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Abstract

This paper deals with the problem of allocating costs in set covering situations. In particular, we focus on set covering situations where the optimal covering is given in advance. Thus, we take into account only the facilities that have to be opened and look for rules distributing their cost. We define a cooperative game and study the core and the nucleolus. We also introduce two new rules: the equal split rule on facilities and the serial rule. We axiomatically characterize the core, the nucleolus, and the two rules. Finally, we study several monotonicity properties of the rules.

Keywords: set covering problems; cost sharing rules; cooperative games

JEL Classification: C71; D61; D70

1 Introduction

An interesting logistic problem arises when there is a set of agents who wish to (or must) be provided with a service which has to be located at several different points in order to serve all those agents. This is a very well-known problem in operations research, called the "set covering problem" (Berge, 1957; Toregas et al., 1971). In particular, there are a set of potential locations, such that each of them covers a subset of agents to be served. Each potential location has a fixed construction and start-up cost associated with it. Therefore, the goal is to choose a subset of potential locations such that all agents are covered and the cost is minimized. Set covering problems are important and interesting

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not only from the viewpoint of theoretical optimization, but also from that of practice. Thus these problems have been applied in many different fields, see Beasley and Jørnsten (1992). Garcia and Marin (2015) survey different covering location models.

After a set covering problem is solved, the next question to be answered is who must pay the cost or, equivalently, how much each agent benefiting from the service must pay? This question can be answering using cooperative game theory and/or cost sharing theory.

In this paper, we focus on situations in which the set of locations is already established and there is no possibility of modifying the locations where the facilities are to be set up. This can occur for several reasons, e.g. because a reasonable solution has been determined cannot be improved on in a reasonable time (note that the set covering problem is NP-complete (Karp, 1972)); because the cost to be divided is the maintenance cost; or because the interested parties have reached an agreement on the final location of the facilities. Therefore, we take into account only the facilities that have to be opened. For this situation, we pose the following research questions:

- How much does each agent who benefits from the service have to pay?
- What properties do the cost sharing rules satisfy?

In order to answer these questions, we first introduce a cooperative game which summarizes for each coalition the cost of using those locations that cover all agents in the coalition at minimum cost. For this game we analyze the core (Gillies, 1953) and the nucleolus (Schmeidler, 1969). Two more cost sharing rules are also introduced: The equal split rule on facilities, and the serial rule. All these three rules are characterized by means of different reasonable properties. In addition, different monotonicity properties are studied in order to learn more about the behavior of the rules when some changes in the problem are introduced. To the best of our knowledge there is no previous research into this approach for allocating the cost in set covering problems. Therefore, we are filling a gap in the literature on cost sharing and logistics.

The rest of the paper is organized as follows. Section 2 presents a review of the relevant literature. Section 3 describes our model and some preliminary concepts. Section 4 introduces the core and the three rules. Section 5 characterizes the core and the three rules using a reduced set of properties. Section 6 analyzes the monotonicity properties of the rules. Section 7 concludes.

2 Literature Review

In this section we review the literature most closely related to our paper. That literature contains many papers studying operations research (OR) problems from a game theoretic perspective. This is because in many OR problems there may be more than one agent involved, so then a natural question to be answered is how to share the extra profits or the cost savings obtained from cooperation. The classical OR problems studied from a game theoretic perspective include minimum cost spanning tree problems (Claus and Kleitman, 1973; Bird, 1976; Bergantiños and Vidal-Puga, 2007; Bergantiños and Lorenzo-Freire, 2008), assignment problems (Shapley and Shubik, 1972), transportation problems

(Sanchez-Soriano, 2003, 2006), linear production problems (Owen, 1975), source connection problems (Bergantiños et al., 2014) and network flow problems (Kalai and Zemel, 1982). Borm et al. (2001) provides a survey on OR-games.

Regarding location problems, cooperative game theory is used in Granot (1987) to deal with the problem of locating one facility in a tree and generating a cost allocation. Tamir (1992) shows that the class of cost allocation games on a tree graph has a non empty core. Deng et al. (1999) consider a class of combinatorial optimization games, which includes covering games, and state that the core is non empty if and only if the linear relaxation of the corresponding linear program has an integer solution. Puerto et al., (2001) introduce a family of cooperative games arising from continuous single facility location problems and show two sufficient conditions so their location games to have a non-empty core. Goemans and Skutella (2004) prove that for the unconstrained facility location problem decisions as to whether the core is non empty and whether a cost allocation is in the core are NP-complete. Caprara and Letchford (2010) define a more general class of games that contains combinatorial games as a particular case, and compute "good" cost shares for these games by using cutting plane and column generation methods. The several papers that survey location games include that of Fragnelli and Gagliardo (2013), where attention is given to problems that are open in this field.

In the literature several papers can be found that study the set covering problem from different game theoretic approaches. Devanur et al. (2005) introduce a set covering cost sharing mechanism based on the LP-relaxation of the set covering problem and prove that the mechanism is strategyproof but not group strategyproof, and provides a solution close to the optimum set covering. Li et al. (2005) define three generalized set covering games and study cost sharing and strategyproof mechanisms for them. In one of the games, they introduce a cost sharing method which is approximately budget-balanced and belongs to the α-core. In the other two games, they propose strategyproof charging/payment mechanisms for allocating the costs.

Our approach to set covering games differs from these papers in several aspects. First, we consider a fixed solution as given, then define a cooperative game and study its core and nucleolus. Second, we introduce two cost sharing methods based on the structure of the given solution itself. And finally, we axiomatically characterize the proposed cost sharing methods, including the nucleolus and the core. Therefore, our approach and results are completely different and novel in this regard.

Recently, several non cooperative set covering games have been defined in the literature. Escoffier et al. (2010) define a strategic set covering game in which agents are charged a fraction of the cost of the facility that they have chosen. The authors prove the existence of pure strategy Nash equilibria and pure strong strategy equilibria for different reasonable tax functions and study their price of anarchy, which is the worst case ratio between the social cost of an equilibrium and the optimal social cost. Cardinal and Hoefer (2010) define non cooperative set covering games and prove the existence of exact Nash equilibria by using LP-relaxation and duality. They also prove that the price of stability, which measures the best Nash equilibrium in terms of the optimum cost instead of the worst equilibrium, is 1. Balcan et al. (2011) define non-cooperative covering games with learning dynamics for studying convergence towards different equilibria which are close to the optimum. And Piliouras et al. (2015) define non cooperative set covering games and prove the existence of Nash equilibria for them, also by using standard LP-relaxation. They further determine the price of anarchy for those games. All these papers approach the set covering problem from a non cooperative perspective, while we do so from a cooperative perspective. Thus, our approach and results are completely different from those of the above-mentioned papers.

In conclusion, as far as we know, our game theoretic approach to set covering problems and the results obtained are new and fill a gap in the literature on a relevant operations research problem. We provide different cost sharing methods which are analyzed axiomatically to show their interest and relevance in terms of reasonable properties that a cost sharing method should satisfy.

3 Set covering model and preliminaries

A set of agents $N = \{1, ..., n\}$ need a service that could be provided by a set of facilities M. Each facility $k \in M$ is characterized by a pair (c_k, A_k) where $c_k \geq 0$ denotes the cost of opening, maintaining, constructing, activating, etc. a facility and $A_k \subset N$ is the set of agents that are served when that facility is opened.

The set covering problem (SCP) consists of opening a subset of M such that all agents in N are served by at least one facility and the total cost is minimized. Formally, this involves looking for $T^N \subset M$ such that

$$\begin{aligned} \bigcup_{k \in T^N} A_k &= N \text{ and} \\ \sum_{k \in T^N} c_k &\leq \sum_{k \in T} c_k, \text{ for each } T \subset M : \bigcup_{k \in T} A_k = N. \end{aligned}$$

It is well-known that SCP is an NP-complete problem, i.e. there is no known algorithm which always reaches an optimal solution of the problem in polynomial time.

We focus on set covering situations where a reasonable cover is given in advance. This approach is meaningful in several circumstances. The first is related to situations where the facilities have been open for some time, there is now a need to divide the maintenance costs of the open facilities among the agents covered by them. The second one is related to situations where the related set covering problem has been solved either by means of an exact algorithm or by a heuristic or, the agents have simply reached an agreement on what facilities are to be opened or built. Moreover, we seek only to take into account the facilities already open so as to distribute the cost of the given cover among the agents.

A *set covering cost sharing problem* (a "problem", for short) is a 4-tuple $P = (N, M, c, A)$ where:

- $N = \{1, \ldots, n\}$ is the set of agents.
- $M = \{1, \ldots, m\}$ is the set of facilities open.
- $c = (c_k)_{k \in M}$ is the vector of costs associated with the facilities.

  We assume that $c_k \geq 0$ for each $k \in M$.

- $A = \{A_k\}_{k \in M}$ with $A_k \subset N$ for each $k \in M$ denotes the agents covered by each facility.

  We assume that all agents are covered, *i.e.* $\bigcup\limits_{k \in M} A_k = N$, and there are no redundant facilities, *i.e.* for each $l \in M$, $\bigcup\limits_{k \in M \setminus \{l\}} A_k \neq N$.

Let $\mathcal{P}$ be the set of all set covering sharing cost problems. Denote by $c(P)$ the total cost of the facilities in M, *i.e*, $c(P) = \sum\limits_{k \in M} c_k$.

We now introduce some notation that will be used later.

Given a problem $P = (N, M, c, A)$ and $i \in N$, let H_i denote the set of facilities covering agent i. Namely,

$$H_i = \{k \in M : i \in A_k\}.$$

Moreover, let N_1 denote the set of agents covered by a single facility. Namely,

$$N_1 = \{i \in N : |H_i| = 1\}.$$

A *cost sharing solution* (a "solution", for short) is a mapping $f \colon \mathcal{P} \to \mathbb{R}^N$ satisfying that for all $P \in \mathcal{P}$, $f(P) \subset \mathbb{R}^N$ and for each $x \in f(P)$, $\sum\limits_{i \in N} x_i = c(P)$.

This means that we only consider allocations which distribute all the cost associated with the facilities among agents served by them (i.e., the classical efficiency condition holds).

When a cost sharing solution is always a single allocation, then we call it a *cost sharing rule* (a "rule").

Given $X, Y \subset \mathbb{R}^N$, we define the set $X + Y$ as

$$X + Y = \{z = x + y : x \in X \text{ and } y \in Y\}.$$

4 Cost sharing methods for set covering problems

In this section we introduce some rules for the set covering cost sharing problem which are either based on a particular cooperative game associated with the problem or are related to the structure of the problem.

4.1 Solutions based on game theory: the core and the nucleolus

Assume that to allocate the total cost of the facilities, the most relevant criterion is considered to be the effect that each subset of agents has on that total cost. An approach to the problem from the perspective of cooperative game theory would then be reasonable. Thus, the first step is to define a cooperative game associated with the problem which measures that effect on the total cost. Secondly a cooperative game solution must be used to allocate the total cost among the agents.

Following the idea above, we first associate with each problem $P = (N, M, c, A)$ a cost game (N, c_P) where for each $S \subset N$,

$$c_P(S) = \min_{T \subset M} \left\{ \sum_{k \in T} c_k : S \subset \bigcup_{k \in T} A_k \right\}.$$

Notice that $c_P(S)$ measures the cost of covering agents in S and that $c_P(N) = c(P)$ and $c_P(\emptyset) = 0$.

The core of a cost game (N, c) is defined as

$$core\,(N, c) = \left\{ (x_i)_{i \in N} : \sum_{i \in N} x_i = c(N) \text{ and} \sum_{i \in S} x_i \leq c(S) \text{ for each } S \subset N \right\}.$$

Given a problem P, the core of P, $core\,(P)$, is the core of the cost game c_P associated with P.

Our first result characterizes the core of a problem P.

Proposition 1 *For each problem $P = (N, M, c, A)$ we have that*

$$\begin{aligned} core\,(P) &= \{x \in \mathbb{R}_+^N : \sum_{i \in N} x_i = c(P);\ \textit{for each } i \notin N_1, x_i = 0, \\ \textit{and for each } k &\in M, \sum_{i \in A_k \cap N_1} x_i = c_k\} \end{aligned}$$

Proof. We first prove $\subset$. Let $x \in core\,(P)$. Then $x \in core\,(N, c_P)$ and hence

$$\sum_{i \in N} x_i = c_P(N) = c(P).$$

By definition of the core of c_P, for each $k \in M$,

$$\sum_{i \in A_k \cap N_1} x_i \leq c_P(A_k \cap N_1) = c_k.$$

Suppose that for some $k \in M$, $\sum\limits_{i \in A_k \cap N_1} x_i < c_k$. Let $S = N \backslash (A_k \cap N_1)$, then

$$\sum_{i \in S} x_i = c(P) - \sum_{i \in N \backslash S} x_i > c(P) - c_k = c_P(S)$$

where the last equality holds because $A \backslash A_k$ is the only covering for S. But $\sum\limits_{i \in S} x_i > c_P(S)$ contradicts that $x \in core\,(P)$. Hence, $\sum_{i \in A_k \cap N_1} x_i = c_k, \forall k \in M$.

Since $\sum\limits_{i \in A_k \cap N_1} x_i = c_k$ for all $k \in M$ and $\sum\limits_{i \in N} x_i = c(P) = \sum_{k \in M} c_k$, it follows that $\sum\limits_{i \notin N_1} x_i = 0$. Assume that $x_i > 0$ for some $i \in N_1$. Let $k \in M$ be such that $i \in A_k$.

Let $S = \{i\} \cup (A_k \cap N_1)$. Then $\sum_{j\in S} x_j = x_i + c_k > c_k = c_P(S)$, which contradicts that $x \in core\,(P)$. Hence, $x_i = 0$, $\forall i \notin N_1$.

Finally, we check that $x_i \geq 0$ for all $i \in N$. Given $i \in N_1$ there is a single $k(i) \in M$ such that $i \in A_{k(i)}$. If $A_{k(i)} \cap N_1 = \{i\}$, then $x_i = c_{k(i)} \geq 0$. Assume that $A_{k(i)} \cap N_1 = \{i\} \cup S$, with $S \neq \emptyset$ and $x_i < 0$. Since, $x_i + \sum_{j\in S} x_j = c_{k(i)}$, it follows that $\sum_{j\in S} x_j > c_{k(i)} = c_P(S)$ which contradicts that $x \in core\,(P)$.

We now prove $\supset$. Let

$$\begin{aligned} x &\in \{x \in \mathbb{R}_+^N : \sum_{i\in N} x_i = c(P);\ \text{for each } i \notin N_1, x_i = 0, \\ \text{and for each } k &\in M, \sum_{i\in A_k \cap N_1} x_i = c_k\} \end{aligned}$$

Let $S \subset N$ and $S_1 = S \cap N_1$. We consider two cases:

- $S_1 = \emptyset$. Then $\sum_{i\in S} x_i = 0 \leq c_P(S)$
- $S_1 \neq \emptyset$. Let $K = \{k \in M : i \in A_k \text{ for some } i \in S_1\}$. Note that

$$\sum_{i\in S} x_i = \sum_{i\in S_1} x_i = \sum_{k\in K} c_k.$$

Now, let K' be the optimal covering of S. Thus, $K \subset K'$. Therefore,

$$\sum_{i\in S} x_i = \sum_{k\in K} c_k \leq \sum_{k\in K'} c_k = c_P(S).$$

Therefore the result holds. ■

Since there are no redundant facilities, each facility has at least one agent connected only to it. Thus, the core of these games is always nonempty. This is good news because it means that coalitionally stable allocations such as the nucleolus can always be proposed.

Consider the following example.

Example 2 *Let* $N = \{1, 2, 3\}$; $A = \{A_1, A_2\}$ *where* $A_1 = \{1, 3\}$ *and* $A_2 = \{2, 3\}$; $c_1 = 6$ *and* $c_2 = 12$.

The core contains only one allocation given by $(6, 12, 0)$ *which just coincides with the nucleolus.*

It is well-known that the nucleolus, an outstanding cooperative rule, is always in the core when the core is nonempty. In general the nucleolus is not easy to calculate, but for these problems it is easy.

We define the **nucleolus** of a problem P as the nucleolus of the associated cost game (N, c_P).

The nucleolus satisfies the symmetry property. Namely, symmetric agents in the cost game (N, c) receive the same allocation under the nucleolus. For each $k \in N$ all agents in $N_1 \cap A_k$ are symmetric in (N, c_P) and the nucleolus belongs to the core. Thus, it can be deduced that the nucleolus of a problem P is given by,

$$\eta_i(P) = \begin{cases} 0 & \text{if } i \notin N_1 \\ \frac{c_k}{|N_1 \cap A_k|} & \text{if } i \in N_1 \cap A_k \end{cases} \tag{1}$$

Note that the nucleolus distributes the cost of a facility equally among all agents who are connected only to it. Moreover, agents connected to more than one facility pay nothing. Thus, the nucleolus can be considered as quite *"unfair"* because agents covered by more than one facility pay nothing.

In the next subsection we define two rules that overcome this shortcoming.

4.2 Rules based on the cost structure of the problem

As mentioned in the previous section, the allocations in the core of the problem are quite extreme. Agents covered by more than one facility pay nothing. The cost of each facility is paid only by those agents who are covered by it alone. There are situations where being covered by more than one facility is clearly better than being covered by just one. In such situations, the core would provide unfair allocations of the total cost.

For this reason we consider other rules that do not belong to the core of the problem, but provide fairer allocations in many cases. In particular, we introduce two rules which are based on the structure of the problem: The equal split rule on facilities and the serial rule.

The **equal split rule on facilities** (EF) allocates the cost of each facility equally among the agents covered by that facility. Formally,

$$EF_i(P) = \sum_{k \in M : i \in A_k} \frac{c_k}{|A_k|}, \text{ for all } i \in N. \tag{EF rule}$$

In Example **??** the equal split rule allocates the cost of facility A_1 equally between agents 1 and 3 and the cost of facility 2 equally between 2 and 3. Thus, agent 1 pays 3, agent 2 pays 6 and agent 3 pays $3 + 6 = 9$.

The equal split rule on facilities could be appropriate for problems where being covered by more than one facility is clearly better than being covered by only one, for example when the facilities are parks or hospitals.

Given a problem $P = (N, M, C, A)$ we assume, without loss of generality, that $c_1 \leq c_2 \leq .... \leq c_m$. By convention, we take $c_0 = 0$ and $A_0 = \emptyset$.

For each $i \in N$, let $m^P(i)$ denote the facility with the lowest index covering agent i. Namely, $m(i) = \min\{k : i \in A_k\}$. When no confusion arises we write $m(i)$ instead of $m^P(i)$

Now we define the **serial rule** (S) as

$$S_i(P)=\sum_{k=1}^{m(i)}\frac{(m-k+1)(c_k-c_{k-1})}{\left|N\backslash\left(\bigcup_{l=0}^{k-1}A_l\right)\right|},\ \text{for all } i\in N. \qquad \text{(serial rule)}$$

This rule is inspired by the serial cost sharing method introduced in Moulin and Shenker (1992). The idea behind of the serial rule is to divide the cost of each facility as follows:

$$\begin{array}{rccccc}
c_1= & c_1 & & & & \\
c_2= & c_1 & c_2-c_1 & & & \\
c_3= & c_1 & c_2-c_1 & c_3-c_2 & \dots & \\
\dots & \dots & \dots & \dots & \dots & \dots \\
c_m= & c_1 & c_2-c_1 & c_3-c_2 & \dots & c_n-c_{n-1} \\
\sum_{k\in M}c_k= & mc_1 & (m-1)(c_2-c_1) & (m-2)(c_3-c_2) & & c_n-c_{n-1}
\end{array}$$

First, mc_1 is allocated equally among all agents. Therefore, the cost of facility 1 (c_1) is fully paid. Now, all agents covered by that facility (A_1) are removed from the procedure. The costs of the remaining facilities are reduced by exactly c_1.

Second, $(m-1)(c_2-c_1)$ is divided equally among all agents in $N\backslash A_1$. Therefore, the cost of facility 2 (c_2) is fully paid. Now all agents covered by facility 2 (A_2) are removed from the procedure. The costs of the remaining facilities are reduced by exactly c_2-c_1.

Third, $(m-2)(c_3-c_2)$ is divided equally among all agent in $N\backslash(A_1\cup A_2)$. The procedure continues as above.

In Example **??** the serial rule divides $2c_1=12$ equally among all agents. The remaining (12-6) is assigned to agent 2, who is not covered by facility 1. Thus, agent 1 pays 4, agent 2 pays $4+6=10$ and agent 3 pays 4.

The serial rule could be appropriate for problems where being covered by more than one facility is the same as being covered by only one, e.g. when facilities are telephone antennas.

5 Some properties and characterization of the rules

Cost sharing solutions may contain many possible allocations, but only one must be chosen to be implemented. Therefore, a rule must somehow be applied. For this reason, from now on we focus mainly on rules. To select a rule a number of criteria must be taken into account. Two of them are the kind of service provided by the facilities and the properties satisfied by the various candidate rules. We distinguish various groups of properties which reflect different ideas of reasonable fairness that should be satisfied by a rule or solution.

The first group of properties is connected with what agents should pay after a rule is applied according to their characteristics.

The first property says that no agent can obtain a profit.

A rule f satisfies **non-subsidy** if for all $P=(N,M,c,A)$ and for all $i\in N$, $f_i(P)\geq 0$.

The second property says that each agent i cannot pay more than the amount that he/she would pay if i were the only agent.

A rule f satisfies **individual rationality** if for all $P = (N, M, c, A)$ and for all $i \in N$,

$$f_i(P) \leq c_P(i) = \min\{c_k : i \in A_k\}.$$

The third property says that if an agent is covered by a facility with no cost, then that agents should pay nothing.

Agent $i \in N$ is said to be *null* in $P = (N, M, c, A)$ if the cost of providing the service to agent i is 0, *i.e.* there is a facility $k \in M$ such that $i \in A_k$ and $c_k = 0$.

A rule f satisfies **null agent** if for all $P = (N, M, c, A)$ and all null agent $i \in N$, $f_i(P) = 0$.

The next property says that if all facilities covering agent i have cost 0, then agent i pays nothing.

Agent $i \in N$ is said to be *totally null* in $P = (N, M, c, A)$ if the cost of every facility providing the service to agent i is 0, *i.e.* $c_k = 0$ for each facility $k \in M$ such that $i \in A_k$.

A rule f satisfies **totally null agent** if for all $P = (N, M, c, A)$ and all totally null agent $i \in N$, $f_i(P) = 0$.

Note that if a rule satisfies null agent, then it also satisfies totally null agent. Therefore, the totally null agent property is weaker than the null agent property.

The second group of properties indicates how a rule should treat agents who are equals in terms of their situations in the set covering problem.

Agents $i, j \in N$ are said to be *symmetric* in $P = (N, M, A, c)$ if for each $A_k \in A$ such that $i \in A_k$ and $j \notin A_k$ it holds that $(A_k \setminus \{i\}) \cup \{j\} = A_{k'}$ for any $A_{k'} \in A$ and $c_k = c_{k'}$.

A rule f satisfies **symmetry** if for each $P = (N, M, c, A)$ and each pair of symmetric agents $i, j \in N$, $f_i(P) = f_j(P)$.

A weaker version of symmetry can be considered by only taking into account the cost of the cheapest facility that covers each agent. The idea is to consider that two agents are symmetric when their corresponding cheapest facilities have the same cost.

Agents $i, j \in N$ are said to be *minimum cost symmetric* in $P = (N, M, c, A)$ if $c_P(i) = c_P(j)$.

A rule f satisfies **minimum cost symmetry** if for each $P = (N, M, c, A)$ and each pair of minimum cost symmetric agents $i, j \in N$, $f_i(P) = f_j(P)$.

The following properties indicate how the allocation of the total cost changes when changes occur in the parameters defining the set covering problem.

The next property states that if the cost of a facility increases, then all agents covered by that facility are affected in the same way.

Given a problem $P = (N, M, c, A)$, $A_k \in A$ let $P' = (N, M, c', A)$ be such that $c'_l = c_l$ for all $l \neq k$ and $c'_k > c_k$. A rule f satisfies **equal treatment on facilities** if for each $i, j \in A_k$,

$$f_i(P') - f_i(P) = f_j(P') - f_j(P).$$

Additional properties of this type, in particular monotonicity properties which we consider relevant for problems of this kind, are analyzed below. They are not included in this section because they are not needed to characterize the rules introduced.

The last group of properties that we consider is related to the possibility of adding up the cost allocation of two set covering problems when they have same "physical" structure but different building costs.

A rule f satisfies **additivity** if for all $P=(N,M,c,A)$, $P'=(N,M,c',A)$, and $P''=(N,M,c+c',A)$,

$$f(P'')=f(P)+f(P').$$

A weaker version of additivity can be considered by relaxing the set covering situations in which the additivity holds.

A rule f satisfies **cone-wise additivity** if for all (N,M,c,A), (N,M,c',A) and $P''=(N,M,c+c',A)$ such that there is an order $\sigma: M\rightarrow\{1,...,m\}$ satisfying the requirement that if $\sigma(k)<\sigma(k^*)$, then $c_{\sigma(k)}\leq c_{\sigma(k^*)}$ and $c'_{\sigma(k)}\leq c'_{\sigma(k^*)}$,

$$f(P'')=f(P)+f(P').$$

5.1 Characterization of the core

We prove that the core is the maximal solution satisfying non-subsidy, null agent, and additivity. Therefore, any rule or solution that satisfies those properties will be located in the core of the set covering game.

We define the properties of non-subsidy, null agent and additivity for rules, but the core is a solution, not a rule. The definition of additivity for solutions is exactly the same as for rules. The formulations of non-subsidy and null agent for solutions are the following.

A solution f satisfies **non-subsidy** if for all $P=(N,M,c,A)$, all $x\in f(P)$ and all $i\in N$, $x_i\geq 0$.

A solution f satisfies **null agent** if for all $P=(N,M,c,A)$, all $x\in f(P)$ and all null agent $i\in N$, $x_i=0$.

We now present the main result of this subsection.

Proposition 3 *The core is the maximal solution that satisfies non-subsidy, null agent, and additivity.*

Proof. From Proposition **??**, it follows immediately that the core satisfies non-subsidy and null agent. Now we show that it also satisfies additivity.

We first prove the following:

$$core(N,M,c,A)+core(N,M,c',A)\subset core(N,M,c+c',A).$$

Let $(N,M,c,A),(N,M,c',A)$, $x\in core(N,M,c,A)$ and $x'\in core(N,M,c',A)$. Thus:

- $x,x'\in\mathbb{R}^N_+$ and hence $x+x'\in\mathbb{R}^N_+$.

- $\sum\limits_{i\in N} x_i = \sum\limits_{k\in M} c_k$ and $\sum\limits_{i\in N} x'_i = \sum\limits_{k\in M} c'_k$. Hence $\sum\limits_{i\in N}(x_i + x'_i) = \sum\limits_{k\in M}(c+c')_k$.
- Let $i \notin N_1$. Then $x_i = x'_i = 0$ and therefore $x_i + x'_i = 0$.
- For each $A_k \in A$, $\sum\limits_{i\in A_k\cap N_1} x_i = c_k$ and $\sum\limits_{i\in A_k\cap N_1} y_i = c'_k$. Hence

$$\sum_{i\in A_k\cap N_1} x_i + y_i = c_k + c'_k.$$

Therefore, $x + x' \in core(N, M, c + c', A)$.

We now prove that

$$core(N, M, c + c', A) \subset core(N, M, c, A) + core(N, M, c', A).$$

Let $z \in core(N, M, c + c', A)$. Then $z \in \mathbb{R}^N_+$, $\sum\limits_{i\in N} z_i = \sum\limits_{k\in M}(c_k + c'_k)$, $z_i = 0$ for all $i \notin N_1$ and $\sum\limits_{i\in A_k\cap N_1} z_i = c_k + c'_k$.

Let $x \in \mathbb{R}^N$ be defined in the following way:

- $x_i = 0$, for all $i \notin N_1$.
- For each $k \in M$, let $A_k \cap N_1 = \{i_1, \ldots, i_{h(k)}\}$. Define x on $A_k \cap N_1$ in the following recursive way:
  - $x_{i_1} = \min\{c_k, z_{i_1}\}$,
  - $x_{i_q} = \min\{c_k - \sum_{j=1}^{q-1} x_{i_j}, z_{i_q}\}$, for $2 \le q \le h(k)$.

Note that $x \in \mathbb{R}^N_+$. Since $\sum\limits_{i\in A_k\cap N_1} z_q = c_k + c'_k$, it follows that $\sum\limits_{i\in A_k\cap N_1} x_q = c_k$. Therefore $x \in core(N, M, c, A)$. Let $x' = z - x$. It can be easily checked that $x' \in core(N, M, c', A)$. Therefore,

$$core(N, M, c + c', A) \subset core(N, M, c, A) + core(N, M, c', A).$$

We now prove that if f is a solution satisfying all three properties, then f selects allocations in the core. Let $P = (N, M, c, A)$ and $x \in f(P)$. Since f satisfies non-subsidy, $x \in \mathbb{R}^N_+$. Obviously, $\sum\limits_{i\in N} x_i = c(P)$.

For each $k \in M$ we define the problem $P^k = (N, M, c^k, A)$ where c^k is given by

$$c^k_h = \begin{cases} c_k & h = k, \\ 0 & h \neq k. \end{cases} \tag{2}$$

Since f satisfies additivity, $f(N, M, c, A) = \sum\limits_{k\in M} f(N, M, c^k, A)$. Thus, there is $\{x^k\}_{k\in M}$ such that for all $k \in M$, $x^k \in f(N, M, c^k, A)$ and $x = \sum\limits_{k\in M} x^k$.

Given $i \notin N_1$ and $k \in M$ it follows that i is a null agent in (N, M, c^k, A). Since f satisfies null agent, $x_i^k = 0$ for all $k \in M$. Therefore $x_i = \sum_{k \in M} x_i^k = 0$.

Let $k \in M$. Agents in $A_k \cap N_1$ are null agents in (N, M, c^j, A) for each $j \neq k$. Since f satisfies null agent, $x_i^j = 0$ for each $i \in A_k \cap N_1$ and $j \neq k$.

Each $i \in N \backslash (A_k \cap N_1)$ is a null agent in (N, M, c^k, A). Since f satisfies null agent, $x_i^k = 0$ for each $i \in N \backslash (A_k \cap N_1)$. Thus,

$$\sum_{i \in A_k \cap N_1} x_i^k = \sum_{i \in N} x_i^k = c_k.$$

Now,

$$\sum_{i \in A_k \cap N_1} x_i = \sum_{j \in M} \sum_{i \in A_k \cap N_1} x_i^j = \sum_{i \in A_k \cap N_1} x_i^k = c_k.$$

Therefore, $x \in core(N, M, c, A)$. ■

5.2 Characterization of the nucleolus

In this subsection we give a characterization of the nucleolus with the properties of null agent, symmetry and additivity.

Theorem 4 *The nucleolus is the only rule that satisfies null agent, symmetry, and additivity.*

Proof. First, we prove that the nucleolus satisfies additivity, null agent and symmetry.

Since the nucleolus belongs to the core, it satisfies null agent.

Let $i, j \in N$ be symmetric in $P = (N, M, c, A)$. We distinguish the following cases:

- $i \notin N_1$. By symmetry $j \notin N_1$. Therefore $\eta_i(P) = \eta_j(P) = 0$.
- There is $k \in M$ such that $i \in A_k \cap N_1$. Since i and j are symmetric, two situations can arise:
  - $j \in A_k \cap N_1$. Then $\eta_i(P) = \eta_j(P) = \frac{c_k}{|A_k|}$.
  - $j \notin A_k \cap N_1$. Then $j \in A_{k'} \cap N_1$, where $A_{k'} = A_k \backslash \{i\} \cup \{j\}$ with $c_{k'} = c_k$. Therefore, $A_k \cap N_1 = \{i\}$ and $A_{k'} \cap N_1 = \{j\}$. Thus, $\eta_i(P) = c_k = c_{k'} = \eta_j(P)$.

Let $P = (N, M, c, A)$, $P' = (N, M, c', A)$ and $i \in N$. We distinguish two cases:

- $i \notin N_1$, then $\eta_i(P) = \eta_i(P') = \eta_i(P + P') = 0$.
- There is $k \in M$ such that $i \in A_k \cap N_1$. Then $\eta_i(P) + \eta_i(P') = \frac{c_k}{|A_k \cap N_1|} + \frac{c'_k}{|A_k \cap N_1|} = \frac{c_k + c'_k}{|A_k \cap N_1|} = \eta_i(P + P')$.

We now prove that every rule f satisfying null agent, symmetry, and additivity satisfies non-subsidy. Since f is additive the proof can be restricted to problems (N, M, c, A) such that there is $k \in M$ with $c_j = 0$ for every $j \neq k$. Note that every $i \notin A_k \cap N_1$ is a null agent, so $f_i(P) = 0$. Moreover, all agents in $A_k \cap N_1$ are symmetric, thus they share equally $c_k \geq 0$. Therefore, the rule satisfies non-subsidy.

Now, if a rule satisfies null agent, symmetry, and additivity, as a result of Proposition **??** it belongs to the core. Furthermore, there is only one symmetric allocation in the core and it is the nucleolus. Therefore, the result holds. ■

We now prove that the properties used in Theorem **??** are independent.

Remark 5 (a) *The equal distribution rule, given by* $\varphi_i(P) = \frac{c(P)}{|N|}, \quad \forall i \in N$, *satisfies symmetry and additivity, but not null agent.*

(b) *Consider the following rule,* α*:*

$$\alpha_i(N, M, c, A)_i = \begin{cases} 0 & \text{if } |H_i| > 1 \\ c_k & \text{if } |H_i| = 1 \text{ and } i = min\{j : j \in A_k\}. \end{cases}$$

α *satisfies null agent and additivity but not symmetry.*

(c) *The rule that distributes the total cost* $c(P)$ *equally among the non null agents satisfies null agent and symmetry, but not additivity.*

5.3 Characterization of the equal split rule on facilities

The equal split rule on facilities provides cost allocations which are not in the core in general, so properties other than those satisfied by the core must be used in order to characterize it. The theorem below shows that such properties are totally null agent, equal treatment on facilities, and additivity.

Theorem 6 *The equal split rule on facilities is the only rule that satisfies totally null player, equal treatment on facilities, and additivity.*

Proof. It is straightforward to prove that the equal split rule on facilities satisfies totally null agent and additivity. Now we prove that it also satisfies equal treatment on facilities.

Let $P = (N, M, c, A)$, $A_k \in A$, and $P' = (N, M, c', A)$ such that $c'_k > c_k$ and $c'_l = c_l$ for all $l \neq k$. Let $i, j \in A_k$. Then,

$$\begin{aligned} EF_i(N, M, c', A) - EF_i(N, M, c, A) &= \frac{c'_k - c_k}{|A_k|} \\ &= EF_j(N, M, c', A) - EF_j(N, M, c, A). \end{aligned}$$

Therefore, EF satisfies equal treatment on facilities.

Let f be a rule satisfying totally null agent, equal treatment on facilities, and additivity. Let $P = (N, M, c, A)$ and for all $k \in M$, $P^k = \left(N, M, c^k, A\right)$ defined as in formula (**??**). Since f satisfies additivity,

$$f(N, M, c, A) = \sum_{k \in M} f(N, M, c^k, A).$$

Now, it suffices to prove that for each $k \in M$,

$$f_i\left(N, M, c^k, A\right) = \begin{cases} \frac{c_k}{|A_k|} & i \in A_k, \\ 0 & i \notin A_k. \end{cases}$$

Let $k \in M$. Since f satisfies totally null agent, $f_i(N, M, c^k, A) = 0$ for all $i \notin A_k$.

Consider the problem (N, M, c^0, A) where $c_j^0 = 0$ for all $j \in M$. Since f satisfies totally null agent, $f_i(N, M, c^0, A) = 0$ for all $i \in N$. Now, since f satisfies equal treatment on facilities, for each $i, j \in A_k$ it follows that

$$f_i(N, M, c^k, A) - f_i(N, M, c^0, A) = f_j(N, M, c^k, A) - f_j(N, M, c^0, A)$$

which turns out into

$$f_i(N, M, c^k, A) = f_j(N, M, c^k, A).$$

Since

$$\sum_{i \in A_k} f_i\left(N, M, c^k, A\right) = \sum_{i \in N} f_i\left(N, M, c^k, A\right) = c\left(N, M, c^k, A\right) = c_k,$$

it follows that $f_i(P^k) = \frac{c_k}{|A_k|}$ for all $i \in A_k$. ■

We now prove that the properties used in Theorem **??** are independent.

Remark 7 (*a*) *The equal distribution rule satisfies equal treatment on facilities and additivity, but not totally null agent.*

(*b*) *The nucleolus satisfies totally null agent and additivity, but not equal treatment on facilities.*

Let $P = (N, M, c, A)$ *where* $N = \{1, 2, 3\}, M = \{1, 2\}, A = \{A_1, A_2\}$ *with* $A_1 = \{1, 2\}$ *and* $A_2 = \{2, 3\}$, $c_1 = c_2 = 1$. *Note that* $\eta(P) = (1, 0, 1)$. *Let* $P' = (N, A, c')$, *where* $c'_1 = 2, c'_2 = c_2 = 1$. *Then* $\eta(P') = (2, 0, 1)$. *Therefore,* $\eta_1(P') - \eta_1(P) \neq \eta_2(P) - \eta_2(P')$.

(*c*) *The rule that distributes the total cost* $c(P)$ *equally among the non totally null agents satisfies null agent and symmetry, but not additivity.*

5.4 Characterization of the serial rule

We now prove that the serial rule can be characterized with the properties of null player, minimum cost symmetry, and cone-wise additivity.

Theorem 8 *The serial rule is the only rule that satisfies null player, minimum cost symmetry, and cone-wise additivity.*

Proof. It is straightforward to prove that the serial rule satisfies null agent.

We now prove that it also satisfies minimum cost symmetry. Let $P = (N, M, c, A)$ and $i, j \in N$ such that $c_P(i) = c_P(j)$. Then, $c_{m(i)} = c_{m(j)}$. We distinguish two cases:

- $m(i) = m(j)$. Then $S_i(P) = S_j(P)$
- $m(i) \neq m(j)$. Assume, without loss of generality, that $m(i) < m(j)$. Then, for all $m(i) \leq k \leq m(j)$, $c_k = c_m(i) = c_m(j)$. Now, the following emerges:

$$
\begin{aligned}
S_j(P) &= \sum_{k=1}^{m(j)} \frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|} \\
&= \sum_{k=1}^{m(i)} \frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|} + \sum_{k=m(i)+1}^{m(j)} \frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|}
\end{aligned}
$$

Since $c_k = c_{k-1}$ for all $k = m(i)+1, ..., m(j)$ it can be concluded that $S_j(P) = S_i(P)$.

In order to prove that the serial rule satisfies cone-wise additivity, let (N, M, c, A) and (N, M, c', A) be such that there is an order σ satisfying the requirement that if $\sigma(k) < \sigma(k^*)$, then $c_{\sigma(k)} \leq c_{\sigma(k^*)}$ and $c'_{\sigma(k)} \leq c'_{\sigma(k^*)}$. Without loss of generality we assume that $c_1 \leq c_2 \leq \cdots \leq c_m$ and $c'_1 \leq c'_2 \leq \cdots \leq c'_m$. Therefore, $c_1 + c'_1 \leq c_2 + c'_2 \leq \cdots \leq c_m + c'_m$. Let $i \in N$, so

$$
\begin{aligned}
S_i(N, M, c+c', A) &= \sum_{k=1}^{m(i)} \frac{(m-k+1)\left((c_k + c'_k) - (c_{k-1} + c'_{k-1})\right)}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|} = \\
&= \sum_{k=1}^{m(i)} \frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|} + \sum_{k=1}^{m(i)} \frac{(m-k+1)\left(c'_k - c'_{k-1}\right)}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|} = \\
&= S_i(N, M, c, A) + S_i(N, M, c', A).
\end{aligned}
$$

Hence, the serial rule satisfies cone-wise additivity.

Now we prove that the serial rule is the only rule that satisfies these properties. Let f be a rule satisfying these properties. Let $(N, M, c, A) \in \mathcal{P}$. Without loss of generality we assume that $M = \{1, 2, \ldots, m\}$ with $c_1 \leq c_2 \leq \cdots \leq c_m$. For each $k \in M$, let (N, M, c^{*k}, A) given by:

$$
c_h^{*k} = \begin{cases} c_k - c_{k-1} & \text{if } h \geq k, \\ 0 & h < k. \end{cases}
$$

Let $i \in N$. Since f satisfies cone-wise additivity, it follows that

$$
f_i(N, M, c, A) = \sum_{k=1}^{m} f_i(N, M, c^{*k}, A).
$$

Note that for all $k > m(i)$, i is a null agent of (N, M, c^{*k}, A). Since f satisfies null agent, $f_i(N, M, c^{*k}, A) = 0$ for all $k > m(i)$.

Let $k \in \{1, \ldots, m(i)\}$. Note that all agents in $\cup_{l=0}^{k-1} A_l$ are null agents in (N, M, c^k, A). Then, $f_j(N, M, c^k, A) = 0$ for all $j \in \cup_{l=0}^{k-1} A_l$. Besides, all agents in $N \backslash \left(\cup_{l=0}^{k-1} A_l\right)$, including player i, are minimum cost symmetric in (N, M, c^{*k}, A). Therefore for all $k \leq m(i)$

$$f_i(N, M, c^{*k}, A) = \frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|}.$$

Then,

$$f_i(N, M, c, A) = \sum_{k=1}^{m(i)} \frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|} = S_i(N, M, c, A).$$

■

We now prove that the properties used in Theorem **??** are logically independent.

Remark 9 (a) *The equal distribution rule satisfies minimum cost symmetry and cone-wise additivity but not null agent.*

(b) *The nucleolus satisfies null agent and cone-wise additivity, but not minimum cost symmetry.*

(c) *The rule that distributes the total cost $c(P)$ equally among the non null agents satisfies null agent and minimum cost symmetry but not cone-wise additivity.*

6 Monotonicity properties for set covering rules

In this section we introduce some appealing monotonicity properties and analyze whether our three rules satisfy those properties.

We consider a situation in which a new agent is taken into account but he/she could be covered by existing facilities. Thus, none of the other agents would be worse off.

Let $P = (N, M, c, A), P' = (N', M, c, A') \in \mathcal{P}$ be such that $N' = N \cup \{n+1\}$, $A'_k \in \{A_k, A_k \cup \{n+1\}\}$ for each $k \in M$, with at least some $h \in M$ such that $A'_h = A_h \cup \{n+1\}$.

A rule f satisfies **population monotonicity** if for each $i \in N$ it holds that $f_i(P') \leq f_i(P)$.

Consider a set covering situation in which the cost of a facility increases. The property below states that agents covered by such a facility cannot be better off.

Formally, let $P = (N, M, c, A), P' = (N, M, c', A)$ be such that there is $k \in M$ with $c'_k > c_k$ and $c'_h = c_h$ for all $h \in M \backslash \{k\}$.

A rule f satisfies **cost monotonicity** if for each $i \in A_k$ it holds that $f_i(P') \geq f_i(P)$.

The next property is a stronger version of the previous one. If the cost of a facility increases no agent can be better off.

A rule f satisfies **strong cost monotonicity** if for each $i \in N$ it holds that $f_i(P') > f_i(P)$.

Obviously, strong cost monotonicity implies cost monotonicity.

Assume that a facility can cover more agents at the same cost but the rest of facilities are still necessary to provide the cover. Thus, the new agents covered by this facility cannot be worse off.

Let $P = (N, M, c, A)$ and $P' = (N, M, c, A')$ be such that there is $k \in M$ with $A_k \subsetneq A'_k$, $A'_h = A_h$ for all $h \in M \backslash \{k\}$ and $N'_1 \cap A'_h \neq \emptyset$ for all $h \in M$.

A rule f satisfies **covering monotonicity** if for each $i \in A'_k \backslash A_k$ it holds that $f_i(P') \leq f_i(P)$.

In the classical optimization model of set covering situations it is assumed that a set of facilities must be opened such that every agent is covered by at least one facility. Applying that philosophy to our model, it seems reasonable to propose that those agents who are covered by more facilities should pay less because, in a sense, they help to decrease the cost of the optimal solution.

On the other hand, there could be real situations in which being covered by more than one facility is better for agents than being covered by a single facility (emergency centers, children's parks,etc.). In that case it seems reasonable to propose that those agents who are covered by more than one facility should pay more than those covered by just one.

We introduce two monotonicity properties related to these situations. Consider a set covering situation in which an agent i is covered by more facilities than agent j. We say that a rule satisfies agent monotonicity 1 when agent i does not pay more than agent j, and a rule satisfies agent monotonicity 2 when agent i does not pay less than agent j.

Formally, let $P = (N, M, c, A)$ and $i, j \in N$ be such that $i \in A_k$ for every $k \in M$ such that $j \in A_k$.

A rule f satisfies **agent monotonicity 1** if $f_i(P) \leq f_j(P)$.

In the same conditions, a rule f satisfies **agent monotonicity 2** if $f_i(P) \geq f_j(P)$.

In the theorem below we discuss which monotonicity properties are satisfied by the three rules.

Theorem 10 *The following statements hold:*

1. *The nucleolus satisfies population monotonicity, cost monotonicity, strong cost monotonicity, covering monotonicity and agent monotonicity 1.*
2. *The equal split rule on facilities satisfies population monotonicity, cost monotonicity, strong cost monotonicity and agent monotonicity 2.*
3. *The serial rule satisfies population monotonicity, cost monotonicity, covering monotonicity and agent monotonicity 1.*

Proof. (1) The nucleolus.

- Population Monotonicity. Let $P = (N, M, c, A)$ and $P' = (N', M, c, A')$ as in the definition of population monotonicity. Let $i \in N$. We distinguish two cases:

- $i \in N \backslash N_1$. Since i is covered by the same facilities in P' as in P, it follows that $i \in N' \backslash N_1'$, and therefore $\eta_i(P') = \eta_i(P) = 0$ for all $i \in N \backslash N_1$.
- $i \in N_1$. Then there is a single $k \in M$ such that $i \in N_1 \cap A_k$. Therefore $\eta_i(P) = \frac{c_k}{|N_1 \cap A_k|}$. Moreover, $i \in N_1' \cap A_k'$. Two cases can arise:
  - $n+1 \in N_1' \cap A_k'$. Then

$$\eta_i(P') = \frac{c_k}{|N_1' \cap A_k'|} = \frac{c_k}{|N_1 \cap A_k| + 1} \leq \eta_i(P).$$

  - $n+1 \notin N_1' \cap A_k'$. Then $N_1' \cap A_k' = N_1 \cap A_k$ and $\eta_i(P') = \eta_i(P)$.
- Strong cost monotonicity. Let $P = (N, M, c, A)$ and $P' = (N, M, c', A)$ be as in the definition of strong cost monotonicity. Note that the amount by which the cost of facility A_k is increased, is shared equally among the among agents in $N_1 \cap A_h$. Moreover, the other agents pay exactly the same as in the previous situation. Now, the proof of the result is straightforward.

- Cost monotonicity. This is obvious because the nucleolus satisfies strong cost monotonicity.

- Covering monotonicity. Let $P = (N, M, c, A)$ and $P' = (N, M, c, A')$ be as in the definition of covering monotonicity. Every $i \in A_k' \backslash A_k$ will be covered in P' by at least two facilities (A_k' and the other facilities which cover i in P). Therefore $\eta_i(P') = 0 \leq \eta_i(P)$.

- Agent monotonicity 1. Let $P = (N, M, c, A)$ and $i, j \in N$ be as in the definition of the property. We distinguish two cases:

  - $j \in N \backslash N_1$. Then $i \in N \backslash N_1$ and $\eta_i(P) = \eta_j(P) = 0$.
  - $j \in A_k \cap N_1$. Then $i \in A_k$. If $i \in A_k \cap N_1$ then

$$\eta_i(P) = \eta_j(P) = \frac{c_k}{|A_k|}.$$

  Otherwise, there is $k' \in M$ such that $i \in A_k \cap A_k'$. Thus

$$\eta_i(P) = 0 \leq \eta_j(P) = \frac{c_k}{|A_k|}.$$

(2) The equal split rule on facilities

- Population monotonicity. Let $P = (N, M, c, A), P' = (N', M, c, A')$ be as in the definition of population monotonicity. The result is straightforward since in every facility of P' there are at least the same agents as in P.

- Strong cost monotonicity. Let $P = (N, M, c, A)$ and $P' = (N, M, c', A)$ be as in the definition of strong cost monotonicity. Note that the amount by which the cost of facility A_k is increased is shared equally among the among agents in $N_1 \cap A_h$. Moreover, the other agents pay exactly the same as in the previous situation. Now, the proof of the result is straightforward.

- Cost monotonicity. This is obvious because the nucleolus satisfies strong cost monotonicity.

- Agent monotonicity 2. The result is straightforward. If an agent i is covered by every facility covering agent j, then agent i pays the same amount as j for every facility covering j. For the facilities covering agent i but not agent j, if any, agent i will pay but agent j will not, so agent i will pay at least the same as agent j,

(3) The serial rule

- Population monotonicity. Let $P = (N, M, c, A)$ and $P' = (N', M, c, A')$ be as in the definition of the property. Without loss of generality, we assume that $c_1 \leq c_2 \leq \cdots \leq c_m$. Note that

$$\frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|} \leq \frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A'_l \right) \right|} \text{ when } 1 \leq k \leq m(n+1),$$

and

$$\frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|} = \frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A'_l \right) \right|} \text{ when } m(n+1) < k \leq m.$$

Therefore, $S_i(P) \leq S_i(P')$ for all $i \in N$.

- Cost monotonicity. Let $P = (N, M, c, A)$, $P' = (N, M, c', A)$ and $k \in M$ be as in the definition of the property. Assume, without loss of generality, that $c_1 \leq c_2 \leq \cdots \leq c_m$. Let $i \in A_k$. We distinguish the following cases:

  - $m^P(i) < k$. Then $m^{P'}(i) = m^P(i) < k$. Since $c'_h = c_h$ for all $h \leq m^P(i)$, it follows that $S_i(P) = S_i(P')$.
  - $m^P(i) = k$. We distinguish the following cases:
    * $c'_k \leq c_{k+1}$. Then $m^{P'}(i) = k$. Since $c'_h = c_h$ for all $h \leq k-1$, it follows that

$$\begin{aligned} S_i(P) &= \sum_{h=1}^{k-1} \frac{(m-h+1)(c_h - c_{h-1})}{\left| N \backslash \left(\bigcup_{l=0}^{h-1} A_l \right) \right|} + \frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|} \\ &\leq \sum_{h=1}^{k-1} \frac{(m-h+1)\left(c'_h - c'_{h-1}\right)}{\left| N \backslash \left(\bigcup_{l=0}^{h-1} A_l \right) \right|} + \frac{(m-k+1)\left(c'_k - c'_{k-1}\right)}{\left| N \backslash \left(\bigcup_{l=0}^{k-1} A_l \right) \right|} \\ &= S_i(P'). \end{aligned}$$

* $c'_k > c_{k+1}$. Then

$$\begin{aligned} S_i(P) &= \sum_{h=1}^{k-1} \frac{(m-h+1)(c_h - c_{h-1})}{\left| N \backslash \left(\bigcup\limits_{l=0}^{h-1} A_l \right) \right|} + \frac{(m-k+1)(c_k - c_{k-1})}{\left| N \backslash \left(\bigcup\limits_{l=0}^{k-1} A_l \right) \right|} \\ &\leq \sum_{h=1}^{k-1} \frac{(m-h+1)(c_h - c_{h-1})}{\left| N \backslash \left(\bigcup\limits_{l=0}^{h-1} A_l \right) \right|} + \frac{(m-k+1)(c_{k+1} - c_{k-1})}{\left| N \backslash \left(\bigcup\limits_{l=0}^{k-1} A_l \right) \right|} \\ &= \sum_{h=1}^{k-1} \frac{(m-h+1)(c_h - c_{h-1})}{\left| N \backslash \left(\bigcup\limits_{l=0}^{h-1} A_l \right) \right|} + \frac{(m-k+1)\left(c'_{k+1} - c'_{k-1}\right)}{\left| N \backslash \left(\bigcup\limits_{l=0}^{k-1} A_l \right) \right|} \\ &\leq S_i\left(P'\right). \end{aligned}$$

- Covering monotonicity. Let $P = (N, M, c, A)$ and $P' = (N, M, c, A')$ be as in the definition of the property. Let $i \in A'_k \backslash A_k$. Note that if $c_h \geq c_P(i)$, then $m^P(i) = m^{P'}(i)$. Hence $S_i(P) = S_i(P')$. If $c_h < c_P(i)$, then $m^P(i) > m^{P'}(i) = k$. Hence $S_i(P) > S_i(P')$.

- Agent monotonicity 1. Let $P = (N, M, c, A)$ and $i, j \in N$ be as in the definition of the property. Note that if $i \in A_k$ for every $k \in M$ such that $j \in A_k$, then $c_P(i) \leq c_P(j)$ and therefore, $S_i(P) \leq S_j(P)$.

■

The table below summarizes the properties satisfied by each rule. The asterisk (∗) denote the properties used in the characterizations.

	Nucleolus	Equal split	Serial
Non-subsidy	YES	YES	YES
Indiv. Ration.	YES	YES	YES
Null	YES *	NO	YES *
Totally Null	YES	YES *	YES
Symmetry	YES *	YES	YES
Minim. cost symm.	NO	NO	YES *
Equal treat. facil.	NO	YES*	NO
Additivity	YES *	YES *	NO
Cone-wise addit.	YES	YES	YES*
Population monot.	YES	YES	YES
Cost monotonicity	YES	YES	YES
Strong cost monot.	YES	YES	NO
Covering monot.	YES	NO	YES
Agent monot. 1	YES	NO	YES
Agent monot. 2	NO	YES	NO

We finish this section by proving the statements of the previous table that have not yet been proved.

We start with the nucleolus. From the definition we deduce that the nucleolus satisfies non-subsidy and individual rationality. Since it satisfies null agent, we deduce that it also satisfies totally null agent. Since it satisfies additivity, we deduce that it also satisfies cone-wise additivity.

Consider Example **??**. Agents 1 and 3 are minimum cost symmetric but the nucleolus is $(6, 12, 0)$. Thus, the nucleolus does not satisfy minimum cost symmetry.

Let $P = (N, M, c, A)$ as in Example **??**. Let $P' = (N, M, c', A)$ be such that $c'_1 = 8$ and $c'_2 = c_2 = 12$. Since the nucleolus of P' is $(8, 12, 0)$ we deduce that it does not satisfy equal treatment on facilities.

Consider Example **??**. Taking $j = 1$ and $i = 3$ revels that the nucleolus does not satisfy agent monotonicity 2.

We now study the equal split rule on facilities. From the definition we deduce that it satisfies non-subsidy, individual rationality, and symmetry.

Let $P = (N, M, c, A)$ be such that N, M, and A are defined as in Example **??**. Moreover $c_1 = 0$ and $c_2 = 12$. Thus, $EF(P) = (0, 6, 6)$. Since agent 3 is a null agent we deduce that EF does not satisfy null agent. Since agents 1 and 3 are minimum cost symmetric, we deduce that EF does not satisfy minimum cost symmetry.

The following example shows that EF does not satisfy covering monotonicity.

Example 11 *Let* $P = (N, M, c, A)$ *where* $N = \{1, 2, 3\}$, $M = \{1, 2\}$, $A = \{A_1, A_2\}$ *with* $A_1 = \{1, 2\}$ *and* $A_2 = \{3\}$, $c_1 = 2, c_2 = 2$. *Note that* $EF(P) = (1, 1, 2)$. *Let* $P' = (N, M, A', c)$, *where* $A'_1 = A_1 = \{1, 2\}$ *and* $A'_2 = \{2, 3\}$. *Then* $EF(P') = (1, 2, 1)$. *Note that* $2 \in A'_2 \backslash A_2$, *but* $1 = EF_2(P) < EF_2(P') = 2$.

Consider Example **??**. Then $EF(P) = (3, 6, 9)$. Taking $j = 1$ and $i = 3$ revels that EF does not satisfy agent monotonicity 1.

Finally we study the serial rule. From the definition we deduce that it satisfies non-subsidy, individual rationality, and symmetry. Since it satisfies null agent, we deduce that it also satisfies totally null agent.

Let $P = (N, M, c, A)$ as in Example **??**. Let $P' = (N, M, c', A)$ be such that $c'_1 = c_1 = 6$ and $c'_2 = 14$. Since $S(P) = (4, 10, 4)$ and $S(P') = (4, 12, 4)$ we deduce that S does not satisfy equal treatment on facilities.

Let $P = (N, M, c, A)$ and $P' = (N, M, c', A)$ be such that N, M, and A are defined as in Example **??**. Moreover $c_1 = c'_2 = 0$, $c_2 = c'_1 = 6$. Since $S(N, M, c, A) = (0, 6, 0)$, $S(N, M, c', A) = (6, 0, 0)$, and $S(N, M, c + c', A) = (4, 4, 4)$ we deduce that S does not satisfy additivity.

Let $P = (N, M, c, A)$ and $P' = (N, M, c', A)$ be such that N, M, and A are defined as in Example **??**. Moreover $c_1 = 6$ and $c_2 = c'_1 = c'_2 = 12$. Since $S(N, M, c, A) = (4, 10, 4)$, $S(N, M, c', A) = (8, 8, 8)$ we deduce that S does not satisfy strong cost monotonicity.

Consider Example **??** where $S(N, M, c, A) = (4, 10, 4)$. Taking $j = 2$ and $i = 3$ revels that S does not satisfy agent monotonicity 2.

7 Conclusions

This paper looks at the problem of allocating costs in set covering situations. In particular, we address the distribution of costs when the facilities to be opened have been decided in advance. Three rules are proposed. The first one is based on a well-known solution concept of game theory: The nucleolus. The other two rules, called the equal split on facilities and serial rules, are based on the structure of the problem itself. Each rule has its own characteristics, so each is suitable for different contexts.

The nucleolus and the serial rule work well when being covered by a single facility is the same as being covered by more than one. The nucleolus assigns the total cost to agents covered by only one facility. In the serial rule agents covered by only one facility pay less, but agents covered by several facilities pay something. The equal split rule on facilities can be used in those situations in which being covered by a single facility is worse than being covered by more than one. For this reason, agents covered by more than one facility pay more than agents covered by only one.

The three rules are characterized by properties that are reasonable in each case. Finally, several monotonicity properties are studied for these rules.

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