Demand and equilibrium with inferior and Giffen behaviors

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Abstract

We introduce a class of differentiable, strictly increasing, strictly concave utility functions exhibiting an explicit demand of a good which may have Giffen behavior. We provide a necessary and sufficient condition (bases on prices and consumers' preferences and income) under which this good is normal, inferior or Giffen good. Interestingly, with this utility, the equilibrium price of a good may increase in the aggregate supply for this good.

JEL Classifications: D11, D50.
Keywords: Inferior good, Giffen good, equilibrium price.

1 Introduction

Inferior and Giffen goods have been mentioned in most microeconomics textbooks (see Mas-Colell et al. (1995), Jehle and Reny (2011), Varian (2014) for instance).[1] However, they are usually illustrated by pictures. In this paper, we present a class of differentiable, strictly increasing, strictly concave utility functions exhibiting an explicit demand of a good which may have Giffen behavior. In our example, the consumption set is $\mathbb{R}^2_+$, and the demand function generated by our simple utility function has a closed-form. Thanks to this tractability, we provide a necessary and sufficient condition (based on prices and consumers' preferences and income) under which this good is normal, inferior or Giffen good. This helps us to analytically study income and prices

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[1] Jensen and Miller (2008) provide real evidences (in two provinces of China: Hunan and Gansu) of Giffen behavior.

effects. In particular, we show that the Giffen behavior arises when the price is not so high and the consumer's income is at the middle level. This is supported by empirical evidences in Jensen and Miller (2008): when the price of a staple good increases, the poor people responds by decreasing their demand of this good while the group in the middle increases demand.

The second part of our paper focuses on the general equilibrium effects. Our utility function leads to an interesting point in general equilibrium context: the price of a good may be an increasing function of the aggregate supply of this good. Moreover, we show that the Giffen behavior may arise in equilibrium when preferences or/and endowments of agents change.

In the existing literature, several examples with Giffen behavior have been proposed. However, in most of the cases, utility functions are piecewise-defined or demand functions are not explicit or the consumption set is restricted. Heijman and von Mouche (2012) provide a collection of papers studying Giffen goods, including the paper of Doi, Iwasa, and Shimomura (2009).

Here, we just mention two recent papers (Haagsma, 2012; Biederman, 2015). Haagsma (2012) presents a separable utility function generating Giffen behavior.[2] In this example, the consumption set is restricted (precisely, it is $(\gamma_1, \infty) \times [0, \gamma_2)$ with $\gamma_1 > 0, \gamma_2 > 0$) and the utility function is quasi-concave but not concave. Moreover, in Haagsma (2012), the good 1 demand c_1 is always decreasing in the income, denoted by w, whatever the prices and the consumer's income. However, in our model, the sign of $\frac{\partial c_1}{\partial w}$ depends on the prices and the consumer's income. Recently, Biederman (2015) provides a concave utility function[3] and gives some numerical examples where Giffen behavior arises. However, the demand function is not explicit. In our paper, we can explicitly derive the demand function.

2 Individual demand

Assume that there are two goods and the consumption set is $\mathbb{R}^2_+$. Given prices $p_1 > 0$, $p_2 > 0$ and income $w > 0$, the consumer maximizes her utility $U(c_1, c_2)$ subject to the budget constraint $p_1c_1 + p_2c_2 \leq w$. We will study how the demand c_1 changes when the consumer's income w or/and price p_1 change.

Assume that the solution is unique and interior, then it is determined by $p_1c_1 + p_2c_2 = w$ and the first order condition

$$\frac{U_1\left(c_1, \frac{w-p_1c_1}{p_2}\right)}{p_1} = \frac{U_2\left(c_1, \frac{w-p_1c_1}{p_2}\right)}{p_2} \tag{1}$$

where $U_i(c_1, c_2) \equiv \frac{\partial U}{\partial c_i}(c_1, c_2)$ for $i = 1, 2$. From this, we obtain the following result.

[2] The utility function is $u(c_1, c_2) = \alpha_1 ln(c_1 - \gamma_1) - \alpha_2 ln(\gamma_2 - c_2)$ where $0 < \alpha_1 < \alpha_2$ and $\gamma_1, \gamma_2 > 0$, with the domain $c_1 > \gamma_1$ and $0 \leq c_2 < \gamma_2$.

[3] Biederman (2015) considers the following utility function

$$u(c_1, c_2) = \begin{cases} \frac{(c_1+\alpha c_2)^{1-\sigma}}{1-\sigma} - Ae^{-\beta c_1} & \text{for } \sigma > 0, \sigma \neq 1 \\ ln(c_1 + \alpha c_2) - Ae^{-\beta c_1} & \text{for } \sigma = 0 \end{cases}.$$

Lemma 1. *Assume that U is strictly concave, strictly increasing and in C^2. Assume that (c_1, c_2) is an interior solution. Then, we have that:*

$$\frac{\partial c_1}{\partial w} < 0 \quad \text{if and only if} \quad \frac{p_1}{p_2} U_{22}(c_1, c_2) - U_{21}(c_1, c_2) > 0 \tag{2a}$$

$$\frac{\partial c_1}{\partial p_1} > 0 \quad \text{if and only if} \quad \Big(\frac{p_1}{p_2} U_{22}(c_1, c_2) - U_{21}(c_1, c_2)\Big) c_1 > U_2(c_1, c_2). \tag{2b}$$

Consequently, $\frac{\partial c_1}{\partial p_1} > 0$ implies $\frac{\partial c_1}{\partial w} < 0$ (if good 1 is Giffen, then it must be inferior).

Proof. See Appendix A.1 □

We now introduce a class of utility function generating demand with Giffen behavior. Suggesting by (2b), we choose a function such that U_{21}/U_{22} is constant.

Proposition 1. *We assume that*

$$U(c_1, c_2) = c_1 + bc_2 + A\frac{(ac_1 + c_2)^{1-\lambda}}{1-\lambda} \tag{3}$$

where $a, b, \lambda, A > 0$, $\lambda \neq 1$, and $ab \neq 1$.[4] *This function is strictly increasing, strictly concave and differentiable. The demand function for good 1 is given by*

$$c_1 = \begin{cases} 0 & \text{if } A(ap_2 - p_1) \leq (bp_1 - p_2)\big(\frac{w}{p_2}\big)^\lambda \\ \frac{p_2\left(A\frac{ap_2 - p_1}{bp_1 - p_2}\right)^{\frac{1}{\lambda}} - w}{ap_2 - p_1} & \text{if } (bp_1 - p_2)\big(\frac{w}{p_2}\big)^\lambda < A(ap_2 - p_1) < (bp_1 - p_2)\big(\frac{aw}{p_1}\big)^\lambda \\ \frac{w}{p_1} & \text{if } A(ap_2 - p_1) \geq (bp_1 - p_2)\big(\frac{aw}{p_1}\big)^\lambda. \end{cases} \tag{4}$$

Moreover, the demand function is continuous.[5]

Proof. See Appendix A.2. □

We now provide intuitions explaining the form (4) of the demand function. It is easy to see that

$$A(ap_2 - p_1) \leq (bp_1 - p_2)\big(\frac{w}{p_2}\big)^\lambda \Leftrightarrow \frac{a + \frac{1}{A}\big(\frac{w}{p_2}\big)^\lambda}{1 + b\frac{1}{A}\big(\frac{w}{p_2}\big)^\lambda} p_2 \leq p_1 \tag{5a}$$

$$A(ap_2 - p_1) \geq (bp_1 - p_2)\big(\frac{aw}{p_1}\big)^\lambda \Leftrightarrow \frac{a + \frac{1}{A}\big(\frac{aw}{p_1}\big)^\lambda}{1 + b\frac{1}{A}\big(\frac{aw}{p_1}\big)^\lambda} p_2 \geq p_1 \tag{5b}$$

Consider the function $f(x) = \frac{a+x}{1+bx}$. We have $f'(x) = \frac{1-ab}{(1+bx)^2}$. By consequence, $\min(a, 1/b) \leq f(x) \leq \max(a, 1/b)\ \forall x \geq 0$. So, according to (5a), the consumer does not buy good 1 if the relative price of good 1 is high (in the sense that $p_1/p_2 \geq \max(a, 1/b)$).

[4] Indeed, when $ab = 1$, maximizing this utility function is equivalent to maximizing the utility function $c_1 + bc_2$, which does not correspond to our perspective.

[5] Moreover, the demand function is differentiable in $(w, p_1, p_2, a, b, \lambda)$ except points satisfying $A(ap_2 - p_1) = (bp_1 - p_2)\big(\frac{w}{p_2}\big)^\lambda$ and $A(ap_2 - p_1) = (bp_1 - p_2)\big(\frac{aw}{p_1}\big)^\lambda$.

According to condition (5b), the consumer does not buy good 2 if the relative price of good 2 is high (in the sense that $p_2 \min(a, 1/b) \geq p_1$).

Under the second condition in (4), the solution is interior. Notice that, this condition implies that $(ap_2 - p_1)(bp_1 - p_2) > 0$, and so, the relative price has a middle level: $\min(a, 1/b) \leq p_1/p_2 \leq \max(a, 1/b)$. The second condition in (4) also requires that the income w of the consumer must be bounded from below and above.

Proposition 1 allows us to identify conditions under which good 1 is normal, inferior or Giffen.

Proposition 2. *Let assumptions in Proposition 1 be satisfied. Consider the case of interior solution.*

1. *Good* 1 *is normal (i.e.,* $\partial c_1/\partial w > 0$*) if and only if* $ap_2 < p_1$.
2. *Good* 1 *is inferior (i.e.,* $\partial c_1/\partial w < 0$*) if and only if* $ap_2 > p_1$.
3. *Good* 1 *has Giffen behavior (i.e.,* $\partial c_1/\partial p_1 > 0$*) if and only if*

$$(bp_1 - p_2)\Big(\frac{w}{p_2}\Big)^{\lambda} < A(ap_2 - p_1) < (bp_1 - p_2)\Big(\frac{aw}{p_1}\Big)^{\lambda} \tag{6a}$$

$$p_2\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}}\Big(1 - \frac{p_2(ab-1)}{\lambda(bp_1 - p_2)}\Big) - w > 0. \tag{6b}$$

Moreover, there exists a positive list $(p_1, p_2, a, b, \lambda, A, w)$ *such that (6b) and (6a) hold.*

Proof. See Appendix A.3. □

By combining Propositions 1 and 2, good 1 is normal if (1) the consumer only buys this good ($c_1 = w/p_1$) or (2) the solution is interior (condition (6a) holds) and the relative price is quite high (i.e., $ap_2 < p_1$). When the solution is interior, good 1 is inferior if and only if the relative price p_1/p_2 is low

We now look at conditions under which Giffen behavior arises. Condition (6a) is to ensure that the optimal allocation is interior while condition (6b) means that $\partial c_1/\partial p_1 > 0$. Given a, b, p_1, p_2, A, w such that $bp_1 > p_2 > w > p_1/a$, conditions (6a) and (6b) are satisfied if λ is high enough; in addition, once (6a) is satisfied, condition 6b) tends to hold if w is low and/or A is high. For example, when $p_1 = 2, p_2 = 2, w = 1.1, a = 2, b = 3, A = 3, \lambda = 6$.

Proposition 2 allows us to understand the role of income in the existence of Giffen behavior. Indeed, let us assume that $bp_1 > p_2 > p_1/a$. According to (4), the consumer only buys the good 1 if her income is very low; in this case, the good 1 consumption is increasing in the income and the good 1 can be viewed as the most basic good that the consumer can buy. Once the income exceeds a threshold $\underline{w}$ determined by $A(ap_2 - p_1) = (bp_1 - p_2)\big(\frac{aw}{p_1}\big)^{\lambda}$ but still lower the upper bound $\bar{w}$ determined by $A(ap_2 - p_1) \leq (bp_1 - p_2)\big(\frac{\bar{w}}{p_2}\big)^{\lambda}$, she buys both goods (interior solution). In this case, the good 1 consumption is firstly decreasing in the income w if $w \in (\underline{w}, w^*)$, where w^* such that the left hand side of (6b) is zero. By consequence, the Giffen behavior arises when the income is at the middle level. This property is supported by the empirical evidences in Jensen and Miller (2008).

We end this section by providing some useful observations when finding utility functions generating inferior goods as well as Giffen behavior.

1. Assume that the utility function is separable, i.e., $U(c_1, c_2) = u(c_1) + v(c_2)$. If u and u are concave, then c_1 is increasing in income w. Indeed, we have $U_{12} = 0$. So, Lemma 1 implies that: $\frac{\partial c_1}{\partial w} < 0$ if and only if $\frac{p_2^2}{p_1^2}\frac{u''(c_1)}{v''(c_2)} + 1 < 0$. This cannot happen because both u and v are concave.

   However, If u or v is not concave, we can obtain inferior good and Giffen behavior. Indeed, assume that the consumption set is $\mathbb{R}^2_+$ and $U(c_1, c_2) = Aln(c_1) + \frac{c_2^2}{2}$. In this case, one can prove that the demand for good 1 is

$$c_1 = \begin{cases} \frac{w}{p_1} & \text{if } w^2 \leq 4Ap_2^2 \\ \frac{w - \sqrt{w^2 - 4Ap_2^2}}{2p_1} & \text{if } w^2 > 4Ap_2^2 \end{cases} \tag{7}$$

   So, the good 1 is normal if $w^2 \leq 4Ap_2^2$. When $w^2 > 4Ap_2^2$, the good 1 is inferior but not Giffen.

   Haagsma (2012) considers a separable function $u(c_1, c_2) = \alpha_1 ln(c_1 - \gamma_1) - \alpha_2 ln(\gamma_2 - c_2)$ where the second term is convex in c_2. In this case, he shows that Giffen behavior may arise. Note that the consumption set is $(\gamma_1, \infty) \times [0, \gamma_2)$ which is restricted.

2. We can also obtain Giffen behavior with simple utility functions by restricting the consumption set in another way. Indeed, assume that $U(c_1, c_2) = c_1 + bc_2$ with $b > 0$ and the consumption set is $\{(c_1, c_2) \in \mathbb{R}^2_+ : c_1 + c_2 \geq 1\}$. $c_1 + c_2 \geq 1$ is interpreted as survival condition. We can verify that: if $p_1 < p_2 < bp_1$ and $w < p_2$, then $c_1 = \frac{p_2 - w}{p_2 - p_1}$ which is increasing in price p_1 and decreasing in income w.

3. In the case of Leontief utility $U(c_1, c_2) = \min(u(c_1), v(c_2))$ where u, v are increasing, c_1 is increasing in w. However, Sorensen (2007) considers the function $U(c_1, c_2) = \min(u(c_1, c_2), v(c_1, c_2))$ and show that this function may generate Giffen behavior.

3 Equilibrium

We now look at equilibrium properties. We consider a pure exchange economy with two goods. Assume that there are m agents with the same utility function $U(c_1, c_2) = c_1 + bc_2 + A\frac{(ac_1+c_2)^{1-\lambda}}{1-\lambda}$, where $a > 0, b > 0, \lambda > 0,\ \lambda \neq 1$. The consumption set is $\mathbb{R}^2_+$ and the endowments of agent i are $w_1^i > 0, w_2^i > 0$ for goods $1, 2$, respectively.

We firstly investigate the equilibrium prices. The income of agent i is $w^i \equiv p_1 w_1^i + p_2 w_2^i$. We focus on interior equilibrium: $c_1^j \in (0, w^i/p_1)\ \forall i$. According to Proposition 1, we have that

$$(ap_2 - p_1)c_1^i = p_2\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} - w^i. \tag{8}$$

From this and the market clearing condition $\sum_i c^i_j = \sum_i w^i_j \ \forall j = 1, 2$, we can compute the relative price $\bar{p}_1 \equiv p_1/p_2$.

Proposition 3. *Assume that $ab \neq 1$. If (w^i_1, w^i_2) is closed to (w_1, w_2) for any i,*[6] *where $w_j \equiv \sum_{i=1}^m w^i_j/m$ for $j = 1, 2$, then there exists an interior equilibrium with the relative price*

$$\frac{p_1}{p_2} = \frac{a + \frac{(aw_1+w_2)^\lambda}{A}}{1 + b\frac{(aw_1+w_2)^\lambda}{A}}. \text{ Moreover, } \frac{\partial \bar{p}_1}{\partial b} < 0 < \frac{\partial \bar{p}_1}{\partial a} \tag{9}$$

1. *If $ab > 1$, then $p_1/p_2 \in (1/b, a)$ and is decreasing in w_1, w_2 but increasing in A.*

2. *If $ab < 1$, then $p_1/p_2 \in (a, 1/b)$ and is increasing in w_1, w_2 but decreasing in A.*

Proof. See Appendix A.4. □

According to Proposition, our utility function (3) generates an interesting property: the price of good 1 (resp., good 2) is increasing in its aggregate supply $W_1 \equiv \sum_i w^i_1$ (resp., $W_2 \equiv \sum_i w^i_2$) if $ab < 1$ (resp., $ab > 1$).

We now look at the demand for good 1 of agent i to understand when Giffen behavior arises. According to (8) and (9), we can compute

$$c^i_1 = \frac{aw_1 + w_2 - w^i_2 - \bar{p}_1 w^i_1}{a - \bar{p}_1}. \tag{10}$$

By consequence, we have the following result.

Corollary 1. *We have*

$$\frac{\partial c^i_1}{\partial \bar{p}_1} = \frac{aw_1 - aw^i_1 + w_2 - w^i_2}{(a - \bar{p}_1)^2}. \tag{11}$$

This result leads to an interesting implication: the Giffen behavior arises when preferences of agents change. Indeed, without the loss of generality, assume that $ab > 1$. We also assume that agent i's endowments are low in the sense that $aw_1+w_2 > aw^i_1+w^i_2$. In this case, when A increases or b decreases, the relative price $\bar{p}_1$ increases. By consequence, the demand for good 1 of this agent increases in the relative price p_1/p_2.

Notice that the Giffen behavior can also arise when agents' endowments change. Indeed, let us consider a simple case where there are identical agents and $ab < 1$. In this case, $c^i_1 = w_1 \ \forall i$ and the relative price p_1/p_2 is increasing in w_1. So, the good 1 consumption $c^i_1 = w_1$ is increasing in p_1/p_2. In this case, the good 2 consumption is decreasing in p_2/p_1.

[6] We require (w^i_1, w^i_2) to be closed to (w_1, w_2) to ensure that with the equilibrium price given by (9), the optimal allocation is interior, i.e., $c^j_1 \in (0, w^i/p_1) \ \forall i$.

A Appendix

A.1 Proof of Lemma 1

The FOC can be rewritten as

$$p_2 U_1\Big(c_1, \frac{w - p_1 c_1}{p_2}\Big) = p_1 U_2\Big(c_1, \frac{w - p_1 c_1}{p_2}\Big). \tag{A.1}$$

By taking the derivatives of both sides of this equation with respect to w, we get that

$$\begin{aligned}&\Big(p_2 U_{11}(c_1, c_2) - p_1 U_{12}(c_1, c_2)\Big)\frac{\partial c_1}{\partial w} + U_{12}(c_1, c_2)\\ =&\Big(p_1 U_{21}(c_1, c_2) - \frac{p_1^2}{p_2} U_{22}(c_1, c_2)\Big)\frac{\partial c_1}{\partial w} + \frac{p_1}{p_2} U_{22}(c_1, c_2)\end{aligned}$$

which implies that

$$\Big(p_2^2 U_{11}(c_1, c_2) - 2p_1 p_2 U_{12}(c_1, c_2) + p_1^2 U_{22}(c_1, c_2)\Big)\frac{\partial c_1}{\partial w} = p_1 U_{22}(c_1, c_2) - p_2 U_{12}(c_1, c_2).$$

Since U is strictly concave, we have $p_2^2 U_{11}(c_1, c_2) - 2p_1 p_2 U_{12}(c_1, c_2) + p_1^2 U_{22}(c_1, c_2) < 0$. Therefore, we get (2a).

By taking the derivatives of both sides of (A.1) with respect to p_1, we get

$$\begin{aligned}&\Big(p_2 U_{11}\Big(c_1, \tfrac{w - p_1 c_1}{p_2}\Big) - p_1 U_{12}\Big(c_1, \tfrac{w - p_1 c_1}{p_2}\Big)\Big)\tfrac{\partial c_1}{\partial p_1} - c_1 U_{12}(c_1, \tfrac{w - p_1 c_1}{p_2})\\ = \quad & U_2(c_1, \tfrac{w - p_1 c_1}{p_2}) + p_1\Big(U_{21}(c_1, \tfrac{w - p_1 c_1}{p_2}) - \tfrac{p_1}{p_2} U_{22}(c_1, \tfrac{w - p_1 c_1}{p_2})\Big)\tfrac{\partial c_1}{\partial p_1} - c_1 \tfrac{p_1}{p_2} U_{22}(c_1, \tfrac{w - p_1 c_1}{p_2})\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}&\tfrac{\partial c_1}{\partial p_1}\Big(p_2 U_{11}(c_1, c_2) - 2p_1 U_{12}(c_1, c_2) + \tfrac{p_1^2}{p_2} U_{22}(c_1, c_2)\Big)\\ &= U_2(c_1, c_2) + c_1 U_{12}(c_1, c_2) - c_1 \tfrac{p_1}{p_2} U_{22}(c_1, c_2).\end{aligned}$$

which implies (2b).

A.2 Proof of Proposition 1

The budget constraint must be binding: $p_1 c_1 + p_2 c_2 = w$. Since the feasible set is convex, concave and the function U is strictly concave and strictly increasing, there exists a unique solution. We write FOCs

$$U_1(c_1, c_2) + \lambda_1 = p_1 \mu, \quad \lambda_1 \geq 0, \lambda_1 c_1 = 0 \tag{A.2a}$$

$$U_2(c_1, c_2) + \lambda_2 = p_2 \mu, \quad \lambda_2 \geq 0, \lambda_2 c_2 = 0. \tag{A.2b}$$

We have $U_1(c_1, c_2) = 1 + aA(ac_1 + c_2)^{-\lambda}$ and $U_2(c_1, c_2) = b + A(ac_1 + c_2)^{-\lambda}$.
We consider different cases.

1. $c_1 = 0, c_2 = w/p_2$. In this case, $\lambda_2 = 0$ and then $\frac{U_2(c_1,c_2)}{p_2} = \mu \geq \frac{U_1(c_1,c_2)}{p_1}$. This means that

   $$p_2\Big(1 + aA(ac_1 + c_2)^{-\lambda}\Big) \leq p_1(b + A(ac_1 + c_2)^{-\lambda}) \Leftrightarrow (\frac{w}{p_2})^{-\lambda}(ap_2 - p_1)A \leq bp_1 - p_2$$

   $$\Leftrightarrow A(ap_2 - p_1) \leq \Big(bp_1 - p_2\Big)\Big(\frac{w}{p_2}\Big)^{\lambda}.$$

   It is easy to verify that: this condition holds if and only if $(c_1, c_2) = (0, w/p_2)$.

2. $c_1 = w/p_1, c_2 = 0$. In this case, $\lambda_1 = 0$ and then $\frac{U_2(c_1,c_2)}{p_2} \leq \mu = \frac{U_1(c_1,c_2)}{p_1}$. This means that

$$p_2\Big(1 + aA(ac_1 + c_2)^{-\lambda}\Big) \geq p_1(b + A(ac_1 + c_2)^{-\lambda}) \Leftrightarrow (\frac{aw}{p_1})^{-\lambda}(ap_2 - p_1)A \geq bp_1 - p_2$$
$$\Leftrightarrow A(ap_2 - p_1) \geq \Big(bp_1 - p_2\Big)\Big(\frac{aw}{p_1}\Big)^{\lambda}$$

It is easy to verify that: this condition holds if and only if $(c_1, c_2) = (w/p_1, 0)$.

3. Let us consider an interior solution $0 < c_1 < w/p_1$. We will prove that this is the case if and only if condition (6a) hold, i.e.,

$$(bp_1 - p_2)(\frac{w}{p_2})^{\lambda} < A(ap_2 - p_1) < (bp_1 - p_2)(\frac{aw}{p_1})^{\lambda}. \tag{A.3}$$

The FOC becomes $\frac{U_1}{p_1} = \frac{U_2}{p_2}$, or equivalent

$$p_2\Big(1 + aA(ac_1 + c_2)^{-\lambda}\Big) = p_1(b + A(ac_1 + c_2)^{-\lambda})$$
$$\Leftrightarrow A(ap_2 - p_1) = \Big(bp_1 - p_2\Big)\Big(\frac{(ap_2 - p_1)c_1 + w}{p_2}\Big)^{\lambda} \tag{A.4}$$

Since $ab \neq 1$, we have $ap_2 - p_1 \neq 0$ and $bp_1 - p_2 \neq 0$. So, the equation (A.4) has a unique solution (because $\lambda > 0$).

(a) Case 1: $ap_2 - p_1 > 0$ which implies that $bp_1 - p_2 > 0$. The above equation has a unique solution c_1 in $(0, w/p_1)$ if and only if (A.3) holds.

(b) Case 2: $ap_2 - p_1 < 0$ which implies that $bp_1 - p_2 < 0$. The right hand side is an increasing function of c_1. So, the equation (A.4) has a unique solution c_1 in $(0, w/p_1)$ if and only if (A.3) holds.

Summing up, the equation (A.4) has a unique solution c_1 in $(0, w/p_1)$ if and only if (A.3) holds. In such case, we find that

$$\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} = \frac{(ap_2 - p_1)c_1 + w}{p_2} \Leftrightarrow (ap_2 - p_1)c_1 = p_2\Big(A\frac{ap_2 - p_1}{bp_1 - p_2}\Big)^{\frac{1}{\lambda}} - w. \tag{A.5}$$

We now prove the continuity of the demand function. Observe that the utility function is continuous and the budget correspondence

$$B(p_1, p_2) \equiv \{(c_1, c_2) \in \mathbb{R}^2_+ : p_1c_1 + p_2c_2 \leq w\}$$

is continuous. From the maximum theorem, the demand correspondence is upper semi continuous. Since we have proven above that it is single valued, it is in fact a continuous function.

We can also prove the continuity of the demand function by using the following properties

$$\lim_{A(ap_2-p_1)-(bp_1-p_2)(\frac{w}{p_2})^{\lambda}\to 0} \frac{p_2\big(A\frac{ap_2-p_1}{bp_1-p_2}\big)^{\frac{1}{\lambda}} - w}{ap_2 - p_1} = 0,$$
$$\lim_{A(ap_2-p_1)-(bp_1-p_2)(\frac{aw}{p_1})^{\lambda}\to 0} \frac{p_2\big(A\frac{ap_2-p_1}{bp_1-p_2}\big)^{\frac{1}{\lambda}} - w}{ap_2 - p_1} = \frac{w}{p_1}.$$

A.3 Proof of Proposition 2

Points 1 and 2 are obvious. We now look at the Giffen behavior. We have

$$c_1 = \frac{1}{ap_2 - p_1}\Big(p_2\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}} - w\Big)$$

$$\frac{\partial c_1}{\partial p_1} = \frac{1}{(ap_2-p_1)^2}\Big(p_2\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}} - w\Big) + \frac{A^{\frac{1}{\lambda}}p_2}{ap_2-p_1}\frac{1}{\lambda}\Big(\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}-1}\frac{p_2(1-ab)}{(bp_1-p_2)^2}.$$

Therefore, we get that

$$(ap_2-p_1)^2\frac{\partial c_1}{\partial p_1} = p_2\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}} + \Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}}\frac{1}{\lambda}\frac{p_2^2(1-ab)}{bp_1-p_2} - w \tag{A.6}$$

which implies point 3. We now prove that there exists a positive list $(p_1, p_2, a, b, \lambda, w, A)$ such that (6b) and (6a) hold, i.e., $\frac{\partial c_1}{\partial p_1} > 0$. It suffices to prove that there exists a positive list $(p_1, p_2, a, b, \lambda, w, A)$ such that the following conditions are satisfied

$$\frac{1}{ap_2-p_1}\Big(p_2\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}} - w\Big) \in (0, w/p_1) \tag{A.7}$$

$$ap_2 - p_1 > 0, bp_1 - p_2 > 0 \tag{A.8}$$

$$p_2\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}} + \Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}}\frac{1}{\lambda}\frac{p_2^2(1-ab)}{bp_1-p_2} - w > 0. \tag{A.9}$$

Indeed, let $ap_2 - p_1 > 0, bp_1 - p_2 > 0$ and $Ap_2 > w > \frac{Ap_1}{a}$. These conditions imply that $0 < \frac{Ap_2-w}{ap_2-p_1} < \frac{w}{p_1}$. When $\lambda \to \infty$, we have

$$\frac{1}{ap_2-p_1}\Big(p_2\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}} - w\Big) \longrightarrow \frac{p_2-w}{ap_2-p_1} \in \Big(0, \frac{w}{p_1}\Big) \tag{A.10}$$

$$p_2\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}} + \Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}}\frac{1}{\lambda}\frac{p_2^2(1-ab)}{bp_1-p_2} - w \longrightarrow Ap_2 - w > 0. \tag{A.11}$$

So, the above conditions are satisfied.

A.4 Proof of Proposition 3

We have proved that if equilibrium is interior, the relative price must be

$$\frac{p_1}{p_2} = \frac{a + \frac{(aw_1+w_2)^\lambda}{A}}{1 + b\frac{(aw_1+w_2)^\lambda}{A}}. \tag{A.12}$$

We have to now prove that with this price, the allocation (c^i, c_2^i) given by

$$(ap_2-p_1)c_1^i = p_2\Big(A\frac{ap_2-p_1}{bp_1-p_2}\Big)^{\frac{1}{\lambda}} - w^i, \quad p_1c_1^i + p_2c_2^i = p_1w_1^i + p_2w_2^i \tag{A.13}$$

is optimal for the agent i. To do so, it suffices to check the following condition (we apply Proposition 1),

$$\Big(bp_1 - p_2\Big)\Big(\frac{w^i}{p_2}\Big)^\lambda < A(ap_2-p_1) < \Big(bp_1-p_2\Big)\Big(\frac{aw^i}{p_1}\Big)^\lambda. \tag{A.14}$$

If this condition is satisfied for the case where $w^i_j = w_j$ with $j = 1, 2$, then so is for the case where (w^i_1, w^i_2) is enough closed to (w_1, w_2) for any i.

We now prove this condition when $w^i_j = w_j$ with $j = 1, 2$. We present a proof for the case $ab > 1$. The case $ab < 1$ is similar. Suppose $ab > 1$. Consider the function $f(x) = \frac{a+x}{1+bx}$. We have $f'(x) = \frac{1-ab}{(1+bx)^2} < 0$. Condition $a > 1/b$ implies that $\frac{p_1}{p_2} = \frac{a+\frac{(aw_1+w_2)^\lambda}{A}}{1+b\frac{(aw_1+w_2)^\lambda}{A}} \in (\frac{1}{b}, a)$. Since $ab > 1$, condition (A.14) is equivalent to

$$\begin{cases} bp_1 - p_2 > 0, ap_2 - p_1 > 0 \\ \left(\frac{w}{p_2}\right)^\lambda < A\frac{ap_2-p_1}{bp_1-p_2} < \left(\frac{aw}{p_1}\right)^\lambda \end{cases} \Leftrightarrow \begin{cases} bp_1 - p_2 > 0, ap_2 - p_1 > 0 \\ \frac{p_1w_1+p_2w_2}{p_2} < aw_1 + w_2 < \frac{a(p_1w_1+p_2w_2)}{p_1} \end{cases} \tag{A.15}$$

$$\Leftrightarrow \begin{cases} bp_1 - p_2 > 0, ap_2 - p_1 > 0 \\ \frac{p_1}{p_2} < a \end{cases} \Leftrightarrow \frac{1}{b} < \frac{p_1}{p_2} < a \tag{A.16}$$

It means that condition (A.14) is satisfied.

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