Incentive and normative analysis on sequencing problem

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5 July 2013

INCENTIVE AND NORMATIVE ANALYSIS ON SEQUENCING PROBLEMS

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ABSTRACT. In this paper we have analyzed sequencing problem from both incentive and normative aspects. We have identified unique class of VCG mechanisms that ensures egalitarian equivalence and we also have shown the possibility result with identical costs lower bound in that unique class of VCG mechanisms. Sequencing game imposes a stronger restriction on the possible set of "reference position", compared to queuing game and that, in turn results into the failure of having a feasible VCG mechanism along with egalitarian equivalence. Although we found the necessary and sufficient condition for the above mentioned unique class of egalitarian equivalent VCG mechanism to satisfy identical costs lower bound when the number of participating agents is two, necessary condition for the same for more than two agents remains an open question. Lastly, we contemplate a situation where the restriction that sequencing problem imposes on "reference position" is overlooked, that is, we assume almost no restriction (except the fact that it must be positive) on reference waiting time and identify the class of VCG mechanism that is egalitarian equivalent.

JEL Classifications: D63, D71, D82;

Keywords: egalitarian equivalence, identical costs lower bound, outcome efficiency, feasibility, budget balance, strategyproofness.

1. INTRODUCTION

In this paper, we analyze normative aspects of the sequencing problem. The main features of a sequencing problem are as follows: (1) there are n agents and a single server, (2) the server has multi-functional capability but can process one particular service at a time, that is the server can serve one agent at a time (3) jobs may not be identical across agents, so their service processing time may differ but is common knowledge. We assume agents have quasi-linear preferences over positions in queue and monetary transfers. Many real life phenomenon has this structure. A diagnostic center, installed with a multi-functional machine (due to space shortage), where a certain number of

enlisted patients visits for diagnosis, software installation problem to PCs of a set of agents can be examples. Many other comparable situations can be found in Maniquet (2003), Hashimoto and Saitoh (2012), Mukherjee (2013).

In case of sequencing problem, outcome efficiency is a widely studied allocation rule[1]. In sequencing or queuing context outcome efficiency implies minimal aggregate waiting cost. The seminal works of Vickrey (1961), Clarke (1971), Groves(1973) shown that outcome efficiency can be harnessed with stratefyproofness or non-manipulability. Holmström(1979)'s result in context of sequencing problem implies a mechanism satisfies strategyproofness and outcome efficiency if and only if it is a Vickerey-Clarke-Groves (VCG) mechanism [2]. We focus on the compatibility of a fairness axiom, egalitarian equivalence, introduced by Pazner and Schmeidler (1978) and get a subclass of Vickerey-Clarke-Groves (VCG) mechanism that satisfies egalitarian equivalence. Chun et al. (2014) has identified similar type of subclass of Vickerey-Clarke-Groves (VCG) mechanism for queuing problems. Their main assumption regarding the egalitarian equivalent reference bundle is; the reference position can only be an element from the set $\{1, 2, \ldots, n\}$. Hence, the reference waiting time is similarly restricted. In our work, the only restriction on the reference waiting time is, it must be positive. For two agents, we have identified a necessary and a sufficient condition to achieve egalitarian equivalent VCG mechanism . For more than two agents, we have identified an interesting sufficient condition that guarantees egalitarian equivalent VCG mechanism. Although we show that it is quite possible to have non-constant reference waiting time function, we argue why we need further restriction on the reference waiting time function.

Mitra (2002) showed among more general and natural class of sequencing problems, sequencing problems with linear cost structure is the only class for which outcome efficiency, budget balancedness and strategyproofness (known as first best) can be achieved. We find that egalitarian equivalence is incompatible with first best situation. In this context Chun et al. (2014) attained feasibility along with egalitarian

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[1]See Suijs (1996), Mitra (2002), Maniquet (2003) in this context
[2]See Vickrey (1961), Clarke (1971), Groves (1973)

equivalent VCG mechanism in case of queuing problem when the reference position was the first position of the queue (but we get an impossibility result in case of sequencing problem, that is, no mechanism is efficient, feasible, non-manipulable and egalitarian equivalent.

Sequencing problem have also been analyzed from the perspective of group manipulability . In this context we must mention the work of Mitra and Mutuswami (2011) that shows there does not exist any mechanism that satisfies outcome efficiency and strong group strategy-proofness in a single machine queuing context. Similarly in sequencing Kayi and Ramaekers (2008) has shown that no rule satisfies outcome efficiency and coalitional strategy-proofness. Whereas we show that, no mechanism satisfies outcome efficiency, egalitarian equivalence and pair-wise weak group strategyproofness which is weaker than group strategyproofness.

A mechachnism however good in itself, is incomplete unless the agent's participation constrain is satisfied. To address this, we use an appropriate and context specific condition known as identical costs lower bound[3] which is based on the idea of identical preference lower bound, first introduced by Moulin (1990) which he termed as egalitarian lower bound. This concept was applied in queuing problem by Maniquet (2003), Chun (2006) and others. We find that the fairness notions identical costs lower bound and egalitarian equivalence are not compatible if we add feasibility with outcome efficiency and strategyproofness. Once feasibility is relax, we can have situation where identical preference lower bound and egalitarian equivalence are compatible with outcome efficiency and strategyproofness.

This paper has been arranged in the following way. In Section 2, we formally introduce the model and add necessary definitions. In Section 3, we state and prove characterization results regarding egalitarian equivalent VCG mechanism. In Section 4, we focus on feasibility and group staretegyproofness issues of egalitarian equivalent VCG mechanism . In Section 5, we analyze the possibility of identical costs lower bound property of egalitarian equivalent VCG mechanism. In Section 6, we again go back to analyze the compatibility of egalitarian equivalence and VCG mechanism

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[3]See Chun and Yengin (2017) [19]

when the reference waiting time is a non-constant function of the type profile of the agents. Lastly, in Section 7, we conclude the paper.

2. THE MODEL

We consider the set of agents $N = \{1, \ldots, n\}$ with a single machine. Each individual has a different kind of work to be executed by the machine. The machine can process one job at a time. Let $\forall i \in N$, $s_i \in \mathfrak{R}_{++}$ where s_i denotes the processing time of ith agent and we assume $s_1 \geq s_2 \geq \ldots \geq s_{n-1} \geq s_n$ without loss of generality. Each agent is identified with a positive waiting cost $\theta_i \in \mathfrak{R}_{++}$, the cost of waiting per unit of time. The profile of waiting costs of the set of all agents is typically denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \mathfrak{R}_{++}^n$. For any $i \in N$, θ_{-i} denotes the profile $(\theta_1 \ldots \theta_{i-1}, \theta_{i+1}, \ldots \theta_n) \in \mathfrak{R}_{++}^{n-1}$.

A sequence is an onto function $\sigma : N \rightarrow \{1, \ldots, n\}$. An allocation of n jobs can be done in many ways. An allocation rule is a mapping $\sigma : \mathfrak{R}_{++}^n \rightarrow \Sigma(N)$ that specifies for each profile $\theta \in \mathfrak{R}_{++}^n$ an allocation (rank) vector $\sigma(\theta) \in \Sigma(N)$. Agent i's position is denoted by $\sigma_i(\theta)$ which is an input of the vector $\sigma(\theta)$. Let $\Sigma(N)$ denote the set of all possible sequence of agents in N. Given $\sigma \in \Sigma(N), \forall \in N, P_i(\sigma) = \{j \in N | \sigma_j(\theta) < \sigma_i(\theta)\}$ denotes the set of predecessors of i and similarly $P_i'(\sigma) = \{j \in N | \sigma_j(\theta) > \sigma_i(\theta)\}$ denotes the set of successors of i. Agent i's waiting time is denoted by $S_i(\sigma(\theta))$ and corresponding waiting cost is $S_i(\sigma(\theta))\theta_i$. A transfer rule is a mapping $t : \mathfrak{R}_{++}^n \rightarrow \mathfrak{R}^n$ that specifies for each profile $\theta \in \mathfrak{R}_{++}^n$ a transfer vector $t(\theta) = (t_i(\theta), \ldots, t_n(\theta)) \in \mathfrak{R}^n$. We assume that the utility function of each agent $i \in N$ is quasi-liner and is of the form $U_i(\sigma(\theta), t_i(\theta), \theta_i) = -S_i(\sigma(\theta)(\theta_i) + t_i(\theta)$, where $t_i(\theta)$ is the monetary transfer of agent to i.

Definition 1. For all $\theta \in \mathfrak{R}_{++}^n$, a sequence $\sigma \in \Sigma(N)$ is outcome efficient if $\sigma \in E(\theta) = \text{argmin}_{\sigma \in \Sigma(N)} \sum_{i=1}^n S_i(\sigma)\theta_i$.

The implication of outcome efficiency is that agents are ranked according to the non-increasing order of their relative waiting costs (that is, if $\theta_i/s_i \geq \theta_j/s_j$ under a profile

θ, then $S_i(\sigma(\theta)) \leq S_i(\sigma(\theta))$). Moreover, there are profiles for which more than one rank vector is efficient. For example, in case of queuing problem if all agents have the same waiting cost, then all rank vectors are efficient. In the context of sequencing problem, if the profile of waiting cost is $(s_1, s_2, \ldots, s_n)$, then all agents have the same relative waiting cost. Therefore, we have an efficiency correspondence. In this paper, we choose a particular outcome efficient rule (that is, a single valued selection from the outcome efficiency correspondence) using a tie breaking rule. For our outcome efficient rule, we use the following tie breaking rule: if $i < j$ and $\theta_i/s_i = \theta_j/s_j$ then $S_i(\sigma(\theta)) < S_i(\sigma(\theta))$. This tie breaking rule guarantees that, given a profile $\theta \in \mathfrak{R}^n_{++}$, the efficient rule selects a single rank vector from $\Sigma(N)$.

A mechanism is (σ, t) constitutes of an allocation rule σ and a transfer rule t. We are interested in strategy proof mechanism for the sequencing problem.

Definition 2. A mechanism (σ, t) is strategy-proof (SP) if for all $i \in N$, for all $\theta_i, \theta_i' \in \mathfrak{R}_{++}$ and for all $\theta_{-i} \in \mathfrak{R}^{(n-1)}_{++}$, $-S_i(\sigma(\theta_i, \theta_{-i}))\theta_i + t_i(\theta_i, \theta_{-i}) \geq -S_i(\sigma(\theta_i', \theta_{-i}))\theta_i + t_i(\theta_i', \theta_{-i})$.

It means, for every agents, reporting the true type weakly dominates reporting false type. Hence, stragyproofness restricts any kind of unilateral deviation.

Definition 3. A mechanism (σ, t) is outcome efficient (OE) if for all announced profile $\theta \in \mathfrak{R}^n_{++}$, $\sigma(\theta) \in E(\theta) = \operatorname{argmin}_{\sigma \in \Sigma(N)} \sum_{i=1}^n S_i(\sigma)\theta_i$.

The main results of this paper that we derive in the next section is based on VCG mechanism that we define next.

Definition 4. A mechanism (σ, t) is a VCG mechanism if for all θ, $\sigma(\theta) \in E(\theta)$, and the transfers are given by,

$$(1) \qquad \forall i \in N : t_i(\theta) = -\sum_{j \neq i} \theta_j S_j(\sigma(\theta)) + h_i(\theta_{-i}).$$

When the preferences are quasi-linear and the domain of type is convex then a mechanism is OE and SP if and only if it is a VCG mechanism. This result was due to Holmström (1979).

Definition 5. A mechanism (σ, t) satisfies egalitarian equivalence (EE) if for all $\theta \in \mathfrak{R}^n_{++}$ there exist $(\bar{S}(\theta), t(\theta))$ such that for all $i \in N$, $-S_i(\sigma(\theta))\theta_i + t_i(\theta) = -\bar{S}(\theta)\theta_i + \bar{t}(\theta)$.

Here $(\bar{S}(\theta), \bar{t}(\theta))$ denotes the reference bundle, where $\bar{S}(\theta)$ is the reference waiting time and $\bar{t}(\theta)$ is the reference transfer. Egalitarian equivalence was introduced by Pazner and Schmeidler (1978) and is based on the idea that all individuals should be placed in a situation which is Pareto-indifferent to a perfectly egalitarian allocation. In case of sequencing problem $(\bar{S}(\theta), \bar{t}(\theta))$ is such a reference bundle, where, if the agent is placed remains indifferent to the original bundle that he receives under VCG mechanism.

Definition 6. A mechanism (σ, t) satisfies budget balancedness (BB) if for all $\theta \in \mathfrak{R}^n_{++}$, $\sum_{i=1}^n t_i(\theta) = 0$.

Definition 7. A mechanism (σ, t) satisfies feasibility (FSB) if for all $\theta \in \mathfrak{R}^n_{++}$, $\sum_{i=1}^n t_i(\theta) \leq 0$.

The profile θ and θ' are S-variants if for all $i \in N \setminus S$, $\theta_i = \theta_i'$.

Definition 8. A mechanism (σ, t) is weak group strategyproof (WSP) if for all S-variants θ, θ' $U_i(\sigma(\theta), t_i(\theta), \theta_i) \geq U_i(\sigma(\theta'), t_i(\theta'), \theta_i)$ for at least one $i \in S$.

This implies as long as all the group member are not strictly better off by deviating from their true profile, such group will not be formed.

Definition 9. A mechanism (σ, t) is pair-wise weak group strategyproof (PWSP) if for all *S-variants* θ, θ' where $|S| = 2$, $U_i(\sigma(\theta), t_i(\theta), \theta_i) \geq U_i(\sigma(\theta'), t_i(\theta'), \theta_i)$ for at least one $i \in S$.

This implies pair of agents deviates from their true profile by jointly misreporting if an only if they are both strictly better off from the situation when they truthfully reports.

Consider any agent $i \in N$. If the agent i a-priori perceives that he is not different form any agent $j(\neq i) \in N$ (in terms of relative waiting cost) then he will consider every feasible allocation $\sigma \in \sum(N)$ as a possible outcome. Hence, we need to consider the a-priori excepted cost perceived by agent i. Note that for any $j(\neq i) \in N$ at position r in the queue, waiting time imposed by agent j is $(r-1)(n-2)!s_j$. So in total any agent $j(\neq i) \in N$ imposes $\sum_{r=1}^{n}(r-1)(n-2)!s_j = n(n-1)!s_j/2$ amount of waiting time on agent i and agent i can except all the $n!$ allocations. Hence, the average waiting cost perceived by agent i would be $(s_i + \sum_{j\neq i} s_j/2)\theta_i$ where θ_i is the per unit time waiting cost of agent i.

Definition 10. A mechanism (σ, t) satisfies identical costs lower bound (ICLB) if for all $\theta \in \Re_{++}^n$, for all $i \in N$, $U_i(\sigma(\theta), t_i(\theta), \theta_i) \geq -(s_i + \sum_{j\neq i} s_j/2)\theta_i$.

This concept was first introduced by Moulin (1990) and is based on the idea that an agent's welfare is at least as that of consuming his equal share of resources. In the context of sequencing problem agents are considered identical as long as their relative waiting cost's are same. That is for all $i, j \in N$, if $\theta_i/s_i = \theta_j/s_j$ then agents are considered identical. identical costs lower bound implies that any agent's utility should be at east as that of average or expected utility of that agent when he/she perceives all the other agents as identical to himself.

3. EGALITARIAN EQUIVALENT VCG MECHANISM

In this section, we examine the implication of egalitarian equivalence on a strategy-proof mechanism that satisfies efficient allocation rule. We will use slightly different notation to refer an agent. An agent at i-th position of the queue is denoted as agent (i). So the true waiting cost profile is $\theta = (\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(n)})$ is such that $\lambda_{(1)} \geq \lambda_{(2)} \geq \ldots \geq \lambda_{(n)} > 0$ where $\forall i \in N$, $\lambda_{(i)} = \theta_{(i)}/s_{(i)}$. Hence, $\forall \theta = (\theta_{(i)}, \theta_{(-i)}) \in \Re_{++}^n$, $\forall i \in N$, $s_i \neq s_{(i)}$.

The crucial fact behind the idea of egalitarian equivalent allocation where everyone consumes the same "reference bundle" and derives same utility as they get with

the initially allocated bundle. In case of queuing problem, Chun et al. (2014) have completely characterized EE, SP and OE mechanisms. They restricted the reference position, that can vary with type profile, on the set $\{1, 2, \ldots, n\}$ as these are the only positions available in queuing problem with $|N|$ agents. Hence, Chun et al. (2014) avoided any arbitrary reference waiting time to keep the analysis natural for queuing context.

Our modification in this context is the following: unlike queuing, in sequencing problem agents differs in job processing time simply because different agents have different jobs to process. Hence, it is not possible to contemplate all the position $\{1,2,\ldots,n\}$ as a potential reference bundle. For example, when $N = \{1, 2, 3\}$, if we fix the reference position as the second position of the queue then $\bar{S}(\theta)$ is perceived differently by different agents as $\bar{S}(\theta) \in \{(s_1 + s_2), (s_1 + s_3), (s_2 + s_3)\}$. Agent 1 in second position may face $(s_1 + s_2)$ or $(s_1 + s_3)$ as waiting time, agent 2 can face $(s_1 + s_2)$ or $(s_2 + s_3)$ and similarly agent 3 can perceive $(s_1 + s_3)$ and $(s_2 + s_3)$. The only feasible reference position is the last position as whatever be the allocation and whoever is the agent, the reference waiting time for the last position is always $(s_1 + s_2 + s_3)$. Hence, to keep our analysis natural in sequencing context, we will assume the only feasible reference position is the last position of the queue. Therefore, we will have $\bar{S}(\theta) = \bar{S} = \sum_{i=1}^{n} s_i$. With this set up we get the following result:

Proposition 1. A mechanism (σ, t) satisfies EE, SP and OE if and only if the reference bundle for the profile $\theta \in \mathfrak{R}^n_{++}$ (where θ is a non-zero profile) is of the form $(\bar{S}(\theta), \bar{t}(\theta))$ where $\forall \theta \in \mathfrak{R}_{++}$ and $\bar{t}(\theta) = \sum_{i \in N} \{\bar{S} - S_{(i)}(\sigma(\theta))\} \theta_{(i)} + \bar{k}$ when $\bar{S}(\theta) = \bar{S}$.

Proof: Let us consider an announcement profile $\theta = (\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(n)}) \in \mathfrak{R}^n_{++}$. Therefore, given the OE allocation rule and the tie breaking rule, we can arrange agents uniquely i.e. $\sigma_{(i)}(\theta) = i$. Since the domain of preference is quasi-linear and type spaces for the agents are convex, it follows from Hölmstrom's result on efficient and strategy-proof mechanisms that (σ, t) must be a VCG mechanism. This implies

that the transfer is given by

$$(2) \qquad \forall i \in N : t_{(i)}(\theta) = - \sum_{j \neq (i)} \theta_{(j)} S_{(j)}(\sigma(\theta)) + h_{(i)}(\theta_{-(i)})$$

If we set $h_{(i)}(\theta_{-(i)}) = \sum_{j \neq (i)} S_{(j)} \theta_{(j)}(\sigma(\theta_{-(i)})) + g_{(i)}(\theta_{-(i)})$ in equation (2) we get

$$(3) \qquad \forall i \in N : t_{(i)}(\theta) = -s_{(i)} \sum_{j \in P'_{(i)}(\sigma(\theta))} \theta_{(j)} + g_{(i)}(\theta_{-(i)})$$

As the mechanism (σ, t) satisfies EE, SP and OE the following condition must hold

$$\forall i \in N : -\theta_{(i)} S_{(i)}(\sigma(\theta)) + t_{(i)}(\theta) = -\theta_{(i)} \bar{S} + \bar{t}(\theta)$$

Where the left side of the above equation is the utility from a VCG mechanism and the right hand side is the utility from EE requirement.The above expression can alternatively be written as

$$(4) \qquad \bar{t}(\theta) = -\theta_{(i)} S_{(i)}(\sigma(\theta)) - s_{(i)} \sum_{j \in P'_{(i)}(\sigma(\theta))} \theta_{(j)} + g_{(i)}(\theta_{-(i)} + \theta_{(i)} \bar{S}$$

Putting $i = 1$ into equation (4) we get,

$$\bar{t}(\theta) = -\theta_{(1)} S_{(1)}(\sigma(\theta)) - s_{(1)} \sum_{j \in P'_{(1)}(\sigma(\theta))} \theta_{(j)} + g_{(1)}(\theta_{-(1)}) + \theta_{(1)} \bar{S}$$

Similarly for $i = 2$ we have,

$$\bar{t}(\theta) = -\theta_{(2)} S_{(2)}(\sigma(\theta)) - s_{(2)} \sum_{j \in P'_{(2)}(\sigma(\theta))} \theta_{(j)} + g_{(2)}(\theta_{-(2)}) + \theta_{(2)} \bar{S}$$

Equating the expressions for $\bar{t}(\theta)$ we get, $-s_{(1)}\theta_{(1)} - s_{(1)}\theta_{(2)} + g_{(1)}(\theta_{-(1)}) = -\theta_{(2)}(s_{(1)} + s_{(2)}) + (s_{(1)} - s_{(2)}) \sum_{j \in P'_{(2)}(\sigma(\theta))} \theta_j + g_{(2)}(\theta_{-(2)}) - \bar{S}(\theta_{(1)} - \theta_{(2)})$.
Since $g_{(1)}(\theta_{-(1)})$ is independent of $\theta_{(1)}$ and $g_{(2)}(\theta_{-(2)})$ is independent of $\theta_{(2)}$ we get $g_{(1)}(\theta_{-(1)}) = (\bar{S} - s_{(2)})\theta_{(2)} + f_{(1)}(\theta_{N \setminus \{(1),(2)\}})$ and $g_{(2)}(\theta_{-(2)}) = (\bar{S} - s_{(1)})\theta_{(1)} + f_{(2)}(\theta_{N \setminus \{(1),(2)\}})$. Now comparing the expression for $\bar{t}(\theta)$ for $i = 1$ and $i = 3$ and using the expression of $g_{(1)}(\theta_{-(1)})$ we have,

$(\bar{S} - s_{(1)})\theta_{(1)} + (\bar{S} - s_{(2)})\theta_{(2)} + f_{(1)}(\theta_{N\setminus\{(1),(2)\}}) = -\theta_{(3)}(s_{(1)} + s_{(2)} + s_{(3)}) + s_{(1)}\theta_{(2)} + s_{(1)}\theta_{(3)} + (s_{(1)} - s_{(2)})\sum_{j\in P'_{(3)}(\sigma(\theta))}\theta_{(j)} + g_{(3)}(\theta_{-(3)}) + \bar{S}\theta_{(3)}.$

Comparing the expressions on both sides in the similar fashion we get $g_{(1)}(\theta_{-(1)}) = (\bar{S} - s_{(2)})\theta_{(2)} + \{\bar{S} - (s_{(2)} + s_{(3)})\}\theta_{(3)} + f'_{(1)}(\theta_{N\setminus\{(1),(2),(3)\}})$ and $g_{(3)}(\theta_{-(3)}) = (\bar{S} - s_{(1)})\theta_1 + \{\bar{S} - (s_{(1)} + s_{(2)})\}\theta_{(2)} + f_{(3)}(\theta_{N\setminus\{(1),(2),(3)\}}).$

By using the same argument recursively we get

$$g_{(1)}(\theta_{-(1)}) = \sum_{j\neq(1)}^{n} \{\bar{S} - S_{(j)}(\sigma(\theta_{N\setminus\{(1)\}}))\}\theta_{(j)} + k_{(1)}$$

In fact (it can easily be shown that) the above expression, holds not only for $i = 1$ but for all $i \in N$;

$$g_{(i)}(\theta_{-(i)}) = \sum_{j\neq(i)}^{n} \{\bar{S} - S_{(j)}(\sigma(\theta_{N\setminus\{(i)\}}))\}\theta_{(j)} + k_{(i)}$$

Now we further get $\forall i, j \in N$, $k_i = k_j = \bar{k}$ by using the above expression of $g_{(i)}(\theta_{-(i)})$ into $\bar{t}(\theta)$ in equation (4) and equating them. Hence

$$(5)\qquad \forall i \in N: \quad g_{(i)}(\theta_{-(i)}) = \sum_{j\neq(i)}^{n} \{\bar{S} - S_{(j)}(\sigma(\theta_{N\setminus\{(i)\}}))\}\theta_{(j)} + \bar{k}$$

Using the above expression of $g_i(\theta_{(-i)})$ in equation (4) we have, $\bar{t}(\theta) = \sum_{i\in N}\{\bar{S}\} - S_{(i)}(\sigma(\theta))\}\theta_{(i)} + \bar{k}$.

Therefore it follows that when $\bar{S}(\theta) = \bar{S}$, a mechanism satisfies EE, SP and OE only if the reference bundle for the profile θ i.e. $(\bar{S}(\theta), t(\theta))$ is of the form $\bar{t}(\theta) = \sum_{i\in N}\{\bar{S} - S_i(\sigma(\theta))\}\theta_i + \bar{k}$.

Sufficiency is fairly obvious, hence omitted.   ∎

Using the expression of $g_{(i)}(\theta_{-(i)})$ given by equation (5), we get the expression of $t_{(i)}(\theta)$ as follows:

$$(6)\qquad t_{(i)}(\theta) = \sum_{j\neq i}\{\bar{S} - S_{(j)}(\sigma(\theta))\}\theta_{(j)} + \bar{k}.$$

4. FEASIBILITY AND PAIR WISE WEAK GROUP STRATEGYPROOFNESS

Proposition 2. In a sequencing problem no mechanism satisfies OE, SP, EE and FSB.

Proof: For all $i \in N$ and for all $\theta = (\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(n)})$ we have $t_{(i)}(\theta) = \sum_{j \neq i} \{\bar{S} - S_{(j)}(\sigma(\theta))\} \theta_{(j)} + \bar{k}$. If FSB holds then for all $\theta = (\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(n)})$, $\sum_{i \in N} t_{(i)}(\theta) \leq 0$. Therefore, we have the following: $\sum_{i \in N} \{S_{(i)}(\sigma(\theta)) - \bar{S}\} \theta_{(i)} \geq n\bar{k}/(n-1)$. If $\bar{k} \geq 0$, consider the profile $\theta = (\theta_{(n)}, \theta_{-(n)})$ such that for all $j \neq n, \theta_{(j)} = 1$. Since, $\bar{S} = \sum_{i \in N} s_{(i)} = S_{(n)}$, we have $\sum_{i \in N} \{S_{(i)}(\sigma(\theta)) - \bar{S}\} \theta_{(i)} < n\bar{k}/(n-1)$. If $\bar{k} < 0$, consider the profile $\theta = (\theta_{(1)}, \theta_{-(1)})$ such that for all $j \neq 1, \theta_{(j)} = 1$ and $\theta_{(1)} = (1 + 2\bar{k}/(S_{(1)} - \bar{S}))$. Then as $\bar{S} = \sum_{i \in N} s_{(i)} = S_{(n)}$, we have $\sum_{i \in N} \{S_{(i)}(\sigma(\theta)) - \bar{S}\} \theta_{(i)} < n\bar{k}/(n-1)$.

Hence, FSB is violated. ∎

Remark 1. The consequence of the above proposition is, in case of sequencing problem, no mechanism satisfies OE, SP, EE and BB.

Proposition 3. Consider a sequencing problem such that $|N| > 2$. Then no mechanism satisfies OE, PWSP, EE.

Proof: If a mechanism (σ, t) satisfies EE, SP and OE then for all $\theta \in \mathfrak{R}_{++}^n$ the allocation of an agent $i \in N$ is given by $(\sigma(\theta), t_{(i)}(\theta) = \sum_{j \neq i} \{\bar{S} - S_{(j)}(\sigma(\theta))\} \theta_{(j)})$. Suppose the true waiting cost profile is $\theta = (\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(n)})$ is such that $\lambda_{(1)} > \lambda_{(2)} > \ldots > \lambda_{(n)} > 0$ where for all $i \in N$, $\lambda_{(i)} = \theta_{(i)}/s_{(i)}$. Consider $\forall i \in N$, $\theta'_{(i)} = \theta_{(i)} + \epsilon_1$ such that $\epsilon_1 = \min(s_{(j)}\lambda_{(j-1)} - \theta_{(j)})/2$, $j \in \{2, 3, \ldots, n\}$. Let agents (1) and (2) jointly mis-reports as $\theta'_{(1)} = \theta_{(1)} + \epsilon_1$ and $\theta'_{(2)} = \theta_{(2)} + \epsilon_1$. The basic idea is to construct a new profile , such that, under this new misreported profile relative queue position is unaltered. Notice that, under this new profile $\theta^* = (\theta'_{(1)}, \theta'_{(2)}, \theta_{(3)}, \theta_{(4)}, \ldots, \theta_{(n)})$, $t_{(2)}(\theta^*) > t_{(2)}(\theta)$ and $t_{(1)}(\theta^*) > t_{(1)}(\theta)$ since $\epsilon_1 > 0$ by construction. Hence, profitable group deviation exists for agents (1) and (2). Therefore, PWSP is impossible along with OE and EE. ∎

Remark 2. In a sequencing problem with exactly two agents the class of mechanisms that satisfy EE, OE and SP is also PWSP. That is, group manipulation is impossible in that case.

5. IDENTICAL COSTS LOWER BOUND(ICLB) AND EGALITARIAN EQUIVALENT VCG MECHANISM

Proposition 4. In case of two agents consider a mechanism that satisfies OE,SP,EE then it also satisfies ICLB if and only if $\bar{k} \geq -s_{(2)}\theta_{(1)}/2$

Proof: In two agent case a typical profile is $(\theta_{(1)}, \theta_{(2)}) \in \Re^2_{++}$. Hence, in the efficient allocation $\lambda_{(1)} \geq \lambda_{(2)}$, that is, $s_{(2)}\theta_{(1)} > s_{(1)}\theta_{(2)}$. For sequencing problem with two agents, the reference waiting time is $\bar{S} = s_{(1)} + s_{(2)}$.

ICLB is compatible with egalitarian equivalent VCG mechanism if for all $i \in N$ and for all $\theta = (\theta_{(1)}, \theta_{(2)}) \in \Re^2_{++}$, $U_{(i)}(\bar{S}, \bar{t}(\theta)) \geq C_{(i)}(\theta)$ or

$$
(7) \qquad -\bar{S}\theta_{(i)} + \sum_{j \neq i}(\bar{S} - S_{(j)})\theta_{(j)} \geq -(s_{(i)} + \sum_{j \neq i} \frac{s_{(j)}}{2})\theta_{(i)}
$$

Consider, $i = 1$. Then following equation (7) we have, $-\{s_{(1)} + s_{(2)}\}\theta_{(1)} + \{(s_{(1)} + s_{(2)} - s_{(1)}\}\theta_{(1)} + \{s_{(1)} + s_{(2)} - s_{(1)} - s_{(2)}\}\theta_{(2)} \geq -\{s_{(1)} + s_{(2)}/2\}\theta_{(1)}$. Solving the above equation we get, (i).. $\bar{k} \geq -s_{(2)}\theta_{(1)}/2$.

Similarly, for $i = 2$, following equation (7) we have, $-\{s_{(1)} + s_{(2)}\}\theta_{(2)} + \{(s_{(1)} + s_{(2)} - s_{(1)}\}\theta_{(1)} + \{s_{(1)} + s_{(2)} - s_{(1)} - s_{(2)}\}\theta_{(2)} \geq -\{s_{(2)} + s_{(1)}/2\}\theta_{(2)}$. Solving the above equation we get,(ii).. $s_{(2)}\theta_{(1)} + \bar{k} \geq -s_{(1)}\theta_{(2)}/2$.

Notice that, if (i) holds then (ii) holds trivially. Therefore, condition (i), that is, $\bar{k} \geq -s_{(2)}\theta_{(1)}/2$ is necessary and sufficient condition[4] for ICLB along with egalitarian equivalent VCG mechanism for two agents. ∎

Proposition 5. Consider a mechanism (σ, t) that satisfies OE, SP, EE if $\bar{k} \geq 0$ then it satisfies ICLB.

Proof: If a mechanism (σ, t) EE, OE, SP and ICLB then $\forall i \in N$, $\forall \theta \in \Re^n_{++}$ we have,

$$
(8)
$$

$$
U_{(i)}(\sigma(\theta)) - C_{(i)}(\theta) = \left(\sum_{q \in P'_{(i)}(\sigma(\theta))} s_q - \sum_{r \in P_{(i)}(\sigma(\theta))} s_r \right) \theta_{(i)} + \sum_{j \neq i}(\bar{S} - S_{(j)})\theta_{(j)} + \bar{k} \geq 0
$$

---

[4]It is the average negative externality that is imposed on the first served agent by the last served agent.

Note that, $\sum_{j \neq i}(\bar{S} - S_{(j)})\theta_{(j)} = \sum_{r \in P_{(i)}(\sigma(\theta))}(\bar{S} - S_r)\theta_r + \sum_{q \in P'_{(i)}(\sigma(\theta))}(\bar{S} - S_q)\theta_q$.

Also, $\sum_{r \in P_{(i)}(\sigma(\theta))}(\bar{S} - S_r)\theta_r = s_{(i)}\sum_{r \in P_{(i)}(\sigma(\theta))}\theta_r + \sum_{r \in P_{(i)}(\sigma(\theta))}\left(\sum_{q \in P'_{(i)}(\sigma(\theta))}s_q\right)\theta_r$

$+ \sum_{r \in P_{(i)}(\sigma(\theta))}\left(\sum_{m=r+1}^{(i)-1}s_m\right)\theta_r$. But, $s_{(i)}\sum_{r \in P_{(i)}(\sigma(\theta))}\theta_r - \theta_r\sum_{r \in P_{(i)}(\sigma(\theta))}s_r \geq 0$,

because agents with higher $\lambda_{(\cdot)}$ are placed in the earlier positions of the queue (since

allocation is outcome efficient). Therefore, $\left(\sum_{q \in P'_{(i)}(\sigma(\theta))}s_q - \sum_{r \in P_{(i)}(\sigma(\theta))}s_r\right)\theta_{(i)} +$

$\sum_{j \neq i}(\bar{S} - S_{(j)})\theta_{(j)} > 0$. Since $\bar{k} \geq 0$, therefore $U_{(i)}(\sigma(\theta)) - C_{(i)}(\theta) > 0$.

Hence ICLB holds. ∎

6. Egalitarian equivalent VCG mechanism revisited

This section is a diversion from the natural condition of sequencing problem. Ideally

there should be an one to one correspondence between reference position and refer-

ence waiting time that had been in the context of queuing problem (See Chun et al.

(2014)). The same is true with sequencing if and only if reference position is the last

position of the queue. But now we assume any positive reference waiting time is pos-

sible. We have already seen that if reference position is assumed to be constant then

egalitarian equivalent VCG mechanism is achievable. Now we ask the following ques-

tion: what if the reference position is explicitly a function of the type profile θ where

$\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathfrak{R}^n_{++}$? The answer that we have found is a sufficient one, al-

though not necessary. Our hunch about the necessary condition is that the reference

position function $\bar{S}(\theta)$ should be symmetric in nature, that is, $\forall \theta, \theta'$ where θ' is some

permutation of θ we need $\bar{S}(\theta) = \bar{S}(\theta')$ for symmetry.

Lemma 1. A mechanism (σ, t) satisfies EE, SP and OE only if $\forall i, j (i \neq j) \in N$, $\forall \theta \in$

$\mathfrak{R}^n_{++}: \ h_i(\theta_i) - h_j(\theta_j) = \bar{S}(\theta)(\theta_j - \theta_i)$.

Proof: The general form of VCG transfer is the followed form equation (2) and in this

case is of the following form.

$$(9) \qquad \forall i \in N : t_i(\theta) = -\sum_{j \neq i}\theta_j S_j(\sigma(\theta)) + h_i(\theta_{-i})$$

So a VCG mechanism is egalitarian equivalent if $\forall \theta \in \mathfrak{R}_{++}^n$ the following holds:

$$(10) \qquad \forall i \in N : -\theta_i S_i(\sigma(\theta)) + t_i(\theta) = -\theta_i \bar{S}(\theta) + \bar{t}(\theta)$$

Using equations (9) and (10) we have

$$-\theta_i S_i(\sigma(\theta)) - \sum_{j \neq i} \theta_j S_j(\sigma(\theta)) + h_i(\theta_{-i}) = -\theta_i \bar{S}(\theta) + \bar{t}(\theta)$$

$$(11) \qquad \qquad or \quad C(\sigma(\theta)) + h_i(\theta_{-i}) = -\theta_i \bar{S}(\theta) + \bar{t}(\theta)$$

where $C(\sigma(\theta))$ denotes the cost under efficient allocation when the type profile is θ. For any $\theta \in \mathfrak{R}_{++}^n$ and any $i \neq j \in N$, using equation (11) we get

$$(12) \qquad \qquad h_i(\theta_i) - h_j(\theta_j) = \bar{S}(\theta)(\theta_j - \theta_i).$$

Thus the lemma is proved.   ∎

Proposition 6. If $N = \{1, 2\}$, a mechanism (σ, t) satisfies EE, SP and OE only if $\bar{S}(\theta)$ is symmetric.

Proof: Consider, $\theta = (\theta_1, \theta_2)$ and $\theta' = (\theta_1', \theta_2')$ where $\theta_1' = \theta_2$ and $\theta_2' = \theta_1$. Using equation (12) we get the following: When the type profile is $\theta = (\theta_1, \theta_2))$ then

$$h_1(\theta_2) - h_2(\theta_1) = \bar{S}(\theta_1, \theta_2)(\theta_2 - \theta_1) ...(I)$$

and when the type profile is $\theta' = (\theta_1', \theta_2')$ then

$$h_1(\theta_1) - h_2(\theta_2) = \bar{S}(\theta_2, \theta_1)(\theta_1 - \theta_2)...(II)$$

Since equation (12) holds for all $\theta \in \mathfrak{R}_2^{++}$, from (I) and (II) we have $h_1(\theta_2) = h_2(\theta_1) = h(\bar{\theta})$ when $\theta_1 = \theta_2 = \bar{\theta}$. Hence, the functional form of $h_1(\cdot) = h_2(\cdot) = h(\cdot)$. Hence, the equation (12), in this case, can be rewritten as $h(\theta_2) - h(\theta_1) = \bar{S}(\theta_1, \theta_2)(\theta_2 - \theta_1)...(1)$ when $\theta = (\theta_1, \theta_2)$. If $\theta' = (\theta_1', \theta_2')$ then $h(\theta_1) - h(\theta_2) = \bar{S}(\theta_2, \theta_1)(\theta_1 - \theta_2)...(2)$. Form (1) and (2) we have, $\bar{S}(\theta) = \bar{S}(\theta')$. Hence, $\bar{S}(\theta)$ is symmetric.   ∎

Proposition 7. If $\forall \theta \in \mathfrak{R}_{++}^n$, $\bar{S}(\theta) = \Sigma(\prod_{(1 \leq t \leq n)} \theta_t^{k_t})$ where $\sum_{t=1}^n k_t = m \in \mathbb{N}$ (set of natural numbers) and $h_i(\theta_{-i}) = \Sigma(\prod_{(t \neq i)} \theta_t^{k_t'})$ where $\sum_{t \neq i} k_t' = (m+1)$ then a mechanism mechanism (σ, t) satisfies EE, SP and OE.

Proof: If $\forall \theta \in \mathfrak{R}^n_{++}$, $\bar{S}(\theta) = \Sigma(\prod_{(1 \le t \le n)} \theta_t^{k_t})$ where $\sum_{t=1}^n k_t = m \in \mathbb{N}$ and $h_i(\theta_{-i}) = \Sigma(\prod_{(t \ne i)} \theta_t^{k'_t})$ where $\sum_{t \ne i} k'_t = (m+1)$ then it can be easily verified that lemma (1) holds. Hence the proposition is proved. ∎

Remark 3. Consider $N = \{1, 2\}$. Assume $\forall \theta \in \mathfrak{R}^2_{++}$, $\bar{S}(\theta_{(1)}, \theta_{(2)}) = \sum_{i=1}^m (\theta_{(1)}^{(m-i)} \theta_{(2)}^{(i-1)})$ where $m \in \mathbb{N}$. Then with $h_1(\theta_{-1}) = \theta_2^m$ and $h_2(\theta_{-2}) = \theta_1^m$ we can see that egalitarian equivalent VCG mechanism is compatible.

Notice that, in particular, a sequencing problem with $s_1 = s_2 = s$ (=1 assumed in the literature of queuing) is also a queuing problem . Therefore, unlike Chun et al. (2014), in this situation $\bar{S}(\theta_1, \theta_2)$ can be of the above form that we have just mentioned and hence not of constant value.

7. CONCLUSION

In this paper we have analyzed sequencing problem from both incentive and normative approaches. We have identified unique class of VCG mechanisms that ensures egalitarian equivalence and we also have shown the possibility result with identical preference lower bound in that unique class of VCG mechanisms. Sequencing game imposes a stronger restriction on the possible set of "reference position", compared to queuing game and that in turn results into the failure of having a feasible VCG mechanism along with egalitarian equivalence.

Although found the necessary and sufficient condition for the above mentioned unique class of egalitarian equivalent VCG mechanism to satisfy identical costs lower bound when the number of participating agents is two, necessary condition for the same when the number of participating agent is more than two remains an open question.

Lastly, we contemplate the situation where the restriction that sequencing problem imposes reference position is overlooked, that is, we assume the almost no restriction (except the fact that it must be positive) reference waiting time and identify the class of VCG mechanism that is egalitarian equivalent.

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