Model and measure the relative efficiency of a four-stage production process. An NDEA multiplier relational model under different systems of resource distribution preferences between sub-processes

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Abstract

Measuring the relative efficiency of a production process with the DEA considers the production process as a "black box" that uses inputs to transform them into outputs. In reality, many production processes are carried out by carrying out several interconnected activities that are usually grouped into phases that are in turn interconnected. For this reason, measuring the relative efficiency of a production process within the DEA technique requires shaping it as a network system (in others words to consider the production process as interconnected sub-process). In the case of network systems, the NDEA approach has developed many models to measure their relative efficiency: independent models, connected models and relational models. In particular, the relational model allows to measure at the same time both the efficiency of the system and the efficiency of the sub-process once the operations between the latter have been considered. In our opinion, many real production processes can be modelled as a network of four sub-processes that are differently interconnected with each other. In this paper we will model a production process as a network of four sub-processes with shared variables and fixed preferences about the allocation of system resources between them. To measure the relative efficiency of the process and its parts we will develop an input-oriented NDEA model in the multiplier version. To solve the model we will use virtual data under several resources allocation preference's structure. Then we will conclude that 1) a production process with four interconnected sub-processes can represent a large number of real production processes, so the NDEA model developed here can potentially be used for many applications, 2) the resource allocation preference system inter-sub-process influences the measurement of relative efficiency.

Keywords: network DEA, performances management, internal structure, inputs-outputs system.

Jel classification: C61,C67,D2

1. Introduction

Inside Operation Research (O.R.) (Hiller & Lieberman, 2001) and Management Science (M.S.) tradition the Data Envelopment Analysis (DEA)[(Cooper, et al., 2007)] occupies a central place as a non-parametric frontier technique to measure the relative efficiency of a production process and is at the same time a technique to model it as an input-output system that uses input that transforms into output without considering its internal structure. In real situations, production processes are instead sets of operations and activities interconnected with each other, we can think for example of the production of cars, or the production of jewelry or clothes, wine and so on. This therefore suggests that a production process should be modelled differently from a "black box". In the applied literature DEA applications cover several sectors [for a review of DEA applications see (Seiford, 1996), (Emrouznejad, et al., 2008); (Emrouznejad, et al., 2017); (Liu, et al., 2013); (Hollingsworth, 2008)], but in all these cases the production process are modelled as "black box" that uses inputs to transform in outputs. More or less recently several authors proposed inside the DEA approach to modelling the production process as a network system [i.e. (Fare & Grosskopf, 2000); (Fa¨re, 1996b); (Kao, 2009(a)); (Castelli, et al., 2010); (Castelli, et al., 2001)] where the stages/operations of a production process are interconnect among them. The measurement of the relative efficiency of a network system within the DEA approach can be conducted with different models for example using independent models [i.e. (Wang, et al., 1997); (Seiford & Zhu, 1999); (Sexton & Lewis, 2003)], or connected models [i.e (Fa¨re & & Grosskopf, 1996a) (Fare & Grosskopf, 2000)]. Unlike the independent model, the connected model allows the researcher to take into account the interrelationships within the production process/organization and thus measure its relative efficiency, but does not allow to derive the relative efficiency of the individual stages. Since the works of [(Kao, 2009(a)); (Kao, 2009(b))] a new class of NDEA models are developed to measure the relative efficiency of the whole process as well as the efficiency of its parties using the same model. The author labelled his approach relational NDEA models. The feature of the relational NDEA model is in that it apply the same weights to the same variables, comprising the relational variables, in a way to consider the operations among its parts and calculate the efficiency of it using the decomposition efficiency formula[(Kao, 2014)]. Kao's work begins by developing a simple two-stage network model with sub-processes differently interconnected among them: in particular Kao consider series and parallel models as basic model. A more general classification of network models within the NDEA approach is in [(Castelli, et al., 2010)]. In general, the characteristic of the NDEA approach is that NDEA models[1] can be adapted to model and measure the relative efficiency of specific network systems. The applications of the NDEA models are relatively recent relatively to the applications of the DEA model but is increasing [for some example we remember (Chodakowska & Nazarko, 2017); (Kao & Hwang, 2008); (Kawaguchi, et al., 2014); (Pinto, 2016); (Prieto & Zofio, 2007); (Sexton & Lewis, 2003); (Wanke & Barros, 2014); (Wanke, et al., 2017); (Despotis, et al., 2015); (Chilingerian & Sherman, 2004), (Cook, et al., 2010)]. In this paper we develop a multiplier relational NDEA model under CRS assumption for a production process of four sub-process. The paper is structured as follow: in the section 2 we show a review of the NDEA applications, in section 3 we show a general modelling in the case of a network system of four sub-process (subsection 3.1) and the relative multiplier NDEA models to measure its efficiency(subsection 3.2.), in the section 4 we offer an application with not real data and finally discussion and conclusions are showed in the section 5.

2. A review of applied empirical literature

The consideration of the internal structure of a production process/organization inside DEA context is relatively recent. A first application can be considered those in (Fa¨re & Whittaker, 1995). In their work the authors adopt a connected NDEA model and their network DEA approach can involve more than two stages and the operations among them. Their work consider an application of an input oriented two-stage DEA model to dairy farms and compare the results obtained with the ones obtained with a standard DEA model. Since the work of (Fa¨re & Whittaker, 1995) other works appeared in the applied economic and managerial literature. For example (Sexton & Lewis, 2003) apply a two-stages NDEA model to Major League Baseball considering both inputs and outputs orientation and each one under constant and variable return to scale assumptions. In their paper the authors using the envelopment form demonstrate as it is possible to make different assumptions for each sub-process in a context of network system. (Chilingerian & Sherman, 2004) apply the NDEA approach in a two stage process in measuring a physicians care. (Prieto & Zofio, 2007) applied network efficiency analysis within an input– output model initiated by (Koopmans, 1951). In their work the authors optimized primary input allocations, intermediate products and final demand products by way of Network DEA techniques and succeeded in applying their models to input–output database of OECD countries. (Kao & Hwang, 2008) introduced for the first time in the context of two-stage network DEA the relational approach. They develop an multiplier input-oriented two-stage DEA model and apply it to measure the relative efficiency of non-life insurance companies in Taiwan. (Wanke & Barros, 2014) apply the network-DEA centralized efficiency model to optimize efficiency in Brazilian banking in both modelled stages simultaneously. (Wanke, et al., 2017) apply a network DEA approach to compute the impact of contextual variables on several types of efficiency scores of the resulting virtual merged banks: global (merger), technical (learning), harmony (scope), and scale (size) efficiencies. (Kawaguchi, et al., 2014) employed a dynamic-network data envelopment analysis model (DN model) to perform the evaluation of the policy effect of the current reform of Japan’s municipal hospitals. (Despotis, et al., 2015) apply a network DEA approach to deal with efficiency assessments in two-stage processes and apply their approach to the assessment of the academic performance of forty faculty members in a Greek University. (Pinto, 2016) apply both constant and variable returns to scale Data Envelopment Analysis network model to estimates the relational and sub-process efficiency of the production process of the hospital’s acute care services. (Chodakowska & Nazarko, 2017) apply a network DEA models in evaluating courier and messenger companies. In their work the authors consider an hypothetical chains consisting of two types of members, i.e. nine leaders of the courier and messenger sector that had the share of nearly 95% of total revenue generated from provision of courier services in 2014. The second member was the bigger enterprise in the

[1] As in the DEA we can have a multiplier, envelopment and slacked NDEA models

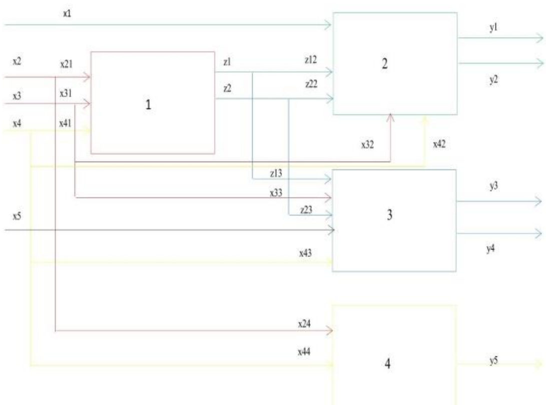
electronic shopping sector that does not have stationary shops e.g. Bonprix Sp. z o.o. Their approach consider envelopment form. For a review of applications of NDEA see (Kao, 2014) and (Cook, et al., 2010).

3. Modelling a production process with four sub-process and the relative NDEA model

3.1 A general four stages production process

In this subsection we are modelling a general production process of four sub-process (see Figure 1).

Figure. 1 *A network system of a production process with four sub-process*



The production process in Figure 1 consider four sub-process interconnected among them. The outputs of the process are five $(y_1, y_2, y_3, y_4, y_5)$ and the inputs of the production process are 5 $(x_1, x_2, x_3, x_4, x_5)$, while the intermediate (Fa¨re & Whittaker, 1995) variables are two (z_1, z_2). In particular the first sub-process produce two outputs (the intermediate variables z_1 *and* z_2) where a proportion of it became the inputs of the second sub-process (z_{12}, z_{22}) and the remaining proportions are the inputs of the third sub-process (z_{13}, z_{23}). To produce his outputs (y_5) the fourth sub-process use a proportion of two inputs system (x_{24}, x_{44}), while the second and third sub-process use an exogenous variable for each one (x_1*and* x_5 respectively). Summarizing, the second sub-process to produce his two outputs (y_1, y_2) use an exogenous input variable (x_1), two shared relational variables (z_{12}, z_{22}) and two shared inputs system (x_{32}, x_{42}). The third sub-process to produce his two outputs (y_3, y_4) use one exogenous variable (x_5), two shared relational variables (z_{13}, z_{23}) and two shared inputs variables (x_{33}, x_{43}), and finally the fourth sub-process to produce its output (y_5) use two shared inputs variables (x_{24}*and* x_{44}). The system have not feedback variables and do not have shared outputs [(Cook, et al., 2010),(Castelli, et al., 2010), (Chen, et al., 2010 b)]. The model is very general and any other configurations of inputs, outputs and intermediate variables are possible. So, for example we can add a second exogenous variable to the second sub-process, or add a third intermediate variable and so on. In the following sub-section we build the relative NDEA model to estimate the relative efficiency of the production process in the Figure 1.

3.2 The NDEA model

In this sub-section we build the multiplier input-oriented relational NDEA model to measure the relational efficiency for the production process of four stages as those displayed in Figure 1 above. The input version of the model, once adopted the linearization suggested by (Charnes & Cooper, 1962), can be write as follow:

$$max\sum_{r=1}^{p} u_r\, Y_{ro}$$

$$s.t.\ \sum_{s=1}^{q} v_i X_{io} = 1\ ,$$

$$\sum_{r=1}^{p} u_r\, Y_{rk} - \sum_{s=1}^{q} v_i\, X_{ik} \leq 0, \qquad (1)$$

$$\sum_{R=1}^{m} w_{r1k}\, Z_{r1k} - \sum_{s=1}^{n} v_{rk}\, X_{r1k}^{S} \beta_j \leq 0\ ,$$

$$\sum_{r=1}^{s} u_r Y_{rk2} - \sum_{r=1}^{t} \alpha_r w_{rk2} Z_{r2k}^{S} - \sum_{r=1}^{T} \dot{v}_i X_{r2} - \sum_{s=1}^{U} \beta_s v_i X_{rk2}^{S} \leq 0$$

$$\sum_{r=1}^{z} Y_{rk3} u_r - \sum_{r=1}^{l} \beta_s v_i X_{ik3}^{S} - \sum_{r=1}^{L} (1-\alpha_r) w_{rk3} Z_{rk3}^{S} - \sum_{r=1}^{M} v_i X_{ik3} \leq 0$$

$$\sum_{r=1}^{o} u_r Y_{rk4} - \sum_{r=1}^{O} v_r X_{rk4}^{S} \leq 0$$

$$u, w, v \geq 0$$

Where:

Y_{rk}= are the outputs of the system with $r \in R^{p=5}$ for p outputs system $(y_1, y_2, y_3, y_4, y_5)$.
X_{ik}= are the inputs of the system with $i \in R^{q=5}$ for q inputs system $(x_1, x_2, x_3, x_4, x_5)$ including the exogenous variables of the sub-process (in this case the variables x_1, x_5).
Z_{h1k}= are the outputs of the first sub-process with $h \in R^{m=2}$ $(z_1\ and\ z_2)$. The pedices 1 indicate the first sub-process.
X_{i1k}=are the inputs of the first sub-process (x_{21}, x_{31}, x_{41}). In our NDEA model all these variables are shared variables and for this reason we adopt the notation X_{r1k}^{S}.
Y_{rk2}=are the outputs of the second sub-process (=Y_{rk})(y_1, y_2). The pedices 2 indicate the second sub-process.
Z_{p2k}^{S}= are the relational shared inputs of the second sub-process (z_{12}, z_{22})
X_{r2k} = are the exogenous variables of the second subprocess (x_1).
X_{rk2}^{s} =are the shared inputs variables of the second subprocesss (x_{32}, x_{42}).
Z_{rk3}^{S} = are the relational shared variables of the third subprocess (z_{13}, z_{23}).
X_{rk3}^{S}= are the shared inputs variables of the subprocess three (z_{13}, z_{23})
X_{ik3} = are the inputs of the third subprocess (x_{33}, x_{43})
Y_{rk3} = are the outputs of the third subprocess(y_3, y_4)
Y_{rk4} = are the outputs of the fourty subprocess(y_5)
X_{i4k} =are the inputs are of the fourth subprocess$(x_{24,}x_{44})$
k= is the number of production units (k=1……N)
u, v, w ,$\dot{w}$= are the weight of the model's variables
$\boldsymbol{\beta}(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$=is the vector of inputs sharing proportions $(\beta_1 for\ inputs\ 2, \beta_2\ and\ \beta_3\ for\ inputs\ 3, \beta_4 and\ \beta_5\ for\ \ inputs\ 4\ respectively)$
$\boldsymbol{\alpha}(\alpha_1, \alpha_2)$ =is the vector of the intermediate variables sharing proportions $(\alpha_1 for\ z1, \alpha_2 for\ z2\ intermediate\ variables)$

The system have *p* outputs $(y_1, y_2, y_3, y_4, y_5)$ and *q* inputs $(x_1, x_2, x_3, x_4, x_5)$. Some of the inputs system (in particular x_2, x_3, x_4) are shared among the four subprocess as follow: the first subprocess have *n=3 shared* inputs (x_{21}, x_{31}, x_{41}) and *m=2* outputs (z_1, z_2), the second subprocess have *s=2* outputs (y_1, y_2), *t* =2intermediate shared variables (z_{12}, z_{22}), *U=2* shared inputs (x_{32}, x_{42}) and T=1 exogenous variables (x_1), the third subprocess have *z=2* outputs (y_3, y_4), *l=1* exogenous variables (x_5), *L=2* intermediate shared variables (z_{13}, z_{23}) and *M=2* shared inputs variables (x_{33}, x_{43}) (with the first subprocess), the fourth subprocess have *o=1* outputs (y_5) and *O=2* shared inputs (x_{24}, x_{44}) (with the first subprocess). The first constraint is the normalization constraint (Charnes & Cooper, 1962). The second constraint is the system constraint, the third is the constraint of the first sub-process and finally, the fourth and five constraint are the constraints of the third and fourth stages. To consider the relationship between the stages/sub-process and treat the model as a relational NDEA model all variables, comprising the relational variables (Z), have the same weigths [(Kao, 2009(a))], and obviuosly we attach the same weigths a the same shared relational variables as well as at the same not relational shared variables (this later solution have an evident computational gain for us). The sharing's proportion is based on the vectors of scalars ($\boldsymbol{\beta}, \boldsymbol{\alpha}$). The output version of the NDEA model above is the following:

$$\min \sum_{s=1}^{q} v_i X_{io}$$

$$s.t. \sum_{r=1}^{p} u_i Y_i = 1$$

$$-\sum_{r=1}^{p} u_r\, Y_{rk} + \sum_{s=1}^{q} v_i\, X_{ik} \geq 0^2,$$

$$-\sum_{R=1}^{m} w_{r1k}\, Z_{r1k} + \sum_{s=1}^{n} v_{rk}\, X_{r1k}\beta_j \geq 0\,, \quad (2)$$

$$-\sum_{r=1}^{s} u_r Y_{rk2} + \sum_{r=1}^{t} \alpha_r w_{rk2} Z_{r2k}^{S} + \sum_{r=1}^{T} \dot{w}_j X_{r2} + \sum_{s=1}^{U} \beta_s v_i X_{rk2}^{S} \geq 0$$

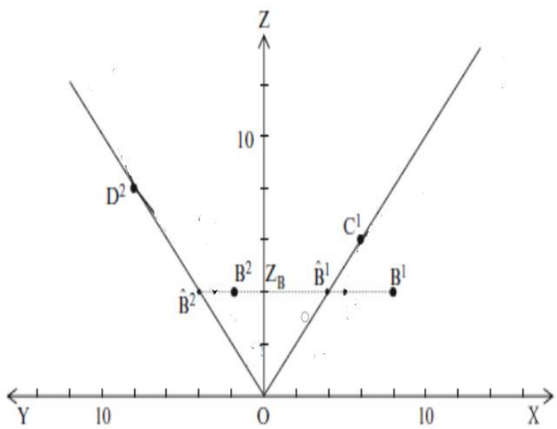
$$-\sum_{r=1}^{z} Y_{rk3} u_r + \sum_{r=1}^{l} \beta_s v_i X_{ik3}^{S} + \sum_{r=1}^{L} (1-\alpha_r) w_{rk3} Z_{rk3}^{S} + \sum_{r=1}^{M} v_i X_{ik3} \geq 0$$

$$-\sum_{r=1}^{o} u_r Y_{rk4} + \sum_{r=1}^{O} v_r X_{rk} \geq 0$$

$$u, w, v, \widetilde{w}, \dot{w} \geq 0$$

The multiplier output-oriented NDEA model in (2) give us the measure of how much the output of the process can be improved once considered the relationship among the sub-process inside it. As it is possible to note (1) and (2) assume constant return to scale for the entire system as well as for each subprocess. Figure 1a below depicts the CRS efficient frontier for the sub-process 1 (the straight line OC^1) in the right side and the CRS efficient frontier for the sub-process 2 (straight line OD^2) in the left side.

Figure. 1a *CRS efficient forntier of first and second sub process*



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[2] To write: $\sum_{r=1}^{p} u_r\, Y_{rk} - \sum_{s=1}^{q} v_i\, X_{ik} \leq 0$ is the same that to write: $-\sum_{r=1}^{p} u_r\, Y_{rk} + \sum_{s=1}^{q} v_i\, X_{ik} \geq 0,$

In the right side of the Figure 1a sub-process 1 apply an input vector X (x_{21}, x_{31}, x_{41}) to produce a vector of intermediate product Z $(z_{1,}z_2)$, and the left side show a sub-process 2 that applies intermediate product Z (z_1, z_2) to produce a vector of outputs Y (y_1, y_2). The superscripts associated with the DMU indicate the subprocess. For example the DMU B is overall inefficient in the first as in the second subprocess. These measure are $\left(\widehat{B^1}/Z_B\right)/(B^1/Z_B)$ and$(B^2/Z_B)/\left(\widehat{B^2}/Z_B\right)$, respectively.

4. *The data and application*

In this section we will proceed with the application of the model to a virtual inputs/outputs vector data generated as random number using uniform distribution[3]. Once fixed in R the set.seed=4 we proceed with the calculation of the descriptive statistics of it (see Table 1 below)[4]. Our virtual dataset is of 5 inputs (x), 5 outputs (y) and 2 relational variables (z)

Table 1 The data description

Variables	*Descriptive statistic*	
Inputs	*mean*	*s.d.*
X1	*10,41*	*2,86*
X2	*72,55*	*36,61*
X3	*103,98*	*60,88*
X4	*89,37*	*51,34*
X5	*27,21*	*15,25*
Outputs		
Y1	*58,73*	*22,01*
Y2	*131,41*	*44,75*
Y3	*176,3*	*90,71*
Y4	*385,77*	*200,1*
Y5	*30,53*	*16,26*
Relational variables		
Z1	*87,09*	*46,71*
Z2	*54,67*	*31,04*

Once applied the NDEA model with different systems of preferences by researchers/managers on the resources allocation among the sub-process (see Table 2) we obtain the following results (see Table 3,4,5 and 6).

Table 2 Managerial/researchers preferences on the allocation of the resources among the sub-process

Scalar	*System of preferences*			
	1	*2*	*3*	*4*
β_1	*0.5*	*0.85*	*0.35*	*0.85*
β_2	*0.3*	*0.5*	*0.2*	*0.6*
β_3	*0.3*	*0.2*	*0.5*	*0.3*
β_4	*0.3*	*0.4*	*0.1*	*0.6*
β_5	*0.2*	*0.2*	*0.3*	*0.1*
β_6	*0.2*	*0.3*	*0.1*	*0.1*
α_1	*0.5*	*0.4*	*0.7*	*1*
α_1	*0.5*	*0.4*	*0.4*	*1*

[3] The function used in R is runif().
[4] We decided to do so because our work at this step is not purely applicative.

The first system of preferences allocates through the parameter α_1 the relational variable z_1 in the same proportion between the sub-process 2 (z_{12}) and 3 (z_{13}) in the proportion of (50/50), through the parameter α_2 the relational variable z_2 is allocated between the sub-process 2 (z_{22}) and 3 (z_{23}) in the same proportion (50/50). The parameter β_1 allocates the variable x_2 in the same proportions (50/50) between the sub-process 1 (x_{21}) and 4 (x_{24}), the parameter β_2 and β_3 allocates the variable x_3 among the sub-process 1(x_{31}), 2(x_{32}) and 3(x_{33}) in the following proportions 0.3,0.3 and (1-0.3-0.3)=0.4, respectively, finally the parameters β_4, β_5 *and* β_6 allocates the variable x_4 among the sub-process 1(x_{41}), 2(x_{42}), 3(x_{43}) and 4(x_{44}) in the following proportions 0.3, 0.2,0.2 and (1-0.3-0.2-0.2)=0.3, respectively. The second system of preferences allocates through the parameter α_1 the relational variable z_1 in the proportions of (40,60) between the sub-process 2 (z_{12}) and 3 (z_{13}), through the parameter $\alpha_2$allocates the relational variable z_2 between the sub-process 2 (z_{23}) and 3 (z_{33}) in the proportions of (40/60). The parameter β_1 allocates the variable in the proportions (0.85/0.15) between the sub-process 1 (x_{21}) and 4 (x_{24}), the parameters β_2 *and* β_3 allocate the variable x_3 among the sub-process 1(x_{31}),2(x_{32}) and 3(x_{33}) in the following proportions 0.3,0.3 and (1-0.3-0.3)=0.4, respectively finally the parameters β_4, β_5 *and* β_6 allocates the variable $x_4$among the sub-process 1(x_{41}),2(x_{42}),3(x_{43}) and 4(x_{44}) in the following proportions 0.3, 0.2,0.2 and (1-0.3-0.2-0.2)=0.3, respectively. The third system of preferences allocates through the parameter α_1 the relational variable z_1 in the proportions of (0.7/0.3) between the sub-process 2 (z_{12}) and 3 (z_{13}), through the parameter $\alpha_2$allocates the relational variable z_2 between the sub-process 2 (z_{22}) and 3 (z_{23}) in the proportions of (0.4/0.6). The parameter β_1 allocates the variable x_2 in the proportions of (0.35/0.65) between the sub-process 1 (x_{21}) and 4 (x_{24}), the parameters β_2*and* β_3 allocates the variable x_3 among the sub-process 1(x_{31}),2(x_{32}) and 3(x_{33}) in the following proportions 0.2,0.5 and (1-0.2-0.5)=0.2, respectively. Finally the parameters β_4, β_5 *and* β_6 allocates the variable $x_4$among the sub-process 1(x_{41}),2(x_{42}),3(x_{43}) and 4(x_{44}) in the following proportions 0.1, 0.3,0.1 and (1-0.1-0.3-0.1)=0.5, respectively. In the first system of preferences the managers express a more equilibrate allocation of the outputs of the first sub-process among the sub-process 3 and 4 than in the third system. In others words the managers evaluate with equilibrium the interrelationship among the sub-process 1,2 and 3, while in the third the managers consider the second sub-process as a sub-process with a more high consumption of the relational resources produced by the first sub-process. A moderate disequilibrium in the allocations of the relational variables characterize the second system of preferences. Instead the second is characterized to the a disequilibrium in the allocation of the variable 2 between the sub-process 2 and 4, and the third to the a disequilibrium in the allocation of the variable x_4, that privilege the sub-process 4 in its allocation. Finally the fourth system of preferences modify radically the role of the relational variables, now the outputs of the first sub-process are totally intermediate variable for the second sub-process, hence lost the characteristic of shared variables with the third sub-process, that now will use only a proportion of the system's inputs (the variables x_3*and* x_4. Following the four system of preferences the relative efficiency of the system and of its sub-process will be those reported in the Tables 3, 4 ,5 and 6.

Table 3 Efficiency estimation (Preferences system 1)

		Descriptive statistics				
Efficiency		*mean*	*s.d.*	*median*	*I Q(25%)*	*IIIQ(75%)*
	Obs	*Input oriented measurement*				
Relational efficiency	150	0,830525	0,186378	0,884463	0,681019	1
Calculated relational efficiency	150	0,830525	0,186378	0,884463	0,681019	1
First sub process efficiency	147	148,6852	230.9413	70,87663	21,81416	194,2922
Second sub-process efficiency	150	0,333558	0,685407	0,076669	0,026964	0,212165
Third sub-process efficiency	150	0,160628	0,187252	0,084133	0,038834	0,211517
Fourth sub-process efficiency	150	0,233376	0,362507	0,123434	0	0,288108
		Output oriented measurement				
Relational efficiency	150	0,131051	0,083731	0,113904	0,070589	0,172138
Calculated relational	150	0,131051	0,083731	0,113904	0,070589	0,172138

efficiency						
First sub process efficiency	150	0,078706	0,133399	0,033797	0,017131	0,092712
Second sub-process efficiency	141	2,596546	6,536574	0,729086	0,457538	1,141824
Third sub-process efficiency	148	2,104779	5,713056	0,398516	0	1,71706
Fourth sub-process efficiency	76	0,018975	0,044954	0,003759	0	0,018601

Table 4 Efficiency estimation (Preferences system 2)

		Descriptive statistics				
Efficiency	*Obs*	*mean*	*s.d.*	*median*	*I Q (25%)*	*IIIQ (75%)*
		Input oriented measurement				
Relational efficiency	150	0,829052	0,186871	0,884463	0,680454	1
Calculated relational efficiency	150	0,829052	0,186871	0,884463	0,680454	1
First sub process efficiency	147	92,799	179,178	34,314	12,0,93	90,659
Second sub-process efficiency	150	0,36926	0,722343	0,099617	0,034076	4,349306
Third sub-process efficiency	150	0,161123	0,194164	0,081659	0,039466	0,937963
Fourth sub-process efficiency	150	0,241008	0,359038	0,13812	0	2
		Output oriented measurement				
Relational efficiency	*150*	0,131051	0,083731	0,113904	0,070589	0,172138
Calculated relational efficiency	*150*	0,131051	0,083731	0,113904	0,070589	0,172138
First sub process efficiency	*150*	0,078706	0,133399	0,033797	0,017131	0,092712
Second sub-process efficiency	*138*	1,05	2,144	0,772125	0,788	1,927
Third sub-process efficiency	*82*	0,049982	0,075322	0,021215	0	0,0788
Fourth sub-process efficiency	*76*	0,018975	0,044954	0,003759	0	0,0186

Table 5 Efficiency estimation (Preferences system 3)

		Descriptive statistics				
Efficiency	*Obs.*	*mean*	*s.d.*	*median*	*I Q (25%)*	*IIIQ (75%)*
		Input oriented measurement				
Relational efficiency	150	0,831071	0,185908	0,884463	0,68126	1
Calculated relational efficiency	150	0,831071	0,185908	0,884463	0,68126	1
First sub process efficiency	147	158,8529	299,333	67,9760	24,092	181,448
Second sub-process efficiency	150	0,250094	0,435956	0,084716	0,033912	0,248641
Third sub-process efficiency	150	0,148958	0,171433	0,078124	0,043175	0,208608
Fourth sub-process efficiency	150	0,242129	0,355655	0,139277	0	0,307079
		Output oriented measurement				
Relational efficiency	*150*	0,131051	0,083731	0,113904	0,070589	0,172138
Calculated relational efficiency	*150*	0,131051	0,083731	0,113904	0,070589	0,172138
First sub process efficiency	*150*	0,078706	0,133399	0,033797	0,004885	0,02357
Second sub-process efficiency	*150*	0,7608	2,010	0,39914	0,18386	0,69647

Third sub-process efficiency	*150*	0,050295	0,075452	0,02217	0	0,072741
Fourth sub-process efficiency	*150*	0,018975	0,044954	0,003759	0	0,018601

Table 6 Efficiency estimation (Preferences system 4)

		Descriptive statistics				
Efficiency		*mean*	*s.d.*	*median*	*I Q(25%)*	*IIIQ(75%)*
	Obs.	*Input oriented measurement*				
Relational efficiency	150	0,733114	0,227461	0,721897	0,574061	0,984066
Calculated relational efficiency	150	0,733114	0,227461	0,721897	0,574061	0,984066
First sub process efficiency	142	68,876	136.879	21,014	4,985	63,499
Second sub-process efficiency	150	0,421303	0,945323	0,110512	0,052987	0,313501
Third sub-process efficiency	150	0,115892	0,184034	0,028536	0,000974	0,154049
Fourth sub-process efficiency	150	0,270046	0,386234	0,156098	0	0,375528
		Output oriented measurement				
Relational efficiency	*150*	0,131051	0,083731	0,113904	0,070589	0,172138
Calculated relational efficiency	*150*	0,131051	0,083731	0,113904	0,070589	0,172138
First sub process efficiency	*150*	0,078706	0,133399	0,033797	0,017131	0,0927
Second sub-process efficiency	*82*	0,256	0,16520	0,2322	0,12992	0,37127
Third sub-process efficiency	*141*	0,0943	0,0847	0,07131	0,03457	0,13199
Fourth sub-process efficiency	*150*	0,034205	0,056005	0,0161	0,005218	0,0328

The cell "Calculated relational efficiency" contain the relative efficiency of the whole process calculated the multiplicative formula (Kao, 2009(a)) once solved the model NDEA in (1). While the cell with "Calculated …sup-process" contain the result of the application of the decomposition formula for calculating the efficiency of each sub-process (Kao, 2009(a)) once solved the NDEA model in (1). The differences in the measurements are outlined to the comparison among the empirical cumulative distribution of the efficiency scores in the Figures 1,2,3 and 4. The graph refer to the relational efficiency (not calculated) of the whole process and the efficiency of the four sub-process 1,2 under the four system of preferences.

***Figure 2** Ecdf of the efficiency scores in the case of CRS-Input orientation*

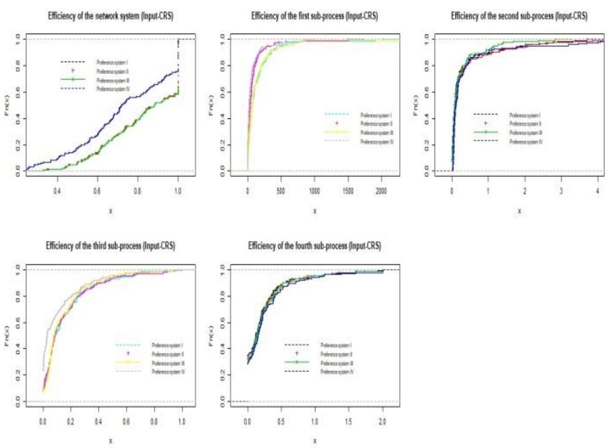
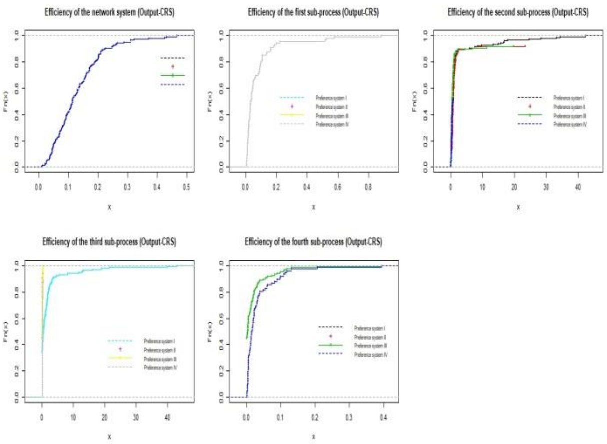


__Figure 3__ Ecdf of the efficiency scores in the case of CRS-Output orientation



How much the efficiency of each sub-process is correlated with every other can be related to the resource's allocation among them. In the following Tables we report the correlation analysis[5].

Table 7 CRS- efficiency scores correlations

	Correlation					
Efficiency	Relational efficiency	Calculated relational efficiency	First sub process efficiency	Second sub-process efficiency	Third sub-process efficiency	Fourth sub-process efficiency
	INPUT ORIENTATION					
	I					
Relational efficiency	1	1	-0,02413	0,263478	0,179014	0,321939
Calculated relational efficiency	1	1	-0,02413	0,263478	0,179014	0,321939
First sub process efficiency	-0,02413	-0,02413	1	-0,08982	-0,06575	-0,05966
Second sub-process efficiency	0,263478	0,263478	-0,08982	1	0,35512	0,196161
Third sub-process efficiency	0,179014	0,179014	-0,06575	0,35512	1	0,235418
Fourth sub-process efficiency	0,321939	0,321939	-0,05966	0,196161	0,235418	1
	II					
Relational efficiency	1	1	-0,01367	0,268164	0,128934	0,281771
Calculated relational efficiency	1	1	-0,01367	0,268164	0,128934	0,281771
First sub process efficiency	-0,01367	-0,01367	1	-0,0987	-0,09887	-0,08982
Second sub-process efficiency	0,268164	0,268164	-0,0987	1	0,145396	0,225273
Third sub-process efficiency	0,128934	0,128934	-0,09887	0,145396	1	0,177881
Fourth sub-process efficiency	0,281771	0,281771	-0,08982	0,225273	0,177881	1
	III					
Relational efficiency	1	1	-0,01367	0,268164	0,128934	0,281771
Calculated relational efficiency	1	1	-0,01367	0,268164	0,128934	0,281771
First sub process efficiency	-0,01367	-0,01367	1	-0,0987	-0,09887	-0,08982

[5]To run the correlation analysis in R we use the function "cor" with the option" use="complete.obs"", that eliminate all missing values in the observations

Second sub-process efficiency	0,268164	0,268164	-0,0987	1	0,145396	0,225273
Third sub-process efficiency	0,128934	0,128934	-0,09887	0,145396	1	0,177881
Fourth sub-process efficiency	0,281771	0,281771	-0,08982	0,225273	0,177881	1
	IV					
Relational efficiency	1	1	-0,0094	0,2068	0,170763	0,203507
Calculated relational efficiency	1	1	-0,0094	0,2068	0,170763	0,203507
First sub process efficiency	-0,0094	-0,0094	1	-0,14678	-0,08184	-0,10184
Second sub-process efficiency	0,2068	0,2068	-0,14678	1	0,191021	0,047469
Third sub-process efficiency	0,170763	0,170763	-0,08184	0,191021	1	-0,01198
Fourth sub-process efficiency	0,203507	0,203507	-0,10184	0,047469	-0,01198	1
	OUTPUT ORIENTATION					
	I					
Relational efficiency	1	1	0,258596	0,092688	0,099812	0,243368
Calculated relational efficiency	1	1	0,258596	0,092688	0,099812	0,243368
First sub process efficiency	0,258596	0,258596	1	-0,1991	-0,20888	0,0949
Second sub-process efficiency	0,092688	0,092688	-0,1991	1	0,608915	-0,20888
Third sub-process efficiency	0,099812	0,099812	-0,20888	0,608915	1	-0,15114
Fourth sub-process efficiency	0,243368	0,243368	0,0949	-0,20888	-0,15114	1
	II					
Relational efficiency	1	1	0,258596	#NUM!	0,701031	0,243368
Calculated relational efficiency	1	1	0,258596	#NUM!	0,701031	0,243368
First sub process efficiency	0,258596	0,258596	1	#NUM!	0,407389	0,0949
Second sub-process efficiency	#NUM!	#NUM!	#NUM!	1	#NUM!	#NUM!
Third sub-process efficiency	0,701031	0,701031	0,407389	#NUM!	1	0,352086
Fourth sub-process efficiency	0,243368	0,243368	0,0949	#NUM!	0,352086	1
	III					
Relational efficiency	1	1	0,307872	#NUM!	0,927164	#NUM!
Calculated relational efficiency	1	1	0,307872	#NUM!	0,927164	#NUM!
First sub process efficiency	0,307872	0,307872	1	#NUM!	0,347914	#NUM!
Second sub-process efficiency	#NUM!	#NUM!	#NUM!	1	#NUM!	#NUM!
Third sub-process efficiency	0,927164	0,927164	0,347914	#NUM!	1	#NUM!
Fourth sub-process efficiency	#NUM!	#NUM!	#NUM!	#NUM!	#NUM!	1
	IV					
Relational efficiency	1	1	0,307872	0,118829	0,927164	0,295091
Calculated relational efficiency	1	1	0,307872	0,118829	0,927164	0,295091
First sub process efficiency	0,307872	0,307872	1	0,015969	0,347914	0,036728
Second sub-process efficiency	0,118829	0,118829	0,015969	1	0,116996	0,110918
Third sub-process	0,927164	0,927164	0,347914	0,116996	1	0,269173

efficiency						
Fourth sub-process efficiency	0,295091	0,295091	0,036728	0,110918	0,269173	1

The correlation analysis reported in the Table 7, among other possible observations, outline that the efficiency of the first sub-process is, albeit slightly, negatively correlated with the efficiency scores of all remaining sub-process under all the system preferences. While the efficiency of the second sub-process is positively correlated with the efficiency of the third sub-process although the correlation of it with the third sub-process is greater under the first system (0.35) of preferences than in all others system of preferences (0.145,0.145 and 0.191 under the second, third and fourth, respectively). This latter is also the higher positive correlation that we observe in the Table 7. While the lower positive correlation is $\cong 0.0475$ and occur between the fourth and second sub-process under the fourth system of preferences. In the Figure 2 below we report the path of the efficiencies.

4.1 Which system of preferences?

The objective of empirical application is to measure the relative efficiency of the whole system and its parts. Another objective is to compare the effect of preference systems on the measurement of relative efficiency. In principle line, for researchers/managers, a system of preference for the allocation of resources among sub-process that allows to have a higher relative efficiency of process and sub-process is preferable to any other. Here, we will show the results of the Wilcoxon-Mann-Whitney test[6] once tested that the data (the vectors of our efficiency scores) do not belong to the normal distribution[7] and that their variance are homogeneous[8]. All test results allow to infer that the calculated efficiency scores do not belong to a normal distribution, that their variances are homogeneous, that the relational efficiency scores under the system of preference 1, 2 and 3 come from the same distribution and that the relational efficiency scores under the system of preference 4 do not belong to the same distributions of systems 1, 2 and 3. The results are in the Table 7 below.

Table 7 Wilcoxon-Mann-Whitney test

	Comparison between the four preference systems (I ,II, III, IV)					
	INPUT					
	Two side test					
	I vs II	I vs III	I vs IV	II vs III	II vs IV	III vs IV
Calculated Statistic Value	11412	11454	14302	11316	14196	14161
p-value	0.8285	0.7858	4.734e-05	0.9292	8.529e-05	0.0001047
	One side $\mu_{iv} > \mu_i, \mu_{ii}, \mu_{ii}$					
Calculated Statistic Value			8198.5		8304	8339
p-value			1		1	0.9999
	OUTPUT					
	Two side test					
	I vs II	I vs III	I vs IV	II vs III	II vs IV	III vs IV
Calculate Statistic Value	11250	11250	11250	11250	11250	11250
p-value	1	1	1	1	1	1

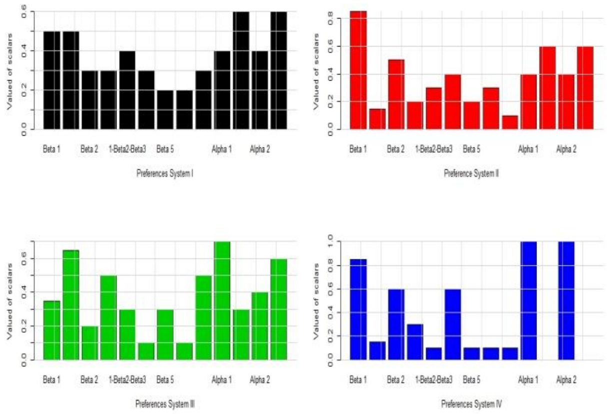
[6] The Wilcoxon-Mann-Whitney assume that the population distribution to which the sample are extract are not normally distributed, that the true population's variance is unknown and that the sample variance are homogeneous. This later hypothesis can be tested using the test F of Fisher.

[7] At this end we conduct five tests: 1) Jarque-Brera normality test 2) Shapiro-Wilk normality test 3)L illiefors (Kolmogorov-Smirnov) normality test 4) Shapiro-Francia normality test 5) Pearson chi-square normality test.

[8] At this end we conduct the test F di Fisher.

As we can see, in the case of input measurement, the preference system IV would be preferred because the average of the relational efficiency of the whole process (see Figure 2) is larger than the average of the relational efficiency estimates under the remaining preference systems (see the values of the p-values in the case of One side test[9] in Table 7). Instead, in the case of output measurement there are no statistically significant differences in the relative efficiency scores of the entire process. Differences appear at level of sub-process efficiency (see the plot on ecdf Figures 2). Yet, in the case of output orientation measurement there are no differences in the distributions (see Figure 3).

***Figure 4** Barplot of the scalars*



The plot of the efficiency scores are in Figure 5 and 6 , while the comparison of the system preferences are in Figure 4.

***Figure 5** Simple plot input CRS*

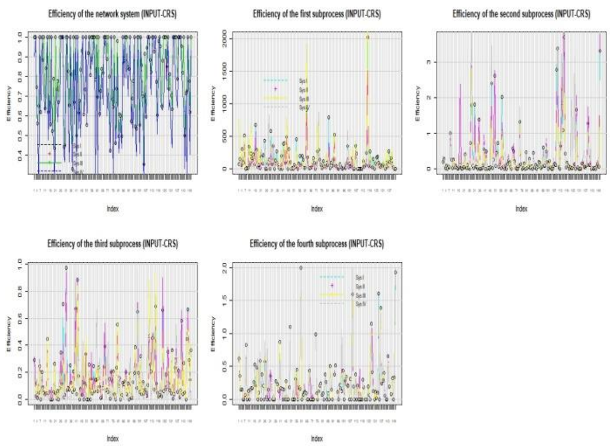
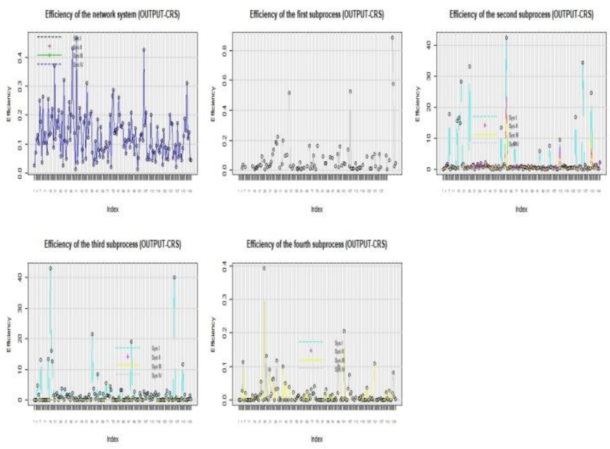


***Figure 6** Simple plot output CRS*

[9] The null hypothesis in this case is: the mean of the relational efficiency under the preference system IV (μ_{IV}) is greater (>) that the mean of the relational efficiency under all others preference systems.



5. Discussion and conclusions

The paper developed a multiplier NDEA model to measure the relative efficiency of a network system of four stages. The measurements have been carried out both in the input and output orientation, assuming constant return to scale (CRS) for the entire system and for each part of it. The type of interrelationships between the sub-processes of the system proposed here have been defined by the researcher at his discretion without any reference to real production processes. The solved network model considers the existence of relational variables shared between sub-processes 1 and 2 (z_{12}, z_{23}) and between sub-processes 1 and 3 (z_{13}, z_{23}). This configuration can also be modified by acting on the system of resource allocation preference between the sub-processes as happen for example in the case of the preferences system IV. In this last system of preference (IV) the sub-process 3 does not have shared relational variables but it is sharing two inputs with the sub-process 1. When the objective is to increase the outputs of the system the output-NDEA model should be solved. All the NDEA models presented here have been solved with the simplex method. The application show that when we adopt the same weights for the same variables in the model (in other words we estimates a relational NDEA model [(Kao, 2009(a)), (Kao, 2009(b)) (Kao, 2014)]) the relative efficiency of the sub-process can be calculated applying the multiplicative decomposition formula proposed by (Kao, 2009(a)) and the sources of inefficiency can be individuate at sub-process level. The characteristic of our model is that the proportion of resources that sub-process are sharing is defined *a-priori* to the researcher or/and manager of the organization where the process is running. This is traducing in the *a-priori* fixing of the vector parameters $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ of our model. This leaves the researcher/manager to define his preferences about the proportions of resources to be allocated to each sub-process. In other words, the latter parameters are assigned exogenously rather than being calculated endogenously inside the NDEA model. The interpretation of the optimal weights is the same as in that given in the standard DEA models, they represent the contribution of the resource to the improvement of the relative efficiency of the sub-process such as that of the process. The estimation of relative efficiency (see Table 3,4,5 and 6) through our NDEA model is strictly dependent on the preferences (see Table 2) on the distribution of resources among the sub-processes defined by the researcher. Differently to the literature on the topics the paper develop a four stages system process, sharing all the limits and advantages of the relational NDEA approach. However, we believe that the model proposed here cover a wide range of real production processes, so the NDEA model developed here can be applied for many applications with real data.

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