History versus Expectations in Economic Geography Reconsidered

Oyama, Daisuke

27 September 2006

History versus Expectations in Economic Geography Reconsidered*

Daisuke Oyama

Graduate School of Economics, Hitotsubashi University
Naka 2-1, Kunitachi, Tokyo 186-8601, Japan
oyama@econ.hit-u.ac.jp

September 27, 2006; revised June 19, 2008

Abstract

This paper studies global stability of spatial configurations in a dynamic two-region model with quadratic adjustment costs where rational migrants make migration decisions so as to maximize their discounted future utilities. A global analysis is conducted to show that, except for knife-edge cases with symmetric regions, there exists a unique spatial configuration that is absorbing and globally accessible whenever the degree of friction is sufficiently small, and such a configuration is characterized as the unique maximizer of the potential function of the underlying static model. *Journal of Economic Literature* Classification Numbers: C61, C62, C73, F12, R12, R23.

Keywords: economic geography; multiple equilibria; forward-looking expectation; global stability; potential.

---

*I would like to thank Josef Hofbauer and Gerhard Sorger for valuable discussions and suggestions. I am also grateful for their comments to Shun-ichiro Bessho, Takatoshi Tabuchi, two anonymous referees, and seminar/conference participants at Otaru University of Commerce and 2006 Mathematical Economics at the Research Institute for Mathematical Sciences, Kyoto University. This work is supported by JSPS Grant-in-Aid for Young Scientists (B).

Web page: www.econ.hit-u.ac.jp/~oyama/papers/hist-vs-exp.html.

Contents

1 Introduction

This paper addresses the issue of 'history versus expectations' in the context of 'new economic geography' (Krugman (1991a)). This literature, typically using a two-region general equilibrium framework with monopolistic competition, demonstrates how the interplay of pecuniary externalities, market competition, and trade costs determines the spatial distribution of mobile production factor. In particular, when trade costs are low enough, agglomeration forces arising from scale economies dominate dispersion forces due to market competition effects, giving rise to multiple equilibria, two 'core-periphery' equilibria with full agglomeration of mobile factor in each region as well as an interior equilibrium. In studying locational adjustment dynamics, most models in the literature abstract from the possibility of forward-looking behavior of migrants: instead, migrants are assumed to be myopic and base their migration decisions on current utility differences, so that core-periphery equilibria are all locally stable under the myopic dynamics.[1]

In the present paper, we consider a class of adjustment dynamics with forward-looking migrants in a new economic geography model with two regions based on Ottaviano (2001) but incorporating exogenous asymmetries in trade costs and market size. Specifically, we employ the equilibrium dynamics due to Krugman (1991b) and Fukao and Benabou (1993) (KFB dynamics, in short), where migration requires moving costs which depend on the size of the current flow of migrants, so that migrants care about the future migration behavior of the economy. An equilibrium path of this dynamics is characterized by a no-arbitrage condition, that migrants are indifferent between staying in the current region and paying the cost to move to the other. The dynamics has stationary states, which correspond to the equilibria of the underlying static model.

Our main goal is to identify a state that is *absorbing* (i.e., if the initial condition is in a neighborhood of this state, then any equilibrium path converges to it) and *globally accessible* (i.e., for any initial condition, there exists an equilibrium path that converges to this state) for small frictions (i.e., when the migration cost is small and/or the rate of time discounting is small). We show that such a state *generically* exists (which is unique by definition) and is characterized as a *unique* maximizer of a potential function (Monderer and Shapley (1996) and Sandholm (2001)) of the static model. Even if all the agents are initially located in one region, there exists a set of self-fulfilling expectations that leads the economy toward full agglomeration in the other region, provided that the latter configuration is the potential maximizer, whenever the degree of friction is small, and once a large fraction of agents have been located there, no self-fulfilling expectation can reverse this outcome. This may be seen as an equilibrium selection result which dis-

[1] See Fujita *et al.* (1999) and Baldwin *et al.* (2003), among others.

criminates a unique equilibrium from others based on its distinctive stability properties under the KFB dynamics.

This result is to be contrasted with that by Ottaviano (2001), who, as many others in the literature, considers the case with completely symmetric regions. He shows that, when agglomeration economies are strong, both core-periphery equilibria are absorbing under the KFB dynamics for any (positive) degree of friction (indeed they both are maximizers of the potential function by symmetry).[2] It should be noted that one of the aims of early studies in the literature has been to explore when the symmetric spatial configuration over exogenously identical regions becomes unstable while an asymmetric one endogenously emerges as a (locally) stable long run outcome (see, e.g., Fujita *et al.* (1999)), and that *under myopic dynamics*, the local stability properties are in fact not altered by introduction of small exogenous asymmetries. By contrast, our equilibrium selection result demonstrates that *when one incorporates forward-looking expectations*, the case of perfect symmetry should be considered as a knife-edge case, and insights obtained may not be robust to exogenous asymmetries between regions.

The proof strategy for our result follows that of Hofbauer and Sorger (1999), who study stability under a different class of perfect foresight dynamics due to Matsuyama (1991) and Matsui and Matsuyama (1995) (MM dynamics, in short)[3] in potential games. First, we show that optimal solutions to an associated optimal control problem, whose objective functional is, roughly, a 'dynamical extension' of the potential function of the static model, are equilibrium paths of our dynamics and that those solutions, regardless of the initial condition, must visit small neighborhoods of the unique maximizer of the potential function for sufficiently small degrees of friction. Together with the absorption property below, this proves the global accessibility of the potential maximizer. Second, we show that the maximized Hamiltonian of the above optimal control problem serves as a Lyapunov function for equilibrium trajectories, from which the absorption of the potential maximizer follows.

In comparison with the MM dynamics, the KFB dynamics involves extra technical complications due to the assumption that agents are assumed to be able to migrate at any point in time (with the migration costs). This assumption implies that feasible paths of the aggregate spatial configuration may hit the boundary of the state space (the one-dimensional simplex) in finite time, which can make binding the constraint that the state variable must be contained in the simplex. This fact considerably complicates the

[2] Baldwin (2001) considers a related dynamics in the original core-periphery model of Krugman (1991a) with symmetric regions and obtains the same conclusion by numerical simulation analyses.

[3] See also, among others, Matsuyama (1992), Kaneda (2003), and Oyama (2006) for applications in economic contexts and Hofbauer and Sorger (2002), Oyama (2002), and Oyama *et al.* (2008) for studies in random-matching game frameworks.

formal definition of equilibrium paths of the dynamics: we have to carefully incorporate transition between the phases, one in which the constraint does not bind and the other in which it does.[4] When considering the associated optimal control problem in our proofs, moreover, we need to rely on non-standard techniques for problems with state-variable inequality constraints (Hartl *et al.* (1995)). Accordingly, the KFB dynamics requires a mathematically subtle treatment compared to the MM dynamics, while, as the results by Hofbauer and Sorger (1999) and the present paper show, these classes of dynamics share the same stability property when the underlying model admits a potential.

The rest of the paper is organized as follows. Section 2 presents our static model. Section 3 formally defines our equilibrium dynamics. Section 4 states our main theorems, while their proofs are given in Section 5. Section 6 concludes.

2 Static Model

In this section, we present our static model which will be embedded in the dynamic context in Section 3. Subsection 2.1 introduces a non-atomic game with binary actions as a canonical framework and defines its potential function, while Subsection 2.2 outlines how a two-region general equilibrium model à la Krugman (1991a) reduces to such a non-atomic game.

2.1 Canonical Framework

The economy consists of two regions, 0 and 1. There are a continuum of entrepreneurs with mass one, who are mobile between the regions. We denote by $x \in [0, 1]$ the fraction of entrepreneurs who are located (to establish manufacturing firms) in region 1. State $i \in \{0, 1\} \subset [0, 1]$ thus corresponds to the core-periphery state where all entrepreneurs are located in region i. For $i \in \{0, 1\}$, we will write $-i \in \{0, 1\} \setminus \{i\}$. The (indirect) utility for an entrepreneur located in region i which depends on a given state $x \in [0, 1]$ is denoted by $f_i(x)$. We assume that the function $f_i \colon [0, 1] \to \mathbb{R}$ is Lipschitz continuous. Let $f \colon [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = f_1(x) - f_0(x).$$

Location choice exhibits strategic complementarity (substitutability, resp.) if the function f is increasing (decreasing, resp.).

The pair of functions (f_0, f_1) in fact defines a non-atomic game in which a continuum of players choose between two actions, 0 and 1, and the payoffs are determined solely by the fraction x of players choosing action 1 as well

[4] This is the source of the error in Krugman (1991b) pointed out by Fukao and Benabou (1993).

as one's own choice. Note that any two games (f_0, f_1) and (f_0', f_1') are equivalent if they share the same payoff difference function f, i.e., $f_1(x) - f_0(x) = f_1'(x) - f_0'(x)$ for all $x \in [0,1]$. A state $x^* \in [0,1]$ is an *equilibrium state* if $x^* > 0 \Rightarrow f(x^*) \geq 0$ and $x^* < 1 \Rightarrow f(x^*) \leq 0$; and x^* is a *strict equilibrium state* if $x^* > 0 \Rightarrow f(x^*) > 0$ and $x^* < 1 \Rightarrow f(x^*) < 0$. The existence of an equilibrium state immediately follows from the continuity of the function f. We further impose the following regularity assumption.

Assumption 2.1. There are finitely many equilibrium states.

A sufficient assumption for this is that f be a real analytic function that is not identically zero.

We will invoke the concept of *potential* from game theory (Monderer and Shapley (1996) and Sandholm (2001)).

Definition 2.1. $F\colon [0,1] \to \mathbb{R}$ is a *potential function* of f if

$$\frac{dF}{dx}(x) = f(x)$$

for all $x \in [0,1]$.

Note that such a function F is unique up to constant. Note also that if x^* is a global maximizer of F over $[0,1]$ (i.e., $F(x^*) \geq F(x)$ for all $x \in [0,1]$), then x^* is an equilibrium state (but not vice versa in general). We will be interested in the generic case where F has a *unique* maximizer over $[0,1]$ (i.e., a x^* such that $F(x^*) > F(x)$ for all $x \in [0,1] \setminus \{x^*\}$).

While we proceed in the context of economic geography, the above abstract framework is more general and captures many other economic scenarios such as sectoral adjustment as in Krugman (1991b) and Matsuyama (1991). The reader most interested in the dynamic analysis independent of the context may skip the next subsection and go directly to Section 3.

2.2 Analytically Solvable Core-Periphery Model

We briefly review how the (indirect) utility difference $f(x)$ is derived from a general equilibrium model of trade and migration. We employ the model by Ottaviano (2001), often referred to as the 'footloose entrepreneur' model, but with asymmetries in trade cost and market size (see also Forslid and Ottaviano (2003) and Baldwin *et al.* (2003, Chapter 4)). This is an analytically solvable version of the original core-periphery model of Krugman (1991a).

In addition to entrepreneurs, there are a mass L of unskilled workers, who are immobile inter-regionally. Denote by L_i the exogenously given mass of workers in region $i = 0, 1$, so that $L = L_0 + L_1$. There are two consumption goods, a modern good and a traditional good, where the traditional good is chosen to be the numeraire. All individuals share the same preference given by the Cobb-Douglas utility function

$$U(M_i, A_i) = \alpha \log M_i + (1-\alpha)\log A_i, \qquad 0 < \alpha < 1$$

with

$$M_i = \left[\int_0^{n_i} d_{ii}(z)^{\frac{\sigma-1}{\sigma}} dz + \int_0^{n_j} d_{ji}(z)^{\frac{\sigma-1}{\sigma}} dz\right]^{\frac{\sigma}{\sigma-1}}, \qquad \sigma > 1$$

where M_i and A_i are the consumption (in region i) of the CES composite of modern varieties and the consumption of the traditional good, respectively, $d_{ji}(z)$ is the consumption (in region i) of a variety z that is produced in j, and n_j is the mass of varieties produced in j.

The modern good is produced in a monopolistically competitive sector. Production of a variety of the modern good involves a fixed input of one entrepreneur and a marginal input of β units of labor, and thus the total cost of production of m_i units is given by $r_i + w^{\mathrm{L}}\beta m_i$, where r_i and w_i^{L} are the wages for an entrepreneur and a worker, respectively. This implies that an entrepreneur and a manufacturing firm correspond one to one with each other, so that $n_0 = 1-x$ and $n_1 = x$ in the market equilibrium. Firms in the traditional sector produce the traditional good under perfect competition and constant returns to scale, involving a marginal input of one unit of labor. The traditional good is freely traded between the regions, so that the nominal labor wage is equalized in the two regions.[5] Consequently, we have $w_i^{\mathrm{L}} = 1$ in equilibrium due to our choice of numeraire.

Trade in the modern good, on the contrary, is costly due to trade barriers which are modeled by iceberg costs: for one unit of the modern good produced in j to reach i, $\tau_{ji} > 1$ units must be shipped.[6] Let $\rho_i = \tau_{ji}^{1-\sigma} \in (0,1)$, which we call the *trade openness* of region i. It increases as trade cost τ_{ji} decreases.

For a given $x \in [0,1]$, the key variables in the market equilibrium with free entry and exit are determined as follows; for their derivation, see the references cited above. In the following, we denote

$$x_0 = 1-x, \quad x_1 = x.$$

The domestic and the foreign prices of any variety produced in region i are respectively

$$p_{ii} = \frac{\sigma\beta}{\sigma-1}, \qquad p_{ij} = \frac{\tau_{ij}\sigma\beta}{\sigma-1},$$

[5]Wage equalization holds as long as the freely tradable, traditional good is produced in both regions. The condition for this, which is called the 'non-full-specialization' condition (Baldwin *et al.* (2003, Section 4.2.2)), is $\max\{L_0/L, L_1/L\} < (1-\alpha)[1-(\alpha/\sigma)]$ in our environment, and it is assumed to hold.

[6]This is the element that distinguishes the two alternatives 0 and 1 in the reduced nonatomic game (f_0, f_1) obtained below, in the sense that if $\tau_{01} = \tau_{10} = 1$, then $f_0(x) = f_1(x)$ for all $x \in [0,1]$.

and thus the CES price index in i is

$$P_i = \frac{\sigma\beta}{\sigma - 1}(x_i + \rho_i x_j)^{-\frac{1}{\sigma-1}}.$$

The reward to an entrepreneur located in i is given by

$$r_i = \frac{(\alpha/\sigma)L}{1 - (\alpha/\sigma)} \frac{\rho_j x_i + \psi_i x_j}{D},$$

where $D = (x_0 + \rho_0 x_1)(x_1 + \rho_1 x_0) - (\alpha/\sigma)(1 - \rho_0\rho_1)x_0 x_1$ and

$$\psi_i = \frac{L_i}{L}\left[1 + \frac{L - L_i}{L_i}\rho_0\rho_1 - (1 - \rho_0\rho_1)\frac{\alpha}{\sigma}\right].$$

Since the indirect utility for an entrepreneur in i is

$$f_i(x) = \alpha \log\left(\alpha \frac{r_i}{P_i}\right) + (1 - \alpha)\log\left((1 - \alpha)r_i\right),$$

the utility difference function f is given by

$$f(x) = \log\left(\frac{\rho_0 x_1 + \psi_1 x_0}{\rho_1 x_0 + \psi_0 x_1}\right) + \frac{\alpha}{\sigma - 1}\log\left(\frac{x_1 + \rho_1 x_0}{x_0 + \rho_0 x_1}\right) \tag{2.1}$$

(compare to equation (13) in Ottaviano (2001, p.58)). Observe that, considered as a function defined on an open interval containing $[0, 1]$, this function f is real analytic (and not identically zero), so that Assumption 2.1 is satisfied.

Finally, we obtain the potential function F as follows:[7]

$$\begin{aligned} F(x) = \sum_{i=0,1} \Bigg[& \frac{1}{\rho_j - \psi_i}\left\{(\rho_j - \psi_i)x_i + \psi_i\right\}\log\left((\rho_j - \psi_i)x_i + \psi_i\right) \\ & + \frac{\alpha}{\sigma - 1}\frac{1}{1 - \rho_i}\left\{(1 - \rho_i)x_i + \rho_i\right\}\log\left((1 - \rho_i)x_i + \rho_i\right)\Bigg]. \end{aligned} \tag{2.2}$$

We note that this function coincides (up to constant) with the function F defined in Ottaviano (2001, p.65) in the degenerate case where the two regions are completely symmetric, i.e., $\rho_1 = \rho_2 = \rho$ and $L_1 = L_2 = L/2$ (and thus $\psi_1 = \psi_2 = \psi$). The graph of the potential function for this degenerate case is depicted for three ranges of trade openness in Figures 2(a), 3(a), and 4(a) in Ottaviano (2001, pp.67–69).

One can show that if ρ_0 and ρ_1 are sufficiently close to one,[8] then f is increasing so that F is convex, while if they are sufficiently close to zero

[7] With only two locations, a potential function trivially exists. See Oyama (2006) for potentials in a new economic geography model with (finitely) many locations.

[8] A sufficient (but not necessary) condition is that $(\rho_0\rho_1)^{1/2} > (\sigma - \alpha)/(\sigma + \alpha)$.

(with an assumption that $\alpha < \sigma - 1$), then f is decreasing so that F is concave and single-peaked in $(0, 1)$ (the interior of $[0, 1]$). In the former (latter, resp.) case, location choice of firms exhibits strategic complementarity (substitutability, resp.). Intuition behind this is well discussed in the literature: With high trade barriers, competition is fierce since firms sell largely in their domestic market, which discourages spatial clustering of firms. With lowered trade barriers, in contrast, this market competition effect is relaxed and the effect of scale economies becomes dominant, fostering agglomeration.

In the former case, F is maximized at either $x = 0$ or $x = 1$. One can verify that $F(i) > F(j)$ holds if $\rho_i < \rho_j$ when $L_i = L_j$ or if $L_i > L_j$ when $\rho_i = \rho_j$. That is, the potential maximizer is the core-periphery state with full agglomeration in the region that is relatively protected or has a larger market size. Located in such a region, firms can have better access (in terms of trade costs) to the markets than otherwise.

3 Equilibrium Dynamics

Given a pair of utility functions (f_0, f_1) as described in the previous section, we consider in this section the dynamics due to Krugman (1991b) and Fukao and Benabou (1993). Entrepreneurs can move between regions at any time instant with moving costs, which depend on the size of the flow of moving entrepreneurs in the economy. Specifically, for a given path $x\colon [0, \infty) \to [0, 1]$, the moving cost is given by $|\dot{x}(t)|/\gamma$, where $\gamma > 0$. The (common) rate of time preference is denoted by $\theta > 0$.

We need to impose a regularity condition on paths $x(\cdot)$. We say that a path $x\colon [0, \infty) \to [0, 1]$ is *feasible* if it is continuous and piecewise continuously differentiable. We choose $\dot{x}(\cdot)$ to be right-continuous: i.e., we define $\dot{x}(t)$ for t at which $x(\cdot)$ is not differentiable by $\dot{x}(t) = \lim_{s \downarrow t} \dot{x}(s)$.

An interval $(\tau_1, \tau_2) \subset [0, \infty)$ with $\tau_1 < \tau_2$ is called an *interior interval* of $x(\cdot)$ if $x(t) \in (0, 1)$ for all $t \in (\tau_1, \tau_2)$.[9] An interval $[\tau_1, \tau_2] \subset [0, \infty)$ with $\tau_1 < \tau_2$ is called a *boundary interval* if $x(t) \in \{0, 1\}$ (and hence $\dot{x}(t) = 0$) for all $t \in [\tau_1, \tau_2]$. A time instant τ_1 is called an *entry time* if an interior interval ends and a boundary interval starts at τ_1; and τ_2 is called an *exit time* if a boundary interval ends at τ_2. If the trajectory is in the interior just before and just after τ, then τ is called a *contact time*. Entry, exit, and contact times are called *junction times*.

We would now like to define equilibrium paths. To motivate our definition below, let a feasible path $x(\cdot)$ be given. For each $t \in [0, \infty)$ and for $\Delta t > 0$, a migration strategy on $[t, t + \Delta t]$ of an agent who is currently located in region $i \in \{0, 1\}$ is characterized by a set of switching times $\{t_1, t_2, \ldots, t_n\} \subset [t, t + \Delta t]$, $t \leq t_1 < \cdots < t_n \leq t + \Delta t$: at each time t_k

[9] If $(0, \tau_2)$ is an interior interval, we say that $[0, \tau_2)$ is an interior interval even when $x(0) \in \{0, 1\}$.

$(k = 1, \dots, n)$, the agent moves from i_{k-1} to i_k, where $i_k = -i_{k-1}$ with $i_0 = i$. The value of locating in region i, V_i, then satisfies

$$
\begin{aligned}
V_i(t) = \sup_{\{t_1,\dots,t_n\}\subset[t,t+\Delta t]} \Biggl\{ & \int_t^{t_1} e^{-\theta(s-t)} f_i(x(s))\, ds \\
&+ \sum_{k=1}^{n} \left(\int_{t_k}^{t_{k+1}} e^{-\theta(s-t)} f_{i_k}(x(s))\, ds - e^{-\theta(t_k-t)} \frac{|\dot{x}(t_k)|}{\gamma} \right) \\
&\qquad\qquad + e^{-\theta\Delta t} V_{i_n}(t+\Delta t) \Biggr\}, \qquad (3.1)
\end{aligned}
$$

where $t_{n+1} = t + \Delta t$. Equilibrium behavior on interior intervals is characterized by a non-arbitrage condition. That is, along an equilibrium path $x(\cdot)$, at any time in interior intervals, if $\dot{x}(t) \geq 0$ ($\dot{x}(t) \leq 0$, resp.), then agents must be indifferent between staying at region 0 (1, resp.) and moving to 1 (0, resp.) by incurring the moving cost $|\dot{x}(t)|/\gamma$.[10] On boundary intervals for the boundary $x = i$ ($i = 0, 1$), on the other hand, agents must weakly prefer to stay at region i, so that $f_i(i) \geq f_{-i}(i)$ must hold (note that, since $\dot{x}(t) = 0$ on boundary intervals, agents can move between the regions with no cost, so that the current location is irrelevant, and hence $V_0(t) = V_1(t)$). Thus, if the system is in the interior or at the boundary $x = i$, then staying at region i is at least weakly optimal, until the system hits the other boundary $x = -i$.

Definition 3.1. A feasible path $x\colon [0, \infty) \to [0, 1]$ is an *equilibrium path* from $x^0 \in [0, 1]$ if $x(0) = x^0$, and for each $i = 0, 1$ there exists a function $V_i\colon [0, \infty) \to \mathbb{R}$ that is right-continuous with left-hand limits and satisfies (3.1) and the following conditions:

(a) for all $t \in [0, \infty)$,

$$\dot{x}(t) \leq 0 \Rightarrow V_0(t) - \frac{|\dot{x}(t)|}{\gamma} = V_1(t), \qquad (3.2)$$

$$\dot{x}(t) \geq 0 \Rightarrow V_1(t) - \frac{|\dot{x}(t)|}{\gamma} = V_0(t), \qquad (3.3)$$

(b-0) if (t, T) is such that $x(s) < 1$ for all $s \in (t, T)$, then

$$V_0(t) = \int_t^T e^{-\theta(s-t)} f_0(x(s))\, ds + e^{-\theta(T-t)} V_0(T^-), \qquad (3.4)$$

and if $[t, T]$ is such that $x(s) = 1$ for all $s \in [t, T]$, then

$$V_0(t) = \frac{1}{\theta} \left(1 - e^{-\theta(T-t)}\right) f_1(1) + e^{-\theta(T-t)} V_0(T^-), \qquad (3.5)$$

[10]To see this, consider for example the case where $\dot{x}(t) \geq 0$, so that agents at least weakly prefer to move to region 1. Since agents have the option to move at any time instant, if the preference were strict, then all the agents would move simultaneously, which in turn makes the moving cost infinity. Clearly, this cannot be supported by an equilibrium.

(b-1) if (t,T) is such that $x(s) > 0$ for all $s \in (t,T)$, then

$$V_1(t) = \int_t^T e^{-\theta(s-t)} f_1(x(s))\, ds + e^{-\theta(T-t)} V_1(T^-), \tag{3.6}$$

and if $[t,T]$ is such that $x(s) = 0$ for all $s \in [t,T]$, then

$$V_1(t) = \frac{1}{\theta}\left(1 - e^{-\theta(T-t)}\right) f_0(0) + e^{-\theta(T-t)} V_1(T^-). \tag{3.7}$$

The existence of equilibrium paths will be shown later (in Corollary 5.3).

While it follows from the definition that V_i is continuous at junction times for $x = i$ as well as on interior and boundary intervals, it is not assumed to be continuous at junction times for $x = -i$; yet, it turns out that it is in fact the case. The following proposition characterizes equilibrium paths in terms of the "shadow price" that represents the difference in value between locating in region 1 rather than in region 0:

$$q(t) = V_1(t) - V_0(t).$$

It shows, in particular, that $q(\cdot)$ is continuous at any junction time. This is precisely the point made by Fukao and Benabou (1993).

Proposition 3.1. *A feasible path $x\colon [0,\infty) \to [0,1]$ is an equilibrium path from x^0 if and only if $x(0) = x^0$, and there exists a function $q\colon [0,\infty) \to \mathbb{R}$ that is continuous and piecewise differentiable and satisfies the following conditions:*

(i) *for any time t in an interior interval,*

$$\dot{x}(t) = \gamma q(t), \tag{3.8a}$$

$$\dot{q}(t) = \theta q(t) - f(x(t)), \tag{3.8b}$$

(ii) *for any time t in a boundary interval and for any contact time t,*

$$q(t) = 0, \tag{3.9}$$

and

$$x(t) = 0 \Rightarrow f(0) \le 0, \tag{3.10a}$$

$$x(t) = 1 \Rightarrow f(1) \ge 0. \tag{3.10b}$$

Furthermore, such a function $q(\cdot)$ is bounded.

Proof. See Appendix. ▮

Condition (i) says that on interior intervals, the law of motion of $(x(t), q(t))$ is governed by the system of differential equations (3.8), while (ii) implies that if the equilibrium path hits the boundary of $[0,1]$ at $x = i$, then $q(t) = 0$

(by (3.9)), and $x = i$ must be an equilibrium state (by (3.10)).[11] Nonetheless, $q(\cdot)$ must be continuous, while satisfying

$$q(t) = \int_t^\tau e^{-\theta s} f(x(s))\, ds$$

if (t, τ) is an interior interval with τ being an entry time or $\tau = \infty$, and

$$q(t) = 0$$

if t is in a boundary interval.

Clearly, the behavior of the dynamics depends on the values of the parameters θ and γ. We note that it is fully captured by the ratio between θ and $\sqrt{\gamma}$. This is easily verified by applying the change of variables: $x'(s) = x(s/\sqrt{\gamma})$ and $q'(s) = \sqrt{\gamma} q(s/\sqrt{\gamma})$. Then, the system (3.8) is written as

$$\dot{x}'(s) = q'(s), \tag{3.11a}$$

$$\dot{q}'(s) = \frac{\theta}{\sqrt{\gamma}} q'(s) - f(x'(s)). \tag{3.11b}$$

We thus view $\delta = \theta/\sqrt{\gamma} > 0$ as the *degree of friction*. It is smaller when the future is more important (i.e., θ is smaller) and/or migration is less costly so that the adjustment is faster (i.e., γ is larger).

It is immediate to see that the stationary states of our dynamics are precisely the equilibrium states of the static model.

Observation 3.2. *The feasible path $x(\cdot)$ such that $x(t) = x^*$ for all $t \geq 0$ is an equilibrium path if and only if x^* is an equilibrium state.*

In general, there may exist multiple equilibrium states, and this is indeed the case when agglomeration forces are strong so that the utility difference f is increasing. Our main objective is to discriminate among the equilibrium states based on their stability properties under the equilibrium dynamics. We employ the following stability concepts, which formalize the argument of 'history versus expectations'.

Definition 3.2. (a) $x^* \in [0, 1]$ is *absorbing* if there exists $\varepsilon > 0$ such that any equilibrium path from any $x \in B_\varepsilon(x^*)$ converges to x^*.

(b) $x^* \in [0, 1]$ is *accessible* from $x \in [0, 1]$ if there exists an equilibrium path from x that converges to x^*. x^* is *globally accessible* if x^* is accessible from any x.

[11] In particular, due to (3.10), for $i \in \{0, 1\}$ the constant path $x(\cdot)$ such that $x(t) = i$ for all $t \geq 0$ cannot be an equilibrium path if i is not an equilibrium state.

To give the intuition behind these concepts, let us consider a core-periphery configuration $x^* = i \in \{0, 1\}$, the state in which all the agents are located in region i. If x^* is absorbing, then the following is true: if history sets the initial state to be one in which a sufficiently large amount of agents are already located in i, then any form of self-fulfilling expectations cannot alter the outcome and the agents located in the other region will eventually migrate to i. If x^* is globally accessible, then the following is true: whatever the initial state is, the expectation that all agents will eventually be settled in i may become self-fulfilling.

It is clear that if the degree of friction $\delta = \theta/\sqrt{\gamma}$ is large (i.e., the future is unimportant and/or the adjustment is slow), then the dynamics becomes similar to myopic dynamics, so that any strict equilibrium state, if any, is absorbing. We are interested in a (unique, by definition) state that is both absorbing and globally accessible whenever the degree of friction is sufficiently small. In the sequel, we show that, except for knife-edge cases, such a state exists and coincides with a unique maximizer of the potential function F.

4 Main Results

In this section, we state and illustrate the main theorems of this paper. Their proofs are given in Section 5.

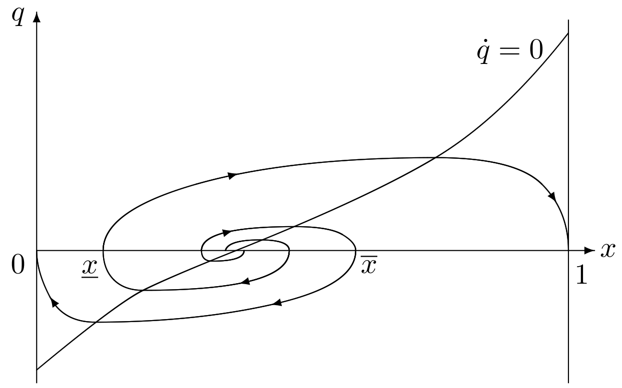
Theorem 4.1. *Assume that x^* is the unique maximizer of F over $[0, 1]$. Then, there exists $\bar{\delta} > 0$ such that x^* is globally accessible whenever $\delta \leq \bar{\delta}$.*

Theorem 4.2. *Assume that x^* is the unique maximizer of F over $[0, 1]$. Then, x^* is absorbing (independently of δ).*

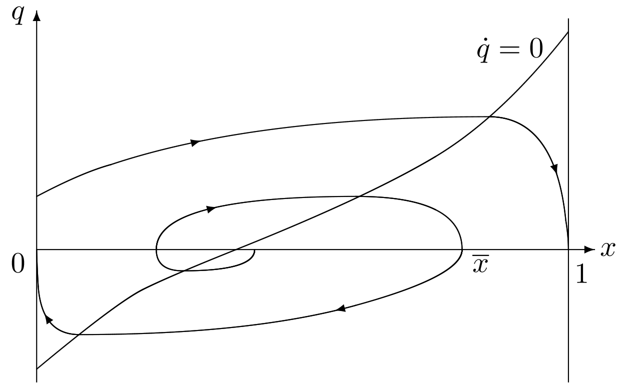
In particular, the potential maximizer x^* is a unique absorbing (and globally accessible) state whenever the friction δ is sufficiently small.

To illustrate our results, we consider in the rest of this section the most interesting case where strong agglomeration economies are present so that the indirect utility difference function f is upward-sloping. In this case, there are two strict equilibrium states, the core-periphery configurations $x = 0$ and $x = 1$, and one mixed equilibrium state. Since the potential function F becomes convex, it is maximized at a vertex of $[0, 1]$ (i.e., $x = 0$ or $x = 1$). Let us assume that $F(1) > F(0)$, so that $x = 1$ is the unique potential maximizer.

If the friction δ is large enough, the behavior of the dynamics is qualitatively the same as that under myopia, so that there is no room for expectations to play a role. For intermediate frictions or smaller, expectations become relevant. Figure 1(a) shows the phase portrait of $(x(\cdot), q(\cdot))$ for intermediate frictions. In this case, if the initial state $x(0)$ lies within the range $[\underline{x}, \overline{x}]$, which Krugman (1991b) refers to as the “overlap”, then there



(a) Intermediate frictions



(b) Small frictions

Figure 1: Phase portraits

are multiple equilibrium paths, some leading to $x = 1$ and others to $x = 0$. If the overlap $[\underline{x}, \overline{x}]$ is strictly contained in $[0, 1]$ (i.e., $0 < \underline{x}$ and $\overline{x} < 1$) as in Figure 1(a), then from a neighborhood of each strict equilibrium state $x = i$ $(i = 0, 1)$, there is a unique equilibrium path, which leads to $x = i$; that is, both states $x = 0$ and $x = 1$ are absorbing.

Intuitively, as the friction becomes smaller, expectations become more likely to be decisive, thus making the overlap wider. What Theorem 4.1 tells us is that the overlap must reach $x = 0$, the endpoint of $[0, 1]$ opposite to the potential maximizer $x = 1$, for small frictions, while Theorem 4.2 says that, however small the friction is, the overlap never contains the potential maximizer $x = 1$ and thus never fills the entire space $[0, 1]$.[12] The phase portrait for this situation is depicted in Figure 1(b), where the overlap is the interval $[0, \overline{x}]$. For any initial condition, there is an equilibrium trajectory

[12]This is true also for $\delta = 0$, in which case $\overline{x}$ is given by $F(\overline{x}) = F(0)$.

that leads to $(x, q) = (1, 0)$, which is the uppermost trajectory in the figure, while if the initial condition is set in $(\overline{x}, 1]$, then the equilibrium trajectory is unique, and the system necessarily leads to $(x, q) = (1, 0)$.

We should mention the work by Ottaviano (2001), who studies the same dynamics in the case of completely *symmetric* regions. In this case, since the utility difference function f is skew symmetric around $x = 1/2$ (i.e., $f(1/2 - z) = -f(1/2 + z)$), the potential function F is symmetric around $x = 1/2$, so that we have $F(0) = F(1)$, thereby violating our assumption that F has a *unique* global maximizer. Ottaviano (2001) shows that for positive friction δ, the overlap is strictly contained in the space $[0, 1]$, and hence, in our terminology, both stationary states $x = 0$ and $x = 1$ are absorbing; and that for $\delta = 0$, the overlap precisely coincides with the whole space $[0, 1]$, so that both stationary states are globally accessible.[13] Our theorems, in contrast, show that, once the regions are asymmetric so that $F(0) \neq F(1)$, there is only one state that becomes absorbing as well as globally accessible for small δ, demonstrating in fact that the results for the knife-edge case of symmetric regions are not robust to exogenous asymmetries between the regions.[14,15]

5 Proofs

We prove Theorems 4.1 and 4.2 in Subsections 5.1 and 5.2, respectively. The proof strategy follows that due to Hofbauer and Sorger (1999).

5.1 Global Accessibility

The proof exploits the relationship between equilibrium paths of the dynamics in consideration and optimal solutions to an associated optimal control problem.

[13] For the case where f is linear in addition to being skew symmetric around $x = 1/2$, Fukao and Benabou (1993) explicitly computes the width of the overlap, which is strictly smaller than one for positive δ and converges to one as δ goes to zero.

[14] This point seems not to have been recognized in the literature. For instance, Baldwin (2001, p.46), who considers the core-periphery (CP) model with symmetric locations, states that "the region of overlapping saddle paths will never include a CP outcome". This statement also appears in the textbook of Baldwin *et al.* (2003, p.60).

[15] This has to be contrasted with stability under myopic dynamics, where *local* maximizers of a potential function are all locally stable, so that introduction of small asymmetries does not alter the local stability.

The optimal control problem is defined as follows:

$$\begin{aligned}
&\text{maximize} \quad J(x(\cdot),u(\cdot)) = \int_0^\infty e^{-\theta t}\left(F(x(t)) - \frac{u(t)^2}{2\gamma}\right) dt, && \text{(5.1a)}\\
&\text{subject to} \quad \dot{x}(t) = u(t), && \text{(5.1b)}\\
&\qquad\qquad\quad x(t) \geq 0, \qquad 1 - x(t) \geq 0, && \text{(5.1c)}\\
&\qquad\qquad\quad x(0) = x^0. && \text{(5.1d)}
\end{aligned}$$

An admissible pair is a pair $(x(\cdot),u(\cdot))$ of an absolutely continuous function $x\colon [0,\infty) \to [0,1]$ and a measurable function $u\colon [0,\infty) \to \mathbb{R}$ that satisfy the constraints (5.1b)–(5.1d). An admissible pair is called an optimal pair if it attains the maximum valued of J over all admissible pairs.

We show in Proposition 5.2 that a solution to this maximization problem is an equilibrium path of our dynamics. This may be seen as a dynamic analog to the fact in the static model that a maximizer of the potential function F is an equilibrium state. We then show in Lemma 5.4 that an optimal path must visit neighborhoods of the potential maximizer when the degree of friction $\delta = \theta/\sqrt{\gamma} > 0$ is sufficiently small. Together with the absorption proved in Subsection 5.2, these prove the global accessibility of the potential maximizer.

We first obtain the existence of optimal solution.

Proposition 5.1. *An optimal pair to problem* (5.1) *exists for each* $x^0 \in [0,1]$.

Proof. Follows from Baum (1976, Theorem 7.1). ∎

The following proposition establishes the relationship between the maximization problem and the equilibrium dynamics.

Proposition 5.2. *If* $(x(\cdot),u(\cdot))$ *is an optimal pair to problem* (5.1)*, then* $x(\cdot)$ *is an equilibrium path from* x^0*.*

Proof. See Appendix. ∎

We have the following as an immediate consequence of the above propositions.

Corollary 5.3. *There exists an equilibrium path for each initial condition* $x^0 \in [0,1]$*.*

Note that the converse of Proposition 5.2 is not true in general.

We here give a heuristic proof of Proposition 5.2 based on "Informal Theorem" 4.1 in Hartl *et al.* (1995), while the formal proof is given in the Appendix. Due to the (pure state variable) *inequality constraints* in (5.1c), which will be binding in particular when F is convex, we need to rely on a non-standard technique for necessary conditions.[16] The current value Hamiltonian H and the Lagrangian L are defined respectively by

$$H(x,u,q) = F(x) - \frac{u^2}{2\gamma} + qu \tag{5.2}$$

and

$$L(x,u,q,\nu_0,\nu_1) = H(x,u,q) + \nu_0 x + \nu_1(1-x). \tag{5.3}$$

By Hartl *et al.* (1995, Informal Theorem 4.1), we have the following necessary conditions for optimality: If $(x(\cdot),u(\cdot))$ is an optimal pair, then there exist a piecewise absolutely continuous function $q\colon [0,\infty) \to \mathbb{R}$ and piecewise continuous functions $\nu_0,\nu_1\colon [0,\infty) \to \mathbb{R}$ such that

$$H_u(x(t),u(t),q(t)) = -\frac{u(t)}{\gamma} + q(t) = 0, \tag{5.4}$$

$$\begin{aligned}\dot q(t) &= \theta q(t) - L_x(x(t),u(t),q(t),\nu_0(t),\nu_1(t))\\ &= \theta q(t) - f(x(t)) - \nu_0(t) + \nu_1(t),\end{aligned} \tag{5.5}$$

$$\nu_0(t) \ge 0, \quad \nu_0(t)x(t) = 0, \tag{5.6}$$

$$\nu_1(t) \ge 0, \quad \nu_1(t)(1-x(t)) = 0, \tag{5.7}$$

and for any time τ in a boundary interval and for any contact time τ, $q(\cdot)$ may have a discontinuity given by the following *jump conditions*:

$$q(\tau^-) = q(\tau^+) + \eta_0(\tau) - \eta_1(\tau), \tag{5.8}$$

$$\eta_0(\tau) \ge 0, \quad \eta_0(\tau)x(\tau) = 0, \tag{5.9}$$

$$\eta_1(\tau) \ge 0, \quad \eta_1(\tau)(1-x(\tau)) = 0 \tag{5.10}$$

for some $\eta_0(\tau),\eta_1(\tau)$ for each τ. Observe first that conditions (5.4)–(5.7) imply that, with the adjoint $q(\cdot)$, the equilibrium conditions in Proposition 3.1 are satisfied for interior intervals.

We then claim that $q(\tau^-) = q(\tau^+) = 0$ for any time τ in a boundary interval and for any contact time τ. Let us verify this in the case where $x(\tau) = 1$, so that $\nu_0(\tau) = \eta_0(\tau) = 0$ by (5.6) and (5.9). First, it must be that $\dot x(\tau^-) \ge 0$ (otherwise, we would have $x(\tau) < 1$), and hence $q(\tau^-) \ge 0$ by (5.1b) and (5.4). Second, it must be that $\dot x(\tau^+) \le 0$ (otherwise, we would have $x(\tau+\varepsilon) > 1$), and hence $q(\tau^+) \le 0$ again by (5.1b) and (5.4). Last, by the jump condition (5.8), $q(\tau^-) = q(\tau^+) - \eta_1(\tau) \le q(\tau^+)$ since $\eta_1(\tau) \ge 0$ as in (5.10). These imply that $q(\tau^-) = q(\tau^+) = 0$.

Finally, we can verify from (5.5)–(5.7) that

$$\nu_0(t) = \begin{cases} -f(0) & \text{if } t \text{ is in a boundary interval for } x = 0,\\ 0 & \text{otherwise,}\end{cases}$$

[16] Here, we follow the "direct adjoining approach". See also Seierstad and Sydsæter (1987, Chapter 5) and Sethi and Thompson (2000, Chapter 4), where the "indirect adjoining approach" is discussed.

and

$$\nu_1(t) = \begin{cases} f(1) & \text{if } t \text{ is in a boundary interval for } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\nu_0(t), \nu_1(t) \geq 0$, it follows that the conditions (3.10) in Proposition 3.1 are satisfied.

Remark 5.1. As noted in Proposition 3.1, the adjoint function $q(\cdot)$ is bounded. Thus, the transversality condition, $\lim_{t\to\infty} e^{-\theta t} q(t) = 0$, holds.

We next have the following lemma, which corresponds to the "visit lemma" in turnpike theory.

Lemma 5.4. *Assume that x^* is the unique maximizer of F over* $[0, 1]$*. For any $\varepsilon > 0$, there exists $\bar{\delta}(\varepsilon) > 0$ such that for all $\delta \leq \bar{\delta}(\varepsilon)$ and for all $x^0 \in [0, 1]$, if $(x(\cdot), u(\cdot))$ is an optimal pair to the problem* (5.1)*, then there exists $t \geq 0$ such that $|x(t) - x^*| < \varepsilon$.*

Proof. See Appendix. ▮

To understand the intuition behind this claim, consider the equivalent maximization problem:

$$\text{maximize} \quad \tilde{J}(y(\cdot), v(\cdot)) = \int_0^\infty \delta e^{-\delta s} \left(F(y(s)) - \frac{v(s)^2}{2} \right) ds,$$

subject to $\dot{y}(s) = v(s)$, $0 \leq y(s) \leq 1$, and $y(0) = x^0$, where $\delta = \theta/\sqrt{\gamma}$, which is obtained by applying to J the change of variables, $y(s) = x(s/\sqrt{\gamma})$ and $v(s) = u(s/\sqrt{\gamma})/\sqrt{\gamma}$, with a positive multiplicative θ. If δ is small, large weights are put on the values of the integrand for far future times s. Therefore, for any small neighborhood of the maximizer of F, if $y(\cdot)$ does not visit this neighborhood, then $(y(\cdot), v(\cdot))$ does not maximize $\tilde{J}$, provided that δ is sufficiently small.

The above claims as well as the absorption imply the global accessibility of x^*.

Proof of Theorem 4.1. Due to Theorem 4.2, x^* is absorbing for any value of $\delta = \theta/\sqrt{\gamma} > 0$, that is, there exists $\varepsilon > 0$ such that any equilibrium path starting from any initial state in $B_\varepsilon(x^*)$ converges to x^*. From the proof of Theorem 4.2, we can take ε independently of δ. Fix this value of ε and assume that $\delta \leq \bar{\delta} = \bar{\delta}(\varepsilon)$ with $\bar{\delta}(\varepsilon)$ as in Lemma 5.4. Consider any initial state $x^0 \in [0, 1]$. From Proposition 5.1, there exists an optimal solution $x(\cdot)$ starting from x^0, and from Proposition 5.2, it is an equilibrium path from x^0. By Lemma 5.4, there exists $T \geq 0$ such that $x(T) \in B_\varepsilon(x^*)$. Since the truncated path $y(t) = x(t + T)$ is also an equilibrium path, it follows from the choice of ε and Theorem 4.2 that $\lim_{t\to\infty} x(t) = x^*$. Since x^0 has been chosen arbitrarily, this proves that x^* is globally accessible when $\delta \leq \bar{\delta}$. ▮

5.2 Absorption

Define the function $H^*\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$H^*(x,q) = F(x) + \frac{\gamma}{2}q^2, \tag{5.11}$$

which is the maximized Hamiltonian, i.e., $H^*(x,q) = \max_u H(x,u,q)$, where H is defined in (5.2). We first show in Lemma 5.5 that H^* works as a Lyapunov function for the system (3.8)–(3.10) which describes the behavior of equilibrium paths. We then show in Lemma 5.6 that if $(x(\cdot),q(\cdot))$ is a solution to (3.8)–(3.10), then we must have $\lim_{t\to\infty} q(t) = 0$, and thus any accumulation point $\hat{x}$ of $x(\cdot)$ must be an equilibrium state and satisfy $H(x(0),q(0)) \le H^*(\hat{x},0)$, which implies $F(x(0)) \le F(\hat{x})$. Hence, if we take a neighborhood of the potential maximizer x^* such that for all x in the neighborhood, $F(x) > F(\hat{x})$ for all equilibrium states $\hat{x} \ne x^*$, then any equilibrium path from this neighborhood converges to x^*, which means the absorption of x^*.

Lemma 5.5. *Let $(x(\cdot),q(\cdot))$ be a solution to* (3.8)–(3.10). *Then, for almost all $t \ge 0$,*

$$\frac{d}{dt}H^*(x(t),q(t)) \ge 0,$$

with equality holding if and only if $q(t) = 0$.

Proof. From (3.8)–(3.10) we have

$$\begin{aligned}\frac{d}{dt}H^*(x(t),q(t)) &= f(x(t))\dot{x}(t) + \gamma q(t)\dot{q}(t)\\ &= \gamma q(t)\big(f(x(t)) + \dot{q}(t)\big)\\ &= \begin{cases}\gamma\theta q(t)^2 \ge 0 & \text{if } t \text{ is in an interior interval}\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

since, by Proposition 3.1, $\dot{q}(t) + f(x(t)) = \theta q(t)$ if t is in an interior interval, while $q(t) = 0$ if t is in a boundary interval or is a contact time. ∎

Lemma 5.6. *Let $x(\cdot)$ be an equilibrium path from x^0. If $\hat{x}$ is an accumulation point of $x(\cdot)$, then*
(1) $F(\hat{x}) \ge F(x^0)$, and
(2) $\hat{x}$ is an equilibrium state.

Under Assumption 2.1, (2) implies that any equilibrium path converges to an equilibrium state.

The proof follows that of Lemma 4 in Hofbauer and Sorger (1999).

Proof. Let $q(\cdot)$ be associated with the equilibrium path $x(\cdot)$. Let $\{t_k\}$ be a sequence such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = \hat{x}$. Let, without loss of generality, $\hat{q} = \lim_{k\to\infty} q(t_k)$. Define $(x^*(\cdot), q^*(\cdot))$ by $t \mapsto \lim_{k\to\infty}(x(t+t_k), q(t+t_k))$, which satisfies (3.8)–(3.10) with $(x^*(0), q^*(0)) = (\hat{x}, \hat{q})$.

We show that $H^*(x^*(t), q^*(t))$ is constant. Assume that for some $t, s \in [0, \infty)$,

$$H^*(x^*(t), q^*(t)) < H^*(x^*(s), q^*(s)).$$

By Lemma 5.5, we must have $t < s$. Since $\lim_{k\to\infty} t_k = \infty$, we may assume without loss of generality that $t_{k+1} > t_k + (s - t)$. Using Lemma 5.5 again, we obtain

$$\begin{aligned} H^*(x^*(t), q^*(t)) &= \lim_{k\to\infty} H^*(x(t_k + t), q(t_k + t)) \\ &= \lim_{k\to\infty} H^*(x(t_{k+1} + t), q(t_{k+1} + t)) \\ &\geq \lim_{k\to\infty} H^*(x(t_k + s), q(t_k + s)) \\ &= H^*(x^*(s), q^*(s)), \end{aligned}$$

which is a contradiction. Thus we have $(d/dt)H^*(x^*(t), q^*(t)) = 0$ for all $t \geq 0$. Hence, $q^*(t) = 0$ for all $t \geq 0$ by Lemma 5.5. Since $(x^*(\cdot), q^*(\cdot))$ satisfies (3.8)–(3.10) and $x^*(0) = \hat{x}$, it follows that $x^*(t) = \hat{x}$ for all $t \geq 0$, and therefore $\hat{x}$ is an equilibrium state by Observation 3.2, which proves (2). Also, we have

$$F(x^0) \leq H^*(x(0), q(0)) \leq H^*(x(t), q(t)) \leq H^*(x^*(t), q^*(t)) = F(\hat{x}),$$

which proves (1). ∎

We are now ready to prove the absorption property of the potential maximizer x^*: that if an equilibrium path starts in a neighborhood of x^*, then it must converge to x^*.

Proof of Theorem 4.2. Since the potential maximizer x^* is isolated from other equilibrium states by Assumption 2.1, we can take $\varepsilon > 0$ such that $F(x) > F(\hat{x})$ for all $x \in B_\varepsilon(x^*)$ and all equilibrium states $\hat{x} \neq x^*$. Lemma 5.6 thus implies that any equilibrium path $x(\cdot)$ from any $x^0 \in B_\varepsilon(x^*)$ satisfies $\lim_{t\to\infty} x(t) = x^*$. ∎

Remark 5.2. Ottaviano (2001) makes use of the function H^* for the case of symmetric regions, and observes that if $\theta = 0$, then along a trajectory $(x(\cdot), q(\cdot))$ satisfying the system (3.8)–(3.10), $(d/dt)H^*(x(t), q(t)) = 0$ for all $t \geq 0$ and, in particular, when $x = 0$ and $x = 1$ are equilibrium states, there are trajectories that connect $(x, q) = (0, 0)$ and $(x, q) = (1, 0)$. By considering perturbation of the system, he concludes that in this case, both $x = 0$ and $x = 1$ are, in our terminology, absorbing for positive small θ. As our result shows for the generic case of asymmetric regions, however, this is a knife-edge result which is not robust to exogenous asymmetries.

6 Concluding Remarks

In this paper, we have addressed the issue of 'history versus expectations' in a two-location new economic geography model, which typically has multiple equilibria, by embedding the model in the class of equilibrium dynamics due to Krugman (1991b) and Fukao and Benabou (1993) (KFB dynamics). Agents are assumed to incur moving costs upon migration which depend on the rate of aggregate migration flow. This, along with agents' impatience (i.e., positive time discounting), constitutes the friction of our dynamic environment. We obtained an equilibrium selection result based on the stability properties under the dynamics: that, except for knife-edge cases, there exists a (unique, by definition) spatial configuration that is absorbing and globally accessible whenever the degree of friction is sufficiently small, and such a configuration is characterized as the unique maximizer of the potential function of the static model. While we proceeded in the specific context of economic geography, we emphasize that our analysis is general enough to capture many situations of social interactions with binary choice.

The fact that forward-looking behavior combined with frictions can lead to equilibrium selection is also observed in the class of equilibrium dynamics as considered by Matsuyama (1991) and Matsui and Matsuyama (1995) (MM dynamics). In fact, the proof strategy for our main theorems follows that of Hofbauer and Sorger (1999) who prove the same stability properties under the MM dynamics in potential games with (finitely) many actions. Here we briefly discuss the differences in the mathematical properties of the KFB and the MM dynamics.

In the MM dynamics, the opportunities for agents to revise their choice arrive only occasionally, according to independent Poisson processes. Therefore, the speed of adjustment at the aggregate level is bounded exogenously, since, during a short interval, only a given fraction of the population is assumed to receive a revision opportunity. An equilibrium path of this dynamics is simply defined to be a feasible path along which each agent, when given a revision opportunity, takes a best alternative that maximizes the expected discounted payoff, which is independent of his current choice.

Agents in the KFB dynamics are allowed to revise their choice at any time instance with adjustment costs which depend on the rate of change in the population state. Equilibrium behavior in the interior of the state space is characterized by the non-arbitrage condition, i.e., the equality between the adjustment cost and the option value of adjustment, while once the economy reaches the boundary, individual agents are able to instantaneously move between the locations with no cost. Accordingly, the formal definition of equilibrium paths in the KFB dynamics must carefully incorporate transition between these phases. This is in fact the source of the error in Krugman (1991b) pointed out by Fukao and Benabou (1993).

Furthermore, while it is conjectured to be the case, it is left as an open question whether, as in the MM dynamics, the stability of a potential maximizer in the KFB dynamics extends to the case of more than two alternatives. On the other hand, using the MM dynamics the companion paper (Oyama (2006)) is able to study forward-looking migration behavior in a new economic geography model with many locations that admits a potential.

It should be noted, however, that when we consider policy issues, the two models may yield different implications. In the KFB dynamics, the speed of aggregate adjustment is directly determined by the no-arbitrage condition and thus can be controlled by governmental instruments such as tax and subsidy, whereas in the MM dynamics, it is bounded by the Poisson parameter and independent of agents' utilities. This point is discussed in the context of infant industry protection in a sectoral choice model by Kaneda (2003), who demonstrates that a subsidy scheme with a shorter duration and a higher rate is not isomorphic to that with a longer duration and a lower rate under the Poisson formulation where the speed of growth in the industry is independent of the subsidy rate, while noting that these schemes are substitutable for each other under the adjustment cost formulation. Conclusive policy implications on such issues should be obtained by developing a more general model that unifies these formulations. We leave this task for future research.

Appendix

Proof of Proposition 3.1. Let $x(\cdot)$ be an equilibrium path and $V_0(\cdot), V_1(\cdot)$ the corresponding value functions. Define $q\colon [0,\infty) \to \mathbb{R}$ by $q(t) = V_1(t) - V_0(t)$, which by definition is continuous on any interior or boundary interval and right-continuous at junction times. We want to show that this function q satisfies the desired conditions in each of cases (i) and (ii).

(i) First, if $\dot{x}(t) \leq 0$, then $V_0(t) + \dot{x}(t)/\gamma = V_1(t)$, while if $\dot{x}(t) \geq 0$, then $V_1(t) - \dot{x}(t)/\gamma = V_0(t)$, so that in each case, we have

$$\dot{x}(t) = \gamma(V_1(t) - V_0(t)) = \gamma q(t).$$

Second, for any sufficiently small $\Delta t > 0$,

$$V_i(t) = \int_t^{t+\Delta t} e^{-\theta(s-t)} f_i(x(s))\, ds + e^{-\theta \Delta t} V_i(t + \Delta t)$$

for each $i = 0, 1$, so that

$$q(t) = \int_t^{t+\Delta t} e^{-\theta(s-t)} f(x(s))\, ds + e^{-\theta \Delta t} q(t + \Delta t).$$

Thus, we have

$$\frac{q(t+\Delta t) - q(t)}{\Delta t} = \frac{1 - e^{-\theta \Delta t}}{\Delta t} q(t + \Delta t) + \frac{1}{\Delta t} \int_t^{t+\Delta t} e^{-\theta(s-t)} f(x(s))\, ds.$$

As $\Delta t \to 0$, the right hand side converges to $\theta q(t) - f(x(t))$. The same argument applies to the limit of $(q(t) - q(t - \Delta t))/\Delta t$.

(ii) If t is in a boundary interval for the boundary $x = 0$ ($x = 1$, resp.), then by definition, $f(0) \leq 0$ ($f(1) \geq 0$, resp.), $x(t) = 0$ ($x(t) = 1$, resp.), and $q(t) = 0$.

We need to show that $q(\cdot)$ is continuous at junction times. The following three lemmata correspond respectively to Lemmata 1–3 in Fukao and Benabou (1993).

Lemma A.1. *If τ is an entry time, then $q(\tau^-) = 0$.*

Proof. We only consider the case where τ is an entry time for the boundary $x = 1$. First, $q(\tau^-) \geq 0$ (otherwise, we would have $x(\tau) < 1$). Next, for sufficiently small $\varepsilon > 0$, we have

$$V_0(\tau - \varepsilon) \geq \int_{\tau-\varepsilon}^{\tau+\varepsilon} e^{-\theta(s-\tau+\varepsilon)} f_0(x(s))\, ds + e^{-2\theta\varepsilon} V_1(\tau + \varepsilon),$$

where the right hand side is the payoff that the agent would obtain if he waited until $\tau + \varepsilon$ to move from 0 to 1. As $\varepsilon \to 0$, we have $V_0(\tau^-) \geq V_1(\tau)$ by the continuity of V_1 at τ, so that $q(\tau^-) = V_1(\tau) - V_0(\tau^-) \leq 0$. ∎

Lemma A.2. *If τ is an exit time, then $q(\tau^+) = 0$.*

Proof. We only consider the case where τ is an exit time for the boundary $x = 1$. First, $q(\tau^+) \leq 0$ (otherwise, we would have $x(\tau + \varepsilon) > 1$). Next, for sufficiently small $\varepsilon > 0$, we have

$$V_1(\tau - \varepsilon) \geq \int_{\tau-\varepsilon}^{\tau+\varepsilon} e^{-\theta(s-\tau+\varepsilon)} f_0(x(s))\, ds + e^{-2\theta\varepsilon} V_0(\tau + \varepsilon),$$

where the right hand side is the payoff that the agent would obtain if he moved from 1 to 0 at $\tau - \varepsilon$. As $\varepsilon \to 0$, we have $V_1(\tau) \geq V_0(\tau^+)$ by the continuity of V_1 at τ, so that $q(\tau^+) = V_1(\tau) - V_0(\tau^+) \geq 0$. ∎

Lemma A.3. *If τ is a contact time, then $q(\tau^-) = q(\tau^+) = 0$.*

Proof. We only consider the case where τ is a contact time for the boundary $x = 1$. First, $q(\tau^-) \geq 0$ and $q(\tau^+) \leq 0$ as previously. Next, for sufficiently small $\varepsilon > 0$, we have

$$V_0(\tau - \varepsilon) \geq \int_{\tau-\varepsilon}^{\tau+\varepsilon} e^{-\theta(s-\tau+\varepsilon)} f_0(x(s))\, ds + e^{-2\theta\varepsilon} V_0(\tau + \varepsilon),$$

where the right hand side is the payoff that the agent would obtain if he remained in 0. As $\varepsilon \to 0$, we have $V_0(\tau^-) \geq V_0(\tau^+)$, so that $q(\tau^-) = V_1(\tau) - V_0(\tau^-) \leq V_1(\tau) - V_0(\tau^+) = q(\tau^+)$. ∎

Finally, we show that $|q(\cdot)|$ is bounded by M/θ, where $M > 0$ is the maximum value of $|f(x)|$ over $x \in [0, 1]$. Suppose that this is not true, say, that $q(T) > M/\theta$ for some T. Then, we would have $\dot{q}(t) > 0$ for all $t \geq T$ by (3.8b), and therefore, $\dot{x}(t) \geq \gamma M/\theta$ for all $t \geq T$ by (3.8a). This implies that $x(T') = 1$ for some finite $T' \geq T$, while $q(T') > M/\theta$, which contradicts (3.9).

To show the converse, let $x(\cdot)$ and $q(\cdot)$ be as in the statement. Then, define $V_i(\cdot)$ by

$$V_i(t) = \int_t^\infty e^{-\theta(s-t)} \tilde{f}_i(x(s))\, ds,$$

where $\tilde{f}_i\colon [0, 1] \to \mathbb{R}$ is defined by

$$\tilde{f}_i(x) = \begin{cases} f_i(x) & \text{if } 0 < x < 1, \\ f_{i^*}(i^*) & \text{if } x = i^*. \end{cases}$$

One can verify that $V_1(t) - V_0(t) = q(t)$ for all t and thus the equilibrium conditions in Definition 3.1 are satisfied. ▌

Proof of Proposition 5.2. Let $(x^*(\cdot), u^*(\cdot))$ be an optimal pair for the problem (5.1). We want to show the existence of a function $q(\cdot)$ that satisfies the conditions in Proposition 3.1.

Consider the finite horizon optimal control problem parameterized by $T > 0$:

$$\begin{aligned} &\text{maximize} && J_T(x(\cdot), u(\cdot)) = \int_0^T e^{-\theta t} \left(F(x(t)) - \frac{u(t)^2}{2\gamma} \right) dt, && \text{(A.1a)} \\ &\text{subject to} && \dot{x}(t) = u(t), && \text{(A.1b)} \\ &&& h(x(t)) \geq 0, && \text{(A.1c)} \\ &&& x(0) = x^*(0), && \text{(A.1d)} \\ &&& x(T) = x^*(T), && \text{(A.1e)} \end{aligned}$$

where $h(x) = (h_0(x), h_1(x)) = (x, 1 - x)$.

Let

$$\tilde{H}(x, u, \lambda_0, \lambda, t) = \lambda_0 e^{-\theta t} \left(F(x) - \frac{u^2}{2\gamma} \right) + \lambda u.$$

Since the restriction of $(x^*(\cdot), u^*(\cdot))$ to $[0, T]$ is optimal for the problem (A.1), it must satisfy the following necessary conditions due to Theorem 4.2 in Hartl *et al.* (1995): there exist a constant $\lambda_0 \in \{0, 1\}$, a right-continuous function $\lambda\colon [0, T] \to \mathbb{R}$, and functions $\tilde{\nu}_i\colon [0, T] \to \mathbb{R}$ $(i = 0, 1)$ that are of bounded variation, nonincreasing, constant on intervals where $h_i(x(t)) > 0$, and right-continuous with left-hand limits everywhere such that

$$(\lambda_0, \lambda(t), \tilde{\nu}_0(T) - \tilde{\nu}_0(0), \tilde{\nu}_1(T) - \tilde{\nu}_1(0)) \neq 0 \qquad \text{(A.2)}$$

for all $t \in [0, T]$,

$$\tilde{H}_u(x^*(t), u^*(t), \lambda_0, \lambda(t), t) = -\lambda_0 e^{-\theta t} \frac{u^*(t)}{\gamma} + \lambda(t) = 0 \tag{A.3}$$

for almost all $t \in [0, T]$, and

$$\begin{aligned}
&\lambda(t_1^+) - \lambda(t_0^+) \\
&\quad = -\lambda_0 \int_{t_0}^{t_1} \tilde{H}_x(x^*(t), u^*(t), \lambda_0, \lambda(t), t)\, dt + \int_{(t_0, t_1]} h_x(x^*(t)) \cdot d\tilde{\nu} \\
&\quad = -\lambda_0 \int_{t_0}^{t_1} e^{-\theta t} f(x^*(t))\, dt \\
&\qquad\qquad + (\tilde{\nu}_0(t_1^+) - \tilde{\nu}_0(t_0^+)) - (\tilde{\nu}_1(t_1^+) - \tilde{\nu}_1(t_0^+))
\end{aligned} \tag{A.4}$$

for $t_0 < t_1$.

We first claim that $\lambda_0 \neq 0$. Indeed, if $\lambda_0 = 0$, then (A.3) and (A.4) would contradict (A.2). Therefore, (A.1b) and (A.3) implies

$$\lambda(t) = e^{-\theta t} \frac{\dot{x}^*(t)}{\gamma}. \tag{A.5}$$

If (t_0, t_1) is an interior interval, then by (A.4), for all $t \in (t_0, t_1)$ we have $\tilde{\nu}_0(t) - \tilde{\nu}_0(t_0^+) = \tilde{\nu}_1(t) - \tilde{\nu}_1(t_0^+) = 0$ and

$$\lambda(t) = \lambda(t_0^+) - \int_{t_0}^{t} e^{-\theta s} f(x^*(s))\, ds.$$

This implies that $\lambda(t)$ is differentiable and $\dot{\lambda}(t) = -e^{-\theta t} f(x^*(t))$ on interior intervals.

We then show that $\lambda(t^-) = \lambda(t^+) = 0$ for any time t in a boundary interval and any contact time t. Let us show this in the case where $x(t) = 1$. First, it must be that $\dot{x}(t^-) \geq 0$ (otherwise, we would have $x(t) < 1$), and hence $\lambda(t^-) \geq 0$ by (A.5). Second, it must be that $\dot{x}(t^+) \leq 0$ (otherwise, we would have $x(t + \varepsilon) > 1$), and hence $\lambda(t^+) \leq 0$ again by (A.5). Last, let $t_0 < t$ be such that $x(s) > 0$ for all $s \in [t_0, t]$, so that $\tilde{\nu}_0(t^+) - \tilde{\nu}_0(t_0^+) = 0$. By (A.4), we have

$$\lambda(t^+) = \lambda(t_0^+) - \int_{t_0}^{t} e^{-\theta s} f(1)\, ds - (\tilde{\nu}_1(t^+) - \tilde{\nu}_1(t_0^+)), \tag{A.6}$$

and therefore

$$\lambda(t^+) = \lambda(t^-) - (\tilde{\nu}_1(t^+) - \tilde{\nu}_1(t^-))$$

in the limit as $t_0 \uparrow t$. Since $\tilde{\nu}_1$ is nonincreasing so that $\tilde{\nu}_1(t^+) - \tilde{\nu}_1(t^-) \leq 0$, we have $\lambda(t^+) \geq \lambda(t^-)$. These imply that $\lambda(t^-) = \lambda(t^+) = 0$.

Furthermore, since (A.6) reduces to

$$\tilde{\nu}_1(t) - \tilde{\nu}_1(t_0^+) = -\int_{t_0}^{t} e^{-\theta s} f(1)\, ds$$

and $\tilde{\nu}_1$ is nonincreasing, we have $f(1) \geq 0$. The same argument applies to boundary intervals for $x = 0$.

Now, let $0 < T^1 < T^2 < \cdots$ be such that $\lim_{k\to\infty} T^k = \infty$. Let $\lambda^k(\cdot)$ be the adjoint for the problem (A.1) with $T = T^k$. Note that if $k < k'$, $\lambda^k(t) = \lambda^{k'}(t)$ for all $t \in [0, T^k]$. Let $\lambda^*(\cdot)$ be the extension of $\lambda^k(\cdot)$'s to $[0, \infty)$: i.e., $\lambda^*(t) = \lambda^k(t)$ where k is such that $t \in [T^{k-1}, T^k)$ (with $T^0 = 0$). Finally, let $q(t) = e^{\theta t}\lambda^*(t)$. Then, verifying that the obtained $q(\cdot)$ satisfies the conditions of Proposition 3.1 completes the proof. ▌

Proof of Lemma 5.4. Assume the contrary: i.e., that there exists $\varepsilon > 0$ such that for all $\bar{\delta} > 0$, there exists an optimal pair $(x(\cdot), u(\cdot))$ for some θ and γ with $\delta = \theta/\sqrt{\gamma} \in (0, \bar{\delta}]$ and some $x^0 \in [0, 1]$ with $x(0) = x^0$ such that $|x(t) - x^*| \geq \varepsilon$ for all $t \geq 0$. Given such an $\varepsilon > 0$, let $c = c(\varepsilon) > 0$ be defined by

$$c = F(x^*) - \max_{x\in[0,1]} \{F(x) \mid |x - x^*| \geq \varepsilon\}$$

and $\bar{\delta} = \bar{\delta}(\varepsilon) > 0$ be such that

$$\left(1 - e^{-\bar{\delta}}\right)\left(2M + \frac{1}{2}\right) < e^{-\bar{\delta}} c,$$

where $M > 0$ is a constant such that $|F(x)| \leq M$ for all $x \in [0, 1]$. Given such a $\bar{\delta} > 0$, let $(x(\cdot), u(\cdot))$ be an optimal pair with $\theta/\sqrt{\gamma} \in (0, \bar{\delta}]$ and $x^0 \in [0, 1]$ such that $|x(t) - x^*| \geq \varepsilon$ for all $t \geq 0$, as assumed. We assume without loss of generality that $x^0 \leq x^*$.

Define $y(\cdot)$ and $v(\cdot)$ by $y(s) = x\big(s/\sqrt{\gamma}\big)$ and $v(s) = u\big(s/\sqrt{\gamma}\big)/\sqrt{\gamma}$. Note that $|y(s) - x^*| \geq \varepsilon$ and therefore $F(y(s)) - F(x^*) \leq -c$ for all $s \geq 0$. Observe that $(y(\cdot), v(\cdot))$ is an optimal pair to the maximization problem for

$$\tilde{J}(y(\cdot), v(\cdot)) = \int_0^\infty \delta e^{-\delta s} \left(F(y(s)) - \frac{v(s)^2}{2}\right) ds$$

subject to $\dot{y}(s) = v(s)$, $0 \leq y(s) \leq 1$, and $y(0) = x^0$, where $\delta = \theta/\sqrt{\gamma}$ $(\leq \bar{\delta})$.

Now define an admissible pair $(y'(\cdot), v'(\cdot))$ by

$$v'(s) = \begin{cases} 1 & \text{if } s < T \\ 0 & \text{if } s \geq T, \end{cases} \qquad y'(s) = \begin{cases} x^0 + s & \text{if } s < T \\ x^* & \text{if } s \geq T, \end{cases}$$

where $T = x^* - x^0$ (≤ 1). Then,

$$
\begin{aligned}
&\tilde{J}(y(\cdot), v(\cdot)) - \tilde{J}(y'(\cdot), v'(\cdot)) \\
&\qquad = \int_0^T \delta e^{-\delta s} \left\{ \left(F(y(s)) - \frac{v(s)^2}{2} \right) - \left(F(y'(s)) - \frac{1^2}{2} \right) \right\} ds \\
&\qquad\qquad + \int_T^\infty \delta e^{-\delta s} \left\{ \left(F(y(s)) - \frac{v(s)^2}{2} \right) - F(x^*) \right\} ds \\
&\qquad \leq \int_0^T \delta e^{-\delta s} \left(F(y(s)) - F(y'(s)) + \frac{1}{2} \right) ds \\
&\qquad\qquad + \int_T^\infty \delta e^{-\delta s} \big(F(y(s)) - F(x^*) \big) ds \\
&\qquad \leq \int_0^T \delta e^{-\delta s} \left(2M + \frac{1}{2} \right) ds - \int_T^\infty \delta e^{-\delta s} c \, ds \\
&\qquad = \left(1 - e^{-\delta T} \right) \left(2M + \frac{1}{2} \right) - e^{-\delta T} c \\
&\qquad \leq \left(1 - e^{-\bar{\delta}} \right) \left(2M + \frac{1}{2} \right) - e^{-\bar{\delta}} c < 0,
\end{aligned}
$$

which contradicts the optimality of $(y(\cdot), v(\cdot))$. ∎

References

Baldwin, R., R. Forslid, P. Martin, G. Ottaviano, and F. Robert-Nicoud (2003). *Economic Geography and Public Policy*, Princeton University Press, Princeton.

Baldwin, R. E. (2001). "Core-Periphery Model with Forward-Looking Expectations," *Regional Science and Urban Economics* **31**, 21-49.

Baum, R. F. (1976). "Existence Theorems for Lagrange Control Problems with Unbounded Time Domain," *Journal of Optimization Theory and Applications* **19**, 89-116.

Forslid, R. and G. I. P. Ottaviano (2003). "An Analytically Solvable Core-Periphery Model," *Journal of Economic Geography* **3**, 229-240.

Fujita, M., P. Krugman, and A. J. Venables (1999). *The Spatial Economy: Cities, Regions, and International Trade*, MIT Press, Cambridge.

Fukao, K. and R. Benabou (1993). "History versus Expectations: A Comment," *Quarterly Journal of Economics* **108**, 535-42.

Hartl, R. F., S. P. Sethi, and R. G. Vickson (1995). "A Survey of the Maximum Principles for Optimal Control Problems with State Constraints," *SIAM Review* **37**, 181-218.

Hofbauer, J. and G. Sorger (1999). "Perfect Foresight and Equilibrium Selection in Symmetric Potential Games," *Journal of Economic Theory* **85**, 1-23.

Hofbauer, J. and G. Sorger (2002). "A Differential Game Approach to Evolutionary Equilibrium Selection," *International Game Theory Review* **4**, 17-31.

Kaneda, M. (2003). "Policy Designs in a Dynamic Model of Infant Industry Protection," *Journal of Development Economics* **72**, 91-115.

Krugman, P. (1991a). "Increasing Returns and Economic Geography," *Journal of Political Economy* **99**, 483-499.

Krugman, P. (1991b). "History versus Expectations," *Quarterly Journal of Economics* **106**, 651-67.

Matsui, A. and K. Matsuyama (1995). "An Approach to Equilibrium Selection," *Journal of Economic Theory* **65**, 415-434.

Matsuyama, K. (1991). "Increasing Returns, Industrialization, and Indeterminacy of Equilibrium," *Quarterly Journal of Economics* **106**, 617-650.

Matsuyama, K. (1992). "The Market Size, Entrepreneurship, and the Big Push," *Journal of the Japanese and International Economies* **6**, 347-364.

Monderer, D. and L. Shapley (1996). "Potential Games," *Games and Economic Behavior* **14**, 124-143.

Ottaviano, G. I. P. (2001). "Monopolistic Competition, Trade, and Endogenous Spatial Fluctuations," *Regional Science and Urban Economics* **31**, 51-77.

Oyama, D. (2002). "p-Dominance and Equilibrium Selection under Perfect Foresight Dynamics," *Journal of Economic Theory* **107**, 288-310.

Oyama, D. (2006). "Agglomeration under Forward-Looking Expectations: Potentials and Global Stability," mimeo.

Oyama, D., S. Takahashi, and J. Hofbauer (2008). "Monotone Methods for Equilibrium Selection under Perfect Foresight Dynamics," *Theoretical Economics* **3**, 155-192.

Sandholm, W. H. (2001). "Potential Games with Continuous Player Sets," *Journal of Economic Theory* **97**, 81-108.

Seierstad, A. and K. Sydsæter (1987). *Optimal Control Theory with Economic Applications*, North-Holland, Amsterdam.

SETHI, S. P. AND G. L. THOMPSON (2000). *Optimal Control Theory: Applications to Management Science and Economics, Second Edition*, Springer, New York.