}_i = \check{\pi}_i - \pi^* \quad \forall i \text{ with } m_i > 0. \quad (23)$$

Therefore, (21) does not hold true for firms' long-run profits  $\pi_i = \check{\pi}_i + \hat{r}_i$ . This property deviates from simple nonatomic potential games in the literature (Sandholm, 2001), where the state space is merely the  $(S - 1)$ -simplex. The subtle difference does not matter as the long-run equilibrium profit of firms is unambiguously determined by the condition (19).

It follows that one can employ the desirable properties of (nonatomic) potential games to analyze the long-run equilibrium problem. The following theorem summarizes the relevant facts in the literature (see, e.g., Sandholm, 2010, and the references therein).

**Theorem 1** (Characterization of equilibria and their stability in nonatomic potential games). *Consider a nonatomic potential game  $\mathcal{G} = (\mathcal{S}, \mathcal{X}, \mathbf{F})$  with the potential function  $p : \mathcal{X} \rightarrow \mathbb{R}$ . Let  $\mathcal{P}$  denote the associated potential maximization problem:  $\max_{\mathbf{x} \in \mathcal{X}} p(\mathbf{x})$ . The following results hold true for  $\mathcal{G}$  and  $\mathcal{P}$ .*

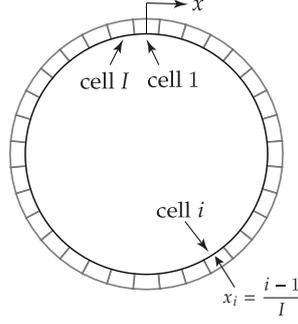
- (1) *The set of KKT points for  $\mathcal{P}$  coincides with that of Nash equilibria of  $\mathcal{G}$ ,  $\text{NE}(\mathcal{G})$ .*
- (2) *The set of local maximizers for  $\mathcal{P}$  coincides with that of locally stable equilibria of  $\mathcal{G}$  under deterministic myopic learning dynamics  $\dot{\mathbf{m}} = \mathbf{V}(\mathbf{m})$  which preserves the total mass of the agents and satisfies  $\dot{\mathbf{m}} \cdot \mathbf{V}(\mathbf{m}) > 0$  for all  $\mathbf{m} \notin \text{NE}(\mathcal{G})$  and  $\dot{\mathbf{m}} = \mathbf{0}$  for all  $\mathbf{m} \in \text{NE}(\mathcal{G})$ .*
- (3) *The set of global maximizers for  $\mathcal{P}$  coincides with that of globally stable equilibria of  $\mathcal{G}$  in the sense of stochastic stability under stochastic learning dynamics with the Logit choice rule.*

As our aim is to neither propose novel behavioral dynamics nor new stability concepts, we proceed by simply accepting the above facts in the literature. In the following, depending on the context, the stability of equilibria indicates either local stability or global stability in the sense summarized in Theorem 1. We tacitly suppose (i) deterministic learning dynamics that satisfies conditions listed in Theorem 1 for local stability claims, and (ii) stochastic learning dynamics with the Logit choice rule for global stability claims.

## 5 Geography

For the remainder of the paper, we focus on a specific stylized geography to elucidate the intrinsic role of the model parameters.<sup>7</sup> To circumvent issues arising from asymmetries in particular, we suppose that the underlying geography is a symmetric circle (Figure 1) as in Papageorgiou and Smith (1983) or Krugman (1993).

<sup>7</sup>Since the potential function  $g(\mathbf{m})$  for our model is neither concave nor convex, a complete study of the model for general set-ups of the parameters  $(\mathbf{D}, \mathbf{t}, \mathbf{a}, \phi)$  is practically impossible. Yet, we also note that the optimization formulation can be utilized to numerically tackle general cases; observe that Propositions 1, 2, and 3 do not impose any restrictions on  $\mathbf{D}$  nor  $\mathbf{t}$  other than Assumption 1.



**Figure 1:** A circular geography

**Assumption 2** (Geography). *The underlying geography is a one-dimensional circular network with circumferential length 1 that is indexed as a unit interval  $[0, 1)$ . In addition the following hold true:*

- (i) *Each cell has the same length  $\frac{1}{I}$  and is equidistantly placed over the circle; the cell  $i \in \mathcal{I}$  is centered at  $x_i = \frac{1}{I}(i - 1) \in [0, 1)$ . For all  $i$ ,  $a_i = a \equiv \frac{1}{I}$  so that  $\sum_{i \in \mathcal{I}} a_i = 1$ .*
- (ii)  *$d_{ij} = e^{-\tau \ell_{ij}}$  and  $t_{ij} = t \ell_{ij}$  with  $\tau, t > 0$ , where  $\ell_{ij} \equiv \min \{|x_i - x_j|, 1 - |x_i - x_j|\}$  is the minimum geographical distance between  $x_i$  and  $x_j$  along the circle.*

Assumption 2 abstracts away any cell-specific advantage arising from greater capacity or proximity to other cells. Following Fujita and Ogawa (1982), we suppose that  $\{d_{ij}\}$  is exponential and that  $\{t_{ij}\}$  is linear in distance. The parameter  $\tau$  determines how fast the technological externalities decrease in distance, whereas  $t$  denotes the commuting cost rate per unit of distance. Note that Assumption 2 (ii) is sufficient for Assumption 1.<sup>8</sup>

Imposing circular geography allows us to isolate the role of endogenous forces in the determination of spatial patterns because it abstracts away the cell-specific advantages induced by the shape of the underlying transportation network. For instance, in a line segment, the locations near the city boundaries have fewer opportunities to access the other cells. By contrast, in a circle, every cell has the same level of accessibility to the other cells under Assumption 2.

It follows that the uniform distribution of firms is always a long-run equilibrium under Assumption 2. The uniform distribution  $\bar{\mathbf{m}}$  is a spatial pattern where  $m_i = \bar{m} \equiv \frac{M}{I}$ ,  $n_{ii} = \phi m_i$ , and  $n_{ij} = 0$  for all  $j \neq i$ . Following the tradition of the economics of agglomeration (Fujita, Krugman and Venables, 1999), we study the spontaneous formation of spatial patterns from *ex-ante* uniform and symmetric distribution of mobile agents. The uniform distribution of agents is, however, never locally stable if the land endowment is larger than the equilibrium mass of agents  $N^* + M = (1 + \phi)M$ .

**Lemma 2.** *Suppose that  $(1 + \phi)M < 1$  and Assumption 2. Then, the uniform distribution of firms  $\bar{\mathbf{m}}$  is a long-run equilibrium, but it is locally unstable.*

Intuitively, if  $(1 + \phi)M < 1$  then excess land supply exists in every cell at the uniform pattern. It implies that some firms can improve their profit by relocating (along with all of its

<sup>8</sup>For the linear commuting cost, equilibrium commuting pattern may not be unique (Berliant and Tabuchi, 2018). The (minimized) total commuting costs is, however, uniquely given; it suffices for stability analysis of long-run equilibria.

employees) to another cell, implying some instability in  $\bar{m}$ .

To allow for the possibility of the uniform pattern being stable, we assume that there is no excess land supply in the city.

**Assumption 3** (No excess land supply).  $(1 + \phi)M = 1$ .

Assumption 3 implies that  $M = \sum_{i \in \mathcal{I}} m_i = \frac{1}{1+\phi} \in (0, 1)$  and  $N = \phi M = \frac{\phi}{1+\phi} \in (0, 1)$ . In the following, we define  $\rho \equiv \frac{1}{1+\phi}$  so that  $M = \rho$  and  $N = 1 - \rho$ ; it is interpreted as the ratio of the mass of firms to the total mass of mobile agents (firms and households).

## 6 Simple examples: The two-, three-, four- and eight-cell city

We first consider cities with a small number of cells ( $I = 2, 3, 4$  and  $8$ ) to illustrate the utility of potential game approach and the basic workings of the model parameters, in particular,  $\tau$  and  $t$ . It is also observed in concrete terms that the potential function is nonconvex.

For the remainder of the paper, we impose Assumptions 2 and 3. For this case, there is a set of useful predictions regarding the local stability of equilibria.

**Lemma 3.** *Suppose Assumptions 2 and 3. Then, the long-run equilibrium has the following properties.*

- (a) *The uniform distribution is locally stable for all  $(\tau, t)$ .*
- (b) *Any interior equilibrium other than the uniform distribution (if it exists) is locally unstable.*
- (c) *No locally stable equilibrium involves more than one “unbalanced” mixed use cells, i.e., cells with  $m_i > 0$ ,  $\sum_{j \in \mathcal{I}} n_{ij} > 0$ , and  $\phi m_i \neq \sum_{j \in \mathcal{I}} n_{ij}$  so that there is commuting from or toward the cell.*

Observe that Lemma 3 allows one to focus on a specific subset of spatial distribution of firms;  $m_i \in \{0, \rho a, a\}$  for all  $i \in \mathcal{I}$  (except for, possibly, one cell) where we recall  $a = \frac{1}{t}$  is the uniform level of land endowment in each cell and  $\bar{m} = \frac{M}{I} = \rho a$ . The fact greatly simplifies the enumeration of spatial configurations that offer the possibility of stability.

### 6.1 The two-cell city

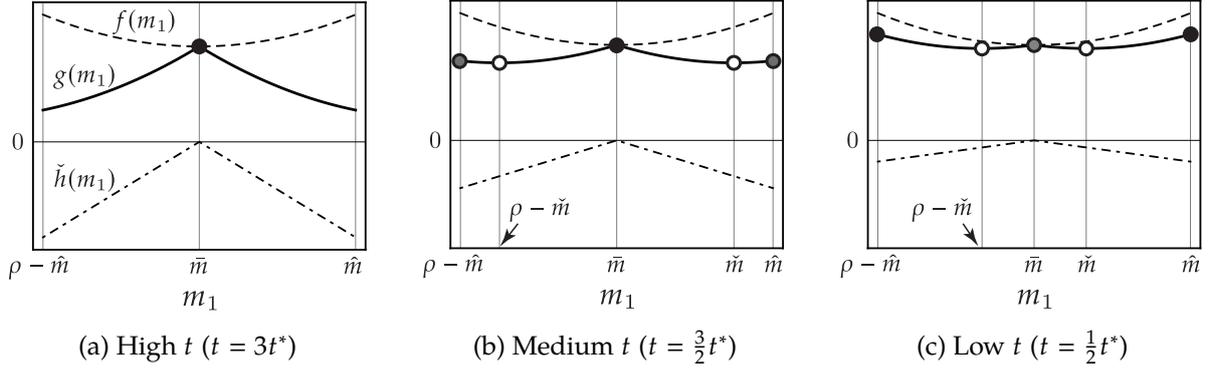
Suppose  $I = 2$ . We take  $m_1$  as the variable as  $m_2 = \rho - m_1$ . To respect that  $0 \leq m_i \leq a$  and  $m_1 + m_2 = \rho$ , we must require that  $m_1 \in [\rho - \hat{m}, \hat{m}]$  where  $\hat{m} = \min\{a, \rho\}$ . Then, solving for the short-run equilibrium,  $\check{h}(\mathbf{m})$  and  $f(\mathbf{m})$  as functions of  $m_1$  are computed as follows:

$$\check{h}(m_1) = \frac{t}{2} \left| \frac{m_1 - \bar{m}}{\rho} \right|, \quad (24)$$

$$f(m_1) = (1 - d)(m_1 - \bar{m})^2 + (1 + d)\bar{m}^2, \quad (25)$$

where  $d \equiv e^{-\frac{\tau}{2}} \in (0, 1)$  is the extent of positive externalities from one cell to the other.

Figure 2 illustrates the graphs of the potential function  $g(m_1) \equiv f(m_1) - \check{h}(m_1)$  as well as  $f(m_1)$  and  $\check{h}(m_1)$  with  $(\rho, d) = (\frac{1}{2}, \frac{1}{2})$  for three different values of  $t$ , namely, high, medium, and low. Observe that  $g$  is neither convex nor concave. Local and global maximizers of  $g$  are indicated by gray and black markers, respectively, whereas white markers indicate local



**Figure 2:** The graphs of  $g(m_1)$  with  $(\rho, d) = (\frac{1}{2}, \frac{1}{2})$  for different values of  $t$ .

minimizers. For the high-commuting-cost case, the uniform distribution is the only local (and global) maximizer. For the medium-commuting-cost case, there are three local maximizers among which the uniform distribution is the global maximizer. For the low-commuting-cost case, there are three local maximizers, where agglomeration toward any one of the cells globally maximize  $g$ .

Theorem 1 implies that the properties of the long-run equilibria are characterized by the maximization problem  $\max_{m_1} g(m_1)$ . Employing the analytic formula for  $g$ , we obtain the following proposition and its corollary.

**Proposition 4.** *Assume that  $I = 2$  and without loss of generality let  $m_1 \geq \bar{m}$ , so that  $m_1 \in \Delta' \equiv [\bar{m}, \hat{m}]$ . The set of the KKT points  $K_2$ , the local maximizer(s)  $L_2$ , and the global maximizer(s)  $G_2$  of the potential function  $g(m_1)$  over  $\Delta'$  are given by the following:*

$$K_2 = \begin{cases} \{\bar{m}\} & \text{if } t > 2t^* \\ \{\bar{m}, \check{m}, \hat{m}\} & \text{if } 0 < t \leq 2t^* \end{cases}, L_2 = \begin{cases} \{\bar{m}\} & \text{if } t > 2t^* \\ \{\bar{m}, \hat{m}\} & \text{if } 0 < t \leq 2t^* \end{cases}, G_2 = \begin{cases} \{\bar{m}\} & \text{if } t > t^* \\ \{\bar{m}, \hat{m}\} & \text{if } t = t^* \\ \{\hat{m}\} & \text{if } 0 < t < t^* \end{cases},$$

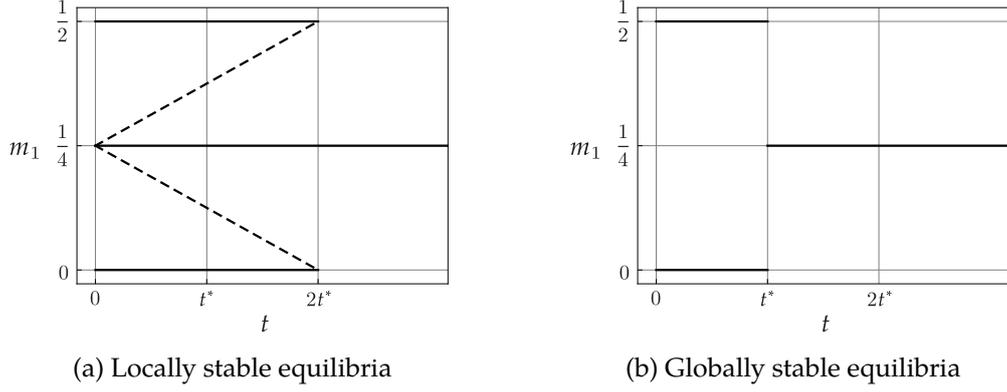
respectively, where  $\check{m} \equiv \bar{m} + \frac{t}{4\rho(1-d)}$ ,  $t^* = \rho^2(1-d)$  if  $0 < \rho \leq a$  and  $t^* = \rho(1-\rho)(1-d)$  if  $a < \rho < 1$ .

**Corollary 3.**  $K_2$ ,  $L_2$ , and  $G_2$  are the set of all possible, locally stable, and globally stable long-run equilibria, respectively, in the sense summarized in Theorem 1.

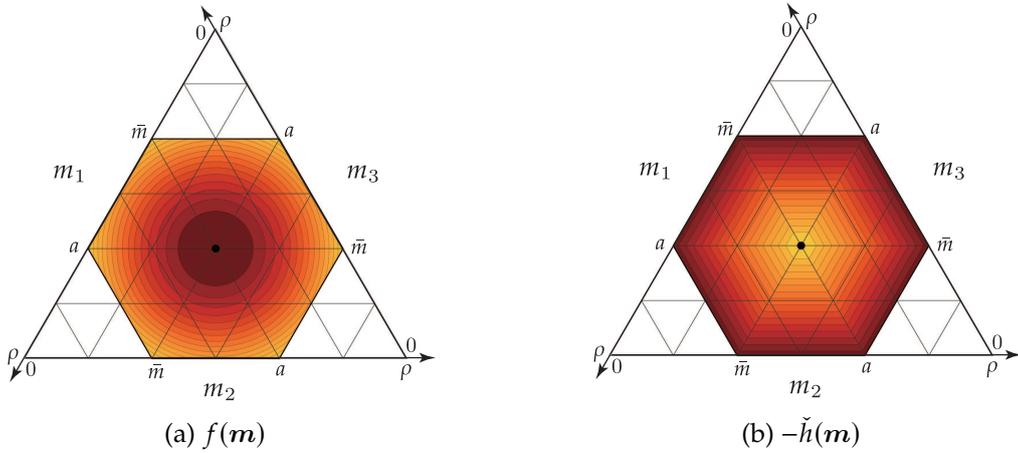
In Proposition 4,  $m_1 = \bar{m}$  is the uniform distribution  $\bar{\mathbf{m}} = (\bar{m}, \bar{m})$ , and  $m_1 = \hat{m}$  agglomeration where one of the cell has a greater mass of firms:  $\mathbf{m} = (\hat{m}, \rho - \hat{m})$ .

Proposition 4 states that the uniform distribution  $\bar{\mathbf{m}}$  is the only possible (locally and globally) stable equilibrium for larger values of  $t$ , whereas agglomeration becomes stable for smaller values of  $t$ . In other words, for any level of  $d$ ,  $\bar{\mathbf{m}}$  should be the only equilibrium if the commuting cost per distance  $t$  is sufficiently high; the costs of commuting overcome the merit of agglomeration. In particular, if  $t > t_0^* \equiv t^*|_{d=0}$ , agglomeration is unstable. On the other hand, provided that  $t < t_0^*$ , agglomeration is stable (unstable) for smaller (larger) values of  $d$  because  $t^*$  is decreasing in  $d$ . If  $d$  is sufficiently large, proximity to other firms matters less.

As Proposition 4 indicates, the (relative) mass of firms  $\rho$  does not affect the qualitative properties of equilibrium *spatial patterns*. To simplify the analysis, we tacitly suppose  $\phi = 1$  for



**Figure 3:** Locally and globally stable equilibria  $t$  ( $I = 2$ ,  $\rho = \frac{1}{2}$ ).



**Figure 4:** Contours of  $f(\mathbf{m})$  and  $\check{h}(\mathbf{m})$  for the three-cell city.

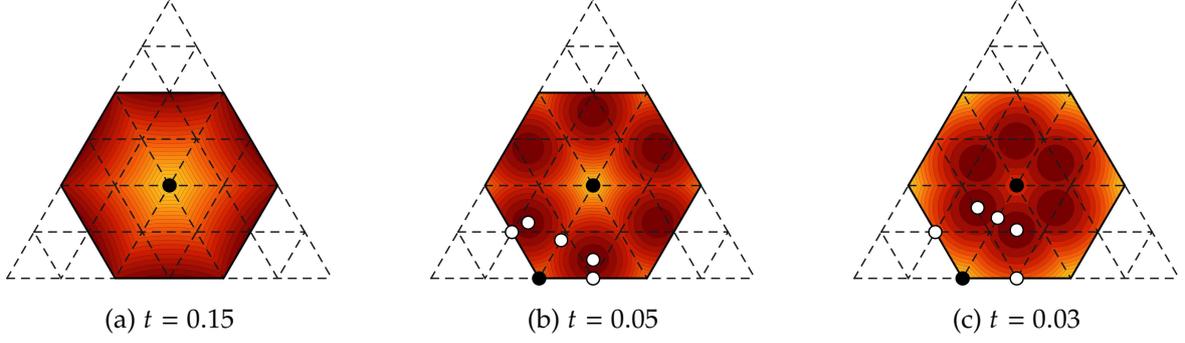
the remainder of this paper; this is also the same parameter setting that was used by [Fujita and Thisse \(2013\)](#).

**Assumption 4.**  $\rho = \frac{1}{2}$ , or equivalently,  $\phi = 1$ .

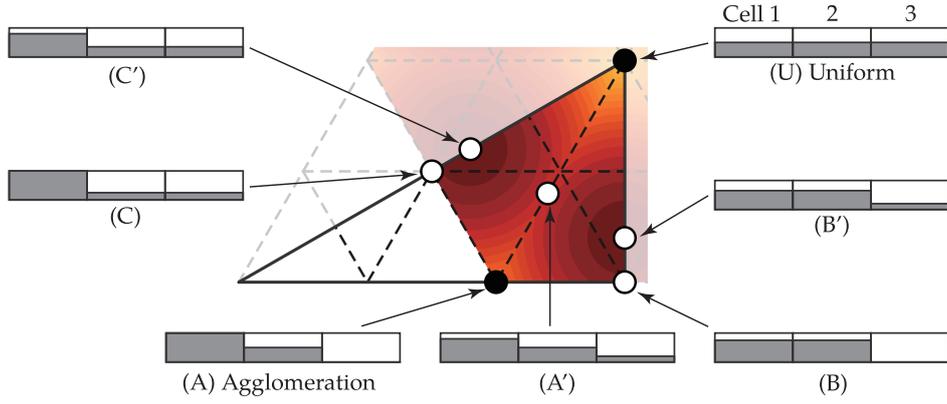
Figure 3a and Figure 3b show the bifurcation diagrams of locally and globally stable equilibria, respectively, in terms of  $m_1$  under Assumption 4 (i.e.,  $\bar{m} = \frac{1}{4}$  and  $\hat{m} = \frac{1}{2}$ ). Observe that agglomeration becomes locally (globally) stable for  $t < 2t^*$  ( $t < t^*$ ). We observe that globally stable equilibria are a subset of locally stable ones; thus Figure 3b may be regarded as a “simplified version” of Figure 3a. Alternatively, if one does not have any *a priori* information regarding locally stable equilibria, Figure 3b provides a sufficient information regarding the effects of changing parameter values. With this observation in mind, we focus on globally stable equilibria for the remainder of this paper.

## 6.2 The three-cell city

We next study the landscape of the potential function  $g(\mathbf{m})$  for  $I = 3$  under Assumption 4. Lemma 3 implies that the uniform pattern  $(\bar{m}, \bar{m}, \bar{m})$  and agglomeration  $(a, \rho - a, 0) = (2\bar{m}, \bar{m}, 0)$  can be stable in the three-cell case, where, without loss of generality, we let  $m_1 \geq m_2 \geq m_3$ .



**Figure 5:** Contours of  $g(m)$  for the three-cell city ( $d = \frac{1}{2}, \rho = \frac{1}{2}$ ).



**Figure 6:** Contours of  $g(m)$  with equilibrium patterns ( $d = \frac{1}{2}, t = 0.05$ ).

Although the simple formulas for  $f(m)$  and  $\check{h}(m)$  are still available for concrete analysis, we focus on graphical intuitions because the basic results are the same with  $I = 2$ . Figure 4 shows the contours of  $f(m)$  and  $-\check{h}(m)$  on  $\Delta$  by the simplex coordinate. In what is similar to the two-cell case,  $f(m)$  and  $\check{h}(m)$  are respectively a parabola and a piecewise affine function, attaining minimum at  $\bar{m}$ .

Figure 5 shows the contours of  $g(m)$  for different values of  $t$  where we let  $(d, \rho) = (\frac{1}{2}, \frac{1}{2})$  where black (white) disks indicate stable (unstable) equilibria, and only those with  $m_1 \geq m_2 \geq m_3$  are drawn for simplicity. Qualitative properties are consistent with the two-cell example. When  $t$  is large,  $\bar{m}$  at the center of simplex is the only maximizer and hence (locally or globally) stable. For lower values of  $t$ , there are a number of KKT points (local maximizers, local minimizers, and saddle points), all of which are equilibria of the model. In particular, the figure illustrates the landscape of the potential function behind Lemma 3.

Figure 6 shows schematic illustrations of the equilibrium patterns associated with the markers in Figure 5b. The gray portion of each cell indicates the mass of firms in the cell. Among equilibrium patterns, only uniform distribution (pattern U) and monocentric agglomeration (pattern A) can be a local maximizer, i.e., a locally stable equilibrium. Patterns C and B are unstable corner solutions. Patterns A', B', and C' are interior equilibria which may be interpreted as the intermediate patterns that connect pattern U to A, B, and C respectively. As we show in Lemma 3, the interior equilibrium patterns A', B', C', B, and C are all unstable.

There is a considerable increase in the (possible) number of equilibrium patterns (from 5 to 25 without symmetry considerations) is detected by merely moving from  $I = 2$  to  $I = 3$ . The majority of equilibria (18 out of 25) are, however, locally unstable and thus less important. This illustrates the fact that stability analysis is crucial for selecting meaningful equilibria within the FO model. To this end, focusing on the global stability of equilibria is an effective remedy. In fact, by comparing the potential values for the dispersion and agglomeration, we obtain the following simple characterization that captures essential insights into the roles of  $\tau$  or  $t$ .

**Proposition 5.** *Suppose  $I = 3$ . Then, the uniform distribution is globally stable if and only if  $t \geq t^*$  and agglomeration is globally stable if and only if  $t \leq t^*$  where  $t^* = \frac{1}{4}(1 - d)$  with  $d \equiv e^{-\frac{\tau}{3}}$ .*

### 6.3 The four- and eight-cell city

The pith and mallow of the FO model is that it can describe the formation of multiple urban centers as an equilibrium outcome. The four-cell city is the minimal set-up where the duo-centric segregated pattern can possibly emerge. Under Assumption 4, in addition to the uniform distribution  $(\bar{m}, \bar{m}, \bar{m}, \bar{m})$ , the possible stable patterns include the monocentric pattern  $(0, 2\bar{m}, 2\bar{m}, 0)$  and the duo-centric pattern  $(2\bar{m}, 0, 2\bar{m}, 0)$ . However, the latter pattern fails to be globally stable. Indeed, from the firms' (and households') perspectives, there is no distinction between monocentric and duo-centric equilibria because, in both cases, households commute to a cell that neighbors their residential cells. The total cost in the short run takes the same value  $t\bar{m}$  for both the monocentric and duo-centric patterns. Therefore, the monocentric pattern is always superior in terms of potential value.

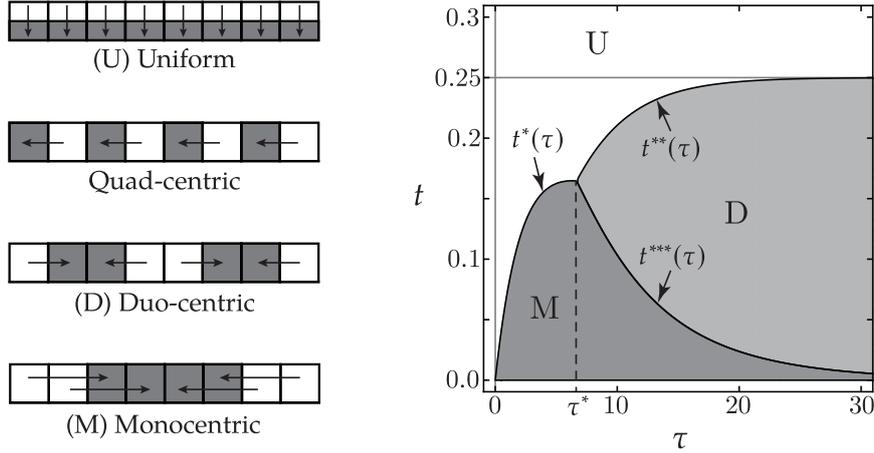
**Proposition 6.** *Suppose  $I = 4$ . Then, the uniform distribution is globally stable if and only if  $t \geq t^*$ , and the monocentric agglomeration is globally stable if and only if  $t \leq t^*$  where  $t^* \equiv \frac{1}{4}(1 - d^2)$  with  $d \equiv e^{-\frac{\tau}{4}}$ . The duo-centric pattern is never globally stable.*

For  $I = 8$ , on the other hand, it is possible that the length of commuting may vary across different spatial configurations. For this case, possible configurations include the uniform distribution (U), the monocentric (M), duo-centric (D), and quad-centric segregated patterns which are illustrated by the schematic diagrams on the left-hand side of Figure 7. In the figures, the arrows on the spatial patterns indicate the short-run equilibrium commuting patterns. The short-run equilibrium commuting costs for the patterns are respectively given by  $0$ ,  $2t\bar{m}$ ,  $t\bar{m}$ , and  $t\bar{m}$ . It implies that the quad-centric pattern cannot be a globally stable equilibrium for the same reason that the two-centric pattern for  $I = 4$  cannot be a globally stable equilibrium.

Employing Lemma 3, one can simply enumerate all spatial configurations that can possibly become locally stable; one only needs to list all  $\mathbf{m}$  such that  $m_i \in \{0, \bar{m}, 2\bar{m}\}$  for all  $i \in \mathcal{I}$  up to symmetry. Then, comparing the potential values for the spatial configurations, one obtains the characterization of globally stable equilibria. The following proposition exhibits the selection results for  $I = 8$ .

**Proposition 7.** *Suppose  $I = 8$ . Then, there exists  $\tau^*$  such that the following conditions are met:*

- (a) *If  $\tau \leq \tau^*$ , there exists  $t^*$  such that (i) the uniform pattern is globally stable for  $t \geq t^*$  and (ii) the monocentric pattern is globally stable for  $t \leq t^*$ .*



**Figure 7:** Globally stable equilibria ( $I = 8$ ).

- (b) If  $\tau > \tau^*$ , there exists  $t^{**}$  and  $t^{***}$  with  $t^{**} > t^{***}$  such that (i) the uniform pattern is globally stable for  $t \geq t^{**}$ , (ii) the duo-centric pattern is globally stable for  $t^{***} \leq t \leq t^{**}$ , and (iii) the monocentric pattern is globally stable for  $t \leq t^{***}$ .

It is far simpler to visualize Proposition 7 in the  $(\tau, t)$ -space. The right-hand side of Figure 7 shows the partition of the  $(\tau, t)$ -space on the basis of the global stability of equilibria. The alphabets (U, M, D) in each region correspond to the spatial patterns on the left-hand side of the figure. When  $t$  is large, there is no possibility of agglomeration; the uniform distribution is globally stable. In what is similar to the two- and three-cell cases, a concentration of firms is possible if  $t$  is sufficiently small so that the merit of proximity overcomes firms' costs. For lower  $\tau$ , the monocentric pattern is chosen. For higher  $\tau$ , one observes a new phenomenon: emergence of duo-centric pattern. When  $\tau$  is sufficiently high, the merit of agglomeration does not sufficiently spill over from one cell to another. While agglomeration is always better for firms than a uniform distribution, the merit of agglomeration does not compensate for larger commuting costs of households incurred by a single business center.

## 7 The continuous city

In this section, we turn our attention to the limiting case  $I \rightarrow \infty$  to approximate the original, continuous-space formulation. To obtain sufficient insights into the properties of the continuous-space model, we restrict our attention to symmetric spatial distribution of firms that can be readily listed, and consider selecting the pattern that maximizes potential value. Throughout Section 7, we suppose that Assumptions 2, 3, and 4 hold true. In particular,  $\rho = \frac{1}{2}$ .

### 7.1 Continuous approximation

We approximate the geography by a continuous circle  $C = [-\frac{1}{2}, \frac{1}{2}]$  where the density of land is unity everywhere. The spatial setup is similar to that employed by Mossay and Picard (2011) or Blanchet et al. (2016). In our case, it is employed as an “approximation” (see Section 2).

The spatial patterns of firms and commuting patterns of households are replaced by non-negative and measurable density functions  $m : C \rightarrow \mathbb{R}_+$  and  $n : C \times C \rightarrow \mathbb{R}_+$ , respectively. The former should be in the following set of bounded measures:

$$\Omega \equiv \left\{ m(x) \geq 0 \mid \int_C m(x) dx = \frac{1}{2}, m(x) \leq 1 \forall x \in C \right\}. \quad (26)$$

$\Omega$  is a continuous analogue of  $\Delta$ . The set of all possible commuting pattern  $n$  of households for given  $m \in \Omega$  is defined by the following:

$$P[m] \equiv \left\{ n(x, y) \geq 0 \mid \begin{array}{l} \int_C n(x, y) dy = 1 - m(x) \quad \forall x \in C \\ \int_C n(x, y) dx = m(y) \quad \forall y \in C \end{array} \right\}. \quad (27)$$

We also let functions  $A : C \rightarrow \mathbb{R}_+$ ,  $\check{r} : C \rightarrow \mathbb{R}_+$ ,  $\check{w} : C \rightarrow \mathbb{R}_+$ , and  $\hat{r} : C \rightarrow \mathbb{R}_+$  be the continuous analogues of the level of production externalities  $A(\mathbf{m}) = (A_i(\mathbf{m}))$ , the short-run land rent  $\check{r} = (\check{r}_i)$ , the short-run wage  $\check{w} = (\check{w}_i)$ , and the long-run additional land rent  $\hat{r} = (\hat{r}_i)$ . In particular,  $A(x)$  is defined by the following:

$$A(x) \equiv \int_C e^{-\tau[x-y]} m(y) dy, \quad (28)$$

where  $[x - y] = \min \{|x - y|, 1 - |x - y|\}$  is the distance between  $x$  and  $y$  along  $C$ . Last, the functions  $g(\mathbf{m})$ ,  $f(\mathbf{m})$ , and  $\check{h}(\mathbf{m})$  are, respectively, replaced by the following functionals:<sup>9</sup>

$$g[m] = f[m] - \check{h}[m], \quad (29)$$

$$f[m] = \frac{1}{2} \iint_{C \times C} e^{-\tau[x-y]} m(x) m(y) dx dy, \quad (30)$$

$$\check{h}[m] = \min_{n \in P[m]} \left\{ t \iint_{C \times C} [x - y] n(x, y) dx dy \right\}. \quad (31)$$

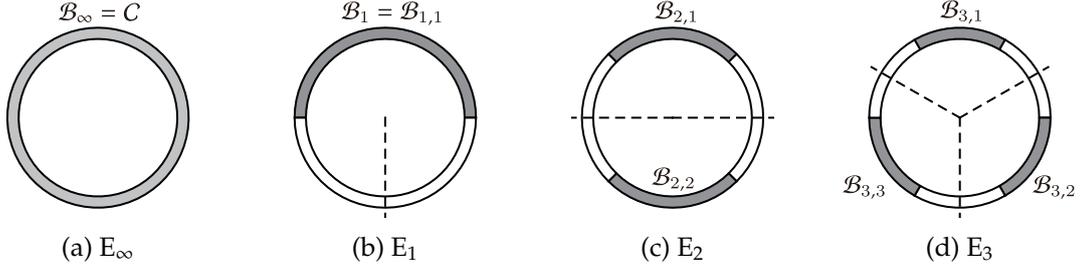
## 7.2 Symmetric patterns

Our aim is to obtain a classification on the  $(\tau, t)$ -space that generalizes Figure 7 to provide a meaningful outline of the global phenomena over the parameter space. To this end, we restrict our attention to a collection of symmetric spatial distributions of firms; Noting that Lemma 3 suggests that there is no room for asymmetric patterns, in which firm density takes nontrivial values other than 0,  $\frac{1}{2}$ , and 1, to become stable, we define the *symmetric patterns*:

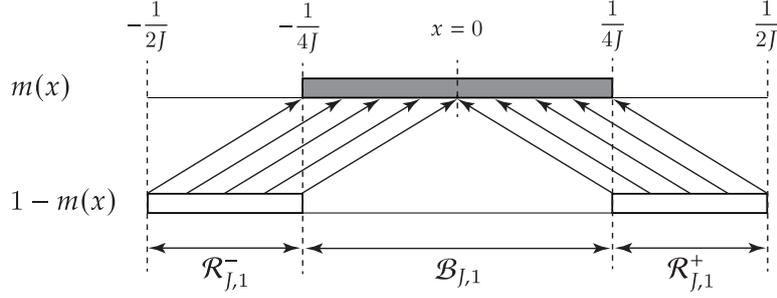
**Definition 5** (Symmetric patterns). The *symmetric patterns*  $\mathcal{E} \equiv \{E_j\}_{j=1}^\infty$  are defined as follows:

- (a) The *integrated pattern*  $E_\infty$ :  $m(x) = \frac{1}{2}$  for all  $x \in \mathcal{B}_\infty = C$ .
- (b) The *segregated patterns*  $E_j$ :  $m(x) = 1$  for all  $x \in \mathcal{B}_j \subset C$  and  $m(x) = 0$  otherwise, where  $\mathcal{B}_j \equiv \bigcup_{i=1}^j \mathcal{B}_{j,i}$  with  $\{\mathcal{B}_{j,i}\}_{i=1}^j$  are intervals of length  $\frac{1}{2j}$  evenly spread over  $C$ .

<sup>9</sup>Observe that  $\check{h}[m]$  is the optimal transport cost functional from source measure  $m$  to target measure  $1 - m$  in the case of distance cost function (see Santambrogio, 2015, Chapter 3).



**Figure 8:** Symmetric patterns  $E_\infty$ ,  $E_1$ ,  $E_2$ , and  $E_3$ .



**Figure 9:** Business district, residential districts, and an equilibrium commuting pattern.

The integrated pattern  $E_\infty$  is uniform distribution in which no firm concentration is present; the business and residential areas are all integrated (Figure 8a). It is an autarky equilibrium in which the market interaction, as well as commuting, are closed at each point in  $C$ . Another interpretation is that there is a continuum of business centers.

The segregated patterns  $E_J$  for finite  $J$  correspond to the formation of business centers, where  $J$  represents the number of business centers in the pattern. Figures 8b, 8c, and 8d illustrate  $E_1$ ,  $E_2$ , and  $E_3$ , respectively. For  $E_2$  and  $E_3$ , there are multiple, disjointed business centers that are solely occupied by firms.<sup>10</sup>

In the short-run equilibrium for the  $J$ -centric segregated pattern, the city is partitioned into a collection of small and identical “zones” in autarky; each business center is associated with two symmetric residential districts. Each business area  $\mathcal{B}_{J,j}$  is surrounded by a couple of residential areas  $\mathcal{R}_{J,j}^-$  and  $\mathcal{R}_{J,j}^+$  from which its workers are supplied. Figure 9 illustrates  $\mathcal{B}_{J,1}$  and its associated residential zones where  $1 - m(x) = \int_C n(x, y) dy$  is the residential density of households. The short-run equilibrium, or the lower-level problem (31), is a standard instance of *optimal transport problems* (see, e.g., Santambrogio, 2015) and analytic solutions are readily available. The short-run equilibrium commuting patterns according to the theory of optimal transport are schematically drawn by arrows;<sup>11</sup> there is no incentive for households (other than those on the boundaries) to commute to different business centers. The minimized total commuting cost, or the short-run equilibrium commuting cost, for each zone is simply computed as  $\frac{t}{8J^2}$ . The analytic expressions for the short-run land rent  $\check{r}$  and wages  $\check{w}$  are also

<sup>10</sup>We note that one may suppose  $l = 2J_0!$  for any finite  $J_0$  so that every symmetric spatial distributions  $E_J \in \mathcal{E}_0 \equiv \{E_J\}_{J=1}^{J_0} \subset \mathcal{E}$  is in fact associated with a segregated spatial distribution over the finite cells.

<sup>11</sup>In fact, it is a short-run equilibrium commuting pattern because for the linear commuting cost the solution of (31) is not unique (see Santambrogio, 2015, Chapter 3).

available, which we relegate to Appendix A since the results are standard.<sup>12</sup>

The symmetric patterns are obviously natural candidates for long-run equilibria. In fact, the following long-run equilibrium condition for each  $E_J$  can be derived.

**Lemma 4.** *Suppose that  $A$  is computed for  $E_J$  where we take the origin  $x = 0$  at the middle of one of the business centers  $\{\mathcal{B}_{j,j}\}_{j=1}^J$ . Let  $A_J(x) = A(x) - A_\infty$  where  $A_\infty \equiv \frac{1}{\tau} (1 - e^{-\frac{\tau}{2}})$  is the uniform level of externalities for the integrated pattern  $E_\infty$ . Then,  $E_J$  is a long-run spatial equilibrium if and only if  $A_J(0) - \frac{t}{2J} \geq 0$ .*

### 7.3 Potential values for the symmetric patterns

We have to derive the values of potential functional  $g[m]$  for the symmetric patterns to select the spatial density profile that maximizes potential function among the symmetric patterns  $\mathcal{E}$ . Let  $f_J$  and  $\check{h}_J$  be the value of  $f[m]$  and  $\check{h}[m]$  for each  $E_J \in \mathcal{E}$ . A direct computation gives the following lemma.

**Lemma 5.** *Let  $g_J$  be the potential value associated for  $E_J \in \mathcal{E}$ .*

(i) *For the integrated pattern,  $g_\infty = f_\infty - \check{h}_\infty$  where  $f_\infty = \frac{1}{4\tau} (1 - e^{-\frac{\tau}{2}})$  and  $\check{h}_\infty = 0$ .*

(ii) *For the segregated patterns,  $g_J = f_J - \check{h}_J$  with the following:*

$$f_J = (1 + \Psi_J) f_\infty \quad \text{and} \quad \check{h}_J = \frac{t}{8J}, \quad (32)$$

where  $\Psi_J \in (0, 1)$ , in fact,  $\Psi_J \equiv \left(1 + \frac{2C(d_J)}{\log(d_J)}\right) \delta_J$  with  $d_J \equiv e^{-\frac{\tau}{2J}}$ ,  $C(z) \equiv \frac{1-z}{1+z}$ , and  $\delta_J \equiv 1$  for an even  $J$  and  $\delta_J \equiv C(e^{-\frac{\tau}{2}})^{-1}$  for an odd  $J$ .

The second term  $\check{h}_J$  is the (minimized) total commuting cost. Evidently and intuitively,  $\check{h}_J$  decreases in the number of business centers  $J$  as the commuting distance decreases with the diminishing of the size of each business center.

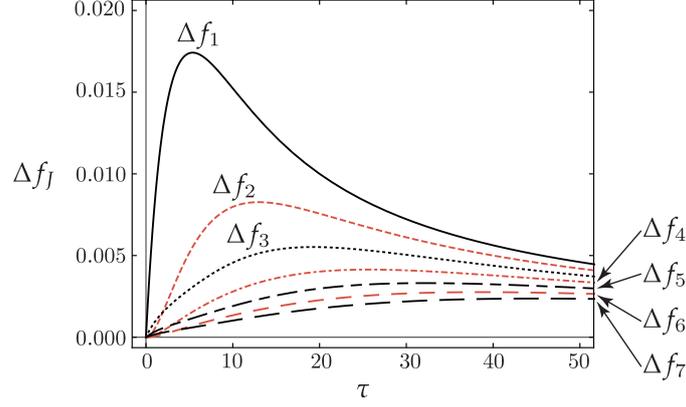
The first term  $f_J$  naturally decreases in the distance decay parameter  $\tau$ . Reflecting the fact that firms prefer concentration,  $f_J$  is always greater than  $f_\infty$  and is (basically) decreasing in the number of business centers  $J$ . Since  $J$ -centric pattern approximates the integrated pattern  $E_\infty$  when  $J \rightarrow \infty$ , the potential values also coincides in the limit.

Figure 10 depicts the relative magnitude  $\Delta f_J \equiv f_J - f_\infty$  for  $J = 1, 2, \dots, 7$ .<sup>13</sup> We observe that  $\Delta f_J$ , and hence  $f_J$ , basically decrease in  $J$ . In particular, one can see that  $f_1$  takes the maximal value for all  $\tau$ . Therefore, if the commuting costs of households are absent so that  $g_J = f_J$ , the monocentric segregated pattern  $E_1$  always maximizes  $g$ .

We clearly observe that the potential values  $\{g_J\}$  encapsulate the trade-off between the merit and demerit of forming multiple business centers. The first term  $f_J$  decreases in the number of business centers,  $J$ , whereas the second one  $-\check{h}_J$  increases in  $J$ . Thus, neither of larger  $J$  nor smaller  $J$  is preferred by maximization across  $\{g_J\}$ . Instead, the ordering of  $g_J = f_J - \check{h}_J$  in  $J$  depends on  $\tau$  and  $t$ .

<sup>12</sup>See the proof of Lemma 4.

<sup>13</sup>Figure 13 in Appendix A shows the graphs of  $\{\Psi_J\}$  and  $\{f_J\}$ , as well as  $\{\Delta f_J\}$ , for interested readers.



**Figure 10:** Graphs of  $\Delta f_J$  for  $J \in \{1, 2, \dots, 7\}$ .

We note that there is a nice sufficient condition that connects the potential values and long-run equilibrium conditions for the symmetric patterns (For interested readers, Figure 14 in Appendix A illustrates the sufficiency on the  $(\tau, t)$ -space).

**Lemma 6.** *Suppose  $g_J \geq g_\infty$ . Then,  $E_J$  is a long-run spatial equilibrium.*

#### 7.4 Potential maximizing patterns

Given a pair of structural parameters  $(\tau, t)$ , one can single out the most meaningful spatial pattern by selecting  $J^* = \arg \max_J \{g_J\}$ . Moreover, Lemma 6 ensures that the spatial pattern selected by the procedure is in fact a long-run equilibrium, as one obviously has  $g_J \geq g_\infty$  if  $g_J = \max_J \{g_J\}$ .

We first investigate the global stability of  $E_\infty$ . For each  $J$ , we have  $g_J \leq g_\infty$  for all  $t \geq t_J^*(\tau)$  where  $t_J^*(\tau)$  is the unique solution of  $g_J = f_J - \check{h}_J = f_J - \frac{t}{8J} = f_\infty = g_\infty$  in terms of  $t$ , i.e.,  $t_J^*(\tau) \equiv 8J\Delta f_J(\tau)$ . If  $t > t_J^*(\tau)$ , then  $E_J$  can never maximize the potential  $g[m]$ . Let  $t^*(\tau)$  be defined as the following:

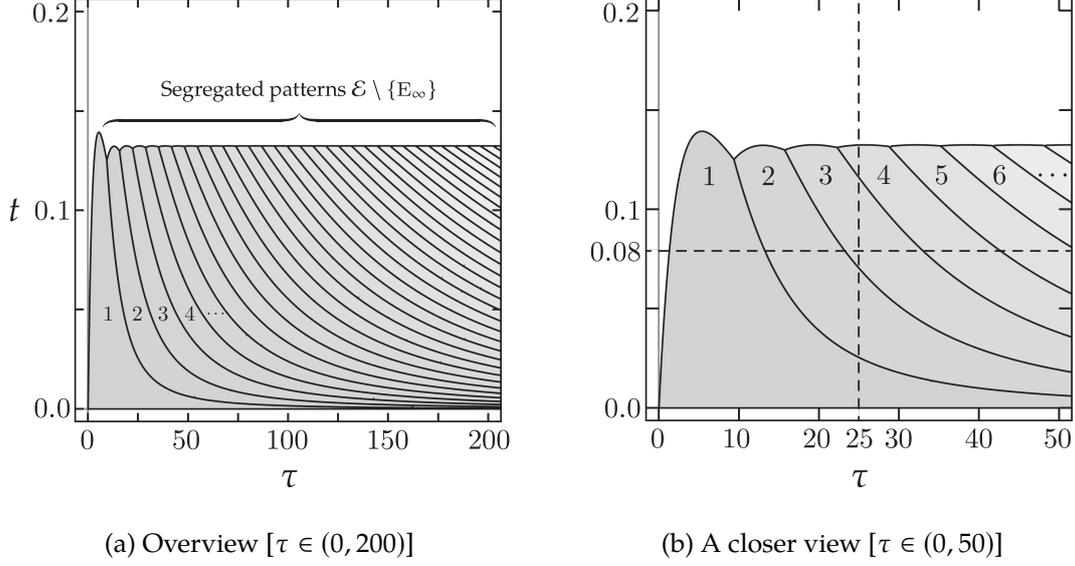
$$t^*(\tau) \equiv \max_J t_J^*(\tau). \quad (33)$$

When  $t \geq t^*(\tau)$ , no segregated patterns can dominate the integrated pattern  $E_\infty$ . In other words, the integrated pattern is selected if the commuting costs of households are sufficiently high, which is consistent with the results found in Section 6 as well as intuition.

**Lemma 7.** *Suppose  $t > t^*(\tau)$ . Then,  $E_\infty$  maximizes  $g$  among  $\mathcal{E}$ .*

For the global stability of the segregated patterns, unfortunately, analytical predictions are limited because of the complicated functional form of  $f_J$  in  $\tau$  and  $J$ . It is not simple to compare the potential values (32) between different  $J$ s (although it might seem so at first glance). We, therefore, look directly into the partition of the  $(\tau, t)$ -space obtained by numerically comparing the potential values  $\{g_J\}$ .

Figure 11 is the resultant partition of the  $(\tau, t)$ -space. The gray regions are where the segregated patterns ( $E_J$  for finite  $J$ ) dominate the integrated pattern  $E_\infty$ , while in the white



**Figure 11:** Potential maximizing patterns (the numbers indicate  $\arg \max_J \{g_J(\tau, t)\}$ ).

region the opposite (i.e.,  $t > t^*(\tau)$ ). Each gray region corresponds to one of the  $J$ -centric patterns, whereas the number on each region corresponds to that of business centers. On the  $\tau$ -axis, gray regions are aligned in the increasing order of  $J$ . Also observe that for larger values of  $\tau$ , there appears to be a threshold value ( $t \approx 0.132$ ) of the commuting cost parameter below which segregated patterns emerge.

## 7.5 Comparative statics

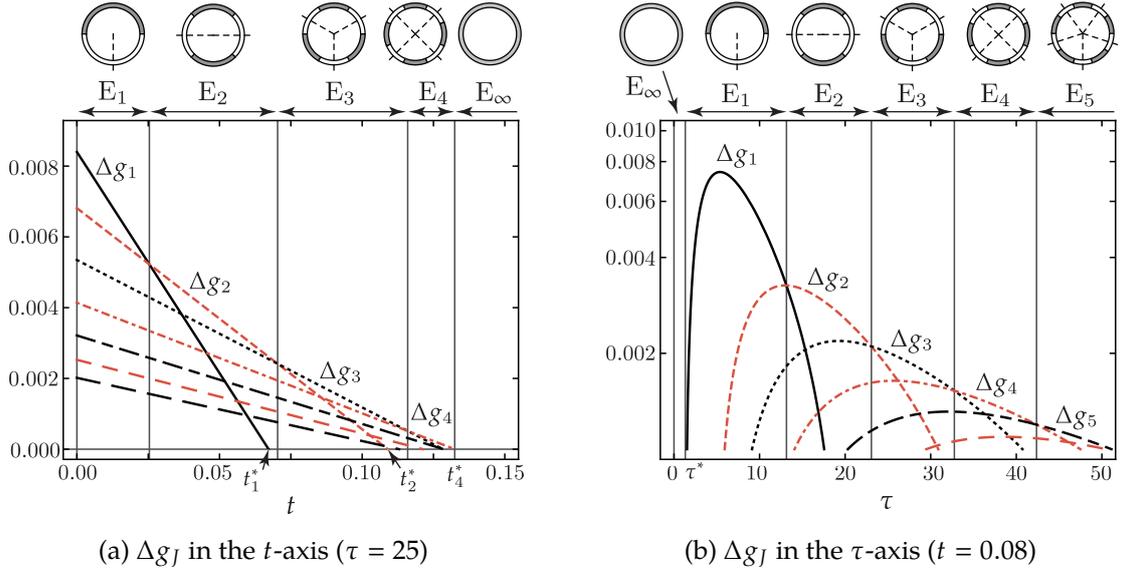
There are two basic implications of Figure 11. One is the effect of the commuting cost parameter  $t$ , and the other is the effect of the distance decay parameter  $\tau$ . First, as  $t$  decreases from larger extremes, one first observes the formation of multiple business centers and then a reduction of their number.

**Observation 1.** Fix  $\tau$  and let  $J^* = \arg \max_J \{t_J^*(\tau)\}$ . Suppose that initially  $t > t^*(\tau)$  and consider a monotonic decrease in  $t$ . Then, the sequence of potential maximizing patterns in  $\mathcal{E}$  is:  $E_\infty \rightarrow E_{J^*} \rightarrow E_{J^*-1} \rightarrow \dots \rightarrow E_2 \rightarrow E_1$ .

Figure 12a illustrates the process for  $\tau = 25$ , which corresponds to the vertical dashed line in Figure 13c. In the figure, the relative potential values  $\Delta g_J \equiv g_J - g_\infty$  are drawn for  $J \leq 7$ . We have  $J^* = 4$  for this case. The upper envelope of the curves corresponds to the global maximizer of the potential functional. For  $t > t^*(25) = t_4^*(25) \approx 0.132$ , we have  $\Delta g_J < 0$  for all segregated patterns and  $E_\infty$  maximizes the potential function. When  $t$  decreases and cuts  $t^*(25)$ ,  $E_4$  emerges as the potential maximizer in  $\mathcal{E}$ . As  $t$  decreases further, the potential maximizer sequentially switches as  $E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1$ .

Next, we consider the process of increasing  $\tau$  under a fixed  $t$ .

**Observation 2.** Fix a sufficiently small  $t$ . Suppose that initially  $\tau \approx 0$  and consider a monotonic increase in  $\tau$ . Then, the sequence of potential maximizing patterns in  $\mathcal{E}$  is:  $E_\infty \rightarrow E_1 \rightarrow E_2 \rightarrow \dots \xrightarrow{\text{limit}} E_\infty$ .



**Figure 12:** Relative potential values  $\Delta g_J \equiv g_J - g_\infty$  and potential maximizing patterns.

Figure 12b illustrates the process for  $t = 0.08$ , which corresponds to the horizontal dashed line in Figure 13c. In that figure, in keeping with Figure 12a, the relative potential values  $\Delta g_J$  are drawn for  $J \leq 7$ . On this line,  $E_\infty$  has the maximal value of potential for small  $\tau$  ( $\tau < \tau^* \approx 1.32$ ) and in the limit  $\tau \rightarrow \infty$ . A steady increase in  $\tau$  induces a repetitive emergence of segregated patterns with increasing  $J$  as the potential maximizer in  $\mathcal{E}$ .

The potential maximizing pattern in the limit  $\tau \rightarrow \infty$  is interpreted to be either the integrated pattern  $E_\infty$  or a *segregated pattern with infinite business centers*. As Lemma 8 indicates, the potential value for the latter coincides with that of  $E_\infty$ . The two extremes of  $\tau$  should be interpreted as having distinct natures: firms at location  $x \in C$  experience positive externalities from other locations  $y \in C$  for small  $\tau$ , whereas they *never* experience it when  $\tau$  approaches infinity. For small  $\tau$ ,  $E_\infty$  is globally stable because the additional merit of agglomeration does not overcome its demerits (i.e., larger commuting costs) of it because positive externalities are already at a sufficiently high level even in dispersion  $E_\infty$ . For  $\tau \rightarrow \infty$ ,  $E_\infty$  is globally stable because there is *no merit* in forming business areas since there is no spatial spillover.

**Remark 2.** Observe that, employing Figure 11, one can infer the partition of the  $(\tau, t)$ -space on the basis of the *social optimum*. By scaling the vertical axis by  $t \mapsto 2t$ , Figure 2 can be read as the classification of welfare maximizing patterns among  $\mathcal{E}$ . To this end, note that the social welfare functional is  $s[m] = 2f[m] - \check{h}[m]$ . For our specification (Assumption 2),

$$g[m] = f[m] - \check{h}[m] = \frac{1}{2} \left( 2f[m] - 2t \frac{\check{h}[m]}{t} \right) = \frac{1}{2} s[m] \Big|_{t=2t}. \quad (34)$$

Thus, if  $E_J$  maximizes the potential functional  $g[m]$  for the parameter pair  $(\tau, t)$ , it maximizes the social welfare functional  $s[m]$  for  $(\tau, 2t)$ . This shows that agglomeration is inadequate in globally stable patterns.

We also note that the symmetric spatial configurations cannot be simply Pareto-ranked

according to the number of business centers; the Pareto ranking depends on the level of  $\tau$  or  $t$ . These findings contrast to the results of [Mossay and Picard \(2011\)](#), which indicate that the formation of a single city is socially optimal in a circular economy. This stems from the fact that our model considers two types of mobile agents, while theirs a single type.

## 8 Concluding remarks

This paper introduces the potential game approach to analyze a canonical urban spatial model for the formation of multiple business centers proposed by [Fujita and Ogawa \(1982\)](#). The model has been known for its intractability, and for the resulting lack of stability analysis of equilibrium patterns. Imposing a circular geography, [Section 6](#) and [Section 7](#) show that the FO model in fact engenders *globally stable* multi-centric patterns. For a continuous circle, selection that is based on global stability implies the emergence of the multistage (dispersion  $\rightarrow$  agglomeration  $\rightarrow$  dispersion) behavior, which is analogous to a “bell-shaped curve” of economic agglomeration in a two-region world ([Fujita and Thisse, 2013](#), Chapter 8). Our results not only provide a concrete answer to the longstanding stability issue of equilibria of the FO model but also demonstrate the effectiveness of the potential game approach in the context of spatial economic models.

There remain several venues for follow-up studies. First, relaxing the simplifying assumptions of the original FO model would be important in providing implications for economic policies and for a robustness check of the results in the present paper. For example, following the original model, we supposed fixed demand for land, which should be relaxed. Toward this goal, the potential game approach is efficacious for any extensions that preserve the symmetry of externalities. For instance, [Negishi’s theorem \(Negishi, 1960\)](#) implies that any competitive assumptions may be reduced to a maximization problem of a specific kind of welfare function, whose optimal value in turn acts as the potential function of the associated game where the mass of agents in each cell is taken as a variable. Also, any externality that acts in each cell does not break the existence condition for the potential function. An example is local amenities that reduce their quality as the mass of agents in the cell grows.

Second, for the specific formulation of FO, the optimization representation  $[\mathcal{P}]$  (or  $[\mathcal{P}_0]$ ) can be employed to study various assumptions on the geographical distances  $\{\ell_{ij}\}$  between cells for the structure of transportation network of the city. It allows one to theoretically investigate specific idealized but important assumptions pertaining to geographies, e.g., a line segment as in the original FO paper, or various stylized but interesting geographies as in [Matsuyama \(2017\)](#). Although one may have to resort to numerical investigations of  $[\mathcal{P}]$  because of asymmetries, the analysis boils down to studying an optimization problem; it is far simpler and more efficient than directly solving the equilibrium conditions.

## A Proofs

*Proof of Proposition 1.* Let  $\langle \mathbf{x}, \mathbf{y} \rangle \equiv \mathbf{x}^\top \mathbf{y} = \sum_{i \in I} x_i y_i$ . With multipliers  $\mathbf{w} = (w_i) \in \mathbb{R}_+^I$ ,  $\mathbf{r} = (r_i) \in \mathbb{R}_+^I$ ,  $v^*$ , and  $\pi^*$ , define the Lagrangian  $L$  for the problem  $[\mathcal{P}_0]$  by

$$L = -f(\mathbf{m}) + h(\mathbf{n}) + \langle \mathbf{r}, -\mathbf{a} + \mathbf{m} + (\mathbf{I} \otimes \mathbf{1}_I^\top) \mathbf{n} \rangle + \langle \mathbf{w}, \phi \mathbf{m} - (\mathbf{1}_I^\top \otimes \mathbf{I}) \mathbf{n} \rangle \\ + \langle v^*, \mathbf{1}_{I^2} \cdot \mathbf{n} - N \rangle + \langle \pi^*, \mathbf{1}_I \cdot \mathbf{m} - M \rangle,$$

where  $\mathbf{I}$  is  $I$ -dimensional identity matrix and  $\mathbf{1}_K$  denotes  $K$ -dimensional all-one vector. The first-order optimality condition with respect to the Lagrangian  $L$  coincides with the equilibrium conditions listed in Definition 1.  $\square$

*Proof of Lemma 1.* The convexity of  $f(\mathbf{m})$  on  $\Delta$  follows from the facts that  $\mathbf{D} = [d_{ij}]$  is non-negative, symmetric and positive definite on the tangent space  $\Delta$  (Assumption 1) and that  $f(\mathbf{m}) = \frac{1}{2} \mathbf{m}^\top \mathbf{D} \mathbf{m}$ . See Theorem 4.4.6 of Bapat and Raghavan (1997).  $\square$

*Proof for Proposition 2.* Define the Lagrangian  $L$  for the problem  $[\mathcal{S}]$  by

$$L = h(\mathbf{n}) + \langle \check{\mathbf{r}}, -\mathbf{a} + \mathbf{m} + (\mathbf{I} \otimes \mathbf{1}_I^\top) \mathbf{n} \rangle + \langle \check{\mathbf{w}}, \phi \mathbf{m} - (\mathbf{1}_I^\top \otimes \mathbf{I}) \mathbf{n} \rangle.$$

Then, the first-order optimality condition with respect to  $L$  coincides with the short-run equilibrium conditions listed in Definition 2. The Lagrangian dual problem  $[\mathcal{D}]$  is obtained by taking  $\inf_{\mathbf{n} \geq \mathbf{0}} L$ .  $\square$

*Proof for Proposition 3.* Define the Lagrangian  $L$  for the problem  $[\mathcal{P}]$  by

$$L = -f(\mathbf{m}) + \check{h}(\mathbf{m}) + \langle \check{\mathbf{r}}, -\mathbf{a} + \mathbf{m} \rangle + \langle \pi^*, \mathbf{1} \cdot \mathbf{m} - M \rangle.$$

Then, the first-order optimality condition with respect to  $L$  coincides with the long-run equilibrium conditions listed in Definition 3.  $\square$

*Proof of Lemma 2.* Define  $\mathcal{O} \equiv \{\epsilon \in T\Delta \mid \max_{i \in I} \{\epsilon_i\} \leq a - (1 + \phi)\bar{m}\} \subset T\Delta$ . Then, for all  $\epsilon \in \mathcal{O}$  we have  $\check{h}(\bar{\mathbf{m}} + \epsilon) = 0$ , so that  $g(\mathbf{m}) = f(\mathbf{m})$  on  $\mathcal{O}$ . Recall that  $f(\mathbf{m})$  is strictly convex in  $\mathbf{m}$ . In particular, the parabola is centered at  $\bar{\mathbf{m}}$  because, noting that the row-sum of  $\mathbf{D}$  is the same value for all row under Assumption 2,  $f(\mathbf{m})$  is rewritten as

$$f(\mathbf{m}) = \frac{1}{2} (\mathbf{m} - \bar{\mathbf{m}})^\top \mathbf{D} (\mathbf{m} - \bar{\mathbf{m}}) + \frac{1}{2} \bar{\mathbf{m}}^\top \mathbf{D} \bar{\mathbf{m}} = f(\mathbf{m} - \bar{\mathbf{m}}) + f(\bar{\mathbf{m}}).$$

Employing the above relation, for  $\epsilon \in T\Delta$  let  $\tilde{g}(\epsilon) \equiv g(\bar{\mathbf{m}} + \epsilon) = f(\epsilon) + f(\bar{\mathbf{m}})$  be the potential function defined on  $T\Delta$ . It follows that  $\nabla \tilde{g}(\mathbf{0}) = \mathbf{0}$  and that  $\nabla^2 \tilde{g}(\epsilon) = \mathbf{D}$  for all  $\epsilon \in \mathcal{O}$ . meaning that  $\epsilon = \mathbf{0}$  is a local minimizer of  $\tilde{g}$  whence  $\bar{\mathbf{m}}$  is a local minimizer of  $g$ . It implies that  $\bar{\mathbf{m}}$  is an unstable equilibrium.  $\square$

*Proof of Lemma 3.* (a)  $\bar{\mathbf{m}}$  is a local minimizer of  $\check{h}$  with  $\check{h}(\bar{\mathbf{m}}) = 0$ . Evidently, we have  $\check{h}(\bar{\mathbf{m}} + \epsilon) > 0$  for any nonzero  $\epsilon \in T\Delta$ . Noting that  $\check{h}(\mathbf{m})$  is a piecewise affine function,  $\bar{\mathbf{m}}$  is an isolated local minimizer of  $\check{h}$  for any  $t > 0$ . Similarly, for any  $\epsilon \in T\Delta$ , letting  $\tilde{f}(\epsilon) \equiv f(\mathbf{m}) = f(\bar{\mathbf{m}} + \epsilon) = f(\epsilon) + f(\bar{\mathbf{m}})$ , we have  $\nabla \tilde{f}(\mathbf{0}) = \mathbf{0}$  (cf., Proof of Lemma 2 above). It follows that  $\bar{\mathbf{m}}$  is a local maximizer of  $g(\mathbf{m}) = f(\mathbf{m}) - \check{h}(\mathbf{m})$  for any finite  $t > 0$ , thereby showing the claim.

(b) Suppose that  $\mathbf{m} \neq \bar{\mathbf{m}}$  is an interior equilibrium. Then, by definition there exists  $\pi$  and  $\mathbf{c} = (c_i)$  such that  $\mathbf{c} = \mathbf{D}\mathbf{m} - \pi \mathbf{1} \geq \mathbf{0}$  and  $g(\mathbf{m}) = f(\mathbf{m}) - \mathbf{c}^\top (\mathbf{m} - \bar{\mathbf{m}})$ , where  $\mathbf{c} > \mathbf{0}$  iff  $\mathbf{m} = \bar{\mathbf{m}}$ . Suppose that  $\mathbf{m} \neq \bar{\mathbf{m}}$  and take  $\mathbf{z} \equiv \epsilon(\mathbf{m} - \bar{\mathbf{m}}) \in T\Delta \setminus \{\mathbf{0}\}$  with some  $\epsilon > 0$ . Then,

$$g(\mathbf{m} + \mathbf{z}) = \frac{1}{2} (\mathbf{m} + \mathbf{z})^\top \mathbf{D} (\mathbf{m} + \mathbf{z}) - \mathbf{c}^\top (\mathbf{m} - \bar{\mathbf{m}} + \mathbf{z})$$

$$\begin{aligned}
&= \frac{1}{2} \mathbf{m}^\top \mathbf{D} \mathbf{m} - \mathbf{c}^\top (\mathbf{m} - \bar{\mathbf{m}}) + \mathbf{z}^\top \mathbf{D} (\mathbf{m} + \mathbf{z}) - \mathbf{c}^\top \mathbf{z} \\
&= g(\mathbf{m}) + \mathbf{z}^\top \mathbf{D} (\mathbf{m} + \mathbf{z}) - \mathbf{z}^\top (\mathbf{D} \mathbf{m} - \pi \mathbf{1}) = g(\mathbf{m}) + \mathbf{z}^\top \mathbf{D} \mathbf{z} > g(\mathbf{m}),
\end{aligned}$$

where we note that  $\mathbf{z}^\top \mathbf{1} = 0$  and thus Assumption 1 (ii) implies  $\mathbf{z}^\top \mathbf{D} \mathbf{z} > 0$  because  $\mathbf{z} \neq \mathbf{0}$ . It shows that any interior equilibrium  $\mathbf{m}$  other than  $\bar{\mathbf{m}}$  is a local minimizer of  $g$  on the subspace spanned by  $\mathbf{z} = \mathbf{m} - \bar{\mathbf{m}}$ ; therefore  $\mathbf{m}$  is either a local minimizer or a saddle point of  $g$ , hence unstable.

(c) Suppose that  $\mathbf{m}$  is a boundary equilibrium with at least two *unbalanced* mixed-use cells  $j$  and  $k$ . Because  $\mathbf{m}$  is an equilibrium, there exist  $\{c_i\}$  and  $\pi$  such that  $c_i = \sum_j d_{ij} m_j - \pi$  for all  $i$  with  $m_i > 0$ . Then, following the same procedure as in (a) with  $\mathbf{z} \equiv \epsilon(e_j - e_k) \in T\Delta$  with  $e_i$  being the  $i$ th standard basis, one shows that  $\mathbf{m}$  is a local minimizer along the extreme line spanned by  $\mathbf{z}$ , thereby showing the claim.

For an equilibrium with more than two *balanced* mixed-use cells, simple extension of this approach fails because the short-run commuting pattern alters for after arbitrarily small deviation so that the cost pattern  $\mathbf{c}$  at the equilibrium does not contain information after a deviation.  $\square$

*Proof of Proposition 4.* We consider  $m_1 = m \in \Delta' = [\bar{m}, \hat{m}]$ . Then,

$$g'(m) = 2(1-d)(m - \bar{m}) - \frac{t}{2\rho}, \quad g'' = 2(1-d) > 0 \quad m \in \Delta' \setminus \{\bar{m}\}, \quad (35)$$

where we note that at the left boundary  $g$  is not differentiable. Because  $g$  is strictly convex over  $\Delta'$ , there is at most two local or global maximizers. For any values of  $(t, d)$ ,  $\check{m} \equiv \bar{m} + \frac{t}{4\rho(1-d)} > \bar{m}$  is the solution for  $g'(m) = 0$ . If  $\check{m} \in \Delta'$ ,  $\check{m}$  it is a KKT point for the problem  $\min_{m \in \Delta'} g(m)$  and is a local minimizer.  $g$  is decreasing for all  $(\bar{m}, \check{m}]$  whereby  $\bar{m}$  is always a local maximizer. If  $g'(\hat{m}) \geq 0$ ,  $\hat{m}$  is a local maximizer.  $g'(\hat{m}) \geq 0$  follows iff  $0 < \rho \leq \frac{1}{2}$  and  $t \leq 2\rho^2(1-d)$ , or  $\frac{1}{2} < \rho < 1$  and  $t < 2\rho(1-\rho)(1-d)$ . The set of possible global maximizers are  $\{\bar{m}, \hat{m}\}$ . Comparing the potential values for the two candidates, the condition for global maximizers are obtained.  $\square$

*Proof of Proposition 5.* The claim follows by simply noting that  $\check{h}(\mathbf{m}) = \sum_{i=1}^3 \max\{0, a - (1+\phi)m_i\}$  and that  $g(\mathbf{m}^*) - g(\bar{\mathbf{m}}) = \frac{1}{12}(1-d) - \frac{t}{3}$ .  $\square$

*Proof of Proposition 6.* Employing Lemma 3, one can enumerate all relevant spatial patterns such that  $m_i \in \{0, \bar{m}, a\} = \{0, \bar{m}, 2\bar{m}\}$  for all  $i \in \mathcal{I}$  as follows:  $(\bar{m}, \bar{m}, \bar{m}, \bar{m})$ ,  $(2\bar{m}, 0, \bar{m}, \bar{m})$ ,  $(2\bar{m}, 2\bar{m}, 0, 0)$ , and  $(2\bar{m}, 0, 2\bar{m}, 0)$ . The proposition follows by just comparing the potential values for those spatial patterns.  $\square$

*Proof of Proposition 7.* Employing Lemma 3, one can enumerate all relevant spatial patterns such that  $m_i \in \{0, \bar{m}, a\} = \{0, \bar{m}, 2\bar{m}\}$  for all  $i \in \mathcal{I}$  (which we refrain from explicitly listing up here). The proposition follows by just comparing the potential values for the spatial patterns. The threshold values  $\tau^*$ ,  $t^*$ ,  $t^{**}$ , and  $t^{***}$  are given by  $\tau^* = -8 \log(d^*)$  with  $d^* \equiv \frac{1}{6}(\sqrt{13} - 1)$  and

$$t^* = \frac{1}{8}(1-d)(1+d)(1+d+d^2), \quad t^{**} = \frac{1}{4}(1-d)^2(1+d)^2, \quad t^{***} = \frac{1}{4}d(1-d)(1+d)(1+2d), \quad (36)$$

where we let  $d \equiv e^{-\frac{\tau}{8}}$ .  $\square$

*Proof of Lemma 4.* First, note that the short-run land rent  $\check{r}$  and wage  $\check{w}$  for each zone are given by the following:

$$\check{r}(x) = \check{w}(x) = \min \{t(x + c_J), -t(x - c_J)\} \quad x \in [-c_J, c_J], \quad (37)$$

where we take the middle of the business center as the origin  $x = 0$  and  $c_J \equiv \frac{1}{2J}$  is the radius of each zone.

To provide a concrete intuition, we first consider the monocentric pattern  $E_1$ . Then, with an auxiliary function  $A_1(x) \equiv \frac{2}{\tau} e^{-\frac{t}{4}} (\cosh(\frac{\tau}{4}) - \cosh(\tau x))$ , the level of externalities  $A : C \rightarrow \mathbb{R}_+$  is given by the following:

$$A(x) = \begin{cases} A_\infty - A_1(x+c) & \forall x \in \mathcal{R}^- = [-c, -b) \\ A_\infty + A_1(x) & \forall x \in \mathcal{B} = [-b, b) \\ A_\infty - A_1(x-c) & \forall x \in \mathcal{R}^+ = [b, c), \end{cases} \quad (38)$$

where  $b = \frac{1}{4}$ ,  $c = \frac{1}{2}$ , and  $A_\infty = \frac{1}{\tau} (1 - e^{-\frac{t}{2}})$  is the uniform level of externality for the integrated pattern  $E_\infty$ . For  $E_1$  to be a long-run equilibrium, there must exist  $\pi^*$  and  $\hat{r}$  such that the following is true:

$$\hat{r}(x) = \underbrace{(A(x) - \check{r}(x) - \check{w}(x))}_{\text{The short-run profit at } x} - \pi^* \geq 0 \quad \forall x \in \mathcal{B} \quad (39)$$

and  $\hat{r}(x) = 0$  for all  $x \in \mathcal{R}^- \cup \mathcal{R}^+ = C \setminus \mathcal{B}$ , which is the continuous version of (23). To ensure continuity of long-run land rent  $\check{r}(x) + \hat{r}(x)$ , we suppose  $\hat{r}(b) = 0$ , and since  $\hat{r}(x) = 0$  for  $x \in C \setminus \mathcal{B}$ , it is implied that  $\pi^* = A(b) - 2\check{r}(b) = A_\infty - \frac{t}{2}$ . It, in turn, gives the following:

$$\hat{r}(x) = A_1(x) - 2(\check{r}(x) - \check{r}(b)) = A_1(x) - 2 \min\{t(x+b), -t(x-b)\} \quad \forall x \in \mathcal{B}. \quad (40)$$

Because  $A_1(x)$  is strictly concave over  $\mathcal{B}$ ,  $\hat{r}(x) \geq 0$  for all  $x \in \mathcal{B}$  and is equivalent to requiring that  $\hat{r}(0) = A_1(0) - \frac{t}{2} \geq 0$ . Therefore, if  $\hat{r}(0) \geq 0$ ,  $E_1$  is a long-run spatial equilibrium.

To generalize, consider  $E_J \in \mathcal{E}$  with a finite  $J$ . Without loss of generality, take one of the business centers in  $E_J$  ( $J = 1, 2, 3, \dots$ ) and let its center be the origin ( $x = 0$ ). Then, the business center is expressed as  $\mathcal{B}_{J,1} \equiv [-b_J, b_J]$  where  $b_J \equiv \frac{1}{4J}$ . We first note that  $A(x)$  is strictly concave on  $\mathcal{B}_{J,1}$ . A direct computation gives

$$A(x) = A_\infty + B_J + C_J \cosh(\tau x) \quad \forall x \in \mathcal{B}_{J,1}, \quad (41)$$

where  $B_J$  and  $C_J$  are  $\tau$  and  $J$ -dependent constants. Obviously,  $A(x)$  is strictly concave (and even). We also compute that  $A(b) = A(-b) = A_\infty$ . Let  $A_J(x) \equiv A(x) - A_\infty = B_J + C_J \cosh(\tau x)$ . Then,  $A_J(b) = A_J(-b) = 0$  and  $A_J(x)$  attains its maximum at  $x = 0$ .

Consider  $\mathcal{B}_{J,1}$  and its associated residential districts which is given by  $\mathcal{Z} \equiv [-c, c]$  where  $c \equiv \frac{1}{2J}$ . The short-run land rent and wage is given by

$$\check{r}(x) = \check{w}(x) = \min\{t(x+c), -t(x-c)\} \quad \forall x \in \mathcal{Z}. \quad (42)$$

For  $E_J$  to be an equilibrium, we must have

$$\hat{r}(x) = (A(x) - \check{r}(x) - \check{w}(x)) - \pi^* \geq 0 \quad \forall x \in \mathcal{Z} \quad (43)$$

with the continuity requirement  $\hat{r}(b) = 0$  where we omit the subscript of  $b_J$  for simplicity. It implies  $\hat{r}(b) = A(b) - 2\check{r}(b) - \pi^* = 0$  and thus  $\pi^* = A_\infty - 2\check{r}(b)$ . Then, we can rewrite (43):

$$\hat{r}(x) = A_J(x) - 2(\check{r}(x) - \check{r}(b)) = A_J(x) - 2 \min\{t(x+b), -t(x-b)\}. \quad (44)$$

Because  $A_J(x)$  is strictly concave and even, the requirement  $\hat{r}(x) \geq 0$  for all  $x \in \mathcal{B}_{J,1}$  is satisfied if and only if  $\hat{r}(0) \geq 0$ . It implies the assertion.  $\square$

*Proof of Lemma 5.* The values of the potential under Assumption 4 is computed as follows.

(i) *The integrated pattern:*  $m(x) = \rho$  for all  $C \equiv [-\frac{1}{2}, \frac{1}{2}]$ .  $n(x, x) = 1 - \rho$  and  $n(x, y) = 0$  for all  $x \neq y$ . It follows that  $\check{h}[m] = 0$  and that

$$g_\infty = f_\infty = \frac{1}{2} \iint_{C \times C} D(x, y) \rho^2 dx dy = \frac{\rho^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\tau|y|} dy = \frac{\rho^2}{\tau} \left(1 - e^{-\frac{\tau}{2}}\right). \quad (45)$$

(ii) *The segregated patterns* ( $J = 1, 2, \dots$ ): given  $J$ ,

$$f_J = \frac{1}{2} \iint_{C \times C} D(x, y) m(x) m(y) dx dy = \frac{1}{2} \sum_{i=1}^J \sum_{j=1}^J \iint_{\mathcal{B}_i \times \mathcal{B}_j} D(x, y) dx dy \quad (46)$$

$$= \frac{J}{2} \sum_{j=1}^J \iint_{\mathcal{B}_1 \times \mathcal{B}_j} D(x, y) dx dy \quad (\text{By symmetry}) \quad (47)$$

$$= \frac{J}{2} \left( \psi_0 + 2 \sum_{j=1}^{\hat{J}} \psi_k + \psi_{J/2} \right) \quad (\hat{J} \equiv \lceil \frac{J-1}{2} \rceil; \psi_{J/2} \text{ exists only for } J \text{ even}) \quad (48)$$

where  $\psi_k \equiv \iint_{\mathcal{B}_1 \times \mathcal{B}_{k+1}} D(x, y) dx dy$  ( $j = 0, 1, \dots, \lfloor \frac{J}{2} \rfloor$ ). Let  $b_J = \frac{\rho}{2J}$  be the radius of each business district. Define  $\omega_J \equiv P(\tau b_J)$  and  $\hat{\omega}_J \equiv P(2\tau b_J)$  with  $P(z) \equiv e^z - e^{-z}$ . Then,  $\{\psi_k\}$  is given by

$$\psi_k = \begin{cases} \tau^{-2} \omega_J^2 - \tau^{-2} (\hat{\omega}_J - 4\tau b_J) & \text{for } k = 0 \\ \tau^{-2} \omega_J^2 e^{-\tau \frac{k}{J}} & \text{for } 1 \leq k \leq \hat{J} \\ \tau^{-2} \omega_J^2 e^{-\frac{\tau}{2}} + \tau^{-2} (\hat{\omega}_J - 4\tau b_J) e^{-\frac{\tau}{2}} & \text{for } J \text{ even and } k = \frac{J}{2} \end{cases}. \quad (49)$$

Employing these formula noting that  $b_J = \frac{\rho}{2J} = \frac{1}{4J}$ , one obtains the formulae for  $f_J$ .

For  $\check{h}_J$ , note that the short-run equilibrium commuting cost is  $\tau b_J$  for every household in the residential districts associated to each business center.

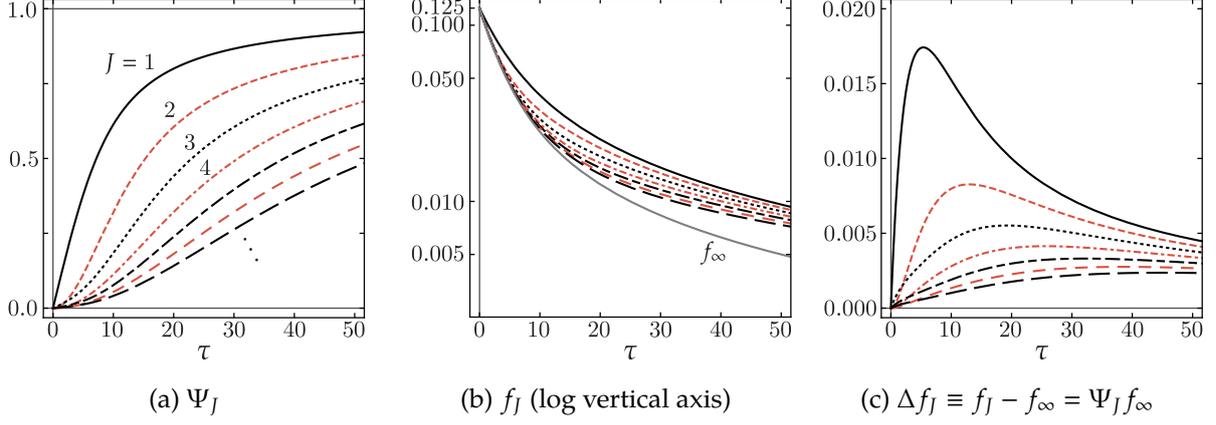
The first term  $f_J$  naturally decreases in the distance decay parameter  $\tau$ . Reflecting the fact that firms prefer concentration,  $f_J$  is always greater than  $f_\infty$  and is (basically) decreasing in the number of business centers  $J$ . Since  $J$ -centric pattern approximates the integrated pattern  $E_\infty$  when  $J \rightarrow \infty$ , the potential values also coincides in the limit. The following lemma summarizes the properties of  $\{f_J\}$  that basically stem from those of  $\{\Psi_J\}$ .

**Lemma 8.**  $\{f_J\}_{J=1}^\infty$  satisfies the following properties:

- (a)  $f_J$  is monotonically decreasing in  $\tau$  for all  $J$ ,
- (b)  $f_J > f_\infty$  for all  $J$ ,
- (c)  $f_J > f_{J+2}$  for all  $J$ ,
- (d)  $f_1 > f_2$ ,
- (e)  $\lim_{J \rightarrow \infty} f_J = f_\infty$ , and
- (f)  $\lim_{\tau \rightarrow 0} f_J = \lim_{\tau \rightarrow 0} f_\infty = \frac{1}{8}$ .

*Proof of Lemma 8.* Studying the analytic formulas for  $\{f_J\}$  one obtain the properties listed in the lemma. For simplicity, we show proofs for even  $J$ . (a) follows by straightforward computation. (b) follows by simply noting that  $\Psi_J > 0$ . Next, for any  $J$ , we have

$$f_J - f_{J+2} = (\Psi_J - \Psi_{J+2}) f_\infty = \left( \frac{2C(d_J)}{\log(d_J)} - \frac{2C(d_{J+2})}{\log(d_{J+2})} \right) \delta_J. \quad (50)$$



**Figure 13:** Graphs of  $\Psi_J$ ,  $f_J$ , and  $\Delta f_J$  for  $J \in \{1, 2, \dots, 7\}$ .

Noting that  $\frac{2C(d)}{\log(d)}$  is decreasing in  $d$  and  $d_J$  is increasing in  $J$ , we see that the RHS is positive, thereby showing the property (c). Next, with  $d \equiv e^{-\frac{\tau}{4}} \in (0, 1)$ , we have

$$f_1 - f_2 = (\Psi_1 - \Psi_2)f_\infty = -\frac{1}{(1-d^2)\log(d)} \left( (1-d)(3-d) - 2d^2 \log(d) \right) > 0, \quad (51)$$

thereby showing the property (d). (e) and (f) follows immediately. Employing the analytic expression for  $\Psi_J$ , one also shows that there is always a value of  $\tau$  such that  $f_J$  is decreasing in  $J$  for all  $J \leq \bar{J}$  with some  $\bar{J}$ .  $\square$

Figure 13 depicts  $\Psi_J$ ,  $f_J$ , and the relative magnitude  $\Delta f_J \equiv f_J - f_\infty = \Psi_J f_\infty$  for  $J = 1, 2, \dots, 7$ . We note that  $\lim_{\tau \rightarrow 0} \Psi_J = 0$ ,  $\lim_{\tau \rightarrow \infty} \Psi_J = 1$ , and  $\lim_{J \rightarrow \infty} \Psi_J = 0$ . We observe that  $\Psi_J$ , and hence  $f_J$  and  $\Delta f_J$ , basically decrease in  $J$ .  $\square$

*Proof of Lemma 6.* Observe that, with the same notations as Proof of Lemma 4, we have

$$\begin{aligned} g_J - g_\infty &= \frac{1}{2} \int m(x)A(x)dx - \frac{t}{8J} - \frac{1}{4}A_\infty = \frac{J}{2} \int_{-b}^b \left( A(x) - \frac{t}{2J} - A_\infty \right) dx \\ &= \frac{J}{2} \int_{-b}^b \left( A_J(x) - \frac{t}{2J} \right) dx, \end{aligned}$$

where we note that  $\int_{-b}^b dx = \frac{1}{2J}$  since  $b = \frac{1}{4J}$ . Because  $A_J(x)$  is strictly concave and even on  $[-b, b]$ ,  $g_J - g_\infty \geq 0$  implies that the integrand in the last expression is positive for some convex interval  $(-\epsilon, \epsilon)$  where  $0 < \epsilon < b$ . Therefore,  $g_J - g_\infty \geq 0$  is sufficient for  $A_J(0) - \frac{t}{2J} > 0$  because  $0 \in (-\epsilon, \epsilon)$ . The condition coincides the equilibrium condition for the symmetric pattern  $E_J$ . Thus,  $g_J \geq g_\infty$  is sufficient for  $E_J$  to be a long-run equilibrium.

Figure 14 depicts the equilibrium condition  $\hat{r}(0) \geq 0$  and the sufficient condition  $g_J \geq g_\infty$  for  $E_1$ ,  $E_2$ , and  $E_3$ . Each symmetric pattern is a long-run equilibrium below the dashed curve which indicates the pairs of  $(\tau, t)$  where  $\hat{r}(0) = 0$ . The condition  $g_J \geq g_\infty$  is satisfied below the dot-dashed curves that indicate  $g_J = g_\infty$ . Observe that the latter is contained by the former, which illustrates the sufficiency. The gray regions indicate where  $g_J = \max_J \{g_J\}$  holds true for each  $J$ , which are extracts from Figure 11.  $\square$

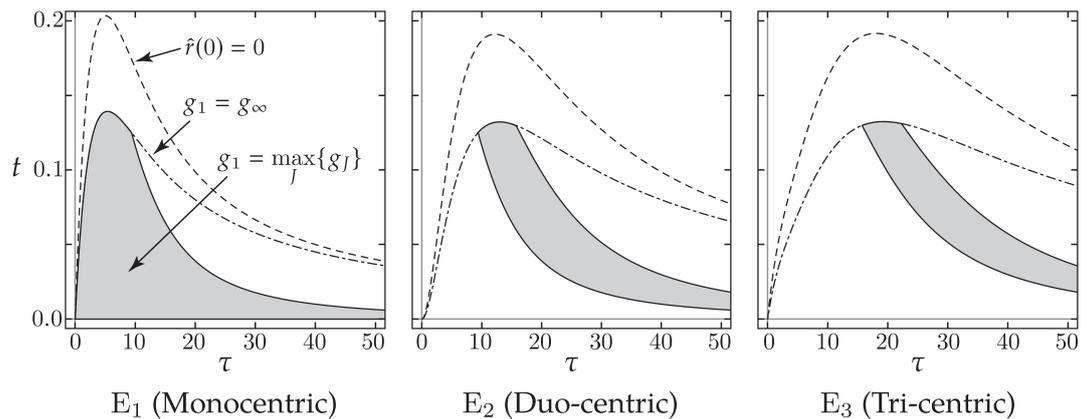


Figure 14: Illustration of Lemma 6.

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