Debasements and Small Coins: An Untold Story of Commodity Money

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Abstract

This paper draws quantitative implications for some historical coinage issues from an existing formulation of a theory that explains the society's demand for multiple denominations. The model is parameterized to match some key monetary characteristics in late medieval England. Inconvenience for an agent due to a shortage of a type of coin is measured by the difference between his welfare given the shortage and his welfare in a hypothetical scenario that the mint suddenly eliminates the shortage. A small coin has a more prominent role than small change. Because of this role, a shortage of small coins is highly inconvenient for poor people and, the inconvenience may extend to all people when commerce advances. A debasement may effectively supply substitutes to small coins in shortage. Large increase in the minting volume, cocirculation of old and new coins, and circulation by weight, critical facts constituting the debasement puzzle, emerge in the equilibrium path that follows the debasement.

JEL Classification Number: E40; E42; N13

Key Words: The debasement puzzle; Gresham's Law; Medieval coinage; Commodity money; Coinage; Shortages of small coins

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1 Introduction

Debasements of coins were not rare in medieval Europe. When a type of coin was debased, i.e., the precious-metal content in the coin was reduced, a person could take bullion or old coins of the type to a mint in exchange for new coins. Rolnick, Velde, and Weber [16] find that a debasement tended to induce unusually large minting volumes and, at least some of the time following the debasement, old and new coins cocirculated by weight; they refer to the finding as the debasement puzzle because people did not receive any additional inducement to bring old coins to the mint in exchange for new coins (people actually paid some minting fees). Interestingly, debasements were often considered and sometimes implemented following the public complaints about inconvenience caused by shortages of small coins. Such complaints were widely recorded, motivating a view that the small-coin provision is a big problem for commodity money (see Cipolla [3], Redish [12], and Sargent and Velde [19]).

What may be a suitable measurement for an individual's inconvenience due to a shortage of some coins? Would a debasement alleviate the shortage? Is the small-coin provision really a big problem for the historical commodity-money system? These issues and the debasement puzzle are the focus of our paper. Our starting point is a folk theory of the society's demand for multiple monetary objects. The theory consists of two ingredients: a wide range of transaction values and a burden of carrying a bulk of monetary objects. To elaborate, suppose there is only one type of coin. If that coin facilitates all transactions, then high-value transactions may require many coins; but if high-value transactions only require a few coins, then even one coin may be too big for low-value transactions.

We build our work on the formalization of the folk theory by Lee, Wallace, and Zhu [9]. We measure the inconvenience for an agent when a type of coin is not supplied by the mint as the change in his welfare from the scenario that the coin is in shortage to a hypothetical scenario that the coin is suddenly supplied by the mint; probably, a real person in history who complained about a shortage of some coins would get his sense of inconvenience from comparing his real experience with his experience in an analogous hypothetical scenario. The hypothetical scenario in the model is an unanticipated shock that adds the coin in shortage into the extant coinage structure. This treatment naturally extends to debasement: the sudden addition of the halfpenny is equivalent to a sudden debasement of the penny by 50%.

We parameterize the model to match key monetary characteristics of England in the fifteenth century. During this period, per capita holdings of silver in money varied but 35 grams may be a useful reference. Pennies (1d) were the mostly used coins; silver per penny declined over time but 1 gram is a good reference. Our baseline coinage consists of pennies, half groats (2d), groats (4d), and shillings (12d); we leave out halfpennies (1/2d) and farthings (1/4d) in the baseline structure as they were targets of public complaints. With these parameters, shortages of small coins are highly inconvenient for at least some agents for a simple reason: the supply of the halfpenny or farthing permits an agent to smooth his consumption by way of spreading his purchasing power previously contained in a penny and used in one transaction into two halfpennies or four farthings. In other words, adding coins smaller than the penny is beneficial because these small coins have a role more prominent than small change, conforming well to that halfpennies and farthings were quite valuable in late medieval England.[1]

The great benefit from small coins (i.e., inconvenience due to their shortage) applies to poor agents even if they spend money once a month; it applies to all agents if monetary transactions are somewhat frequent (e.g., people spend money at least twice a week). Remarkably, the prominent role of small coins may be consistent with frequent usage of large coins. In the model, debasing the penny by 50% has the similar welfare effect as adding the halfpenny; new coins draw agents to the mint and cocirculate with old coins by weight. Debasement, however, cannot ultimately resolve the small-coin problem as long as small coins remain full-bodied. For, as noted by Redish [12], a precious metal is practically indivisible, i.e., there is a practical lower bound on the precious-metal content in a full-bodied coin—a low-fineness coin is easy to counterfeit and a high-fineness but low-content coin is too small to carry. In fact, a coin like the farthing is largely impractical; the weight of a high-fineness farthing is around 0.4 grams while the weight of a modern U.S. cent is 2.5 grams.

Viewing full-bodiedness as a commitment device to prevent over-issuance of money, our study leads to the following story of the historical commodity-money system. Commercial advancement inevitably confers a prominent role on coins like farthings or smaller than farthings, though those small coins (if supplied) may only appear to be small change; thus indivisibility of precious metals is not only constraining but rather costly. The significant indivisibility-constraint cost may contribute to the experimentation with a variety of imperfect substitutes to full-bodied small coins by a society before an alternative commitment device emerged and the final triumph of fiat money after.[2]

Debasements and small coins have drawn a fair amount of attention through the influential monograph of Sargent and Velde [19], *The Big Problem of Small Change*. Sargent and Velde [19] adapt the cash-in-advance model of Lucas and Stokey [11] by replacing cash and credit goods with penny and dollar goods: penny goods can only be bought with pennies (small coins) while dollar goods can be bought with dollars (large coins) and pennies; a shortage of pennies is identified with a binding

[1] In 1490s, a whole pig would cost 33 pennies and one penny could buy 3.73 kg of salt, 3.56 kg of wheat, 1.20 kg of cheese, or 4.35 kg of wool; see Farmer [4, Tables 4, 7].

[2] The imperfect substitutes included billon coins, copper coins, pieces cut from coins, foreign coins with less metal content, etc.; see Redish [12, ch 4] for problems with billon coins and copper coins. The standard formula for the small-coin provision prescribes issuing token coins convertible to some precious metal; see Cipolla [3]. But convertibility needs commitment. When the state commitment was somehow in place, a society adopted presumably convertible token coins and large-denomination notes. The presumed convertibility finally phased out but the state commitment somehow keeps over-issuance in check.

penny-in-advance constraint, occurring when pennies depreciate relative to dollars.[3] In the Sargent-Velde model, a shortage is a demand-side problem; debasing the penny alleviates the shortage because the assumed circulation-by-tale enhances the agent's incentive to hold new pennies; and if a society finances the mint's operation, the small-coin problem is resolved because the zero minting fees eliminate all non-steady-state equilibria. In our model, a shortage is a supply-side problem; debasing the penny alleviates the shortage because new and old pennies circulate by weight and new pennies are smaller than old; and even if a society finances the mint's operation, the small-coin problem may not be resolved.

The rest of the paper is organized as follows. We set up the basic model in section 2. Quantitative results are presented in sections 3 and 4. We discuss the key finding from our model and the related literature in section 5. Section 6 concludes.

2 The basic model

We begin with the physical environment and the monetary institution. Time is discrete, dated as $t \geq 0$. There is a unit measure of infinitely lived agents. There are two stages per period. At the start of the first stage of period t, each agent knows his type at the period—he becomes a buyer or a seller with equal chance. Then agents visit a mint that produces monetary items, referred to as coins, from a durable commodity, called silver. Silver has a fixed stock M; it can also be costlessly converted into and back from a product, called jewelry. There are K types of coins and a unit of coin k contains $m_k > 0$ units of silver, $1 \leq k \leq K$. A unit of jewelry contains m_0 units of silver. Agents choose their wealth portfolios in silver at the mint by the way described below. There is an exogenous upper bound B on each agent's silver wealth. At the second stage, agents carry coins into a decentralized market where each buyer is randomly matched with a seller. In each pairwise meeting, the seller can produce a perishable good that can only be consumed by the buyer. Trading histories are private information, ruling out credits between the two agents. In the meeting, each agent's wealth portfolio is observed by his meeting partner and the buyer makes a take-it-or-leave-it offer.

Let $Y_t = \prod_{k=0}^{K}\{0, 1, ..., B/m_k\}$ so $y = (y_0, ..., y_K) \in Y_t$ represents an agent's generic portfolio of wealth in silver at period t, meaning that the agent holds y_0 units of jewelry and y_k units of coin k, $k \geq 1$. Coins may exist at the start of period 0; that is, $m_0\pi_0(y_0, 0, ..., 0)$ may be less than M, where π_0 is the distribution of wealth portfolios in silver among agents at the start of period 0. If the agent visits the mint with $y \in Y_t$, he can choose a portfolio from the set

$$\Gamma_t(y) = \{y' \in Y_t : m \cdot y' = m \cdot y\}, \tag{1}$$

where $m = (m_0, ..., m_K)$. Here and below, $a \cdot b$ denotes the inner product of vectors a and b. If the agent ends with y' at stage 1 and if he consumes $q_b \geq 0$ (when he is a

[3] As noted by Wallace [27], users of the Lucas-Stokey model usually do not interpret a binding cash-in-advance constraint as a shortage of cash.

buyer) and produces $q_s \geq 0$ (when he is a seller) at stage 2, then his realized utility at period t is

$$u(q_b) - q_s + v(m_0 y_0') - \gamma \cdot y'. \tag{2}$$

Here $\gamma = (\gamma_0, ..., \gamma_K)$, $\gamma_0 = 0$, and $\gamma_k = \gamma_C > 0$ is the disutility to carry a unit of coin k to the decentralized market; the utility functions u and v satisfy $u', v' > 0$, $u'' < 0, v'' \leq 0$, $v(0) = u(0) = 0$, and $u'(0) = \infty$. Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$.

There may be an unanticipated shock to the coinage structure $(m_1, ..., m_K)$. The shock is either a *structure* shock that adds some types of coins into the pre-shock coinage structure or a *debasement* shock that reduces silver content in each of some $J \leq K$ types of coins in the pre-shock coinage structure. The shock is introduced at the start of period 0 (at that time agents only hold coins in the pre-shock coinage structure). The debasement is represented by a one-to-one mapping $j \mapsto d(j)$ such that if $k = d(j)$ for some $j \in \{1, ...J\}$, then coin k is debased. If coin k is debased, the pre-shock coin is called *old coin k* and the post-shock coin is called *new coin k*; the mint does not provide old coin k any more; and old coins can be held up to period $\hat{t} < \infty$ but must be melted at period $\hat{t}$.

A few remarks on the above setup are in order. First, the separation of monetary and non-monetary uses of silver, which is borrowed by Velde and Weber [25], implies that the stock of money is endogenous and, in particular, depends on the coinage structure. Second, while we assume away minting fees, the debasement puzzle is preserved by a positive γ_C (it still incurs an extra cost for an agent to melt one penny in exchange for two halfpennies). Reflecting real people's tendency to avoid bulky wallets, a positive γ_C is a key ingredient in the denomination-structure model of Lee, Wallace and Zhu [9]. Third, a finite $\hat{t}$ for old coins to exit is a simple way to capture that in history, old coins did eventually disappear for a variety of reasons (lost, deteriorated, etc) not considered in our model.

Next we turn to the equilibrium conditions. In the post-shock economy, let Y_t, m, $\Gamma_t(y)$, and γ be defined the same way as in the pre-shock economy for a distinct K following the structure shock and for a distinct $(m_1, ..., m_K)$ following the debasement shock when $t \geq \hat{t}$. Following the debasement shock when $t < \hat{t}$, let m_k^o denote the amount of silver per old coin k, $Y_t = \prod_{k=0}^{K} \{0, 1, ..., B/m_k\} \times \prod_{j=1}^{J} \{0, 1, ..., B/m_{d(j)}^o\}$, $m = (m_0, ..., m_K, m_{d(1)}^o ..., m_{d(J)}^o)$,

$$\Gamma_t(y) = \{y' \in Y_t : m \cdot y' = m \cdot y, \ y_{d(j)}^{o\prime} \leq y_{d(j)}^o\}, \tag{3}$$

and $\gamma = (\gamma_0, ..., \gamma_K, \gamma_{d(1)}^o, ..., \gamma_{d(J)}^o)$ with $\gamma_{d(j)}^o = \gamma_C$ all $j \geq 1$. The equilibrium conditions are described by a same set of constructs for the pre-shock and post-shock economies, with the understanding that the suitable Y_t, m, $\Gamma_t(y)$, and γ are applied.

For each period t, the set of constructs consists of three probability measures on Y_t, denoted π_t, θ_t^b, and θ_t^s, and three value functions on Y_t, denoted w_t, h_t^b, and h_t^s. Here $\pi_t(y)$ is the fraction of and $w_t(y)$ is the value for agents holding the wealth portfolio y before agents know their period-t types; $\theta_t^a(y)$ is the fraction of and $h_t^a(y)$

is the value for buyers (sellers, resp.) holding y right after visiting the mint at t when $a = b$ ($a = s$, resp.) In terms of h_t^a, the portfolio-choice problem for an agent holding y at the mint can be expressed as

$$g(y, h_t^a) = \max_{y' \in \Gamma_t(y)} h_t^a(y') + v(m_0 y_0'),\ a \in \{b, s\}. \tag{4}$$

In terms of w_{t+1}, the trade in a pairwise meeting between a buyer with y_b and a seller with y_s solves the maximization problem

$$f(y_b, y_s, w_{t+1}) = \max_{(q,\iota)} u(q) + \beta w_{t+1}(y_b - \iota) \tag{5}$$

subject to $-q + \beta w_{t+1}(y_s + \iota) \geq \beta w_{t+1}(y_s)$ and $\iota \in L(y_b, y_s)$, where

$$\begin{aligned} L(y_b, y_s) = \{ & \iota \in Y_t : \iota = \iota_b - \iota_s, \iota_b, \iota_s \in Y_t, \iota_{b,0} = \iota_{s,0}, \\ & \text{and } \forall k \geq 1,\ \iota_{b,k} \leq y_{b,k}, \iota_{s,k} \leq y_{s,k} \} \end{aligned} \tag{6}$$

is the set of feasible coin transfers between the buyer and the seller. Given h_t^b and h_t^s, the function w_t satisfies

$$w_t(y) = 0.5 g(y, h_t^b) + 0.5 g(y, h_t^s). \tag{7}$$

As implied by the maximization problem in (5), the function h_t^s satisfies

$$h_t^s(y) = \beta w_{t+1}(y) - \gamma \cdot y. \tag{8}$$

Given w_{t+1} and θ_t^s, the function h_t^b satisfies

$$h_t^b(y) = \sum_{y'} \theta_t^s(y') f(y, y', w_{t+1}) - \gamma \cdot y. \tag{9}$$

Given π_t, the measure θ_t^a satisfies

$$\theta_t^a(y') = \sum_{y} \pi_t(y) \lambda_1^a(y'; y),\ a \in \{b, s\}, \tag{10}$$

for some $\lambda_1^a(.; y) \in \Lambda_1[y, h_t^a]$, where $\Lambda_1[y, h_t^a]$ is the set of measures that represent all randomizations over the optimal portfolios for the maximization problem in (4). Given θ_t^b and θ_t^s, the measure π_{t+1} satisfies

$$\pi_{t+1}(y) = \sum_{(y_b, y_s)} \theta_t^b(y_b) \theta_t^s(y_s) [\lambda_2(y; y_b, y_s) + \lambda_2(y_b - y + y_s; y_b, y_s)] \tag{11}$$

for some $\lambda_2(.; y_b, y_s) \in \Lambda_2[y_b, y_s, w_{t+1}]$, where $\Lambda_2[y_b, y_s, w_{t+1}]$ is the set of measures that represent all randomizations over the optimal transfers of coins for the maximization problem in (5) and $\lambda(y)$ is the proportion of buyers with y_b who leave with y after meeting sellers with y_s.

Definition 1 *In each of the pre-shock and post-shock economies, a monetary equilibrium is a sequence $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ that satisfies (4)-(11) all t and $\sum_{\{y \in Y_t : y_k = 0, k \geq 1\}} m_0[\theta_t^b(y) + \theta_t^s(y)] < 2M$ some t for a given π_0 and for the applicable $(Y_t, m, \Gamma_t(y), \gamma)$; a steady state is a tuple $(w, \theta^b, \theta^s, \pi)$ such that $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ with $(w_t, \theta_t^b, \theta_t^s, \pi_t) = (w, \theta^b, \theta^s, \pi)$ all t is a monetary equilibrium.*

For existence, let $m_* = \min_{k\geq 1} m_k$ and we maintain a simple sufficient condition

$$\frac{B - m_* - 0.5M}{B - m_*} u[\frac{\beta(v(B) - v(B - m_*))}{1-\beta}] > v(B) + \frac{\beta}{1-\beta}[v(B) - v(B - m_*)] + \gamma_C. \tag{12}$$

The condition in (12) says that the silver content m_* in the smallest coin is not too close to the upper bound B on silver wealth and that compared with some utility from consuming produced goods (the u term), the unit cost γ_C to carrying coins is not too great and the utility from jewelry (the v terms) is much limited; notice that there is no monetary equilibrium if $m_* = B$, γ_C is sufficiently great, or the jewelry utility is sufficiently large.

Proposition 1 *In each of the pre-shock and post-shock economies, there exists a monetary equilibrium for a given π_0 and there exists a monetary steady state.*

Proof. See the appendix. ■

3 Quantitative results

To conduct quantitative analysis, we set $M = 35$ and $m_0 = 60$ and let the baseline coinage structure be $(m_1, m_2, m_3, m_4) = (12, 4, 2, 1)$. These parameters are meant to approximate the monetary characteristics of England in the fifteenth century. One unit of silver in the model corresponds to 1 gram. As one penny contained around 1 gram of silver in the fifteenth century, coins 1 to 4 represent the shilling, groat, half groat, and penny, respectively. And $m_0 = 60$ is about 2 times of *troy ounce* (31 grams); a regular tablespoon weighs around 60 grams. With $(m_0, M) = (60, 35)$, agents turn out to hold most of the silver in coins so the per capita silver in money falls in the mid of the estimated range for England in the fifteenth century (see Allen [1, p. 607]). We set $B = 3M$. This upper bound on wealth in silver is not restrictive in that it is reached by a negligible measure of agents.[4]

We set the annual discount factor at 0.9. (This choice is consistent with the choice of Lee and Wallace [8] and Redish and Weber [15]; Lee and Wallace [8] motivate their choice by the study of Kimball [7], which suggests that medieval people have a lower discount factor than modern people.) The relatively low annual discount factor has little influence on our results and, in particular, as is clear below, a higher value of the annual discount factor would only imply a higher welfare cost of shortages of small coins. When people have F rounds of pairwise meetings per year, the discount factor is

$$\beta = 0.9^{1/F};$$

we use $F = 24$ as the baseline value.

[4] Under the baseline parameters, the measure is $4 \times 10^{-12}\%$ at the steady state.

We set $u(x) = x^{1-\sigma}/(1-\sigma)$ and $\sigma = 0.5$. (This choice is consistent with the choice of Lee and Wallace [8] and Redish and Weber [15], too.) We set $v(x) = \varepsilon x/F$. The literal interpretation of jewelry is luxury goods. While there is no obvious reference for the marginal utility of luxury goods, it seems reasonable to be conservative by choosing a small value. Indeed, in our model jewelry covers all silver in non-monetary use and, in history much of silver in this use was hoarded. But if ε is too small, the stock of money moves little following a shock to the coinage structure. We set $\varepsilon = 0.01$. With this value, one unit of silver in jewelry yields a utility equivalent to 0.04% of the steady-state per capita consumption per round and we can observe dependence of the stock of money on the coinage structure. The main patterns of the presented results hold when σ varies from 0.5 to 1, when ε varies from 0.001 to 0.05, and when v has some strict curvature.

The carrying cost of coins is the dominant factor that determines the minting volume following a debasement shock: a larger γ_C tends to induce a larger minting response. As ε, there is no obvious reference for γ_C. But smaller values for γ_C seem more preferable than larger; apparently, one exerts a tiny effort to carry a coin. We guide our choice of γ_C by examining values that are sufficiently close to zero (in both absolute and consumption-equivalent terms) and generate sufficient post-shock minting responses. We present our result at $\gamma_C = 10^{-5}$. This value is equivalent to 0.001% of the steady-state per capita consumption per round. The main patterns of the presented results hold when γ_C varies from 10^{-4} to 10^{-6}.

Given a shock, we compute a monetary steady state $(\tilde{w}, \tilde{\theta}^b, \tilde{\theta}^s, \tilde{\pi})$ in the pre-shock economy, a monetary steady state $(w, \theta^b, \theta^s, \pi)$ in the post-shock economy, and a monetary equilibrium $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ in the post-shock economy that starts with $\pi_0 = \tilde{\pi}$ and converges to $(w, \theta^b, \theta^s, \pi)$. In computation, our algorithm approximates that property by letting $w_T = w$ for a sufficient large T (often $T = 500$ serves the purpose). Details of all algorithms are given in the appendix. Two findings in computation are worth noting. First, Proposition 1 does not tell uniqueness of the monetary steady state in either the pre-shock or post-shock economy. But for any given parameter values we have tested, our algorithm always converges to the same steady state from a variety of initial conditions. Second, we cannot prove that the post-shock steady state is locally stable and even if it is, we cannot prove that there exists the post-shock monetary equilibrium with the desired property of $\lim_{t\to\infty}(w_t, \theta_t^b, \theta_t^s, \pi_t) = (w, \theta^b, \theta^s, \pi)$. In fact, for some given parameter values outside the aforementioned ranges, our algorithm fails to converge for the reason that $\tilde{\pi}$ is too far away from π.

In presenting our computed results, we use

$$\delta_p(y) \equiv w_0(y)/\tilde{w}(y) - 1 \tag{13}$$

to measure the *change in an individual agent's welfare* (expected lifetime utility) following the shock, where y is the agent's pre-shock portfolio; if $w_0(y) = w_0(y')$ whenever the two portfolios y and y' contain the same amount of silver, we use

$$\delta(z(y)) = \delta_p(y) \tag{14}$$

	m_k	0.25	0.5	1	2	4	6 / 8	12	60	Total
Baseline	Stock			1.000	1.182	3.942		28.80	0.075	35
	Circ.			0.486	0.033	0.007		$4e^{-12}$		0.527
	Mint.			0.252	0.395	0.453		0.310		1.409
Adding halfpenny	Stock		0.419	0.936	1.019	3.924		28.70	0.001	35
	Circ.		0.059	0.442	0.038	$1e^{-4}$		$4e^{-5}$		0.540
	Mint.		0.108	0.320	0.328	0.450		0.373		1.579
Adding halfpenny & farthing	Stock	0.250	0.357	0.905	0.991	4.104		28.39	0.001	35
	Circ.	0.061	0.054	0.433	0.043	0.007		$4e^{-4}$		0.599
	Mint.	0.065	0.126	0.323	0.342	0.432		0.318		1.607
Adding sixpence	Stock			1.000	0.526	2.624	1.973	28.80	0.075	35
	Circ.			0.491	0.025	0.009	0.009	$4e^{-12}$		0.536
	Mint.			0.253	0.262	0.708	0.689	0.310		2.221
Adding eightpence	Stock			1.000	1.182	0.017	7.827	24.90	0.075	35
	Circ.			0.489	0.031	0.002	$6e^{-6}$	$2e^{-11}$		0.522
	Mint.			0.252	0.395	0.010	0.974	1.063		2.694

Table 1: Pre-shock and post-shock steady states.

to measure the individual welfare change, where $z(y)$ is the amount of silver in the portfolio y. If the shock is a structure shock, these statistics measure the inconvenience for an individual agent when the coins added by shock are in complete shortage. If the shock is a debasement shock, these statistics measure the improvement for an individual agent due to the debasement. For comparison, we use

$$\Delta \equiv \pi \cdot w / \tilde{\pi} \cdot \tilde{w} - 1 \tag{15}$$

to measure the *change in the aggregate welfare*.[5] To emphasize, $(m_1, m_2, m_3, m_4) = (12, 4, 2, 1)$ is the pre-shock coinage structure and $F = 24$ in an exercise below unless indicated otherwise.

Structure shocks

Here we organize our results around four structure shocks, the *halfpenny, halfpenny-farthing, eightpence,* and *sixpence shocks* that add the halfpenny, halfpenny and farthing, eightpence, and sixpence, respectively, to the coinage structure. The eightpence, sixpence, halfpenny, and farthing are coins with 8, 6, 0.5, and 0.25 units of silver, respectively.

Table 1 provides an overview of the stocks, circulation volumes, and minting volumes of coins measured in silver units of the pre-shock steady state and the four

[5] An alternative aggregate statistic is $\tilde{\pi} \cdot w_0 / \tilde{\pi} \cdot \tilde{w} - 1$. In all our exercises, the two aggregate statistics are in the same order of magnitude. We focus on $\pi \cdot w / \tilde{\pi} \cdot \tilde{w} - 1$ because $\tilde{\pi} \cdot \tilde{w}$ ($\pi \cdot w$, resp.) is the ex-ante welfare for each agent in the pre-shock (post-shock, resp.) economy when he draws his initial portfolio from the distribution $\tilde{\pi}$ (π, resp.).

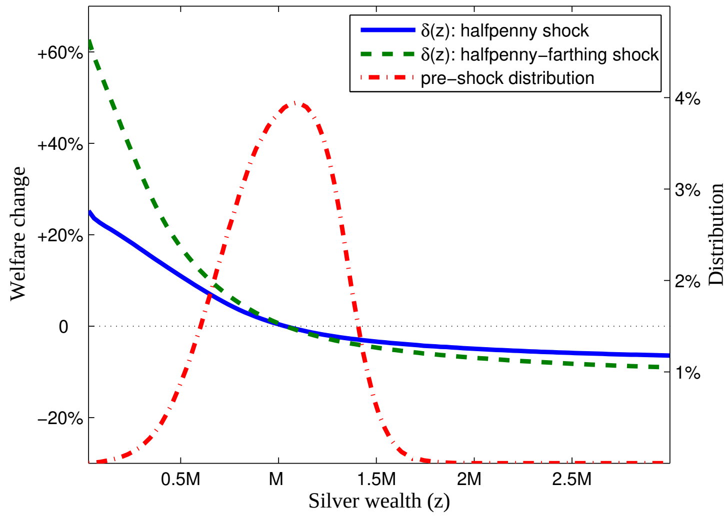

Figure 1: Left axis: changes in individual welfare ($\delta(z)$) under the halfpenny structure shock and the halfpenny-farthing structure shock. Right axis: steady-state distribution before the shocks.

post-shock steady states. A couple of remarks are in order. First, shillings and jewelry together absorb more than 70% of silver and almost all silver in this proportion is not used for the transactional purpose. This proportion does not vary much when we vary m_0 from 60 down to 30 but the split between shillings and jewelry may vary substantially. Second, while coins larger than pennies facilitate less than 3% of the total transaction values, they contribute to more than 70% of the total minting volume; that is, a larger minting volume need not imply that the corresponding coin is more useful in transactions.

The key statistic for each shock is the change in the individual agent's welfare $\delta(z)$ defined by (14). For the eightpence and sixpence shocks, these statistics are positive for all individuals but bounded above by 0.001%. The lifetime improvement of an agent who benefits the most from these shocks is offset by the costs to carrying 70 coins into the decentralized market once. So filling in the gap between the shilling and groat benefits everyone but no one would be bothered much if the gap is left there. For each of these two shocks, the change in the aggregate welfare Δ defined by (15) is around 0.001%, which is largely indicative of inconvenience felt by an individual agent when the coins in concern are in shortage.

Figure 1 displays the two $\delta(z)$ curves for the halfpenny and halfpenny-farthing shocks. The two curves share the very same patterns. The change in an agent's welfare is decreasing in his pre-shock wealth, agents in the poor side get great improvements, and agents in the rich side are worse off. For the halfpenny shock, $\Delta = 1.43\%$ and $\delta(z)$ ranges from 25.21% to -6.43%. For the halfpenny-farthing shock, $\Delta = 1.71\%$ and

$\delta(z)$ ranges from 62.66% to −9.02%. For these two shocks, the aggregate statistics highly underestimate inconvenience felt by poor people when coins in concern are in shortage. The difference between the two $\delta(z)$ curves in Figure 1 gives a measurement of the marginal effect from adding the farthing when the halfpenny is available. For an alternative measurement, we study the *alternative farthing shock* that adds the farthing to the coinage structure $(m_1, m_2, m_3, m_4, m_5) = (12, 4, 2, 1, 0.5)$. The aggregate statistics is $\Delta = 0.27\%$ and the $\delta(z)$ curve is very close the difference between the two $\delta(z)$ curves in Figure 1.

The patterns of $\delta(z)$ in Figure 1 may be explained by the consumption-smoothing effect and a countering effect. To see the former effect, suppose an agent spends one unit of the smallest coin when he is a buyer.[6] If his present wealth is z, then his lifetime utility can be written as

$$\sum_{t=1}^{z/m_*} \left(\frac{0.5}{1-0.5\beta}\right)^t \beta^{t-1} u(c_t), \tag{16}$$

where c_t is his consumption when his wealth is $z - (t-1)m_*$ ($m_* = \min_{k\geq 1} m_k$). One may interpret (16) as that the agent spreads his purchasing power over z/m_* periods. Suppose a shock does not affect the agent's purchasing power. But with a reduction in m_*, the agent benefits because he can spread his consumption over more periods. Because of discounting, a smaller z means a larger consumption-smoothing benefit. To understand the countering effect, note that the amount of goods received by a buyer is decreasing in his partner's reservation value when the buyer transfers the same amount of silver in the payment. Because consumption smoothing benefits all agents, it tends to raise that reservation value, which, in turn, reduces the buyer's surplus from trade. The countering effect may be the dominant one for rich agents as it may not vary much across agents.

The tale of two sides (due to shortages of small coins) in Figure 1 is of great interest. But when agents meet more frequently, i.e., F and β become larger, the consumption-smoothing effect may be strengthened to dominate the countering effect for people in the rich side. A larger F works through two channels. First, it weakens the influence of discounting. Secondly, as is shown in Table 1, when F is at the baseline value, pennies play the dominant role in transactions and, adding coins smaller than pennies has an observable but not dramatic effect on the usage of pennies. So the consumption pattern in (16) only applies to agents in the poor side after the shock. The larger F increases the measure of agents who spend one unit of the smallest coin in the decentralized market after the shock. In other words, the larger F leads to a larger proportion of agents who take full advantage of the spread of consumption permitted by the addition of coins smaller than pennies.

For a shock to induce a positive $\delta(z)$ curve, F needs to exceed some level that depends on the pre-shock m_*. When $F = 48$, the halfpenny shock yields $\Delta = 28.75\%$

[6] This may happen in equilibrium if β is sufficiently close to unity when $\gamma = 0$ and $m = (m_0, m_1) = (\infty, 1)$; see Camera and Corbae [2].

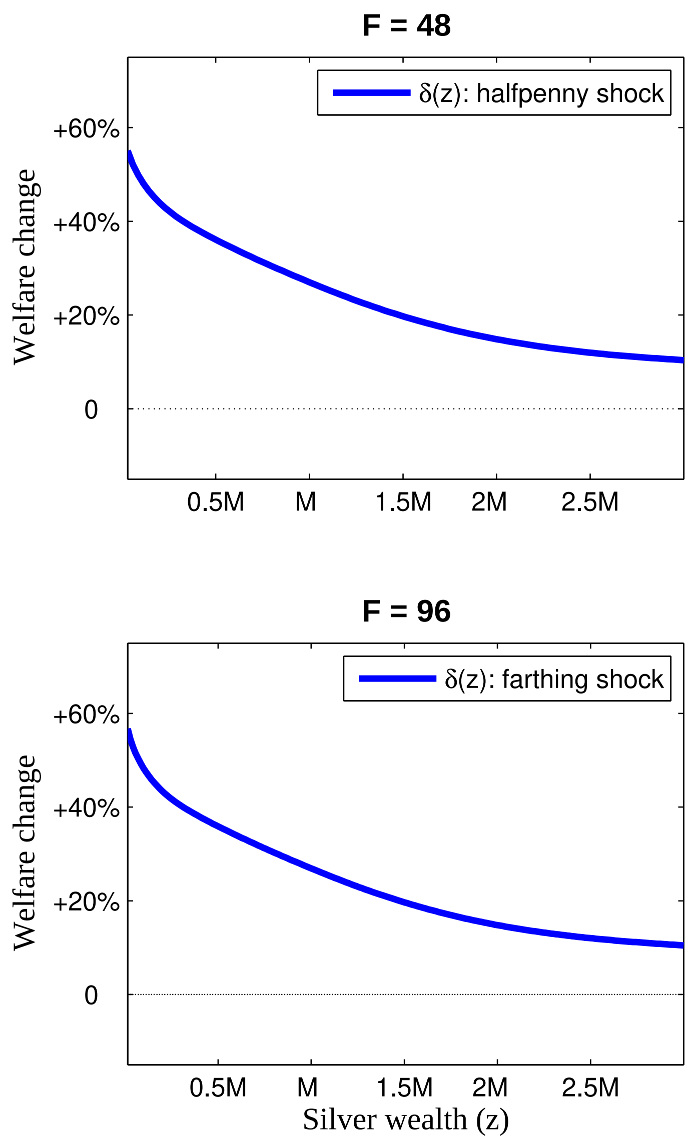

Figure 2: Changes in individual welfare ($\delta(z)$) under the halfpenny structure shock with $F = 48$ (upper); and the alternative farthing structure shock with $F = 96$ (bottom).

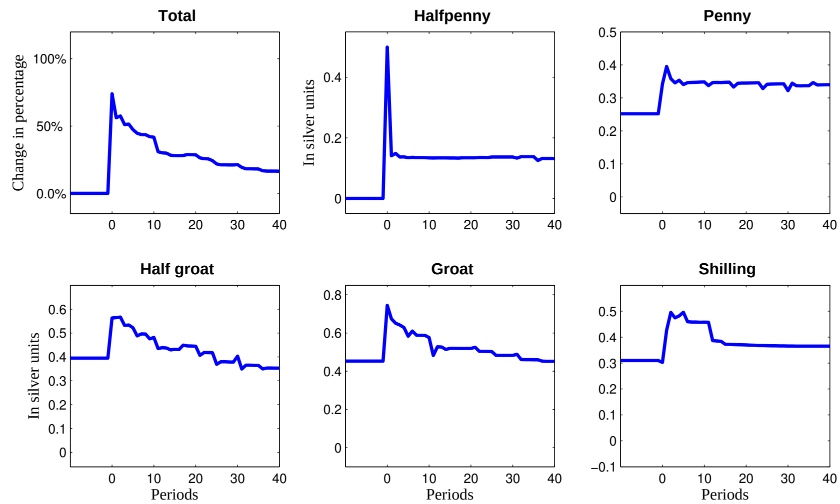

Figure 3: Minting volume responses following the halfpenny structure shock.

and the $\delta(z)$ curve in the upper row of Figure 2 that ranges from 55.06% to 10.37%; when $F = 96$, the alternative farthing shock yields $\Delta = 28.60\%$ and the $\delta(z)$ curve in the bottom row of Figure 2 that ranges from 56.76% to 10.48%. The new tale points to a universal unhappiness. In fact, the universal unhappiness prevails even when the farthing is available, as long as the trade is sufficiently frequent. To make the point, we study the *alternative halffarthing shock* that adds $m_7 = 0.125$ to the coinage structure $(m_1, m_2, m_3, m_4, m_5, m_6) = (12, 4, 2, 1, 0.5, 0.25)$. When $F = 240$, $\delta(z)$ ranges from 66.65% to 19.97% and $\Delta = 39.05\%$.

Structure shocks as special debasement shocks

A structure shock may be viewed as a special debasement shock; for example, the halfpenny shock is equivalent to a shock that debases the penny by 50% while mints the coin with 1 gram of silver as the zenny. From this perspective, we relate the minting and usage of coins in the post-shock equilibrium to the debasement puzzle. First, coins in the pre-shock and post-shock coinage structures cocirculate by weight following each shock and, one can see from Table 1 cocirculation even persists in the long run. Secondly, each shock induces large increases in the minting volume in the post-shock equilibrium (compared to the pre-shock steady state). The increases are presented in Figures 3 and 4 for the halfpenny and sixpence shocks, respectively. Aside from some details, the patterns in Figure 3 apply to the halfpenny-farthing shock and the patterns in Figure 4 apply to the eight-pence shock.

When halfpennies are added, the minting volume in the post-shock steady state increases by 12% and halfpennies contribute to more than 80% of that increase. The transition to the post-shock steady state is gradual. In each of the first 10 periods,

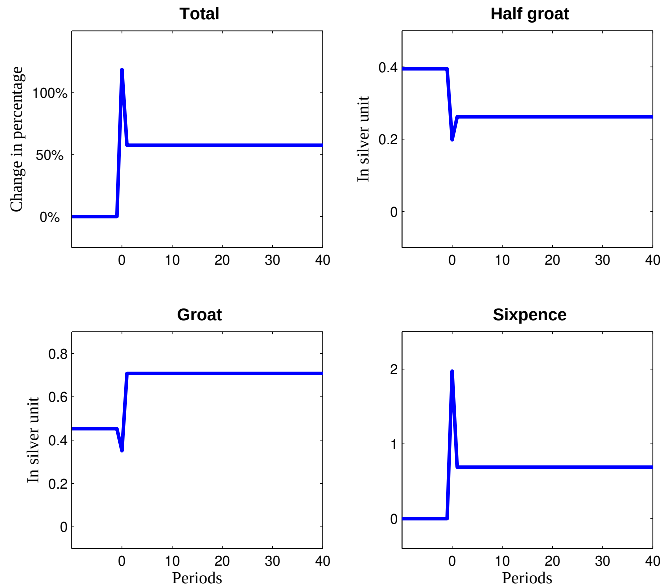

Figure 4: Minting volume responses following the sixpence structure shock.

the minting volume increases by more than 40% and all coins contribute substantially. This transitional process may be explained as follows. In equilibrium, a buyer holds one or two halfpennies for the transactional purpose but a seller holds a halfpenny only when his silver wealth is not an integer. When a buyer melts jewelry or other coins in exchange for halfpennies at period 0, he tends to have extra silver which can only be used to mint other coins. Because many agents need to adjust holdings of halfpennies after period 0, the extra minting of other coins due to the extra silver from minting halfpennies lasts for multiple periods.

When sixpences are added, the minting volume in the post-shock steady state increases by 57% and sixpences contribute to more than 80% of that increase. The transition to the post-shock steady state is almost instantaneous for two reasons. First, an agent can support the period-0 minting of sixpences by half groats, groats, and other coins that are used to mint half groats and groats in the pre-shock steady state; that is, his minting of sixpences at period 0 may only affect the minting of half groats and groats. Second, the agent's choice of sixpences, groats, and half groats at the mint is not sensitive to his type (because those coins are not frequently used in transactions); that is, there is little need for him to adjust holdings of these coins at period 1.

	m_k	0.5	1	2	4	6 / 8	12	60	Total
	Stock		1.000	1.182	3.942		28.80	0.075	35
Baseline	Circ.		0.486	0.033	0.007		$4e^{-12}$		0.527
	Mint.		0.252	0.395	0.453		0.310		1.409
Debasing	Stock	1.317		1.056	3.924		28.70	0.001	35
penny	Circ.	0.485		0.066	0.034		$1e^{-4}$		0.585
(by 50%)	Mint.	0.298		0.367	0.457		0.374		1.496
Debasing	Stock		1.000	0.204	3.911	29.81		0.075	35
shilling	Circ.		0.486	0.021	0.026	0.039			0.572
(by 50%)	Mint.		0.252	0.203	0.974	1.085			2.514
Debasing	Stock		1.000	1.182	2.038	30.70		0.075	35
shilling	Circ.		0.487	0.033	0.005	$6e^{-6}$			0.525
(by 33%)	Mint.		0.252	0.395	0.351	0.376			1.374

Table 2: Steady states before and after the debasement shocks.

Debasement shocks

Compared with a structure shock that is equivalent to a special debasement shock, a debasement shock has the feature that in the post-shock economy the mint does not supply coins with the same silver content as some old coins. As it turns out, this feature imposes a problem for our computation. That is, the values of old coins may be highly sensitive to the measures of old coins in circulation and, as a result, our algorithm may fail to converge. To deal with the problem, we need to choose a not very large $\hat{t}$ for old coins to exit. We present our results with $\hat{t} = 50$.

Our main interest is whether the indicated feature of a debasement shock may substantially alter the welfare effects and the post-shock minting and usage of coins that are observed from a corresponding structure shock. To this end, we study the *penny debasement shock* that debases the penny in the coinage structure by 50%, and the *shilling debasement shocks* that debase the shilling by 50% and 33%.

Table 2 summarizes the stocks, circulation volumes, and minting volumes of coins measured in silver units of the pre-shock steady state and the three post-shock steady states. A notable feature is that following the penny debasement, much of silver occupied by jewelry is released to coins even though a new penny contains a less amount of silver than an old penny (following each shilling debasement only a tiny amount of silver occupied by jewelry is released). In other words, the penny debasement makes holding silver in money more attractable than holding silver in jewelry.

Regarding the post-shock equilibrium outcomes, the penny debasement shock resembles the halfpenny structure shock and the two shilling debasement shocks resemble the sixpence and eightpence structure shocks in the welfare effects and minting activities. For all the debasement shocks, $\delta_p(y) \approx \delta_p(y')$ (see (13)) when $z(y) = z(y')$. When the penny is debased, $\Delta = 1.43\%$ and $\delta_p(y)$ ranges from 25.21% to -6.44%.

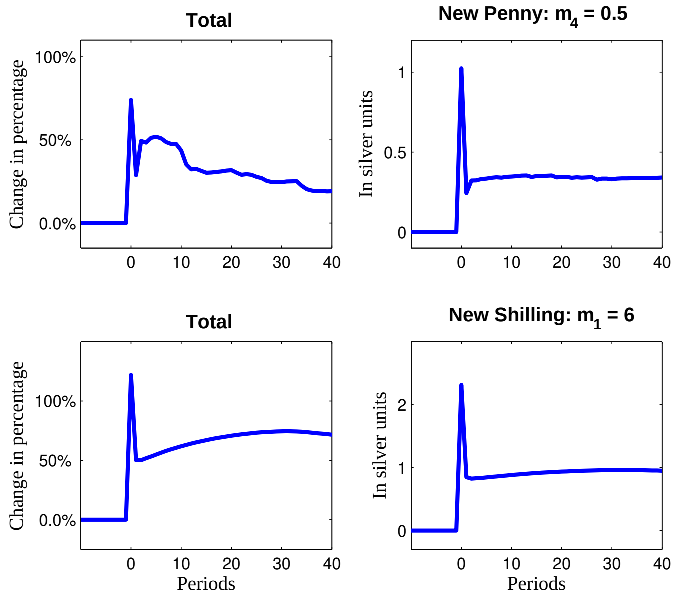

Figure 5: Minting volume responses following the debasement shocks. Upper row: debasing the penny from 1 to 0.5; bottom row: debasing the shilling with from 12 to 6.

When the shilling is debased, the values of $\delta_p(y)$ and Δ are all negative, with $\delta_p(y)$ bounded below by -0.008% and Δ by -0.007%; the negative (but insignificant) effects may be attributed to the fact that old shillings are a more convenient store of value than new shillings. Figure 5 presents increases in the minting volume following the penny debasement and following the 50% shilling debasement.

Following each debasement shock, we observe cocirculation of old and new coins before old coins exit. An interesting finding pertains to the difference between circulation of old shillings and circulation of old pennies. After the shilling is debased, old shillings get more and more circulated because people can only get this convenient store of value from the decentralized-market trade. After the penny is debased, old pennies get less and less circulated because new pennies are good substitutes and more and more old pennies are melted in exchange for new pennies. Figure 6 presents the different patterns when the penny is debased and when the shilling is debased by 50%.

Small-change problem and idiosyncratic shocks

The folk theory in introduction emphasizes the small-change role of small coins. In medieval documents, we may also see the complaints that people had to waste some trading opportunities because of the small-change problem. An often-cited petition to the England king in 1444 asserted that

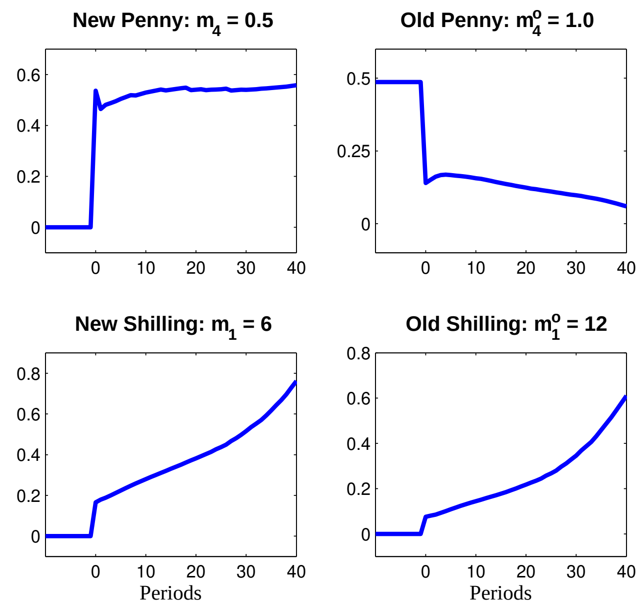

Figure 6: Circulations of coins following the debasement shocks. Upper row: debasing the penny from 1 to 0.5; bottom row: debasing the shilling from 12 to 6.

> people, which would buy such victuals and other small things necessary, may not buy them, for default of half pennies and farthings not had on the part of the buyer nor on the part of the seller. (Ruding [17, p. 275])

A structure shock that adds a small coin may be used to measure wasted trading opportunities as follows. If two agents in a meeting do not trade in the pre-shock steady state but they trade at period 0 in the post-shock equilibrium, then we say that two agents *waste the trading opportunity* in the pre-shock steady state. We use the mass of such meetings to measure wasted trading opportunities because of the small-change problem. In the above exercises, measures of wasted trading opportunities are pretty low. For example, it is 0.04% when we apply the halfpenny shock at the baseline F. This means that for a fixed pair of agents, if the buyer does not spend a penny in the pre-shock steady state, he tends to have no sufficient incentive to spend a halfpenny in the post-shock equilibrium. In other words, no trade in the pre-shock meeting is not so much because a penny is big (relative to a halfpenny).

One may naturally think that small coins would be more like small change if agents experience some idiosyncratic preference or technology shocks in pairwise meetings. Consider that the buyer's utility from consuming x in a meeting is $\alpha u(x)$, where $\alpha \in \{1, ..., \bar{\alpha}\}$ is the realization of an i.i.d. (meeting-specific) random variable at the start of the meeting. For illustration, we experiment with the uniformly-distributed idiosyncratic preference shock and let $\bar{\alpha} = 3$. Applying the halfpenny shock, we find that the wasted trading opportunities due to the shortage of halfpennies are 24%.

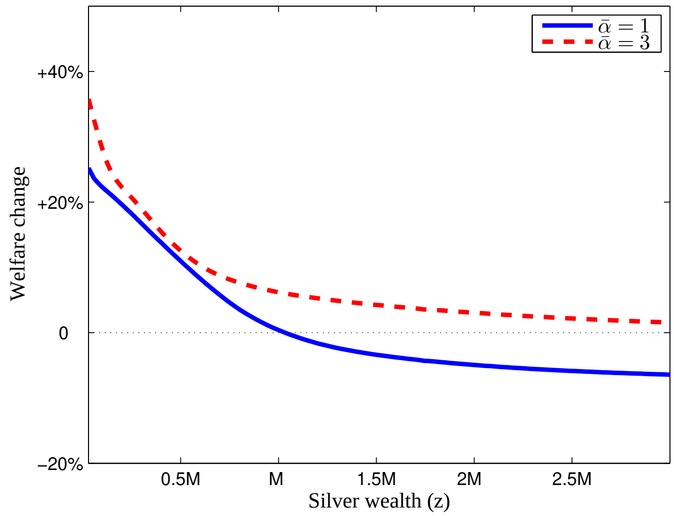

Figure 7: Changes in individual welfare($\delta(z)$) under the halfpenny structure shock with idiosyncratic preference shocks. $\bar{\alpha} \in \{1, 3\}$.

Figure 7 displays the $\delta(z)$ curve for the halfpenny shock (the dashed line). Compared with the version absent of idiosyncratic shocks ($\bar{\alpha} = 1$), the addition of the halfpenny has a much strengthened effect.

To get a better sense of this experiment, we examine pairwise meetings in the pre-shock steady state that agents trade when buyers are more eager to consume (α is large) but do not when buyers are less eager (α is small). The mass of those meetings is about 25%. In such a meeting, the buyer skips the present trading opportunity, anticipating a higher payoff in a future meeting where he is a buyer and becomes more eager to consume. But after the halfpenny is added, the buyer who is less eager to consume tends to have a sufficient incentive to spend the halfpenny to buy some goods from the seller. One may interpret that agents transact with halfpennies in meetings when buyers are less eager to consume as that halfpennies play a small-change role in those transactions.

4 High meeting frequency and usage of large coins

In the last section, we find that the individual inconvenience due of a shortage of small coins increases as the meeting frequency F increases. One may interpret a rise in F as a consequence of commercial advancement. When F increases, agents have a strong tendency to use the smallest coins. For example, with idiosyncratic preference shocks, we observe that buyers revert to halfpennies in most transactions and skip trading opportunities when they are less eager to consume in the above exercise of the halfpenny shock after F rises to 96. That is, when F increases to a certain level, there is little room for small coins to be small change and, in the meanwhile, there is little room for large coins to circulate. But little circulation of large coins would cast some doubt on the finding pertaining to the large welfare loss due to shortages of small coins, motivating us to study the following setup of pairwise meeting.

In this setup, the seller produces in a meeting at t multiple goods by a sequential order and subject to some random breakdown. Specifically, the meeting consists of N phases, indexed by $n \in \{1, ..., N\}$. Conditional on that the two agents stay together at the start of phase n, the sequence of events in that phase is the following. First, both agent observe the realization of an i.i.d. separation shock; the realization takes the value 0 with probability ρ_n and 1 with probability $1 - \rho_n$. Next, each agent chooses to stay or to voluntarily depart. If either agent voluntarily departs, then both agents are separated (implying no production and no consumption) at phases n to N; otherwise, the seller can produce a good that is consumed by the buyer at the phase. Finally, at the end of the phase, if neither agent has voluntarily departed but the realization of the separation shock is equal to 1, then the two agents are separated at the remaining phases of period t; otherwise, they stay together. We let $\rho_n < 1$ for $N > n \geq 1$, and $\rho_N = 0$. The buyer's utility from consuming the bundle $(c_1, ..., c_n)$, $n \leq N$, is $\sum_{i=1}^{n} u(c_i)$; the seller's disutility from producing the bundle is $\sum_{i=1}^{n} c_i$.

The sequential randomness in the meeting captures two scenarios of the buyer-seller relationship with realistic relevance. In one, a period is short and goods in the meeting are physically distinct (e.g., a helper may clean the house, farm the land, prepare food, etc. for his employer within a day); in another, a period may be longer and goods in the meeting are time-indexed goods (e.g., the helper may clean the house for the employer each day in a multiple-day period); and in either scenario, some random event may terminate the buyer-seller relationship before the end of the period (e.g., the helper may be sick after the land farming or after the first-day house cleaning).[7]

We generalize the buyer's take-it-or-leave-it offer in the basic model as follows. The buyer makes an offer before the phase-1 separation shock is realized and the offer is in form of $(c_1, ..., c_N, \iota_1, ..., \iota_N)$. Here, $c_n \geq 0$ is the seller's production of phase-n goods if the agents stay together at the time when the seller can produce at phase n; and $\iota_n \in L(y_b, y_s)$ (see (8)) is the buyer's payment if the buyer and seller are separated (voluntarily or not) after the seller has produced the consumption bundle $(c_1, ..., c_n)$. In terms of w_{t+1}, the offer made by a buyer with y_b when meeting a seller with y_s solves the optimization problem

$$\max_{(c_1,...,c_N,\iota_1,...,\iota_N)} \sum_{n=1}^{N} \mu_n \left[\sum_{i=1}^{n} u(c_i) + \beta w_{t+1}(y_b - \iota_n) \right] \tag{17}$$

[7]In the latter scenario, we may follow Shi [22] to assume that the buyer needs a consumption device to consume and he surrenders the device to the seller at phase n as a collateral if he comes back at phase $n + 1$. In a more complete version for this scenario, there may be another round of matching in the period among agents who depart from their meetings by the end of phase $n < N$.

subject to $\iota_n \in L(y_b, y_s)$ and

$$-c_n + \beta w_{t+1}(y_s + \iota_n) \geq \beta w_{t+1}(y_s + \iota_{n-1}) \tag{18}$$

for $1 \leq n \leq N$, and

$$-c_n + \sum_{j=n+1}^{N} \lambda_{n,j} \left[-\sum_{i=n+1}^{j} c_i + \beta w_{t+1}(y_s + \iota_j) \right] \geq \beta w_{t+1}(y_s + \iota_{n-1}) \tag{19}$$

for $1 \leq n \leq N-1$, where $\iota_0 = 0$, $\mu_n = 1 - \rho_n$ for $n = 1$, $\mu_n = (1-\rho_n)\prod_{i=1}^{n-1}\rho_i$ for $1 < n \leq N-1$, $\mu_n = \prod_{i=1}^{n-1}\rho_i$ for $n = N$, $\lambda_{n,j} = 1 - \rho_j$ for $j = n+1$, $\lambda_{n,j} = (1-\rho_j)\prod_{i=n+1}^{j-1}\rho_i$ for $n+2 \leq j \leq N-1$, and $\lambda_{n,j} = \prod_{i=n+1}^{j-1}\rho_i$ for $j = N$ (when $n \leq N-2$). The constraint (18) means that the seller does not voluntarily depart if the realization of the separation shock of phase n is 1 (separation); the constraint (19) means that the seller does not depart if the realization is 0 (no separation).

We make three observations about the optimization problem (17). First, one can verify by induction that (18) implies (19), i.e., (18) effectively represents the seller's participation constraint.[8] This simplification of the participation constraint eases some computational burden and is the reason for us to assume that the phase-n separation shock is realized before the seller's phase-n production. For the second observation, suppose the buyer's phase-1 offer $(c_1^*, ..., c_N^*, \iota_1^*, ..., \iota_N^*)$ solves the optimization problem (17). Suppose after $(c_1^*, ..., c_{n-1}^*)$ has been carried out and the agents stay together at the start of phase n, the buyer can change the part of the offer that has not been carried out prior to the realization of the phase-n separation shock subject to the seller's participation constraint. Observe that the buyer's choice is $(c_n^*, ..., c_N^*, \iota_n^*, ..., \iota_N^*)$. Hence we may alternatively assume that the buyer makes an offer $(c_n, ..., c_N, \iota_n, ..., \iota_N)$ at each phase n before the phase-n separation shock is realized. The third observation is that μ_n is the ex-ante probability that the buyer consumes the bundle $(c_1^*, ..., c_n^*)$ and pays ι_n^*. As the payment ι_n^* is roughly proportional to n, small coins do not dominate in transactions while the small-coin problem due to a strong consumption-smoothing effect persists.

To illustrate, we experiment with $F = 72$, $N = 3$, and $\rho_1 = \rho_2 = 0.9$; one interpretation is that people meet once every 5 days and 10% of meetings lasts for

[8] First, note $\lambda_{N-1,N} = 1$ and (19) at $n = N-1$ is

$$-c_{N-1} + [-c_N + \beta w_{t+1}(y_s + \iota_N)] \geq \beta w_{t+1}(y_s + \iota_{N-2}),$$

which is implied by (18) at $n = N$ and (18) at $n = N-1$. Next, notice that

$$\begin{aligned} &\sum_{j=n}^{N} \lambda_{n-1,j} \left[-\sum_{i=n}^{j} c_i + \beta w_{t+1}(y_s + \iota_j) \right] \\ &= (1-\rho_n)\left[-c_n + \beta w_{t+1}(y_s + \iota_n)\right] + \rho_n \sum_{j=n+1}^{N} \lambda_{n,j} \left[-\sum_{i=n+1}^{j} c_i + \beta w_{t+1}(y_s + \iota_j) \right]. \end{aligned}$$

Hence (19) at $n-1$ is implied by (18) at n, (18) at $n-1$, and (19) at n.

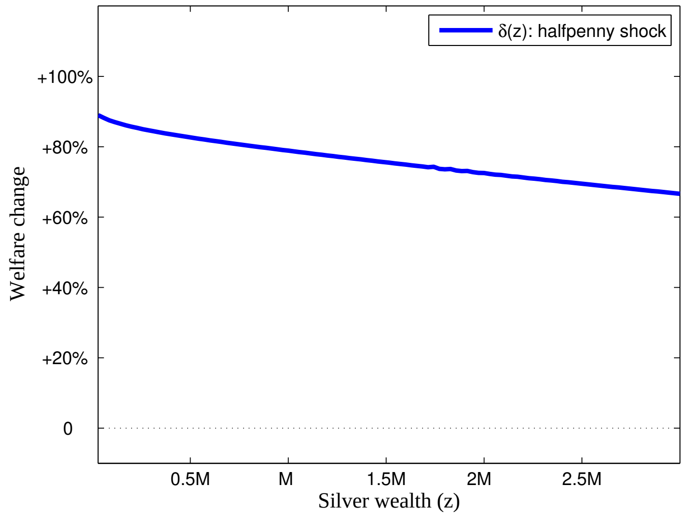

Figure 8: Changes in individual welfare ($\delta(z)$) under the halfpenny structure shock, with $F = 72$, $N = 3$ and $\rho_1 = \rho_2 = 0.9$.

1 day, 9% for 2.5 days, and 81% for 5 days. The steady-state nominal GDP is around 96 pence per year;[9] the circulation volumes of the penny, half groat and groat, respectively, are 0.431, 0.599 and 0.481 units of silver in each period of trade; and the shilling is purely a store of value. Figure 8 displays $\delta(z)$ under the halfpenny shock. We may obtain other distributions of coins used in transactions by varying N and $\{\rho_n\}$ without affecting the welfare implications.

5 Discussion

Here we discuss first the main finding in our model and next the related literature.

[9] For the part of history in concern, the annual nominal GDP per capita in England fell in the range from 200 to 400 pence. Likely, even commerce had advanced in a late medieval economy, a substantial portion of GDP was not realized through the market transactions and, there was some intrinsic heterogeneity that permitted a small class of people to procure a large proportion of GDP. Our model may be better interpreted as the part of economy that excluded that small class of people and excluded the non-market transactions. With this interpretation, the annual nominal GDP at 100 pence seems a reasonable target; if a household has 5 members, this means that the household annually receives monetary incomes around 500 pence, which may be close to the historical data. In general, our model can match any pre-set nominal GDP level. Indeed, if we double the meeting frequency F, we double the nominal GDP.

Our model

We follow Lee, Wallace, and Zhu [9] closely in setting up our basic model. Attractiveness of this modelling choice is that the Lee-Wallace-Zhu model itself is built on a model not first designed for the historical coinage issues. Indeed, if we set $\gamma = 0$ (zero carrying costs), $\varepsilon = 0$ (fiat money), and $m = (m_0, m_1) = (\infty, 1)$ (one denomination), then the basic model turns into a version of the familiar model of Trejos and Wright [23] and Shi [21] (general money holdings with an arbitrary upper bound and take-it-or-leave-it offers by buyers). With some minimal departure from the plain version of the Trejos-Shi-Wright model, our model delivers a rich set of implications for the coinage issues in concern.

The main finding is the high individual inconvenience or welfare loss due to shortages of small coins. The critical parameter for this result is the silver stock M. Our choice of M is explained above. A local change in M, say, from 35 to 40, does not affect the relevant numbers much. We have not found a way that can efficiently redo all the above exercises for a large change in M, say, from 35 to 100; the computational burden increases dramatically in some exercises. To give some idea of what may happen for a larger M, we note that the alternative farthing shock above is almost identical to the structure shock that adds the halfpenny to the coinage structure $(m_1, m_2, m_3, m_4, m_5) = (24, 8, 4, 2, 1)$ when $M = 70$, and the alternative halffarthing shock above is almost identical to the structure shock that adds the halfpenny to the coinage structure $(m_1, m_2, m_3, m_4, m_5, m_6) = (48, 16, 8, 4, 2, 1)$ when $M = 140$.[10]

Our model likely exaggerates the welfare loss because a medieval person's consumption did not all come from monetary transactions. Suppose monetary transactions only contributed to one third of the consumption. Suppose the consumption of goods and services from monetary transactions entered into the person's utility function as an object distinct from the consumption of goods and services from other means (e.g., barter, credits, and self production). Then, one may discount a welfare number by 2/3 to get a more realistic estimation, which is still quite significant.

Our model also misses two important aspects of minting in a late-medieval European economy. First, medieval mints charged people to cover the labor and material costs and collect seigniorage. The minting fees would contribute to shortages of small coins. For, it was much more costly to produce farthings than shillings; thus, given the minting fees permitted by kings, mints might not produce farthings as demanded (see Redish [12, p. 113]). Second, it would be costly for medieval people to visit mints and, hence, people would not visit mints frequently. A low frequency to visit mints implies a low frequency to adjust portfolios, which would tend to reinforce the inconvenience due to shortages of small coins.

Our model can be extended to accommodate these two aspects. For the minting

[10] Also, when $M = 140$, $m = (m_0, m_1) = (\infty, 1)$ and $\gamma = 0$, we work on the shock that reduces m_1 from 1 to 0.5; the $\delta(z)$ curves for $F = 24$ and $F = 120$ are similar to those in Figures 1 and 2, respectively.

fees, one may assume that each agent incurs some amount of disutility to obtain a unit of coin and that there is an upper bound on the aggregate minting volume for each type of coin. The bound may be exogenous but it can be endogenous as the mint's optimal choice; either way, a binding bound describes partial shortages of small coins in history. For the mint-visiting cost, one may assume that each agent incurs some amount of disutility to play a lottery at the start of a date, that the realization of the lottery determines whether he visits the mint or not, and that the probability to visit the mint depends on the amount of disutility.[11]

Although we do not see a reason for either extension to overturn the main finding, we do anticipate new insights from these extensions. Because the state spaces increase dramatically, these extensions are much more challenging to analyze and left for the future work.

The related literature

In the economic literature, a few papers study shortages of coins or small coins with matching models. Wallace and Zhou [28] study a model with a unit upper bound on money holdings and with some agents less productive than other; they identify a shortage of coins with the concentration of wealth in steady state. Kim and Lee [6] compare the steady-state aggregate welfare in a model with one sort of coins in fiat money with the steady-state aggregate welfare in a commodity money version of that model; they identify a shortage of small commodity-money coins with a part of the welfare difference contributed by that commodity-money coins are more valuable than fiat-money coins. Lee and Wallace [8] compare the steady-state aggregate welfare in a model with one sort of coins in fiat money by varying the size of the coin; they include the cost of maintaining monetary objects in their analysis; and they conclude that medieval Europe might set the size of the penny right (we suspect that poor agents in their model should get great improvements if the size of the penny is reduced).

Redish and Weber [13] study a model with multiple monetary objects. They build a bimetallic system into the model of Lee, Wallace, and Zhu [9]: gold and silver coins, respectively, are large and small coins while quantities of gold and silver coins are fixed because gold and silver are distinct metals and because there is no jewelry. Focusing on steady-state comparison, Redish and Weber [13] identify a shortage of small coins with the improvement in the steady-state aggregate welfare when small coins are added. While their model is similar to ours, their parameterization is different. Most importantly, they work with much higher degrees of indivisibility of money and much

[11]Bimetallism was typical in late medieval Europe: gold was largely used in high-value transactions and silver was mostly used in the daily life. The carrying cost of monetary objects may play a more significant role in bimetallism as a trade facilitated by gold may have a much higher nominal value in silver units. To study bimetallism, we may follow Wallace and Zhou [28] by assuming that there are two types of agents who permanently differ in productivity as sellers. Intuitively, agents with high productivity may tend to use gold coins. The equilibrium outcomes in this extension may much depend on the details of how two types of people interact.

lower levels of meeting frequency: averages of coin holdings (small plus large) are no more than 10 and the value of β is 0.9 (i.e., one meeting per year when the annual discount factor is 0.9). While a low average of coin holdings tends to strengthen the welfare loss due to shortages of small coins, the low meeting frequency appears powerfully enough to prevent the effect from standing out in their exercises.[12]

As Kim and Lee [6], Lee and Wallace [8], and Redish and Weber [13], we aim to draw quantitative implications of indivisibility of money for the historical commodity-money system from similar models.[13] Different from these authors, we go beyond steady-state comparison, which is necessary to quantify an individual's inconvenience due to a shortage of some coins. In terms of results, our contribution lies in demonstrating that the individual inconvenience can be strikingly strong under parameters that do not exaggerate the degree of indivisibility; relating the inconvenience to the consumption-smoothing role of small coins and commercial advancement; showing that the inconvenience can persist even when large coins are substantially used in transactions; and offering an explanation for the debasement puzzle.

Sargent and Velde [19] have ignited the recent interest to small coins in specific and commodity money in general. We have indicated above the difference between their work and ours. While it is not our point to argue what may be a superior approach to modelling money,[14] it seems fair to say that we are able to say things differently because our model is better suited to accommodate indivisibility of a precious metal.

There is a small economic literature that tackles the debasement puzzle. In a cash-in-advance model, Sargent and Smith [18] assume that new and old coins circulate by tale. Under this assumption, agents bring all old coins into the mint in exchange for new coins. On the empirical ground, Rolnick, Velde, and Weber [16] argue that by-tale circulation violates facts documented in the debasement puzzle and that by-tale circulation would have induced a much larger minting volume than observed (the data indicates that only a portion of old coins were recoined). In matching models

[12] Redish and Weber [13] assume that there is a probability for a meeting to be a non-trade meeting, i.e., the buyer may not want the good produced by the seller; they vary the trading probability in some exercises but that variation seems not enough to offset the influence of the low meeting frequency (the trading probability is bounded above by unity). In a related working paper, Redish and Weber [14] study the essentially same model as we study. They again focus on steady-state welfare comparison, consider two sorts of silver coins, and apply parameters with a low average holdings of coins and a low frequency of meeting in their exercises. The findings in that working paper are largely in line with findings in Redish and Weber [13].

[13] By setting values of some parameters to zeros, one may turn all these models into the version of Trejos-Wright-Shi model studied by Zhu [29]. While it is proved in Zhu [29] that different degrees of indivisibility imply different real allocations, no much further has been established analytically by the literature.

[14] A reader may observe that jewelry in our model resembles bonds in some cash-in-advance model; that is, jewelry has a higher rate of return than coins but is assumed to be illiquid. This is a valid observation. Nonetheless, different from the Sargent-Velde model, our model does not impose a coin-specific constraint for any sort of coin and is able to endogenously generate different degrees of liquidity for different coins. Moreover, our main finding holds even if jewelry does not yield direct utility (i.e., if money is fiat).

with one unit upper bound on coin holdings, Velde, Weber, and Wright [26] and Li [10] use side payments offered by the mint as incentives for people to bring in old coins in exchange for new coins at a one-to-one rate. None of these three models is suitable to study the demand for multiple monetary objects.

All three models concern Gresham's law. Partly rooted in medieval debasements, Gresham's law says that bad money (new coins) drives out good money (old coins).[15] The renowned law has long been known to be ambiguous at best. On the theoretical ground, it relies on the circulation-by-tale assumption that is effectively imposed from the outset; on the empirical ground, there are numerous counterexamples (see Rolnick and Weber [15] and Velde [24]). In each of the three models, Gresham's law is not universal but good money is driven out by bad money at some parameter space because of asymmetric information (Velde, Weber, and Wright [26]), the government transaction policy (Li [10]), or the circulation-by-tale assumption (Sargent and Smith [18]). In our model, some old coins are melt (i.e., some good money is driven out) but other are kept (up to the exit period $\hat{t}$) following each debasement shock we study; the reason is in line with the aforementioned folk theory.

6 Concluding remarks

Commodity money occupies most part of the monetary history. Compared with the prevailing fiat-money system, the historical commodity-money system is primitive in that its monetary service seems much constrained by the physical properties of precious metals such as scarcity, portability, divisibility, and recognizability. Conventionally thought to be critical, these properties are hard to place in models that many economists are used to and, hence, far from sufficiently explored. Through an off-the-shelf model, our paper demonstrates that the practical indivisibility of precious metals may imply a significant cost for the historical commodity-money system. On the flip side, our paper suggests a reason for one to reconsider commodity money. After all, technological progress would equip an advanced system with a money-making commodity that is practically divisible (e.g., bitcoins) and has all other nice physical properties. Of course, there is always an opportunity cost for a commodity-money system (see Sargent and Wallace [20] and Velde and Weber [25]) and one should ask whether it is worth paying the cost for the commitment to no over-issuing money.

[15] Fetter [5] describes how Gresham's law was reformed in the nineteenth century from a comment on debasements made by Gresham in 1558.

Appendix

A Proof of Proposition 1

The proof applies the standard fixed point argument. For existence of an equilibrium for a given π_0, it is routine to (i) construct a set S that is compact in the product topology and an element of which is a sequence $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$, (ii) construct a mapping F from S to S that is implied by the definition of equilibrium and whose fixed points are equilibria, and (iii) verify that all conditions for the application of Fan's fixed-point theorem are satisfied. So there exists an equilibrium. To show that this equilibrium is a monetary equilibrium, suppose by contradiction the opposite. Without loss of generality, suppose that some agent holds silver wealth B at date 0 and all his wealth is in jewelry. Consider two options of this agent when he is a buyer: minting one unit of the smallest coin and no minting any coin. For the first option, his expected payoff is bounded below by

$$-\gamma_C + (1 - \frac{0.5M}{B - m_*})u[\frac{\beta\,(v(B) - v(B - m_*))}{1-\beta}] + \frac{\beta}{1-\beta}v(B - m_*).$$

Notice that $1 - 0.5M/(B - m_*)$ is a lower bound on the measure of sellers whose wealth levels in silver do not exceed $B - m_*$ and the agent can receive at least $\beta\,(v(B) - v(B - m_*))\,/(1-\beta)$ amount of the good from such a seller. For the second option, his expected payoff is $v(B)/(1-\beta)$. But then (12) implies the first option has a higher payoff, a contradiction. Existence of a monetary steady state can be proof by essentially the same argument.

B Numerical algorithms

B.1 Computing a steady state

To begin with, vectorize the $K+1$-state space into a one-dimensional state, and define the value vectors $\{w, g\}$ and distribution vectors $\{\theta, \pi\}$, $\theta = (\theta^b, \theta^s)$, accordingly. Denote the total possible number of states as S.

1. Begin with an initial guess $\{w^0, h^0, \theta^0, \pi^0\}$, where π^0 and θ^0 are consistent with the total silver stock M.

2. Given end-of-stage-1 value h^i and beginning-of-stage-1 distribution π^i from i-th iteration, solve the problem (4), and use the solution to update beginning-of-stage-1 value w^{i+1} and end-of-stage-1 distribution θ^{i+1} .

3. With w^{i+1} and θ^{i+1}, solve the problem as described in (5). Record the terms of trade of each relevant pairs, and update h^{i+1} and π^{i+1} accordingly.

4. Repeat step 2-3 until the convergence criterion is satisfied: $\|w^{i+1} - w^i\| < 10^{-6}, \|h^{i+1} - h^i\| < 10^{-6}$ and $\|\theta^{i+1} - \theta^i\| < 10^{-8}, \|\pi^{i+1} - \pi^i\| < 10^{-8}$.

B.2 Computing a post-shock equilibrium

The computation for the transition path is essentially about iterations on the series of $\Psi \equiv \{w_t, h_t, \theta_t, \pi_{t+1}\}_{t=1}^T$, $h_t = (h_t^b, h_t^s)$ and $\theta_t = (\theta_t^b, \theta_t^s)$, where T is the number of periods it takes for the economy to reach a new steady state. Before computing the transition paths, we first need to compute the post-shock steady state using an algorithm similar to B.1, with the change that choice of portfolios containing old coins are eliminated at the minting stage. Denote this steady state as $\{w_T, h_T, \theta_T, \pi_{T+1}\}$. We also have to translate the distribution from the pre-shock steady state, into the beginning distribution in the debasement environment, denote the beginning distribution as π_1.

1. Take an initial guess $\Psi^0 \equiv \left\{w_t^0, h_t^0, \theta_t^0, \pi_{t+1}^0\right\}_{t=1}^T$, with $w_T^0 = w_T$.

2. Start from the last period T. Given w_T and θ_T^i, solve the pairwise bargaining problem as described in (5), and get h_T^i. Record the implied Markov transition matrix as Λ_T^i. Use h_T^i and π_T^i, solve the problem of minting, and get w_{T-1}^i accordingly. Record the implied Markov transition matrix as Υ_T^i. Then use w_{T-1}^i and θ_{T-1}^i, repeat the previous procedure for problems in period $T-1$. Finally, we will have a new series $\{w_t^i, h_t^i\}_{t=1}^T$. And then use $\{\Lambda_t^i, \Upsilon_t^i\}_{t=1}^T$ and π_1 and generate a new series of distributions $\left\{\pi_t^{i+1}, \theta_t^{i+1}\right\}_{t=1}^T$.

3. Now use $\left\{\pi_t^{i+1}, \theta_t^{i+1}\right\}_{t=1}^T$ and w_T, repeat Step 2 and get $\left\{\pi_t^{i+2}, \theta_t^{i+2}\right\}_{t=1}^T$.

4. Repeat 2-3 until the convergence criterion is met: $\max_t \left(\left\|\pi_t^{i+1} - \pi_t^i\right\|\right) < 10^{-8}$, $\max_t \left(\left\|\theta_t^{i+1} - \theta_t^i\right\|\right) < 10^{-8}$, $\max_t \left(\left\|w_t^{i+1} - w_t^i\right\|\right) < 10^{-6}$, and $\max_t \left(\left\|h_t^{i+1} - h_t^i\right\|\right) < 10^{-6}$.

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