Cooperative games with externalities and probabilistic coalitional beliefs

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Abstract

We revisit cooperative games with externalities, i.e. cooperative games where the payoff of a coalition depends on the partition of the entire set of players. We define the worth of a coalition assuming that its members have probabilistic beliefs over the coalitional behavior of the outsiders, i.e., they assign various probability distributions on the set of partitions that the outsiders can form. We apply this framework to symmetric aggregative games and derive conditions on coalitional beliefs that guarantee the non-emptiness of the core of the induced cooperative games.

Keywords: cooperative game; aggregative game; probabilistic belief; core

JEL Classification: C71

1 Introduction

Cooperative game theory studies situations where groups or coalitions of players act collectively by signing binding agreements. One of the starting points of the theory is to determine the worth a coalition can achieve. In games with orthogonal coalitions, i.e., coalitions that do not affect one another, this task is quite straightforward, as it suffices to study the actions of the members of that coalition only. However, when orthogonality is absent, or in other words, when there are inter-coalitional externalities, the specification of the worth of a coalition requires the studying of the actions of the players in all coalitions.

Therefore, when a number of players contemplate to form a coalition in an environment with externalities they need to have a theory, or a conjecture, about the actions of the players outside the proposed coalition. Clearly, different conjectures lead to different specifications of the worth of the coalition, which in turn affects the outcome of the game. In particular, these conjectures determine the *core* of the cooperative game. The core is the set of all allocations of the value that the entire society of players generates that are

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immune to coalitional rejections. Non-emptiness of the core means that cooperation among all players in the game is a priori feasible.

The literature on cooperative games with externalities has proposed a number of such coalitional conjectures, each giving rise to a specific core notion. According to *α and β-conjectures* (Aumann 1959), the members of a coalition compute their worth assuming that the outside players select their strategies so as to minimize the payoff of the coalition; the concepts of α and β-core are then defined with respect to the resulting coalitional payoffs. According to *γ-conjectures* (Hart & Kurz 1983; Chander & Tulkens 1997), it is assumed that the outsiders select individual best strategies, i.e., they form only singleton coalitions; the notion of γ-core is then accordingly defined. The same approach can be followed under the additional assumption that each coalition assumes for itself the role of Stackelberg leader (Currarini & Marini 2003). The so called *δ-conjectures* presume that once one or more players depart from a coalition, the remaining ones stay put (Hart & Kurz 1983); this gives rise to δ-core.

The *r-approach* (Huang & Sjostrom 2003; Koczy 2007) proposes that the members of a coalition compute their worth by looking recursively on the sub-games played among the outsiders; the r-core arises when the solution concept employed in these sub-games is the core itself. On another approach, Nax (2014) focuses on the core of cooperative games with multiple sources of externalities, i.e, games where cooperation takes place in more than one spheres or layers. Externalities in a certain sphere are borne out of inter-coalitional interactions within that sphere (intra-sphere externalities) or from interactions in other spheres (inter-sphere externalities). Finally, Bloch & Nouweland (2014) assume that the expectations of a coalition on the reaction of the outsiders are guided by a set of axioms. The core notions that the axioms pick depend on whether expectations depend or not on the initial partition of the players.

Economists often restore to cooperative games with externalities to model various economic environments. Applications include the use of α and β-core concepts in oligopolistic markets (Zhao 1999; Norde et al. 2002; Lardon 2010); the use of γ-core in economies with production externalities (Chander & Tulkens 1997; Chander 2007; Helm 2012), in oligopolies (Rajan 1989; Lardon 2010; Lardon 2012) or in extensive-form games (Chander & Wooders 2012); of sequential γ-core for cooperative games with strategic complements (Currarini & Marini 2003) or for economies with environmental externalities (Marini 2013); of δ-core for oligopolistic markets with vertical differentiation (Gabszewicz et al 2016), etc. The main focus of these papers is to find conditions under which the corresponding core is non-empty.

The current paper focuses too on cooperative games with externalities but takes a different route. It assumes that when a group of players, S, contemplate to break off from the rest of the society, they are uncertain about the partition that the players outside S will form. As a result, they assign various probability distributions on the set of all possible partitions. These probabilistic beliefs do not necessarily reflect the behavior of the outsiders, i.e., beliefs need not be consistent with actual choices. Given the beliefs, no matter how they form, one can compute the expected worth of S and define the core of the resulting cooperative game. The task that arises then is to find conditions on the data of the game, i.e., on payoff functions and probability distributions, that guarantee the non-emptiness of the core.[1]

The motivation of our paper is twofold. First, we are interested in generalizing some of the existing approaches on the definition of core. For example, the notion of γ-core is a special case of our approach that arises when each coalition assigns probability one to the event that the outsiders form only singleton coalitions; likewise, the notion of δ-core corresponds to the case where the probability that the outsiders form one coalition is one. Secondly, our paper could be read as a work on bounded rationality in relation to cooperative games. The assignment of an ad hoc probability distribution on the set of partitions of the outsiders may reflect the cognitive inability of the members of a coalition to accurately deduce the outsiders' equilibrium partition. In this sense, probabilistic beliefs act as a rule of thumb. This approach is particularly relevant for games with a large number of players, where the number of different partitions is very large.

We apply our framework to cooperative games generated by symmetric aggregative normal form games. Imposing certain restrictions on how the beliefs of coalitions evolve when the number of players in the game changes, we first derive a result for the case of three-player games and then, using the above said restrictions and an induction argument, we generalize to the case of any number of players.

The paper is organized as follows. Section 2 introduces the basic framework. Section 3 presents the results and the last section offers brief concluding remarks.

2 Preliminaries

The primitive data in our paper is given by a collection $\{N, (X_i, u_i)_{i \in N}\}$, where $N = \{1, 2, \ldots, n\}$ is a set of players, X_i is the strategy set of player $i \in N$ and u_i is i's payoff function. We assume that the payoff of i is of the form $u_i(x_i, x)$, where $x = \sum\limits_{k \in N} x_k$; i.e., it depends only on his strategy and on the sum of all players' strategies. So, we focus on *aggregative* games.

Before selecting their strategies, the players organize themselves into coalitions by signing binding contracts. The objective of each coalition is to maximize the sum of its members' payoffs (we assume that utility is transferable). The current paper focuses on the formation of the grand coalition, N. This event might be blocked by the formation of smaller coalitions. We denote by S such a candidate deviant coalition. As our analysis will often focus on such an S, let us denote by π_{-S} a partition of the players *outside* S; and by Π_{-S} the set of all partitions that the players outside S can form.

Our main premise is that the members of S assign a probability distribution $h_{n,S}$ over Π_{-S}. So, for $\pi_{-S} \in \Pi_{-S}$, the value $h_{n,S}(\pi_{-S})$ is the probability assigned to partition π_{-S}. Given π_{-S}, denote by $x_j^{\pi_{-S}}$ a strategy of player[2] $j \notin S$ and by x_i a strategy of player $i \in S$; finally denote $x^{\pi_{-S}} = \sum\limits_{j \notin S} x_j^{\pi_{-S}} + \sum\limits_{i \in S} x_i$. Then, to find the worth of S we need to solve the problems

[1] We note that Lekeas & Stamatopoulos (2014) analyzes a specific application of the current framework. In particular, it examines the cooperative game generated in a linear Cournot market where the belief of a coalition is represented by a specific probability distribution, the logit distribution.

[2] For the time being, we define strategies with respect to players only and not with respect to coalitions to economize on notation.

$$\max_{(x_i)_{i\in S}} \sum_{\pi_{-S}\in\Pi_{-S}} h_{n,S}(\pi_{-S}) \sum_{i\in S} u_i(x_i, x^{\pi_{-S}}), \tag{1}$$

and also for each outside coalition $T \in \pi_{-S}$,

$$\max_{(x_j)_{j\in T}} \sum_{j\in T} u_j(x_j^{\pi_{-S}}, x^{\pi_{-S}}) \tag{2}$$

Finally, the problem facing the grand coalition is the standard one, namely it maximizes the sum of the payoff functions of all players.

We restrict attention to a specific class of aggregative games, the symmetric linear aggregative games. A linear aggregative game is an aggregative game in which the payoff of i has a bilinear form. To elaborate, we follow Martimort & Stole (2010) and we consider two linear spaces, V and W. A bilinear form is a mapping $\langle\cdot,\cdot\rangle : V \times W \to \mathbb{R}$, where for every $\psi, \tilde{\psi} \in V$ and $z, \tilde{z} \in W$, and scalar λ,

$$\langle\psi + \tilde{\psi}, \cdot\rangle = \langle\psi, \cdot\rangle + \langle\tilde{\psi}, \cdot\rangle, \quad \langle\cdot, z + \tilde{z}\rangle = \langle\cdot, z\rangle + \langle\cdot, \tilde{z}\rangle$$

$$\langle\lambda\psi, z\rangle = \langle\psi, \lambda z\rangle = \lambda\langle\psi, z\rangle$$

A linear aggregative normal form game arises when there exists a function $\tilde{u}_i$ such that the payoff of player i has the form

$$u_i(x_i, x) = \langle x_i, \tilde{u}_i(x)\rangle, \ \ i = 1, 2, \ldots, n$$

Further we impose symmetry among the players. In our framework, this requires to set $\tilde{u}_i(\cdot) \equiv u(\cdot)$, all $i \in N$. Hence,

$$u_i(x_i, x) = \langle x_i, u(x)\rangle, \ \ i = 1, 2, \ldots, n \tag{3}$$

Many economic models can be represented within the framework dictated by (3), for example, symmetric oligopoly models with linear cost functions, cost and surplus sharing games, contest games with linear costs, etc.

Given (3) we can write the objective function of deviant coalition S in the following form

$$\sum_{\pi_{-S}} h_{n,S}(\pi_{-S}) \sum_{i\in S}\langle x_i, u(x^{\pi_{-S}})\rangle = \sum_{\pi_{-S}} h_{n,S}(\pi_{-S})\langle\sum_{i\in S} x_i, u(\sum_{i\in S} x_i + \sum_{j\notin S} x_j^{\pi_{-S}})\rangle \tag{4}$$

I.e., S selects simply the sum $\sum\limits_{i\in S} x_i \equiv x_S$. A similar observation holds for any outside coalition. Picking some $T \in \pi_{-S}$, we have the following objective function for T,

$$\sum_{j\in T}\langle x_j^{\pi_{-S}}, u(x^{\pi_{-S}})\rangle = \langle\sum_{j\in T} x_j^{\pi_{-S}}, u(\sum_{j\in T} x_j^{\pi_{-S}} + x_{-T}^{\pi_{-S}})\rangle, \tag{5}$$

where $x_{-T}^{\pi_{-S}}$ is the sum of the strategies of all players outside T. Hence T selects the sum $\sum\limits_{j\in T} x_j^{\pi_{-S}} \equiv x_T^{\pi_{-S}}$.

Finally, we can write the objective function of the grand coalition as $\sum_{i\in N}\langle x_i, u(x)\rangle = \langle \sum_{i\in N} x_i, u(x)\rangle$. I.e., the grand coalition simply selects the sum $\sum_{i\in N} x_i \equiv x$. Among other things, this implies that N's worth does not depend on the number of players in the game.

The above allow us to conclude that for any π_{-S}, the worth of S depends only on the number of the coalitions in π_{-S} (and not on, say, how many players each coalition has). This fact will be used to simplify the exposition as follows. Assume S has $|S| = s$ members. Since what matters for S is only the number of coalitions of the outsiders, we define a probability distribution $f_{n,S}$ as follows

$$f_{n,S}(l) \;=\; \sum_{\pi_{-S}:\, |\pi_{-s}|=l} h_{n,S}(\pi_{-S}), \quad l = 1, 2, ..., n-s \tag{6}$$

The value $f_{n,S}(l)$ is the total probability of the event that the players outside S form l coalitions. Denote by π^l_{-S} a generic partition of the outsiders with l coalitions. Let

$$\sum_{T\in\pi^l_{-S}} x_T^{\pi^l_{-S}} \equiv x^l_{-S}, \quad x^l = x_S + x^l_{-S},$$

Then to find the worth of S we need to solve the problems

$$\max_{x_S} \sum_{l=1}^{n-k} f_{n,S}(l)\langle x_S, u(x^l)\rangle, \tag{7}$$

and for each $T \in \pi^l_{-S}$,

$$\max_{x_T}\langle x_T, u(x^l)\rangle \tag{8}$$

We denote the above collection of problems by $\Gamma^{f_n}_S$. We will need the following two conditions.

A1 X is a compact, convex interval of $\mathbb{R}$; and $\langle x_i, u(x)\rangle$ is continuously differentiable.

To state the second condition, denote by $D\langle x_i, u(x))$ the marginal payoff function of player i.

A2 $D\langle x_i, u(x)\rangle$ is decreasing in x_i and in x.

A2 is the strong concavity assumption of Corchón (1994). A1-A2 guarantee that the optimization problems described by $\Gamma^{f_n}_S$ have a solution; moreover A2 implies that coalition formation creates positive externalities[3] (see Lemma A1 in the Appendix).

Given the above, denote the solution of (7)-(8) by x^f_S and $x^{f,l}_T$ respectively, $l \in \{1, 2\ldots, n-s\}$. Let

$$x^{f,l}_{-S} = \sum_{T\in\pi^l} x^{f,l}_T, \quad x^{l,f} = x^f_S + x^{f,l}_{-S} \tag{9}$$

[3]Positive externalities arise when a coalition benefits from the merging of other coalitions (Hafalir 2007).

Then the worth of S is

$$v^{f_n}(S) = \sum_{l=1}^{n-s} f_{n,S}(l)\langle x_S^f, u(x^{f,l})\rangle \tag{10}$$

Denote the resulting cooperative game by (N, v^{f_n}). The core of this game is the set

$$\mathcal{C}^{f_n} = \{(w_1, w_2, \ldots, w_n) \in \mathbb{R}^n : \sum_{i\in N} w_i = v(N) \text{ and } \sum_{i\in S} w_i \geq v^{f_n}(S), \text{ for all } S\}$$

where $v(N)$ denotes the worth of the grand coalition. The following constitute necessary and sufficient conditions for non-emptiness of the core of our symmetric cooperative game:

$$\frac{v(N)}{n} \geq \frac{v^{f_n}(S)}{s}, \text{ for all } S \tag{11}$$

Our approach in this paper will focus on showing the validity of (11).

Remark 1
In what follows we reserve the term $v^{\gamma_n}(S)$ to denote the worth of S under the assumption that it assigns probability one to γ-scenario; likewise, $v^{\delta_n}(S)$ will denote the worth of S under the assumption that it assigns probability one to δ-scenario.

Remark 2
Positive externalities from coalition formation imply that $v^{f_n}(S) \in [v^{\gamma_n}(S), v^{\delta_n}(S)]$ (see the discussion after assumption A2 and Lemma A1 in the Appendix).

3 Results

As benchmark we first present the analysis of three player-games. For this case we only need to specify beliefs for S when $|S| = 1$.

Proposition 1 *Assume that $f_{3,S}(1)$ is sufficiently low, where $|S| = 1$. Then the core of (N, v^{f_3}) is non-empty.*

Proof We will show the validity of (11) for $|S| \in \{1, 2\}$. Consider first the case $|S| = 1$. As said before, the worth of S satisfies the condition[4] $v^{f_3}(S) \in [v^{\gamma_3}(S), v^{\delta_3}(S)]$. Notice that $v^{\gamma_3}(S)$, which corresponds to S's belief that the two outsiders stay separate with probability one, is simply the Nash payoff of the symmetric game where all players stand alone. This payoff clearly cannot exceed $\frac{v(N)}{3}$, i.e., the per capita efficient payoff. Hence under this belief, (11) is satisfied.

As the probability of γ-scenario decreases, or as the probability of the outsiders forming one coalition increases, the payoff of S increases monotonically towards $v^{\delta_3}(S)$. There are two possible cases:

(i) $v^{\delta_3}(S) \leq \frac{v(N)}{3}$: then S does not deviate irrespective of its beliefs.

[4]See Remark 2.

(ii) $v^{\delta_3}(S) > \frac{v(N)}{3}$: then S does not deviate if the probability attached to the scenario where the outsiders form one coalition, i.e., $f_{3,S}(1)$, does not exceed a critical value.

The current model cannot tell us whether (i) or (ii) holds. So, we can only deduce that S does not deviate if the probability that it assigns to the scenario of a sole outside coalition is low enough.

Consider next a coalition with two players, $|S| = 2$. Given our linear aggregative formulation, the payoff of such a coalition is equal to the payoff of a singleton coalition under the γ-scenario. Since the latter payoff cannot exceed, as we said already, $\frac{v(N)}{3}$, so the more is true for half of this payoff. But this means that (11) holds for $|S| = 2$. ∎

We now move to games with arbitrary number of players. We will use an induction argument. To this end, we will establish a certain pattern on the (probabilistic) beliefs of S across games with different number of players. Namely, we will "tie" the beliefs of S in a game with n players with its beliefs in a game with $n+1$ players. This connection will help us in our induction argument: given the pattern, if S does not deviate in a game with a certain number of players, it will not also deviate in a game with a larger number of players.

To begin, take a distribution $f_{n,S}$ and denote by $\tilde{f}_{n+1,S}$ a distribution that satisfies

$$v^{\tilde{f}_{n+1}}(S) = \frac{n}{n+1} v^{f_n}(S)$$

Such an $\tilde{f}_{n+1,S}$ exists as we elaborate in (the proof of) Proposition 2 below. Define the set of distributions

$$B_{f_{n,S}} = \{f_{n+1,S} : v^{f_{n+1}}(S) \leq v^{\tilde{f}_{n+1}}(S)\} \tag{12}$$

The above set is non-empty. For example, it contains the following non-empty set

$$B'_{f_{n,S}} = \{f_{n+1,S} : f_{n+1,S}(l) \leq \tilde{f}_{n+1,S}(l),\ l = 1, 2, \ldots, n-s-1\}$$

Compared to $\tilde{f}_{n+1,S}$, any $f_{n+1,S} \in B'_{f_{n,S}}$ assigns uniformly lower probabilities to the most favorable partitions for S and higher probability to the most unfavorable partition.[5] Hence, for any such f_{n+1,S_k} the inequality $v^{f_{n+1}}(S) \leq v^{\tilde{f}_{n+1}}(S)$ must hold.

We will focus on the following beliefs of S, for $|S| \leq n-2$, which we will define recursively w.r.t. the number of players in the game.

Definition 1 *Consider the following beliefs:*

(i) *Let $n = 3$. Then $f_{3,S}$ is as in Proposition 1.*

(ii) Let $n = 4$. Then $f_{4,S}$ is restricted to be an element of $B_{f_{3,S}}$.

(iii) Let $n > 4$ and assume that $f_{m,S}$ has been defined for all $m \leq n-1$. Then, $f_{n,S}$ is restricted to be an element of $B_{f_{n-1,S}}$.

[5] See Remark 2.

The above beliefs will be called admissible.

We are now ready to state and prove the following.

Proposition 2 *Assume that each S has admissible beliefs. Then the core of (N, v^{f_n}) is non-empty.*

Proof We will show that (11) holds using induction on $n \geq 3$.

Base: Consider a game with 3 players. Proposition 1 then guarantees than no S deviates.

Induction hypothesis: Assume that in a game with n players the following hold

$$\frac{v(N)}{n} \geq \frac{v^{f_n}(S)}{s}, \quad \text{for all } S$$

Induction step: We will show that in a game with $n+1$ players,

$$\frac{v(N)}{n+1} \geq \frac{v^{f_{n+1}}(S)}{s}, \quad \text{for all } S \tag{13}$$

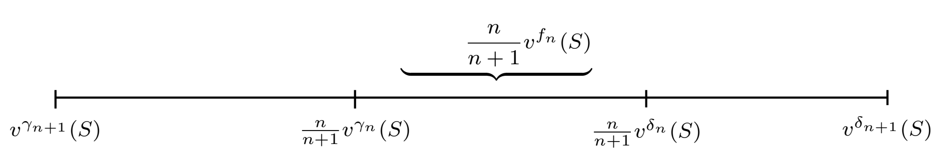
To show (13), it suffices by the induction hypothesis to show

$$(n+1)v^{f_{n+1}}(S) \leq nv^{f_n}(S) \tag{14}$$

To show the above, we recall first that $v^{f_n}(S) \in [v^{\gamma_n}(S), v^{\delta_n}(S)]$. So, the range of values of $\frac{n}{n+1}v^{f_n}(S)$ is given by the interval $[\frac{n}{n+1}v^{\gamma_n}(S), \frac{n}{n+1}v^{\delta_n}(S)]$. Moreover, the range of values of $v^{f_{n+1}}(S)$ is $[v^{\gamma_{n+1}}(S), v^{\delta_{n+1}}(S)]$. In Lemma A2 (in the Appendix) we show that

$$v^{\gamma_{n+1}}(S) \leq \frac{n}{n+1}v^{\gamma_n}(S) \tag{15}$$

Moreover, we notice that $v^{\delta_n}(S) = v^{\delta_{n+1}}(S)$. This is due again to the linear aggregative structure. Hence we have the following picture:



By the above we can conclude the following: For each distribution $f_{n,S}$ different than the distribution that gives probability one to the event that all outsiders form one coalition there exists a (non-unique) distribution $\tilde{f}_{n+1,S}$ such that $v^{\tilde{f}_{n+1}}(S) = \frac{n}{n+1}v^{f_n}(S)$ and $v^{f_{n+1}}(S) \leq v^{\tilde{f}_{n+1}}(S)$, for all $f_{n+1,S} \in B_{f_{n,S}}$ (see (12)). Hence, under admissible beliefs (14) holds. This concludes the induction step. ■

4 Conclusion

This short paper analyzed the core of cooperative games with externalities under the assumption that coalitions have probabilistic beliefs over the partition of the outsiders. The results we obtained rest on a number of assumptions that clearly restrict their applicability. Dropping the bilinear form of payoffs, in particular, or the aggregative structure altogether, can provide the most immediate direction of future research as this will enhance the applicability of our approach.

Finally, the biggest challenge that lies ahead is the analysis of non-symmetric games. For this case, showing (11) is not a valid way to proceed, and hence induction is not also fruitful. The best route of analysis could be to find probabilistic beliefs that lead to balancedness of the induced cooperative game. This is also left as future research.

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Appendix

Lemma A1 $v^{f_n}(S) \in [v^{\gamma_n}(S), v^{\delta_n}(S)]$.

Proof We will define two auxiliary normal form games which we will denote Γ_S^l and Γ_S^{l+1}: In the first game, S faces a partition of the outsiders in l coalitions; in the latter it faces a partition of the outsiders in $l+1$ coalitions. Given our linear aggregative structure, each coalition in Γ_S^l and in Γ_S^{l+1} identical to a player. So the former game has $l+1$ symmetric players and the latter has $l+2$ symmetric players. With a slight abuse of notation, denote by x_i^l the equilibrium strategy of each coalition in Γ_S^l and by x_i^{l+1} the equilibrium strategy of each coalition in Γ_S^{l+1}. Let also $x^l = (l+1)x_i^l$ and $x^{l+1} = (l+2)x_i^{l+1}$. Assumptions A1-A2 imply that we can use the results of Corchón (1994, Proposition 1) or Acemoglu & Jensen (2013, Theorem 7), which show that the entry of a player in an aggregative game increases the sum of the equilibrium strategies of all players. The translation of this result in our context means that $x^{l+1} \geq x^l$.

Observe next that total equilibrium payoffs in either Γ_S^l or Γ_S^{l+1} are of the form

$$U(m) \equiv (m+1)\langle x_i^m, u(x^m)\rangle = \langle x^m, u(x^m)\rangle, \ \ m = 0, 1, 2, \dots, n-s$$

Indeed, for Γ_S^l we have

$$U(l) = (l+1)\langle x_i^l, u(x^l)\rangle = \langle x^l, u(x^l)\rangle$$

whereas for Γ_S^{l+1},

$$U(l+1) = (l+2)\langle x_i^{l+1}, u(x^{l+1})\rangle = \langle x^{l+1}, u(x^{l+1})\rangle$$

Notice that $U(m)$ is maximized for $m = 0$. In other words, $U(m)$ is increasing for $x < x^0$ and decreasing for $x > x^0$. Since $x^{l+1} \geq x^l \geq x^0$ (see first paragraph of the proof), we must have $U(l) \geq U(l+1)$ or $(l+1)\langle x_i^l, u(x^l)\rangle \geq (l+2)\langle x_i^{l+1}, u(x^{l+1})\rangle$. But the last inequality implies that $\langle x_i^l, u(x^l)\rangle \geq \langle x_i^{l+1}, u(x^{l+1})\rangle$, which means that S prefers facing l coalitions over facing $l+1$ coalitions. Introducing now probabilistic beliefs, we have that the worth of S is: (i) maximum when it assigns probability one to δ scenario; (ii) it is minimum when it assigns probability one to γ scenario; (iii) it decreases (increases) the more (less) weight it assigns to partitions with many coalitions. In other words, $v^{f_n}(S) \in [v^{\gamma_n}(S), v^{\delta_n}(S)]$. ∎

Lemma A2 *Inequality (15) holds.*

Proof Consider the symmetric normal form games $\Gamma_S^{\gamma_n}$ and $\Gamma_S^{\gamma_{n+1}}$: the former arises when coalition S assigns probability one to the event that all $n-s$ outsiders form singleton coalitions; the latter arises when coalition S assigns probability one to the event that all $n+1-s$ outsiders form singleton coalitions. Denote by $x_i^{\gamma_n}$ the equilibrium strategy of each player in $\Gamma_S^{\gamma_n}$, and by $x_i^{\gamma_{n+1}}$ the equilibrium strategy of each player in $\Gamma_S^{\gamma_{n+1}}$. Then we can use again Corchón (1994, Proposition 1) or Acemoglu & Jensen (2013, Theorem 7) and derive the inequality $(n-s+1)x_i^{\gamma_n} \leq (n-s+2)x_i^{\gamma_{n+1}}$.

We next observe that total equilibrium payoffs in either $\Gamma_S^{\gamma_n}$ or $\Gamma_S^{\gamma_{n+1}}$ are of the form

$$V(m) \equiv m\langle x_i^{\gamma_m}, u(mx_i^{\gamma_m})\rangle = \langle mx_i^{\gamma_m}, u(mx_i^{\gamma_m})\rangle$$

Notice that $V(m)$ is maximized when $m = 1$. Hence $V(m)$ is monotonically increasing for $x < x^{\gamma_1}$ and monotonically decreasing for $x > x_i^{\gamma_1}$. Since $(n-s+2)x_i^{\gamma_{n+1}} \geq (n-s+1)x_i^{\gamma_n} \geq x_i^{\gamma_1}$, we must have that $V(n) \geq V(n+1)$. But this is tantamount to writing that[6] $(n-s+1)v^{\gamma_n}(S) \geq (n-s+2)v^{\gamma_{n+1}}(S)$ or $v^{\gamma_n}(S) \geq \frac{n-s+2}{n-s+1}v^{\gamma_{n+1}}(S)$. The proof is completed by noticing that $\frac{n-s+2}{n-s+1} \geq \frac{n+1}{n}$.

∎

[6] Symmetry and the linear aggregative structure imply that all players in $\Gamma_S^{\gamma_n}$ have the same equilibrium payoff and the same holds for $\Gamma_S^{\gamma_{n+1}}$.